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A Speech Act Calculus

A Pragmatised Natural Deduction Calculus and its Meta-theory

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Comments welcome!

Building on the work of PETER HINST and GEO SIEGWART, we develop a pragmatised natural deduction calculus, i.e. a natural deduction calculus that incorporates illocutionary operators at the formal level, and prove the equivalence between the consequence relation for the calculus and the classical model-theoretic consequence relation.

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Introductory Remarks

In this text[1], we build on the works of PETER HINST and GEO SIEGWART on the pragmatisation of natural deduction calculi[2] and develop a (classical) speech act calculus[3] of natural deduction that has the following properties: (i) Every sentence sequence $\mathfrak{H}$, which here means: every sequence of assumption- and inference-sentences, is not a derivation of a proposition (i.e. a closed formula) from a set of propositions or there is exactly one proposition Γ and exactly one set of propositions X such that $\mathfrak{H}$ is a derivation of Γ from X, this being determinable for every sentence sequence without recourse to any meta-theoretical means of commentary.[4] (ii) The classical first-order model-theoretic consequence relation is equivalent to the consequence relation for the calculus.

Developing the calculus, we presuppose the grammatical framework of pragmatised first-order languages, which has been developed by PETER HINST and GEO SIEGWART, and supplement it with some additional concepts (1). Then the concept of the availability of propositions is established: In contrast to the calculi developed by HINST and SIEGWART, the formulation of the speech act rules for this calculus does not take recourse to a de-

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1 This text is basically a translation of our German paper: Ein Redehandlungskalkül. Ein pragmatisierter Kalkül des natürlichen Schließens nebst Metatheorie. Version 2.0. Online available at http://hal.archives-ouvertes.fr/hal-00532643/en/.

2 Pragmatised natural deduction calculi are natural deduction calculi that incorporate illocutionary operators at the formal level: For each speech act governed by the calculus (i.e. making an assumption or drawing an inference) there is a specific type of illocutionary operator, called performator, whose application to a proposition yields a sentence (i.e. an assumption or an inference sentence). These performators and the sentences that result from their application to propositions are part of the language of the respective calculus and their use in speech acts is governed by the rules of the respective calculus. Pragmatised calculi thus allow for the formal treatment of the linguistic practice of uttering derivations. More generally, the framework of pragmatised languages developed by HINST and SIEGWART allows for a formal treatment of all kinds of speech acts and linguistic practices. See HINST, P.: *Pragmatische Regeln*, *Logischer Grundkurs*, *Logik*, and SIEGWART, G.: *Vorfragen*, *Denkwerkzeuge* and, in English and most recent, *Alethic Acts*.

3 Our use of the expression 'speech act calculus' (German: Redehandlungskalkül) to designate pragmatised natural deduction calculi follows SEBASTIAN PAASCH.

4 Note that we regulate the predicate '.. is a derivation of .. from ..' in such a way that the set of propositions mentioned at the third place is identical to the set of assumptions which actually occur in the sentence sequence that is named at the first place and which are not eliminated in that sequence. If one regulates the predicate so that the set of propositions named at the third place has to be a superset of the set of assumptions that actually occur in the respective sentence sequence and are not eliminated there, which is not unusual either, the calculus accordingly ensures that every sentence sequence $\mathfrak{H}$ is either not a derivation of a proposition from a set of propositions or that there is a proposition Γ and a set of propositions X, such that for every proposition Δ and set of propositions Y one has: $\mathfrak{H}$ is a derivation of Δ from Y iff $\Delta = \Gamma$ and $X \subseteq Y$.

pendence relation between sets of propositions and propositions, but to an availability relation between propositions, sequences of sentences and positions (natural numbers in the domain of sequences). The concept of availability is inspired by the idea that all propositions in a subproof except the conclusion of the subproof should not be available after the subproof has been closed, which is implemented, for example, in the KALISH-MONTAGUE calculus.[5] Here, however, only subproofs that aim at conditional introduction (CdI), negation introduction (NI) or particular-quantifier elimination (PE), are treated in this way and the calculus is established in such a way that neither graphic means nor meta-theoretical commentaries have to be used: Which propositions are available in a given sentence sequence can be unambiguously determined without recourse to any kind of commentary (2).

Next the Speech Act Calculus is established. As is usual for pragmatised natural deduction calculi, the calculus contains a rule of assumption, which allows one to assume any proposition, and two rules for every logical operator, one regulating its introduction and the other one its elimination. Except for the rule of identity introduction (II), which allows the premise-free inference of self-identity propositions, the introduction and elimination rules always demand that suitable premises have already been gained, i.e. are available. So, for example, the rule of conditional elimination (CdE) allows one to infer Γ if one has already gained Δ and ⌜Δ → Γ⌝, i.e. if Δ and ⌜Δ → Γ⌝ are available. Propositions are gained or made available by being inferred or assumed. One gains a proposition Γ departing from an assumption if this assumption is the last one that has been made before gaining Γ and that is still available.

Three of the rules, CdI, NI and PE, allow one to discharge assumptions one has made: If one has gained a proposition Γ departing from the assumption of a proposition Δ, then one may infer ⌜Δ → Γ⌝ and thus discharge the assumption of Δ (CdI); if one has gained propositions Γ and ⌜¬Γ⌝ departing from the assumption of a proposition Δ, then one may infer ⌜¬Δ⌝ and thus discharge the assumption of Δ (NI), if a particular-quantification ⌜⋁ξΔ⌝ is available and one has gained a proposition Γ departing from the representative instance assumption [β, ξ, Δ], then one may infer Γ and thus discharge the representative

[5] See KALISH, D.; MONTAGUE, R.; MAR, G.: *Logic*. See also LINK, G.: *Collegium Logicum*, p. 299–363.

instance assumption (PE). The discharge of the respective initial assumptions is achieved as each application of CdI, NI and PE closes the whole subproof beginning with the respective assumption. One consequence of this is that the respective initial assumptions are not any more available, but it also makes the intermediate conclusions drawn during the subproof unavailable as premises – these intermediate conclusions only served the purpose of preparing the application of the respective rule and have been gained under the respective assumption. If the assumption is not any more available, then neither should any propositions that one was only able to gain under this assumption be available. One may reflect on this using the example of the pair Γ and ⌜¬Γ⌝ that has to be gained to prepare the application of NI.

After the establishment of the calculus, a derivation and a consequence concept for the calculus are established. A sequence of sentences $\mathfrak{H}$ will then be a derivation of a proposition Γ from a set of propositions X if and only if $\mathfrak{H}$ can be uttered in compliance with the rules of the calculus, Γ is the proposition of the last member of $\mathfrak{H}$ and X is the set of the assumptions available in $\mathfrak{H}$. Accordingly, a proposition Γ will then be a deductive consequence of a set of propositions X if and only if there is a derivation of Γ from a $Y \subseteq X$ (3).

The reflexivity, closure under introduction and elimination, transitivity as well as other properties of the deductive consequence relation have to be shown in order to prepare the proof of the adequacy of the then established concept of deductive consequence (4). Subsequently, a version of the classical model-theoretic consequence concept that fits the grammatical framework is established (5). Then the correctness and the completeness of the deductive consequence concept relative to this model-theoretic concept of consequence are shown (6). We conclude with some remarks on ways to elaborate on the approach taken here (7).

In the development of the calculus, we assume an established set or class-set theory, such as ZF or NBG(U). Since we do not want to restrict our meta-theory to a purely set-theoretical framework, we sometimes have to stipulate additional properties – such as, for example, $X \in \{X\}$ – that are trivial within a pure set theory, but informative within a class-set-theory. The development and meta-theoretical analysis of the Speech Act Calculus employ common set-theoretical and meta-logical instruments and techniques, which are presented in the works listed in the references.

A note concerning the use of this document: All entries in the table of contents link to the respective chapters and are bookmarked. Moreover, all cross-references as well as all mentions of postulates, definitions, theorems and speech-act rules link to the respective item.

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1 Grammatical Framework

The Speech Act Calculus and its meta-theory are developed for denumerable pragmatised first-order languages.[6] To simplify the following presentation, we suppress any reference to specific languages, or, more precisely, we assume an arbitrary but fixed language of this kind with a denumerably infinite vocabulary, the language L. First, the vocabulary and syntax of L are to be specified (1.1). Then the substitution operation is to be developed and some theorems on substitution are to be proved (1.2).

1.1 Vocabulary and Syntax

L is supposed to be an arbitrary, but fixed representative of languages of the desired kind with a denumerably infinite non-logical vocabulary. However, the calculus also works for languages with finitely many descriptive constants. Since L is not an actually constructed language, it is now just stipulated that a suitable vocabulary and a suitable concatenation operation for expressions exist. Which vocabulary is chosen in particular cases or how it is constructed (and how it is set-theoretically modelled, e.g. with recourse to subsets of $\mathbb{N}$ in NBG or ZF, or described, e.g. with recourse to axiomatically characterised (sets of) urelements in NBGU) is left open. The same holds for the concatenation operation for expressions: It is left open how this concatenation operation is established, e.g. with recourse to finite sequences or in some other way. The first postulate demands the existence of suitable sets of basic expressions for the vocabulary of L:

Postulate 1-1. *The vocabulary of* L *(CONST, PAR, VAR, FUNC, PRED, CON, QUANT, PERF, AUX)*

The following sets are well-defined, pairwise disjunct and do not have Ø as an element:

(i) The denumerably infinite set CONST = $\{c_i \mid i \in \mathbb{N}\}$, where for all $i, j \in \mathbb{N}$ with $i \neq j$: $c_i \neq c_j$ and $c_i \in \{c_i\}$, (*the set of individual constants; metavariables:* α, α', α*, …),

(ii) The denumerably infinite set PAR = $\{x_i \mid i \in \mathbb{N}\}$, where for all $i, j \in \mathbb{N}$ with $i \neq j$: $x_i \neq x_j$ and $x_i \in \{x_i\}$, (*the set of parameters; metavariables:* β, β', β*, …),

(iii) The denumerably infinite set VAR = $\{x_i \mid i \in \mathbb{N}\}$, where for all $i, j \in \mathbb{N}$ with $i \neq j$: $x_i \neq x_j$ and $x_i \in \{x_i\}$, (*the set of variables; metavariables:* ξ, ζ, ω, ξ', ζ', ω', ξ*, ζ*, ω*, …),

(iv) The denumerably infinite set FUNC = $\{f_{i.j} \mid i \in \mathbb{N}\setminus\{0\}$ and $j \in \mathbb{N}\}$, where for all $i, k \in \mathbb{N}\setminus\{0\}$ and $j, l \in \mathbb{N}$ with $(i, j) \neq (k, l)$: $f_{i.j} \neq f_{k.l}$ and $f_{i.j} \in \{f_{i.j}\}$, (*the set of function con-*

[6] See the literature mentioned in footnote 2. For a rigorous development oft the grammatical framework see especially HINST, P.: *Logik*, ch. 1.

stants; metavariables: φ, φ', φ*, …),

(v) The denumerably infinite set PRED = {=} ∪ {$P_{i.j}$ | $i \in \mathbb{N}\backslash\{0\}$ and $j \in \mathbb{N}$}, where {=} ⊄ {$P_{i.j}$ | $i \in \mathbb{N}\backslash\{0\}$ and $j \in \mathbb{N}$} and for all $i, k \in \mathbb{N}\backslash\{0\}$ and $j, l \in \mathbb{N}$ with $(i, j) \neq (k, l)$: $P_{i.j} \neq P_{k.l}$ and $P_{i.j} \in \{P_{i.j}\}$, (*the set of predicates; metavariables:* Φ, Φ', Φ*, …),

(vi) The 5-element set CON = {¬, →, ↔, ∧, ∨} (*the set of connectives; metavariables:* ψ, ψ', ψ*, …),

(vii) The 2-element set QUANT = {⋀, ⋁} (*the set of quantificators; metavariables:* Π, Π', Π*, …),

(viii) The 2-element set PERF = {Suppose, Therefore} (*the set of performators; metavariables:* Ξ, Ξ', Ξ*, …), and

(ix) The 3-element set AUX = {(} ∪ {)} ∪ {,} (*the set of auxiliary symbols*).

The meta-theoretical expressions by which the elements of the sets PERF and AUX are *designated* will also be *used* as meta-theoretical performators and auxiliary symbols, the same holds for the identity predicate. To avoid confusion and to enhance intuitive readability, we will therefore use quasi-quotation marks ('⌜', '⌝') if object-language expressions are to be designated. μ, τ, μ', τ', μ*, τ*, … serve as general metavariables for object-language expressions. The vocabulary of L is now simply defined as the set of the sets postulated in Postulate 1-1:

Definition 1-1. *The vocabulary of L (VOC)*
VOC = {CONST, PAR, VAR, FUNC, PRED, CON, QUANT, PERF, AUX}.

The syntax of L contains the categories of terms, quantifiers, formulas and sentences according to the definitions found below. First, however, the set of basic expressions is established:

Definition 1-2. *The set of basic expressions (BEXP)*
BEXP = ⋃VOC.

Now, we demand the existence of a suitable operation with which we can concatenate expressions to form larger expressions. As already remarked above, the way in which this operation is constructed in particular cases is left open. To do this, we first regulate the concatenation of basic expressions, and then, after defining the set of expressions and the expression length function, we regulate the general concatenation of arbitrary expressions.

Postulate 1-2. *Concatenation of basic expressions*[7]

The concatenation of expressions expressed by juxtaposition is well-defined and it holds that:

(i) For all k, $j \in \mathbb{N}\setminus\{0\}$: If $\{\mu_0, \ldots, \mu_{k\text{-}1}\} \subseteq$ BEXP and $\{\mu'_0, \ldots, \mu'_{j\text{-}1}\} \subseteq$ BEXP, then: $\ulcorner\mu_0\ldots\mu_{k\text{-}1}\urcorner = \ulcorner\mu'_0\ldots\mu'_{j\text{-}1}\urcorner$ iff $j = k$ and for all $i < k$: $\mu_i = \mu'_i$,

(ii) If $\mu \in$ BEXP, then there is no $k \in \mathbb{N}\setminus\{0, 1\}$ such that $\{\mu_0, \ldots, \mu_{k\text{-}1}\} \subseteq$ BEXP and $\mu = \ulcorner\mu_0\ldots\mu_{k\text{-}1}\urcorner$, and

(iii) For all $k \in \mathbb{N}\setminus\{0\}$: If $\{\mu_0, \ldots, \mu_{k\text{-}1}\} \subseteq$ BEXP, then $\ulcorner\mu_0\ldots\mu_{k\text{-}1}\urcorner \neq \emptyset$ and $\ulcorner\mu_0\ldots\mu_{k\text{-}1}\urcorner \in \{\ulcorner\mu_0\ldots\mu_{k\text{-}1}\urcorner\}$.

The expression of the concatenation operation by juxtaposition already presupposes the associativity of the concatenation operation. This property can thus be regarded as implicitly stipulated. Now, the set of all expressions, i.e. all concatenations of basic expressions, will be defined. This set will be a superset of all grammatical categories that are to be defined. Then a function that assigns each expression its length will be defined:

Definition 1-3. *The set of expressions (EXP; metavariables:* μ, τ, μ', τ', μ^*, τ^*, *…)*

EXP = $\{\ulcorner\mu_0\ldots\mu_{k\text{-}1}\urcorner \mid k \in \mathbb{N}\setminus\{0\}$ and $\{\mu_0, \ldots, \mu_{k\text{-}1}\} \subseteq$ BEXP$\}$.

Definition 1-4. *Length of an expression (EXPL)*

EXPL = $\{(\mu, k) \mid \mu \in$ EXP, $k \in \mathbb{N}\setminus\{0\}$ and there is $\{\mu_0, \ldots, \mu_{k\text{-}1}\} \subseteq$ BEXP with $\mu = \ulcorner\mu_0\ldots\mu_{k\text{-}1}\urcorner\}$.

Theorem 1-1. *EXPL is a function on EXP*

(i) Dom(EXPL) = EXP and

(ii) For all $\mu \in$ EXP, $k, l \in \mathbb{N}$: If $(\mu, k), (\mu, l) \in$ EXPL, then $k = l$.

Proof: (i) follows directly from Definition 1-3 and Definition 1-4. *Ad (ii)*: Let $\mu \in$ EXP, $k, l \in \mathbb{N}$ and $(\mu, k), (\mu, l) \in$ EXPL. Then there is $\{\mu_0, \ldots, \mu_{k\text{-}1}\} \subseteq$ BEXP with $\mu = \ulcorner\mu_0\ldots\mu_{k\text{-}1}\urcorner$ and there is $\{\mu'_0, \ldots, \mu'_{l\text{-}1}\} \subseteq$ BEXP with $\mu = \ulcorner\mu'_0\ldots\mu'_{l\text{-}1}\urcorner$. According to Postulate 1-2-(i), it then holds that $k = l$. ■

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7 Here and in the following, we assume: If $k \in \mathbb{N}\setminus\{0\}$ and $\{a_0, \ldots, a_{k\text{-}1}\} \subseteq X$, where $X \in \{X\}$, then for all $i < k$: $a_i \in \{a_0, \ldots, a_{k\text{-}1}\}$.

Theorem 1-2. *Expressions are concatenations of basic expressions*
If $\mu \in$ EXP, then there is $\{\mu_0, \ldots, \mu_{\mathrm{EXPL}(\mu)-1}\} \subseteq$ BEXP such that $\mu = \ulcorner\mu_0\ldots\mu_{\mathrm{EXPL}(\mu)-1}\urcorner$.

Proof: Follows directly from Definition 1-3 and Definition 1-4. ■

Theorem 1-3. *Identification of concatenation members*
If $k \in \mathbb{N}\backslash\{0\}$ and for all $i < k$: $\mu_i \in$ EXP, then for all $s < \sum_{j=0}^{k-1}$ EXPL(μ_j):

(i) $s <$ EXPL(μ_0)

or

(ii) EXPL(μ_0) $\leq s$ and there are l, r such that

a) $0 < l < k$ and $r <$ EXPL(μ_l) and $s = (\sum_{n=0}^{l-1}$ EXPL(μ_n))+r, and

b) For all l', r': If $0 < l' < k$ and $r' <$ EXPL($\mu_{l'}$) and $s = (\sum_{n=0}^{l'-1}$ EXPL(μ_n))+r', then $l' = l$ and $r' = r$.

Proof: Suppose $k \in \mathbb{N}\backslash\{0\}$ and that for all $i < k$: $\mu_i \in$ EXP. Now, suppose $s < \sum_{j=0}^{k-1}$ EXPL(μ_j). We have that $s <$ EXPL(μ_0) or EXPL(μ_0) $\leq s$. In the first case, the theorem holds. Now, suppose EXPL(μ_0) $\leq s$. Then we have that $1 < k$, because otherwise we would have $1 = k$ and thus EXPL(μ_0) = $\sum_{j=0}^{k-1}$ EXPL(μ_j) $> s$. Thus, there is at least one i, namely 1, such that $0 < i < k$ and $\sum_{n=0}^{i-1}$ EXPL(μ_n) $\leq s$. Now, let l = max($\{i \mid 0 < i < k$ and $\sum_{n=0}^{i-1}$ EXPL(μ_n) $\leq s\}$). Then we have $0 < l < k$ and $\sum_{n=0}^{l-1}$ EXPL(μ_n) $\leq s$. Then there is an r such that ($\sum_{n=0}^{l-1}$ EXPL(μ_n))+r = s. Suppose for contradiction that EXPL(μ_l) $\leq r$. We have that $l < k$-1 or $l = k$-1. Suppose $l < k$-1. Then we have l+1 $< k$. Then we would have $\sum_{n=0}^{l}$ EXPL(μ_n) = ($\sum_{n=0}^{l-1}$ EXPL(μ_n))+EXPL(μ_l) $\leq$ ($\sum_{n=0}^{l-1}$ EXPL(μ_n))+r = s, which contradicts the maximality of l. Suppose $l = k$-1. Then we would have l-1 = k-2. Thus we would have $\sum_{n=0}^{k-1}$ EXPL(μ_n) = ($\sum_{n=0}^{k-2}$ EXPL(μ_n))+EXPL(μ_{k-1}) $\leq$ ($\sum_{n=0}^{k-2}$ EXPL(μ_n))+r = s, which contradicts the assumption about s. Thus, the assumption that EXPL(μ_l) $\leq r$ leads to a contradiction in both cases. Therefore we have $r <$ EXPL(μ_l). Hence we have $0 < l < k$ and $r <$ EXPL(μ_l) and s = ($\sum_{n=0}^{l-1}$ EXPL(μ_n))+r.

Now, we still have to show b), i.e. that l and r are uniquely determined. For this, suppose $0 < l' < k$ and $r' <$ EXPL($\mu_{l'}$) and s = ($\sum_{n=0}^{l'-1}$ EXPL(μ_n))+r'. Then it holds that $\sum_{n=0}^{l'-1}$ EXPL(μ_n) $\leq s$. From the maximality of l, it then follows that $l' \leq l$. Now, suppose for contradiction that $l' < l$. Then we would have $l' \leq l$-1. Thus we would have ($\sum_{n=0}^{l'-1}$ EXPL(μ_n))+EXPL($\mu_{l'}$) = $\sum_{n=0}^{l'}$ EXPL(μ_n) $\leq$ $\sum_{n=0}^{l-1}$ EXPL(μ_n) $\leq$ s =

$(\sum_{n=0}^{l'-1}$ EXPL$(\mu_n))+r'$. But then we would have EXPL$(\mu_{l'}) \leq r'$, which contradicts our assumption about r'. Thus we have $l' = l$. With this, we then also have $(\sum_{n=0}^{l'-1}$ EXPL$(\mu_n))+r'$ $= (\sum_{n=0}^{l-1}$ EXPL$(\mu_n))+r' = s = (\sum_{n=0}^{l-1}$ EXPL$(\mu_n))+r$ and hence $r' = r$. ■

Postulate 1-3. *Concatenation of expressions*
If $k \in \mathbb{N}\setminus\{0\}$ and if for all $i < k$: $\mu_i \in$ EXP and $\mu_i = \ulcorner\mu^{\mu_i}{}_0\ldots\mu^{\mu_i}{}_{\text{EXPL}(\mu_i)\text{-}1}\urcorner$, where $\{\mu^{\mu_i}{}_0, \ldots, \mu^{\mu_i}{}_{\text{EXPL}(\mu_i)\text{-}1}\} \subseteq$ BEXP, then there are $m \in \mathbb{N}\setminus\{0\}$ and $\{\mu^*{}_0, \ldots, \mu^*{}_{m\text{-}1}\} \subseteq$ BEXP such that for all $i < k$:

$\ulcorner\mu_0\ldots\mu_{k\text{-}1}\urcorner$
$=$
$\ulcorner\mu_0\ldots\mu_{i\text{-}1}\mu^{\mu_i}{}_0\ldots\mu^{\mu_i}{}_{\text{EXPL}(\mu_i)\text{-}1}\mu_{i+1}\ldots\mu_{k\text{-}1}\urcorner$
$=$
$\ulcorner\mu^*{}_0\ldots\mu^*{}_{m\text{-}1}\urcorner$, where

a) $m = \sum_{j=0}^{k-1}$ EXPL(μ_j), and
b) For all $s < m$:
$\mu^*{}_s = \mu^{\mu_0}{}_s$, if $s <$ EXPL(μ_0), and
$\mu^*{}_s = \mu^{\mu_l}{}_r$ for the uniquely determined l, r for which $0 < l < k$ and $r <$ EXPL(μ_l) and $s = (\sum_{n=0}^{l-1}$ EXPL$(\mu_n))+r$, if EXPL$(\mu_0) \leq s$.

As an immediate consequence of Postulate 1-3, we have that every concatenation of expressions is identical to a concatenation of basic expressions and thus itself an expression. Now, we will prove some general theorems on expressions and their concatenations (Theorem 1-4 to Theorem 1-8). Then, we will define the arity of operators and subsequently the categories of terms, quantifiers and formulas.

Theorem 1-4. *On the identity of concatenations of expressions (a)*
If $k \in \mathbb{N}\setminus\{0\}$, for all $i < k$: $\mu_i \in$ EXP and $\mu_i = \ulcorner\mu^{\mu_i}{}_0\ldots\mu^{\mu_i}{}_{\text{EXPL}(\mu_i)\text{-}1}\urcorner$, where $\{\mu^{\mu_i}{}_0, \ldots, \mu^{\mu_i}{}_{\text{EXPL}(\mu_i)\text{-}1}\}$ $\subseteq$ BEXP, then:

(i) $\ulcorner\mu_0\ldots\mu_{k\text{-}1}\urcorner$
$=$
$\ulcorner\mu^{\mu_0}{}_0\ldots\mu^{\mu_0}{}_{\text{EXPL}(\mu_0)\text{-}1}\ldots\mu^{\mu_{k\text{-}1}}{}_0\ldots\mu^{\mu_{k\text{-}1}}{}_{\text{EXPL}(\mu_{k\text{-}1})\text{-}1}\urcorner$,

(ii) EXPL$(\ulcorner\mu_0\ldots\mu_{k\text{-}1}\urcorner) = \sum_{j=0}^{k-1}$ EXPL(μ_j), and

(iii) If $m \in \mathbb{N}\backslash\{0\}$ and $\{\mu'_0, \ldots, \mu'_{m-1}\} \subseteq \text{BEXP}$, then:
$\ulcorner \mu^{\mu_0}_{0} \ldots \mu^{\mu_0}_{\text{EXPL}(\mu_0)-1} \ldots \mu^{\mu_{k-1}}_{0} \ldots \mu^{\mu_{k-1}}_{\text{EXPL}(\mu_{k-1})-1} \urcorner$
$=$
$\ulcorner \mu'_0 \ldots \mu'_{m-1} \urcorner$
iff
$m = \sum_{j=0}^{k-1} \text{EXPL}(\mu_j)$ and for all $s < m$: $\mu'_s = \mu^{\mu_0}_{s}$, if $s < \text{EXPL}(\mu_0)$, and $\mu'_s = \mu^{\mu_l}_{r}$ for the uniquely determined l, r for which $0 < l < k$ and $r < \text{EXPL}(\mu_l)$ and $s = (\sum_{n=0}^{l-1} \text{EXPL}(\mu_n))+r$, if $\text{EXPL}(\mu_0) \leq s$.

Proof: Suppose $k \in \mathbb{N}\backslash\{0\}$, for all $i < k$: $\mu_i \in \text{EXP}$ and $\mu_i = \ulcorner \mu^{\mu_i}_{0} \ldots \mu^{\mu_i}_{\text{EXPL}(\mu_i)-1} \urcorner$, where $\{\mu^{\mu_i}_{0}, \ldots, \mu^{\mu_i}_{\text{EXPL}(\mu_i)-1}\} \subseteq \text{BEXP}$. *Ad (i)*: First, we show, by induction on i, that for all $i < k$:

$\ulcorner \mu_0 \ldots \mu_{k-1} \urcorner$
$=$
$\ulcorner \mu^{\mu_0}_{0} \ldots \mu^{\mu_0}_{\text{EXPL}(\mu_0)-1} \ldots \mu^{\mu_i}_{0} \ldots \mu^{\mu_i}_{\text{EXPL}(\mu_i)-1} \mu_{i+1} \ldots \mu_{k-1} \urcorner$.

Then, this statement also holds for $i = k$-1, and thus we have (i). Now, suppose the statement holds for all $l < i$. Suppose $i < k$. Then we have that $i = 0$ or $0 < i$. Suppose $i = 0$. Because of $\mu_0 = \ulcorner \mu^{\mu_0}_{0} \ldots \mu^{\mu_0}_{\text{EXPL}(\mu_0)-1} \urcorner$, we then have, with Postulate 1-3:

$\ulcorner \mu_0 \ldots \mu_{k-1} \urcorner$
$=$
$\ulcorner \mu^{\mu_0}_{0} \ldots \mu^{\mu_0}_{\text{EXPL}(\mu_0)-1} \mu_1 \ldots \mu_{k-1} \urcorner$.

Now, suppose $0 < i$. Then it holds for all $l < i$ that $l < k$ and thus, according to the I.H., that

$\ulcorner \mu_0 \ldots \mu_{k-1} \urcorner$
$=$
$\ulcorner \mu^{\mu_0}_{0} \ldots \mu^{\mu_0}_{\text{EXPL}(\mu_0)-1} \ldots \mu^{\mu_l}_{0} \ldots \mu^{\mu_l}_{\text{EXPL}(\mu_l)-1} \mu_{l+1} \ldots \mu_{k-1} \urcorner$.

Since i-1 < i, we thus have

$\ulcorner \mu_0 \ldots \mu_{k-1} \urcorner$
$=$
$\ulcorner \mu^{\mu_0}_{0} \ldots \mu^{\mu_0}_{\text{EXPL}(\mu_0)-1} \ldots \mu^{\mu_{i-1}}_{0} \ldots \mu^{\mu_{i-1}}_{\text{EXPL}(\mu_{i-1})-1} \mu_i \ldots \mu_{k-1} \urcorner$.

Because of $\mu_i = \ulcorner \mu^{\mu_i}_{0} \ldots \mu^{\mu_i}_{\text{EXPL}(\mu_i)-1} \urcorner$, we then have, with Postulate 1-3:

$\ulcorner \mu^{\mu_0}_{0} \ldots \mu^{\mu_0}_{\text{EXPL}(\mu_0)-1} \ldots \mu^{\mu_{i-1}}_{0} \ldots \mu^{\mu_{i-1}}_{\text{EXPL}(\mu_{i-1})-1} \mu_i \ldots \mu_{k-1} \urcorner$
$=$
$\ulcorner \mu^{\mu_0}_{0} \ldots \mu^{\mu_0}_{\text{EXPL}(\mu_0)-1} \ldots \mu^{\mu_{i-1}}_{0} \ldots \mu^{\mu_{i-1}}_{\text{EXPL}(\mu_{i-1})-1} \mu^{\mu_i}_{0} \ldots \mu^{\mu_i}_{\text{EXPL}(\mu_i)-1} \mu_{i+1} \ldots \mu_{k-1} \urcorner$
$=$

$\ulcorner\mu^{\mu_0}{}_0\ldots\mu^{\mu_0}{}_{\mathrm{EXPL}(\mu_0)\text{-}1}\ldots\mu^{\mu_i}{}_0\ldots\mu^{\mu_i}{}_{\mathrm{EXPL}(\mu_i)\text{-}1}\mu_{i+1}\ldots\mu_{k\text{-}1}\urcorner$.

Hence we have

$$\ulcorner\mu_0\ldots\mu_{k\text{-}1}\urcorner$$
$$=$$
$$\ulcorner\mu^{\mu_0}{}_0\ldots\mu^{\mu_0}{}_{\mathrm{EXPL}(\mu_0)\text{-}1}\ldots\mu^{\mu_i}{}_0\ldots\mu^{\mu_i}{}_{\mathrm{EXPL}(\mu_i)\text{-}1}\mu_{i+1}\ldots\mu_{k\text{-}1}\urcorner.$$

Ad (ii) and (iii): With Postulate 1-3, there are $m^* \in \mathbb{N}\backslash\{0\}$ and $\{\mu^*_0, \ldots, \mu^*_{m^*\text{-}1}\} \subseteq$ BEXP such that $\ulcorner\mu_0\ldots\mu_{k\text{-}1}\urcorner = \ulcorner\mu^*_0\ldots\mu^*_{m^*\text{-}1}\urcorner$ and $m^* = \sum_{j=0}^{k-1} \mathrm{EXPL}(\mu_j)$ and for all $s < m^*$: $\mu^*_s = \mu^{\mu_0}{}_s$, if $s < \mathrm{EXPL}(\mu_0)$, and $\mu^*_s = \mu^{\mu_l}{}_r$ for the uniquely determined l, r for which $0 < l < k$, $r < \mathrm{EXPL}(\mu_l)$ and $s = (\sum_{n=0}^{l-1} \mathrm{EXPL}(\mu_n))+r$, if $\mathrm{EXPL}(\mu_0) \leq s$. Then we have $\sum_{j=0}^{k-1} \mathrm{EXPL}(\mu_j) = m^* = \mathrm{EXPL}(\ulcorner\mu^*_0\ldots\mu^*_{m^*\text{-}1}\urcorner) = \mathrm{EXPL}(\ulcorner\mu_0\ldots\mu_{k\text{-}1}\urcorner)$. Thus we have (ii). Now, for (iii), suppose $m \in \mathbb{N}\backslash\{0\}$ and $\{\mu'_0, \ldots, \mu'_{m\text{-}1}\} \subseteq$ BEXP. (*L-R*): Suppose $\ulcorner\mu^{\mu_0}{}_0\ldots\mu^{\mu_0}{}_{\mathrm{EXPL}(\mu_0)\text{-}1}\ldots\mu^{\mu_{k\text{-}1}}{}_0\ldots\mu^{\mu_{k\text{-}1}}{}_{\mathrm{EXPL}(\mu_{k\text{-}1})\text{-}1}\urcorner = \ulcorner\mu'_0\ldots\mu'_{m\text{-}1}\urcorner$. With (i), we then have $\ulcorner\mu'_0\ldots\mu'_{m\text{-}1}\urcorner = \ulcorner\mu_0\ldots\mu_{k\text{-}1}\urcorner = \ulcorner\mu^*_0\ldots\mu^*_{m^*\text{-}1}\urcorner$. With Postulate 1-2-(i), we then have $m = m^* = \sum_{j=0}^{k-1} \mathrm{EXPL}(\mu_j)$ and for all $s < m$: $\mu'_s = \mu^*_s$. Thus we have for all $s < m$: $\mu'_s = \mu^{\mu_0}{}_s$, if $s < \mathrm{EXPL}(\mu_0)$, and $\mu'_s = \mu^{\mu_l}{}_r$ for the uniquely determined l, r for which $0 < l < k$, $r < \mathrm{EXPL}(\mu_l)$ and $s = (\sum_{n=0}^{l-1} \mathrm{EXPL}(\mu_n))+r$, if $\mathrm{EXPL}(\mu_0) \leq s$.

(*R-L*): Suppose $m = \sum_{j=0}^{k-1} \mathrm{EXPL}(\mu_j)$ and that it hold for all $s < m$ that $\mu'_s = \mu^{\mu_0}{}_s$, if $s < \mathrm{EXPL}(\mu_0)$, and $\mu'_s = \mu^{\mu_l}{}_r$ for the uniquely determined l, r for which $0 < l < k$, $r < \mathrm{EXPL}(\mu_l)$ and $s = (\sum_{n=0}^{l-1} \mathrm{EXPL}(\mu_n))+r$, if $\mathrm{EXPL}(\mu_0) \leq s$. Then it holds that $m^* = m$ and that for all $s < m$: $\mu'_s = \mu^*_s$. With Postulate 1-2-(i), we then have $\ulcorner\mu'_0\ldots\mu'_{m\text{-}1}\urcorner = \ulcorner\mu^*_0\ldots\mu^*_{m^*\text{-}1}\urcorner$. With (i), we then have $\ulcorner\mu^{\mu_0}{}_0\ldots\mu^{\mu_0}{}_{\mathrm{EXPL}(\mu_0)\text{-}1}\ldots\mu^{\mu_{k\text{-}1}}{}_0\ldots\mu^{\mu_{k\text{-}1}}{}_{\mathrm{EXPL}(\mu_{k\text{-}1})\text{-}1}\urcorner = \ulcorner\mu_0\ldots\mu_{k\text{-}1}\urcorner = \ulcorner\mu^*_0\ldots\mu^*_{m^*\text{-}1}\urcorner = \ulcorner\mu'_0\ldots\mu'_{m\text{-}1}\urcorner$. ■

Theorem 1-5. *On the identity of concatenations of expressions (b)*
If k, $k' \in \mathbb{N}\backslash\{0\}$ and for all $i < k$: $\mu_i \in$ EXP and $\mu_i = \ulcorner\mu^{\mu_i}{}_0\ldots\mu^{\mu_i}{}_{\mathrm{EXPL}(\mu_i)\text{-}1}\urcorner$, where $\{\mu^{\mu_i}{}_0, \ldots, \mu^{\mu_i}{}_{\mathrm{EXPL}(\mu_i)\text{-}1}\} \subseteq$ BEXP, and for all $i < k'$: $\mu'_i \in$ EXP and $\mu'_i = \ulcorner\mu'^{\mu'_i}{}_0\ldots\mu'^{\mu'_i}{}_{\mathrm{EXPL}(\mu'_i)\text{-}1}\urcorner$, where $\{\mu'^{\mu'_i}{}_0, \ldots, \mu'^{\mu'_i}{}_{\mathrm{EXPL}(\mu'_i)\text{-}1}\} \subseteq$ BEXP, and if $\ulcorner\mu_0\ldots\mu_{k\text{-}1}\urcorner = \ulcorner\mu'_0\ldots\mu'_{k'\text{-}1}\urcorner$, then:

(i) $$\ulcorner\mu_0\ldots\mu_{k\text{-}1}\urcorner$$
$$=$$
$$\ulcorner\mu^{\mu_0}{}_0\ldots\mu^{\mu_0}{}_{\mathrm{EXPL}(\mu_0)\text{-}1}\ldots\mu^{\mu_{k\text{-}1}}{}_0\ldots\mu^{\mu_{k\text{-}1}}{}_{\mathrm{EXPL}(\mu_{k\text{-}1})\text{-}1}\urcorner$$
$$=$$
$$\ulcorner\mu'^{\mu'_0}{}_0\ldots\mu'^{\mu'_0}{}_{\mathrm{EXPL}(\mu'_0)\text{-}1}\ldots\mu'^{\mu'_{k'\text{-}1}}{}_0\ldots\mu'^{\mu'_{k'\text{-}1}}{}_{\mathrm{EXPL}(\mu'_{k'\text{-}1})\text{-}1}\urcorner$$

$=$

$\ulcorner\mu'_0\ldots\mu'_{k'\text{-}1}\urcorner$,

(ii) $\mathrm{EXPL}(\ulcorner\mu_0\ldots\mu_{k\text{-}1}\urcorner) = \sum_{j=0}^{k-1} \mathrm{EXPL}(\mu_j) = \sum_{j=0}^{k'-1} \mathrm{EXPL}(\mu'_j) = \mathrm{EXPL}(\ulcorner\mu'_0\ldots\mu'_{k'\text{-}1}\urcorner)$, and

(iii) For all $i < k, k'$: If $\mathrm{EXPL}(\mu_j) = \mathrm{EXPL}(\mu'_j)$ for all $j \leq i$, then:

a) $\ulcorner\mu_0\ldots\mu_i\urcorner$

$=$

$\ulcorner\mu^{\mu_0}{}_0\ldots\mu^{\mu_0}{}_{\mathrm{EXPL}(\mu_0)\text{-}1}\ldots\mu^{\mu_i}{}_0\ldots\mu^{\mu_i}{}_{\mathrm{EXPL}(\mu_i)\text{-}1}\urcorner$

$=$

$\ulcorner\mu'^{\mu'_0}{}_0\ldots\mu'^{\mu'_0}{}_{\mathrm{EXPL}(\mu'_0)\text{-}1}\ldots\mu'^{\mu'_i}{}_0\ldots\mu'^{\mu'_i}{}_{\mathrm{EXPL}(\mu'_i)\text{-}1}\urcorner$

$=$

$\ulcorner\mu'_0\ldots\mu'_i\urcorner$, and

b) For all $j \leq i$: $\mu_j = \mu'_j$.

Proof: Suppose $k, k' \in \mathbb{N}\backslash\{0\}$ and for all $i < k$: $\mu_i \in \mathrm{EXP}$ and $\mu_i = \ulcorner\mu^{\mu_i}{}_0\ldots\mu^{\mu_i}{}_{\mathrm{EXPL}(\mu_i)\text{-}1}\urcorner$, where $\{\mu^{\mu_i}{}_0, \ldots, \mu^{\mu_i}{}_{\mathrm{EXPL}(\mu_i)\text{-}1}\} \subseteq \mathrm{BEXP}$, and for all $i < k'$: $\mu'_i \in \mathrm{EXP}$ and $\mu'_i = \ulcorner\mu'^{\mu'_i}{}_0\ldots\mu'^{\mu'_i}{}_{\mathrm{EXPL}(\mu'_i)\text{-}1}\urcorner$, where $\{\mu'^{\mu'_i}{}_0, \ldots, \mu'^{\mu'_i}{}_{\mathrm{EXPL}(\mu'_i)\text{-}1}\} \subseteq \mathrm{BEXP}$, and suppose $\ulcorner\mu_0\ldots\mu_{k\text{-}1}\urcorner = \ulcorner\mu'_0\ldots\mu'_{k'\text{-}1}\urcorner$. Then clauses (i) and (ii) follow with Theorem 1-4-(i) and -(ii).

Now, for (iii), suppose $i < k, k'$ and suppose $\mathrm{EXPL}(\mu_j) = \mathrm{EXPL}(\mu'_j)$ for all $j \leq i$. First, with Postulate 1-3, we have that there are $m^* \in \mathbb{N}\backslash\{0\}$ and $\{\mu^*_0, \ldots, \mu^*_{m\text{-}1}\} \subseteq \mathrm{BEXP}$ such that $\ulcorner\mu_0\ldots\mu_{k\text{-}1}\urcorner = \ulcorner\mu^*_0\ldots\mu^*_{m\text{-}1}\urcorner$ and $m = \sum_{n=0}^{k-1} \mathrm{EXPL}(\mu_n)$ and for all $s < m$: $\mu^*_s = \mu^{\mu_0}{}_s$, if $s < \mathrm{EXPL}(\mu_0)$, and $\mu^*_s = \mu^{\mu_l}{}_r$ for the uniquely determined l, r for which $0 < l < k$, $r < \mathrm{EXPL}(\mu_l)$ and $s = (\sum_{n=0}^{l-1} \mathrm{EXPL}(\mu_n))+r$, if $\mathrm{EXPL}(\mu_0) \leq s$; and that there are $m' \in \mathbb{N}\backslash\{0\}$ and $\{\mu'^*_0, \ldots, \mu'^*_{m'\text{-}1}\} \subseteq \mathrm{BEXP}$ such that $\ulcorner\mu'_0\ldots\mu'_{k'\text{-}1}\urcorner = \ulcorner\mu'^*_0\ldots\mu'^*_{m'\text{-}1}\urcorner$ and $m' = \sum_{n=0}^{k'-1} \mathrm{EXPL}(\mu'_n)$ und for all $s < m'$: $\mu'^*_s = \mu'^{\mu'_0}{}_s$, if $s < \mathrm{EXPL}(\mu'_0)$, and $\mu'^*_s = \mu'^{\mu'_{l'}}{}_{r'}$ for the uniquely determined l', r' for which $0 < l' < k'$, $r' < \mathrm{EXPL}(\mu'_{l'})$ and $s = (\sum_{n=0}^{l'-1} \mathrm{EXPL}(\mu'_n))+r'$, if $\mathrm{EXPL}(\mu'_0) \leq s$. With (ii), we then have $m = m'$. Furthermore, we have, with (i):

$\ulcorner\mu^*_0\ldots\mu^*_{m^*\text{-}1}\urcorner$

$=$

$\ulcorner\mu^{\mu_0}{}_0\ldots\mu^{\mu_0}{}_{\mathrm{EXPL}(\mu_0)\text{-}1}\ldots\mu^{\mu_{k\text{-}1}}{}_0\ldots\mu^{\mu_{k\text{-}1}}{}_{\mathrm{EXPL}(\mu_{k\text{-}1})\text{-}1}\urcorner$

$=$

$\ulcorner\mu'^{\mu'_0}{}_0\ldots\mu'^{\mu'_0}{}_{\mathrm{EXPL}(\mu'_0)\text{-}1}\ldots\mu'^{\mu'_{k'\text{-}1}}{}_0\ldots\mu'^{\mu'_{k'\text{-}1}}{}_{\mathrm{EXPL}(\mu'_{k'\text{-}1})\text{-}1}\urcorner$

$=$

$\ulcorner\mu'^*_0\ldots\mu'^*_{m'\text{-}1}\urcorner$.

With Postulate 1-2-(i), we then have for all $s < m = m'$: $\mu^*_s = \mu'^*_s$. We have that $i = 0$ or $0 < i$. First, suppose $i = 0$. By hypothesis, we have $\text{EXPL}(\mu_0) = \text{EXPL}(\mu'_0)$. Now, suppose $s < \text{EXPL}(\mu_0)$. Then we have $s < \text{EXPL}(\mu'_0)$ and $s < m = m'$. Then we have $\mu^*_s = \mu^{\mu_0}_s$ and $\mu'^*_s = \mu'^{\mu'_0}_s$. Then we have $\mu^{\mu_0}_s = \mu'^{\mu'_0}_s$. Thus we have for all $s < \text{EXPL}(\mu_0) = \text{EXPL}(\mu'_0)$ that $\mu^{\mu_0}_s = \mu'^{\mu'_0}_s$. Thus we have, with Postulate 1-2-(i), that $\mu_0 = \ulcorner\mu^{\mu_0}_0\ldots\mu^{\mu_0}_{\text{EXPL}(\mu_0)-1}\urcorner = \ulcorner\mu'^{\mu'_0}_0\ldots\mu'^{\mu'_0}_{\text{EXPL}(\mu'_0)-1}\urcorner = \mu'_0$. Thus a) holds for $i = 0$. Also, if $i = 0$, we have for all $j \leq i$ that $j = i = 0$ and thus b) holds as well for $i = 0$.

Now, suppose $0 < i$. By hypothesis, we have $\text{EXPL}(\mu_j) = \text{EXPL}(\mu'_j)$ for all $j \leq i$. From this, we get: $\sum_{n=0}^{i} \text{EXPL}(\mu_n) = \sum_{n=0}^{i} \text{EXPL}(\mu'_n)$. With Postulate 1-3, we have that there are $t \in \mathbb{N}\setminus\{0\}$ and $\{\mu^+_0, \ldots, \mu^+_{t-1}\} \subseteq \text{BEXP}$ such that $\ulcorner\mu_0\ldots\mu_i\urcorner = \ulcorner\mu^+_0\ldots\mu^+_{t-1}\urcorner$ and $t = \sum_{n=0}^{i} \text{EXPL}(\mu_n)$ and for all $s < t$: $\mu^+_s = \mu^{\mu_0}_s$, if $s < \text{EXPL}(\mu_0)$, and $\mu^+_s = \mu^{\mu_{l^\circ}}_{r^\circ}$ for the uniquely determined l°, r° for which $0 < l^\circ < i+1$, $r^\circ < \text{EXPL}(\mu_{l^\circ})$ und $s = (\sum_{n=0}^{l^\circ-1} \text{EXPL}(\mu_n))+r^\circ$, if $\text{EXPL}(\mu_0) \leq s$; and that there are $t' \in \mathbb{N}\setminus\{0\}$ and $\{\mu'^+_0, \ldots, \mu'^+_{t'-1}\} \subseteq \text{BEXP}$ such that $\ulcorner\mu'_0\ldots\mu'_i\urcorner = \ulcorner\mu'^+_0\ldots\mu'^+_{t'-1}\urcorner$ and $t' = \sum_{n=0}^{i} \text{EXPL}(\mu'_n)$ and for all $s < t'$: $\mu'^+_s = \mu'^{\mu'_0}_s$, if $s < \text{EXPL}(\mu'_0)$, and $\mu'^+_s = \mu'^{\mu'_{l'^\circ}}_{r'^\circ}$ for the uniquely determined l'°, r'° for which $0 < l'^\circ < i+1$, $r'^\circ < \text{EXPL}(\mu'_{l'^\circ})$ and $s = (\sum_{n=0}^{l'^\circ-1} \text{EXPL}(\mu'_n))+r'^\circ$, if $\text{EXPL}(\mu'_0) \leq s$. Then we have $t = \sum_{n=0}^{i} \text{EXPL}(\mu_n) = \sum_{n=0}^{i} \text{EXPL}(\mu'_n) = t'$. Because of $\sum_{n=0}^{i} \text{EXPL}(\mu_n) \leq \sum_{n=0}^{k-1} \text{EXPL}(\mu_n)$, we also have $t \leq m = m'$.

Now, suppose $s < t$. Then we have $s < t'$ and $s < m = m'$. We have that $s < \text{EXPL}(\mu_0)$ or $\text{EXPL}(\mu_0) \leq s$. Suppose $s < \text{EXPL}(\mu_0)$. Since $0 < i$, we have, by hypothesis, that $\text{EXPL}(\mu_0) = \text{EXPL}(\mu'_0)$, and thus also that $s < \text{EXPL}(\mu'_0)$. Then we have $\mu^*_s = \mu^{\mu_0}_s = \mu^+_s$ und $\mu'^*_s = \mu'^{\mu'_0}_s = \mu'^+_s$. Because of $\mu^*_s = \mu'^*_s$, we thus have $\mu^+_s = \mu'^+_s$.

Now, suppose $\text{EXPL}(\mu_0) = \text{EXPL}(\mu'_0) \leq s$. Then it holds that

> $\mu^*_s = \mu^{\mu_l}_r$ for the uniquely determined l, r for which $0 < l < k$, $r < \text{EXPL}(\mu_l)$ and $s = (\sum_{n=0}^{l-1} \text{EXPL}(\mu_n))+r$
> and
> $\mu'^*_s = \mu'^{\mu'_{l'}}_{r'}$ for the uniquely determined l', r' for which $0 < l' < k'$, $r' < \text{EXPL}(\mu'_{l'})$ and $s = (\sum_{n=0}^{l'-1} \text{EXPL}(\mu'_n))+r'$
> and
> $\mu^+_s = \mu^{\mu_{l^\circ}}_{r^\circ}$ for the uniquely determined l°, r° for which $0 < l^\circ < i+1$, $r^\circ < \text{EXPL}(\mu_{l^\circ})$ and $s = (\sum_{n=0}^{l^\circ-1} \text{EXPL}(\mu_n))+r^\circ$
> and
> $\mu'^+_s = \mu'^{\mu'_{l'^\circ}}_{r'^\circ}$ for the uniquely determined l'°, r'° for which $0 < l'^\circ < i+1$, $r'^\circ < \text{EXPL}(\mu'_{l'^\circ})$ and $s = (\sum_{n=0}^{l'^\circ-1} \text{EXPL}(\mu'_n))+r'^\circ$.

With $l°$, $l'°$ < i+1, we then have $l°$, $l'° \leq i$. By hypothesis, we thus have that EXPL($\mu_{l'°}$) = EXPL($\mu'_{l'°}$) and $\sum_{n=0}^{l'°-1}$ EXPL(μ_n) = $\sum_{n=0}^{l'°-1}$ EXPL(μ'_n). Then we have 0 < $l'°$ < i+1 and $r'°$ < EXPL($\mu_{l'°}$) and s = ($\sum_{n=0}^{l'°-1}$ EXPL(μ_n))+$r'°$. By Theorem 1-3, we then have $l'°$ = $l°$ und $r'°$ = $r°$. Now, suppose for contradiction that i+1 ≤ l. Then we would have $i \leq l$-1. But then we would have t = $\sum_{n=0}^{i}$ EXPL(μ_n) ≤ $\sum_{n=0}^{l-1}$ EXPL(μ_n) ≤ s. Contradiction! Thus we have l < i+1. From this, we get l = $l°$ und r = $r°$. In the same way, we get l' = $l'°$ and r' = $r'°$. Thus we have l = $l°$ = $l'°$ = l' und r = $r°$ = $r'°$ = r'. With this, we have $\mu^*_s = \mu^{\mu_l}{}_r = \mu^+{}_s$ and $\mu'^*_s = \mu'^{\mu'_l}{}_r = \mu'^+{}_s$. Since $\mu^*_s = \mu'^*_s$, we thus have $\mu^+{}_s = \mu'^+{}_s$. Thus it holds for all $s < t = t'$ that $\mu^+{}_s = \mu'^+{}_s$ and thus, with Postulate 1-2-(i), that $\ulcorner\mu_0\ldots\mu_i\urcorner = \ulcorner\mu^+{}_0\ldots\mu^+{}_{t\text{-}1}\urcorner = \ulcorner\mu'^+{}_0\ldots\mu'^+{}_{t'\text{-}1}\urcorner = \ulcorner\mu'_0\ldots\mu'_i\urcorner$. Moreover, we have, with Theorem 1-4-(i), that $\ulcorner\mu_0\ldots\mu_i\urcorner = \ulcorner\mu^{\mu_0}{}_0\ldots\mu^{\mu_0}{}_{\mathrm{EXPL}(\mu_0)\text{-}1}\ldots\mu^{\mu_i}{}_0\ldots\mu^{\mu_i}{}_{\mathrm{EXPL}(\mu_i)\text{-}1}\urcorner$ and $\ulcorner\mu'_0\ldots\mu'_i\urcorner = \ulcorner\mu'^{\mu'_0}{}_0\ldots\mu'^{\mu'_0}{}_{\mathrm{EXPL}(\mu'_0)\text{-}1}\ldots\mu'^{\mu'_i}{}_0\ldots\mu'^{\mu'_i}{}_{\mathrm{EXPL}(\mu'_i)\text{-}1}\urcorner$. Hence a) also holds for 0 < i.

Now, suppose, for b), that $j \leq i$. For j = 0, we have already shown above that $\mu_j = \mu'_j$. Suppose 0 < $j \leq i$. Now, suppose r < EXPL(μ_j) = EXPL(μ'_j). Then we have ($\sum_{n=0}^{j-1}$ EXPL(μ_n))+r = ($\sum_{n=0}^{j-1}$ EXPL(μ'_n))+r < t = t' ≤ m = m'. With s = ($\sum_{n=0}^{j-1}$ EXPL(μ_n))+r, it then holds that $\mu^+{}_s = \mu^{\mu_j}{}_r$ and $\mu'^+{}_s = \mu'^{\mu'_j}{}_r$. Since $s < t = t'$, we then have, as we have just shown, that $\mu^+{}_s = \mu'^+{}_s$ and thus that $\mu^{\mu_j}{}_r = \mu'^{\mu'_j}{}_r$. Thus it holds for all r < EXPL(μ_j) = EXPL(μ'_j) that $\mu^{\mu_j}{}_r = \mu'^{\mu'_j}{}_r$. Then it holds, with Postulate 1-2-(i), that $\mu_j = \ulcorner\mu^{\mu_j}{}_0\ldots\mu^{\mu_j}{}_{\mathrm{EXPL}(\mu_j)\text{-}1}\urcorner = \ulcorner\mu'^{\mu'_j}{}_0\ldots\mu'^{\mu'_j}{}_{\mathrm{EXPL}(\mu'_j)\text{-}1}\urcorner = \mu'_j$. Hence b) also holds for 0 < i. ■

Theorem 1-6. *On the identity of concatenations of expressions (c)*
If k, $s \in \mathbb{N}\setminus\{0\}$ and $\{\mu_0, \ldots, \mu_{k\text{-}1}\} \subseteq$ EXP and $\{\mu'_0, \ldots, \mu'_{s\text{-}1}\} \subseteq$ EXP and $j < k$ and $\mu_j = \ulcorner\mu'_0\ldots\mu'_{s\text{-}1}\urcorner$, then: $\ulcorner\mu_0\ldots\mu_{k\text{-}1}\urcorner = \ulcorner\mu_0\ldots\mu_{j\text{-}1}\mu'_0\ldots\mu'_{s\text{-}1}\mu_{j+1}\ldots\mu_{k\text{-}1}\urcorner$.

Proof: Suppose k, $s \in \mathbb{N}\setminus\{0\}$ and $\{\mu_0, \ldots, \mu_{k\text{-}1}\} \subseteq$ EXP and $\{\mu'_0, \ldots, \mu'_{s\text{-}1}\} \subseteq$ EXP and $j < k$ and $\mu_j = \ulcorner\mu'_0\ldots\mu'_{s\text{-}1}\urcorner$. With $\{\mu'_0, \ldots, \mu'_{s\text{-}1}\} \subseteq$ EXP and Theorem 1-2, it then holds for all $i < s$ that there is $\{\mu'^{\mu'_i}{}_0, \ldots, \mu'^{\mu'_i}{}_{\mathrm{EXPL}(\mu'_i)\text{-}1}\} \subseteq$ BEXP such that $\mu'_i = \ulcorner\mu'^{\mu'_i}{}_0\ldots\mu'^{\mu'_i}{}_{\mathrm{EXPL}(\mu'_i)\text{-}1}\urcorner$. With Theorem 1-4-(i), we then have $\mu_j = \ulcorner\mu'_0\ldots\mu'_{s\text{-}1}\urcorner = \ulcorner\mu'^{\mu'_0}{}_0\ldots\mu'^{\mu'_0}{}_{\mathrm{EXPL}(\mu'_0)\text{-}1}\ldots\mu'^{\mu'_{s\text{-}1}}{}_0\ldots\mu'^{\mu'_{s\text{-}1}}{}_{\mathrm{EXPL}(\mu'_{s\text{-}1})\text{-}1}\urcorner$. With Postulate 1-3, we then have $\ulcorner\mu_0\ldots\mu_{k\text{-}1}\urcorner = \ulcorner\mu_0\ldots\mu_{j\text{-}1}\mu'^{\mu'_0}{}_0\ldots\mu'^{\mu'_0}{}_{\mathrm{EXPL}(\mu'_0)\text{-}1}\ldots\mu'^{\mu'_{s\text{-}1}}{}_0\ldots\mu'^{\mu'_{s\text{-}1}}{}_{\mathrm{EXPL}(\mu'_{s\text{-}1})\text{-}1}\mu_{j+1}\ldots\mu_{k\text{-}1}\urcorner$. Now, we first show by induction on i that for all $i < s$:

$$\ulcorner\mu_0\ldots\mu_{j\text{-}1}\mu'^{\mu'_0}{}_0\ldots\mu'^{\mu'_0}{}_{\mathrm{EXPL}(\mu'_0)\text{-}1}\ldots\mu'^{\mu'_{s\text{-}1}}{}_0\ldots\mu'^{\mu'_{s\text{-}1}}{}_{\mathrm{EXPL}(\mu'_{s\text{-}1})\text{-}1}\mu_{j+1}\ldots\mu_{k\text{-}1}\urcorner$$
=

$$\ulcorner\mu_0\ldots\mu_{j\text{-}1}\mu'_0\ldots\mu'_i\mu'^{\mu'_{i+1}}_0\ldots\mu'^{\mu'_{i+1}}_{\mathrm{EXPL}(\mu'_{i+1})\text{-}1}\ldots\mu'^{\mu'_{s\text{-}1}}_0\ldots\mu'^{\mu'_{s\text{-}1}}_{\mathrm{EXPL}(\mu'_{s\text{-}1})\text{-}1}\mu_{j+1}\ldots\mu_{k\text{-}1}\urcorner.$$

Then, this also holds for $i = s\text{-}1$ and thus we get

$$\ulcorner\mu_0\ldots\mu_{k\text{-}1}\urcorner$$
$$=$$
$$\ulcorner\mu_0\ldots\mu_{j\text{-}1}\mu'^{\mu'_0}_0\ldots\mu'^{\mu'_0}_{\mathrm{EXPL}(\mu'_0)\text{-}1}\ldots\mu'^{\mu'_{s\text{-}1}}_0\ldots\mu'^{\mu'_{s\text{-}1}}_{\mathrm{EXPL}(\mu'_{s\text{-}1})\text{-}1}\mu_{j+1}\ldots\mu_{k\text{-}1}\urcorner$$
$$=$$
$$\ulcorner\mu_0\ldots\mu_{j\text{-}1}\mu'_0\ldots\mu'_{s\text{-}1}\mu_{j+1}\ldots\mu_{k\text{-}1}\urcorner.$$

Then the theorem holds. Now, suppose the statement holds for all $l < i$. Suppose $i < s$. Then we have that $i = 0$ or $0 < i$. Suppose $i = 0$. Because of $\mu'_0 = \ulcorner\mu'^{\mu'_0}_0\ldots\mu'^{\mu'_0}_{\mathrm{EXPL}(\mu'_0)\text{-}1}\urcorner$, we then have, with Postulate 1-3:

$$\ulcorner\mu_0\ldots\mu_{j\text{-}1}\mu'^{\mu'_0}_0\ldots\mu'^{\mu'_0}_{\mathrm{EXPL}(\mu'_0)\text{-}1}\ldots\mu'^{\mu'_{s\text{-}1}}_0\ldots\mu'^{\mu'_{s\text{-}1}}_{\mathrm{EXPL}(\mu'_{s\text{-}1})\text{-}1}\mu_{j+1}\ldots\mu_{k\text{-}1}\urcorner$$
$$=$$
$$\ulcorner\mu_0\ldots\mu_{j\text{-}1}\mu'_0\mu'^{\mu'_1}_0\ldots\mu'^{\mu'_1}_{\mathrm{EXPL}(\mu'_1)\text{-}1}\ldots\mu'^{\mu'_{s\text{-}1}}_0\ldots\mu'^{\mu'_{s\text{-}1}}_{\mathrm{EXPL}(\mu'_{s\text{-}1})\text{-}1}\mu_{j+1}\ldots\mu_{k\text{-}1}\urcorner.$$

Now, suppose $0 < i$. Then it holds for all $l < i$ that $l < s$ and thus, according to the I.H.:

$$\ulcorner\mu_0\ldots\mu_{j\text{-}1}\mu'^{\mu'_0}_0\ldots\mu'^{\mu'_0}_{\mathrm{EXPL}(\mu'_0)\text{-}1}\ldots\mu'^{\mu'_{s\text{-}1}}_0\ldots\mu'^{\mu'_{s\text{-}1}}_{\mathrm{EXPL}(\mu'_{s\text{-}1})\text{-}1}\mu_{j+1}\ldots\mu_{k\text{-}1}\urcorner$$
$$=$$
$$\ulcorner\mu_0\ldots\mu_{j\text{-}1}\mu'_0\ldots\mu'_l\mu'^{\mu'_{l+1}}_0\ldots\mu'^{\mu'_{l+1}}_{\mathrm{EXPL}(\mu'_{l+1})\text{-}1}\ldots\mu'^{\mu'_{s\text{-}1}}_0\ldots\mu'^{\mu'_{s\text{-}1}}_{\mathrm{EXPL}(\mu'_{s\text{-}1})\text{-}1}\mu_{j+1}\ldots\mu_{k\text{-}1}\urcorner.$$

Since with $0 < i$, we have $i\text{-}1 < i$, we thus have

$$\ulcorner\mu_0\ldots\mu_{j\text{-}1}\mu'^{\mu'_0}_0\ldots\mu'^{\mu'_0}_{\mathrm{EXPL}(\mu'_0)\text{-}1}\ldots\mu'^{\mu'_{s\text{-}1}}_0\ldots\mu'^{\mu'_{s\text{-}1}}_{\mathrm{EXPL}(\mu'_{s\text{-}1})\text{-}1}\mu_{j+1}\ldots\mu_{k\text{-}1}\urcorner$$
$$=$$
$$\ulcorner\mu_0\ldots\mu_{j\text{-}1}\mu'_0\ldots\mu'_{i\text{-}1}\mu'^{\mu'_i}_0\ldots\mu'^{\mu'_i}_{\mathrm{EXPL}(\mu'_i)\text{-}1}\ldots\mu'^{\mu'_{s\text{-}1}}_0\ldots\mu'^{\mu'_{s\text{-}1}}_{\mathrm{EXPL}(\mu'_{s\text{-}1})\text{-}1}\mu_{j+1}\ldots\mu_{k\text{-}1}\urcorner.$$

Since $\mu'_i = \ulcorner\mu'^{\mu'_i}_0\ldots\mu'^{\mu'_i}_{\mathrm{EXPL}(\mu'_i)\text{-}1}\urcorner$, we then have, with Postulate 1-3:

$$\ulcorner\mu_0\ldots\mu_{j\text{-}1}\mu'^{\mu'_0}_0\ldots\mu'^{\mu'_0}_{\mathrm{EXPL}(\mu'_0)\text{-}1}\ldots\mu'^{\mu'_{s\text{-}1}}_0\ldots\mu'^{\mu'_{s\text{-}1}}_{\mathrm{EXPL}(\mu'_{s\text{-}1})\text{-}1}\mu_{j+1}\ldots\mu_{k\text{-}1}\urcorner$$
$$=$$
$$\ulcorner\mu_0\ldots\mu_{j\text{-}1}\mu'_0\ldots\mu'_{i\text{-}1}\mu'^{\mu'_i}_0\ldots\mu'^{\mu'_i}_{\mathrm{EXPL}(\mu'_i)\text{-}1}\ldots\mu'^{\mu'_{s\text{-}1}}_0\ldots\mu'^{\mu'_{s\text{-}1}}_{\mathrm{EXPL}(\mu'_{s\text{-}1})\text{-}1}\mu_{j+1}\ldots\mu_{k\text{-}1}\urcorner$$
$$=$$
$$\ulcorner\mu_0\ldots\mu_{j\text{-}1}\mu'_0\ldots\mu'_i\mu'^{\mu'_{i+1}}_0\ldots\mu'^{\mu'_{i+1}}_{\mathrm{EXPL}(\mu'_{i+1})\text{-}1}\ldots\mu'^{\mu'_{s\text{-}1}}_0\ldots\mu'^{\mu'_{s\text{-}1}}_{\mathrm{EXPL}(\mu'_{s\text{-}1})\text{-}1}\mu_{j+1}\ldots\mu_{k\text{-}1}\urcorner.$$

Hence the statement holds for all $i < s$ and the theorem follows as indicated above. ■

Theorem 1-7. *Unique initial and end expressions*
If $\mu, \mu', \mu^*, \mu^+ \in$ EXP, then:
(i) If $\ulcorner\mu\mu^*\urcorner = \ulcorner\mu\mu^+\urcorner$, then: $\mu^* = \mu^+$,
(ii) If $\ulcorner\mu^*\mu\urcorner = \ulcorner\mu^+\mu\urcorner$, then: $\mu^* = \mu^+$, and
(iii) If $\mu, \mu' \in$ BEXP and $\ulcorner\mu\mu^*\urcorner = \ulcorner\mu'\mu^+\urcorner$, then $\mu = \mu'$.

Proof: Suppose $\mu, \mu', \mu^*, \mu^+ \in$ EXP. Then there are $i \in \mathbb{N}\setminus\{0\}$ such that $\{\mu_0, \ldots, \mu_{i-1}\} \subseteq$ BEXP and $\mu = \ulcorner\mu_0\ldots\mu_{i-1}\urcorner$, and $j \in \mathbb{N}\setminus\{0\}$ such that $\{\mu^*_0, \ldots, \mu^*_{j-1}\} \subseteq$ BEXP and $\mu^* = \ulcorner\mu^*_0\ldots\mu^*_{j-1}\urcorner$, and $k \in \mathbb{N}\setminus\{0\}$ such that $\{\mu^+_0, \ldots, \mu^+_{k-1}\} \subseteq$ BEXP and $\mu^+ = \ulcorner\mu^+_0\ldots\mu^+_{k-1}\urcorner$. Now, suppose for (i) that $\ulcorner\mu\mu^*\urcorner = \ulcorner\mu\mu^+\urcorner$. Then it holds, with Theorem 1-5-(i), that $i+j = i+k$ and hence $j = k$. With Theorem 1-5-(iii), we then have $\mu^* = \mu^+$. (ii) follows analogously. Now, for (iii), suppose $\mu, \mu' \in$ BEXP and $\ulcorner\mu\mu^*\urcorner = \ulcorner\mu'\mu^+\urcorner$. With EXPL($\mu$) = 1 = EXPL($\mu'$) and Theorem 1-5-(iii), we then have $\mu = \mu'$. ■

Theorem 1-8. *No expression properly contains itself*
If $\mu', \mu^*, \mu^+ \in$ EXP, then:
(i) $\mu' \neq \ulcorner\mu'\mu^*\urcorner$,
(ii) $\mu' \neq \ulcorner\mu^*\mu'\mu^+\urcorner$, and
(iii) $\mu' \neq \ulcorner\mu^*\mu'\urcorner$.

Proof: Suppose $\mu', \mu^*, \mu^+ \in$ EXP. Then there are $i \in \mathbb{N}\setminus\{0\}$ such that $\{\mu'_0, \ldots, \mu'_{i-1}\} \subseteq$ EXP and $\mu' = \ulcorner\mu'_0\ldots\mu'_{i-1}\urcorner$, and $j \in \mathbb{N}\setminus\{0\}$ such that $\{\mu^*_0, \ldots, \mu^*_{j-1}\} \subseteq$ EXP and $\mu^* = \ulcorner\mu^*_0\ldots\mu^*_{j-1}\urcorner$, and $k \in \mathbb{N}\setminus\{0\}$ such that $\{\mu^+_0, \ldots, \mu^+_{k-1}\} \subseteq$ EXP and $\mu^+ = \ulcorner\mu^+_0\ldots\mu^+_{k-1}\urcorner$. Assume for contradiction that $\mu' = \ulcorner\mu'\mu^*\urcorner$ or $\mu' = \ulcorner\mu^*\mu'\mu^+\urcorner$ or $\mu' = \ulcorner\mu^*\mu'\urcorner$. With Theorem 1-5-(ii), we would then have $i = i+j$ or $i = j+i+k$ or $i = j+i$ and, on the other hand, with $i, j, k \in \mathbb{N}\setminus\{0\}$: $i \neq i+j$ and $i \neq j+i+k$ and $i \neq j+i$. Contradiction! Therefore $\mu' \neq \ulcorner\mu'\mu^*\urcorner$ and $\mu' \neq \ulcorner\mu^*\mu'\mu^+\urcorner$ and $\mu' \neq \ulcorner\mu^*\mu'\urcorner$. ■

Now, all operators can be assigned an arity, where the category of the operators described in Definition 1-5-(vi) will be defined as the category of quantifiers further below in Definition 1-8. Following the definition of arity, we can also define the categories of terms and formulas and subsequently prove the unique readability for the categories established by then. Afterwards, we will introduce further grammatical concepts up to sentence sequences.

Definition 1-5. *Arity*

μ <u>is</u> i-<u>ary</u>

iff

(i) μ ∈ FUNC and there is $j \in \mathbb{N}$ such that μ = ⌜$f_{i.j}$⌝ or

(ii) μ ∈ PRED and there is $j \in \mathbb{N}$ such that μ = ⌜$P_{i.j}$⌝ or

(iii) μ = ⌜=⌝ and $i = 2$ or

(iv) μ = ⌜¬⌝ and $i = 1$ or

(v) μ ∈ CON\{⌜¬⌝} and $i = 2$ or

(vi) There are Π ∈ QUANT and ξ ∈ VAR and μ = ⌜Πξ⌝ and $i = 1$ or

(vii) μ ∈ PERF and $i = 1$.

Definition 1-6. *The set of terms (TERM; metavariables:* θ, θ', θ*, …*)*

TERM = ∩{R | R ⊆ EXP and

(i) CONST ∪ PAR ∪ VAR ⊆ R, and

(ii) If $\{\theta_0, \ldots, \theta_{n-1}\} \subseteq R$ and φ ∈ FUNC n-ary, then ⌜$\varphi(\theta_0, \ldots, \theta_{n-1})$⌝ ∈ R}.

Note: Here and in the following, blanks only serve the purpose of easing readability, blanks are not a part of the expressions. So, for example, ⌜$f_{3.1}(c_0,\ c_0,\ c_1)$⌝ stands for ⌜$f_{3.1}(c_0,c_0,c_1)$⌝.

Definition 1-7. *Atomic and functional terms (ATERM and FTERM)*

(i) ATERM = CONST ∪ PAR ∪ VAR,

(ii) FTERM = TERM\ATERM.

Definition 1-8. *The set of quantifiers (QUANTOR)*

QUANTOR = {⌜Πξ⌝ | Π ∈ QUANT and ξ ∈ VAR}.

Definition 1-9. *The set of formulas (FORM; metavariables:* A, B, Γ, Δ, A', B', Γ', Δ', A*, B*, Γ*, Δ*, …*)*

FORM = ∩{R | R ⊆ EXP and

(i) If $\{\theta_0, \ldots, \theta_{n-1}\} \subseteq$ TERM and Φ ∈ PRED n-ary, then ⌜$\Phi(\theta_0, \ldots, \theta_{n-1})$⌝ ∈ R,

(ii) If Δ ∈ R, then ⌜¬Δ⌝ ∈ R,

(iii) If $\Delta_0, \Delta_1 \in R$ and ψ ∈ CON\{⌜¬⌝}, then ⌜$(\Delta_0\ \psi\ \Delta_1)$⌝ ∈ R, and

(iv) If Δ ∈ R and ξ ∈ VAR and Π ∈ QUANT, then ⌜ΠξΔ⌝ ∈ R}.

Definition 1-10. *Atomic, connective and quantificational formulas (AFORM, CONFORM, QFORM)*

(i) AFORM = $\{\ulcorner\Phi(\theta_0, \ldots, \theta_{n\text{-}1})\urcorner \mid \Phi \in$ PRED n-ary and $\{\theta_0, \ldots, \theta_{n\text{-}1}\} \subseteq$ TERM$\}$,

(ii) CONFORM = $\{\ulcorner\neg\Delta\urcorner \mid \Delta \in$ FORM$\} \cup \{\ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner \mid \Delta_0, \Delta_1 \in$ FORM and $\psi \in$ CON$\backslash\{\ulcorner\neg\urcorner\}\}$,

(iii) QFORM = $\{\ulcorner\Pi\xi\Delta\urcorner \mid \Delta \in$ FORM and $\Pi \in$ QUANT und $\xi \in$ VAR$\}$.

The following theorem leads directly to the theorems on unique readability.

Theorem 1-9. *Terms resp. formulas do not have terms resp. formulas as proper initial expressions*

(i) If $\theta, \theta' \in$ TERM and $\mu \in$ EXP, then $\theta' \neq \ulcorner\theta\mu\urcorner$, and

(ii) If $\Delta, \Delta' \in$ FORM and $\mu \in$ EXP, then $\Delta' \neq \ulcorner\Delta\mu\urcorner$.

Proof: *Ad (i)*: Suppose $\theta, \theta' \in$ TERM and $\mu \in$ EXP. The proof is carried out by induction on EXPL(θ'). For this, suppose the statement holds for all $\theta^* \in$ TERM with EXPL(θ^*) < EXPL(θ'). For EXPL(θ') = 1, and thus $\theta' \in$ ATERM, the statement holds trivially, because, according to Postulate 1-2-(ii), there are no $\theta, \mu \in$ EXP such that $\theta' = \ulcorner\theta\mu\urcorner$. Now, suppose 1 < EXPL($\theta'$). Then $\theta' \notin$ ATERM and therefore $\theta' \in$ FTERM. Then there are $n' \in \mathbb{N}\backslash\{0\}$ and $\varphi' \in$ FUNC, φ' n'-ary, and $\{\theta'_0, \ldots, \theta'_{n'\text{-}1}\} \subseteq$ TERM such that $\theta' = \ulcorner\varphi'(\theta'_0, \ldots, \theta'_{n'\text{-}1})\urcorner$. Suppose for contradiction that $\theta' = \ulcorner\theta\mu\urcorner$. Now, suppose for contradiction that $\theta \in$ ATERM. Then, we would have $\theta \in$ CONST $\cup$ PAR $\cup$ VAR. According to Theorem 1-7-(iii) and with $\ulcorner\varphi'(\theta'_0, \ldots, \theta'_{n'\text{-}1})\urcorner = \theta' = \ulcorner\theta\mu\urcorner$, we would then have that $\varphi' = \theta \in$ CONST $\cup$ PAR $\cup$ VAR. Contradiction! Therefore $\theta \in$ FTERM and there are thus $n \in \mathbb{N}\backslash\{0\}$ and $\varphi \in$ FUNC, φ n-ary, and $\{\theta_0, \ldots, \theta_{n\text{-}1}\} \subseteq$ TERM such that $\theta = \ulcorner\varphi(\theta_0, \ldots, \theta_{n\text{-}1})\urcorner$. Therefore $\ulcorner\varphi'(\theta'_0, \ldots, \theta'_{n'\text{-}1})\urcorner = \ulcorner\varphi(\theta_0, \ldots, \theta_{n\text{-}1})\mu\urcorner$. Then it holds with Theorem 1-7-(iii) that $\varphi' = \varphi$ and thus, according to Definition 1-5 and Postulate 1-1-(iv), we have $n = n'$. Therefore $\ulcorner\varphi(\theta'_0, \ldots, \theta'_{n\text{-}1})\urcorner = \ulcorner\varphi(\theta_0, \ldots, \theta_{n\text{-}1})\mu\urcorner$. Note that EXPL($\theta'_i$), EXPL($\theta_i$) < EXPL($\theta'$) for all $i < n$.

With $\{\mu\} \cup$ TERM $\subseteq$ EXP, it then holds that there are $\{\mu^*_0, \ldots, \mu^*_{\mathrm{EXPL}(\mu)\text{-}1}\} \subseteq$ BEXP and $\{\mu^{\theta'_0}_0, \ldots, \mu^{\theta'_0}_{\mathrm{EXPL}(\theta'_0)\text{-}1}\} \cup \ldots \cup \{\mu^{\theta'_{n\text{-}1}}_0, \ldots, \mu^{\theta'_{n\text{-}1}}_{\mathrm{EXPL}(\theta'_{n\text{-}1})\text{-}1}\} \subseteq$ BEXP and $\{\mu^{\theta_0}_0, \ldots, \mu^{\theta_0}_{\mathrm{EXPL}(\theta_0)\text{-}1}\} \cup \ldots \cup \{\mu^{\theta_{n\text{-}1}}_0, \ldots, \mu^{\theta_{n\text{-}1}}_{\mathrm{EXPL}(\theta_{n\text{-}1})\text{-}1}\} \subseteq$ BEXP such that $\mu = \ulcorner\mu^*_0\ldots\mu^*_{\mathrm{EXPL}(\mu)\text{-}1}\urcorner$ and for all $i < n$: $\theta'_i = \ulcorner\mu^{\theta'_i}_0\ldots\mu^{\theta'_i}_{\mathrm{EXPL}(\theta'_i)\text{-}1}\urcorner$ and $\theta_i = \ulcorner\mu^{\theta_i}_0\ldots\mu^{\theta_i}_{\mathrm{EXPL}(\theta_i)\text{-}1}\urcorner$. With Theorem 1-5-(i), it then holds that

$$\ulcorner\varphi(\mu^{\theta'_0}_0\ldots\mu^{\theta'_0}_{\mathrm{EXPL}(\theta'_0)\text{-}1}, \ldots, \mu^{\theta'_{n\text{-}1}}_0\ldots\mu^{\theta'_{n\text{-}1}}_{\mathrm{EXPL}(\theta'_{n\text{-}1})\text{-}1})\urcorner$$

$$= \ulcorner\varphi(\mu^{\theta_0}_0\ldots\mu^{\theta_0}_{EXPL(\theta_0)-1}, \ldots, \mu^{\theta_{n-1}}_0\ldots\mu^{\theta_{n-1}}_{EXPL(\theta_{n-1})-1})\mu^*_0\ldots\mu^*_{EXPL(\mu)-1}\urcorner$$

and thus with Theorem 1-7-(i)

$$\ulcorner\mu^{\theta'_0}_0\ldots\mu^{\theta'_0}_{EXPL(\theta'_0)-1}, \ldots, \mu^{\theta'_{n-1}}_0\ldots\mu^{\theta'_{n-1}}_{EXPL(\theta'_{n-1})-1})\urcorner$$
$$=$$
$$\ulcorner\mu^{\theta_0}_0\ldots\mu^{\theta_0}_{EXPL(\theta_0)-1}, \ldots, \mu^{\theta_{n-1}}_0\ldots\mu^{\theta_{n-1}}_{EXPL(\theta_{n-1})-1})\mu^*_0\ldots\mu^*_{EXPL(\mu)-1}\urcorner.$$

Suppose for contradiction that $EXPL(\theta'_i) = EXPL(\theta_i)$ for all $i < n$. With Theorem 1-5-(iii) and Theorem 1-7-(i), we would then have that $\ulcorner)\urcorner = \ulcorner)\mu^*_0\ldots\mu^*_{EXPL(\mu)-1}\urcorner$, whereas, with Postulate 1-2-(ii), we have that $\ulcorner)\urcorner \neq \ulcorner)\mu^*_0\ldots\mu^*_{EXPL(\mu)-1}\urcorner$. Contradiction! Thus there is an $l < n$ with $EXPL(\theta'_l) \neq EXPL(\theta_l)$. Let i be the smallest such l and suppose first that $EXPL(\theta'_i) < EXPL(\theta_i)$. Suppose $i = 0$. It then follows, with Theorem 1-5-(iii), that for all $j < EXPL(\theta'_0)$: $\mu^{\theta'_0}_j = \mu^{\theta_0}_j$ and thus, with Postulate 1-2-(i), we have that $\theta'_0 = \ulcorner\mu^{\theta'_0}_0\ldots\mu^{\theta'_0}_{EXPL(\theta'_0)-1}\urcorner = \ulcorner\mu^{\theta_0}_0\ldots\mu^{\theta_0}_{EXPL(\theta'_0)-1}\urcorner$. Because of $EXPL(\theta'_0) < EXPL(\theta_0)$ it then follows, with Theorem 1-6, that $\ulcorner\theta'_0\mu^{\theta_0}_{EXPL(\theta'_0)}\ldots\mu^{\theta_0}_{EXPL(\theta_0)-1}\urcorner = \ulcorner\mu^{\theta_0}_0\ldots\mu^{\theta_0}_{EXPL(\theta'_0)-1}\mu^{\theta_0}_{EXPL(\theta'_0)}\ldots\mu^{\theta_0}_{EXPL(\theta_0)-1}\urcorner = \ulcorner\mu^{\theta_0}_0\ldots\mu^{\theta_0}_{EXPL(\theta_0)-1}\urcorner = \theta_0$, which contradicts the I.H. Suppose $i > 0$. Then it holds, with Theorem 1-5-(iii), that

$$\ulcorner\mu^{\theta'_0}_0\ldots\mu^{\theta'_0}_{EXPL(\theta'_0)-1}, \ldots, \mu^{\theta'_{i-1}}_0\ldots\mu^{\theta'_{i-1}}_{EXPL(\theta'_{i-1})-1},\urcorner$$
$$=$$
$$\ulcorner\mu^{\theta_0}_0\ldots\mu^{\theta_0}_{EXPL(\theta_0)-1}, \ldots, \mu^{\theta_{i-1}}_0\ldots\mu^{\theta_{i-1}}_{EXPL(\theta_{i-1})-1},\urcorner.$$

Therefore with Theorem 1-7-(i):

$$\ulcorner\mu^{\theta'_i}_0\ldots\mu^{\theta'_i}_{EXPL(\theta'_i)-1}, \ldots, \mu^{\theta'_{n-1}}_0\ldots\mu^{\theta'_{n-1}}_{EXPL(\theta'_{n-1})-1})\urcorner$$
$$=$$
$$\ulcorner\mu^{\theta_i}_0\ldots\mu^{\theta_i}_{EXPL(\theta_i)-1}, \ldots, \mu^{\theta_{n-1}}_0\ldots\mu^{\theta_{n-1}}_{EXPL(\theta_{n-1})-1})\mu^*_0\ldots\mu^*_{EXPL(\mu)-1}\urcorner.$$

With Theorem 1-5-(iii), we then have that for all $j < EXPL(\theta'_i)$ it holds that $\mu^{\theta'_i}_j = \mu^{\theta_i}_j$ and thus, with Postulate 1-2-(i), that $\theta'_i = \ulcorner\mu^{\theta'_i}_0\ldots\mu^{\theta'_i}_{EXPL(\theta'_i)-1}\urcorner = \ulcorner\mu^{\theta_i}_0\ldots\mu^{\theta_i}_{EXPL(\theta'_i)-1}\urcorner$. Because of $EXPL(\theta'_i) < EXPL(\theta_i)$ it then follows, with Theorem 1-6, that $\ulcorner\theta'_i\mu^{\theta_i}_{EXPL(\theta'_i)}\ldots\mu^{\theta_i}_{EXPL(\theta_i)-1}\urcorner = \ulcorner\mu^{\theta_i}_0\ldots\mu^{\theta_i}_{EXPL(\theta'_i)-1}\mu^{\theta_i}_{EXPL(\theta'_i)}\ldots\mu^{\theta_i}_{EXPL(\theta_i)-1}\urcorner = \ulcorner\mu^{\theta_i}_0\ldots\mu^{\theta_i}_{EXPL(\theta_i)-1}\urcorner = \theta_i$, which also contradicts the I.H. In case of $EXPL(\theta_i) < EXPL(\theta'_i)$, a contradiction follows analogously. Hence the assumption that $\theta' = \ulcorner\theta\mu\urcorner$ for a $\theta \in TERM$ leads to a contradiction.

Ad (ii): Now, suppose $\Delta, \Delta' \in FORM$ and $\mu \in EXP$. The proof is carried out by induction on $EXPL(\Delta')$. For this, suppose the statement holds for all $\Delta^* \in FORM$ with

$\text{EXPL}(\Delta^*) < \text{EXPL}(\Delta')$. With $\Delta' \in \text{FORM}$, we have $\Delta' \in \text{AFORM} \cup \{\ulcorner\neg\Delta^*\urcorner \mid \Delta^* \in \text{FORM}\} \cup \{\ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner \mid \Delta_0, \Delta_1 \in \text{FORM}$ and $\psi \in \text{CON}\backslash\{\ulcorner\neg\urcorner\}\} \cup \text{QFORM}$. These *four* cases are now considered separately.

First: Suppose $\Delta' \in \text{AFORM}$. The proof is carried out analogously to the induction step for (i) by applying (i). Suppose $\Delta' = \ulcorner\Delta\mu\urcorner$. With $\Delta' \in \text{AFORM}$ there are $n' \in \mathbb{N}\backslash\{0\}$ and $\Phi' \in \text{PRED}$ and $\{\theta'_0, \ldots, \theta'_{n'-1}\} \subseteq \text{TERM}$ such that $\Delta' = \ulcorner\Phi'(\theta'_0, \ldots, \theta'_{n'-1})\urcorner$. Suppose for contradiction that $\Delta \in \text{CONFORM} \cup \text{QFORM}$. Then there would be $\mu' \in \{\ulcorner\neg\urcorner, \ulcorner(\urcorner\} \cup \text{QUANT}$ and $\mu^* \in \text{EXP}$ such that $\Delta = \ulcorner\mu'\mu^*\urcorner$. Therefore, according to Theorem 1-6, $\ulcorner\Phi'(\theta'_0, \ldots, \theta'_{n'-1})\urcorner = \Delta' = \ulcorner\Delta\mu\urcorner = \ulcorner\mu'\mu^*\mu\urcorner$ and thus, according to Theorem 1-7-(iii), $\Phi' = \mu'$. Thus we would have that $\Phi' \in \{\ulcorner\neg\urcorner, \ulcorner(\urcorner\} \cup \text{QUANT}$. Contradiction! Therefore $\Delta \notin \text{CONFORM} \cup \text{QFORM}$ and thus $\Delta \in \text{AFORM}$. Thus there are $n \in \mathbb{N}\backslash\{0\}$ and $\Phi \in \text{PRED}$, Φ n-ary, and $\{\theta_0, \ldots, \theta_{n-1}\} \subseteq \text{TERM}$ such that $\Delta = \ulcorner\Phi(\theta_0, \ldots, \theta_{n-1})\urcorner$. Therefore $\ulcorner\Phi'(\theta'_0, \ldots, \theta'_{n'-1})\urcorner = \ulcorner\Phi(\theta_0, \ldots, \theta_{n-1})\mu\urcorner$. Then it holds with Theorem 1-7-(iii) that $\Phi' = \Phi$ and thus we have according to Definition 1-5 and Postulate 1-1-(v) that $n = n'$. Therefore $\ulcorner\Phi(\theta'_0, \ldots, \theta'_{n-1})\urcorner = \ulcorner\Phi(\theta_0, \ldots, \theta_{n-1})\mu\urcorner$. From here on, the proof for $\Delta' \in \text{AFORM}$ proceeds analogously to the induction step for (i), while the contradiction resulting here is not with the I.H., but with (i).

Second: Now, suppose $\Delta' \in \{\ulcorner\neg\Delta^*\urcorner \mid \Delta^* \in \text{FORM}\}$. Then there is $\Delta^\# \in \text{FORM}$ such that $\Delta' = \ulcorner\neg\Delta^\#\urcorner$, and also $\text{EXPL}(\Delta^\#) < \text{EXPL}(\Delta')$. Suppose $\Delta' = \ulcorner\Delta\mu\urcorner$ and thus $\ulcorner\Delta\mu\urcorner = \ulcorner\neg\Delta^\#\urcorner$. Suppose for contradiction that $\Delta \in \text{AFORM} \cup \{\ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner \mid \Delta_0, \Delta_1 \in \text{FORM}$ and $\psi \in \text{CON}\backslash\{\ulcorner\neg\urcorner\}\} \cup \text{QFORM}$. Then there would be $\mu' \in \text{PRED} \cup \{\ulcorner(\urcorner\} \cup \text{QUANT}$ and $\mu^* \in \text{EXP}$ such that $\Delta = \ulcorner\mu'\mu^*\urcorner$. Therefore according to Theorem 1-6 $\ulcorner\neg\Delta^\#\urcorner = \ulcorner\Delta\mu\urcorner = \ulcorner\mu'\mu^*\mu\urcorner$ and thus according to Theorem 1-7-(iii) $\ulcorner\neg\urcorner = \mu'$. Then we would have that $\ulcorner\neg\urcorner \in \text{PRED} \cup \{\ulcorner(\urcorner\} \cup \text{QUANT}$. Contradiction! Therefore $\Delta \in \{\ulcorner\neg\Delta^*\urcorner \mid \Delta^* \in \text{FORM}\}$ and there is $\Delta^+ \in \text{FORM}$ such that $\Delta = \ulcorner\neg\Delta^+\urcorner$. Therefore $\ulcorner\neg\Delta^\#\urcorner = \ulcorner\neg\Delta^+\mu\urcorner$. With Theorem 1-7-(i) one then has that $\Delta^\# = \ulcorner\Delta^+\mu\urcorner$, which contradicts the I.H.

Third: Now, suppose $\Delta' \in \{\ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner \mid \Delta_0, \Delta_1 \in \text{FORM}$ and $\psi \in \text{CON}\backslash\{\ulcorner\neg\urcorner\}\}$. Then there are $\Delta'_0, \Delta'_1 \in \text{FORM}$ and $\psi' \in \text{CON}\backslash\{\ulcorner\neg\urcorner\}$ such that $\Delta' = \ulcorner(\Delta'_0\ \psi'\ \Delta'_1)\urcorner$, and also $\text{EXPL}(\Delta'_0) < \text{EXPL}(\Delta')$ and $\text{EXPL}(\Delta'_1) < \text{EXPL}(\Delta')$. Suppose $\Delta' = \ulcorner\Delta\mu\urcorner$ and thus $\ulcorner\Delta\mu\urcorner = \ulcorner(\Delta'_0\ \psi'\ \Delta'_1)\urcorner$. Suppose for contradiction $\Delta \in \text{AFORM} \cup \{\ulcorner\neg\Delta^*\urcorner \mid \Delta^* \in \text{FORM}\} \cup \text{QFORM}$. Then there would be $\mu' \in \text{PRED} \cup \{\ulcorner\neg\urcorner\} \cup \text{QUANT}$ and $\mu^* \in \text{EXP}$ such that $\Delta = \ulcorner\mu'\mu^*\urcorner$, and therefore $\ulcorner(\Delta'_0\ \psi'\ \Delta'_1)\urcorner = \Delta' = \ulcorner\Delta\mu\urcorner = \ulcorner\mu'\mu^*\mu\urcorner$ and thus according to Theorem 1-7-(iii) $\ulcorner(\urcorner = \mu'$. Thus one would have that $\ulcorner(\urcorner \in \text{PRED} \cup \{\ulcorner\neg\urcorner\} \cup \text{QUANT}$.

Contradiction! Therefore $\Delta \in \{\ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner \mid \Delta_0, \Delta_1 \in \text{FORM}$ and $\psi \in \text{CON}\backslash\{\ulcorner\neg\urcorner\}\}$ and there are $\Delta_0, \Delta_1 \in$ FORM and $\psi \in \text{CON}\backslash\{\ulcorner\neg\urcorner\}$ such that $\Delta = \ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner$, and also EXPL($\Delta_0$), EXPL($\Delta_1$) < EXPL($\Delta'$). Therefore $\ulcorner(\Delta'_0\ \psi'\ \Delta'_1)\urcorner = \ulcorner(\Delta_0\ \psi\ \Delta_1)\mu\urcorner$. With Theorem 1-7-(i) it holds that $\ulcorner\Delta'_0\ \psi'\ \Delta'_1)\urcorner = \ulcorner\Delta_0\ \psi\ \Delta_1)\mu\urcorner$. With $\{\mu\} \cup \text{FORM} \subseteq \text{EXP}$ it also holds that there are $\{\mu^*_0, \ldots, \mu^*_{\text{EXPL}(\mu)-1}\} \subseteq$ BEXP and $\{\mu^{\Delta'_0}_0, \ldots, \mu^{\Delta'_0}_{\text{EXPL}(\Delta'_0)-1}\} \cup \{\mu^{\Delta'_1}_0, \ldots, \mu^{\Delta'_1}_{\text{EXPL}(\Delta'_1)-1}\} \subseteq$ BEXP and $\{\mu^{\Delta_0}_0, \ldots, \mu^{\Delta_0}_{\text{EXPL}(\Delta_0)-1}\} \cup \{\mu^{\Delta_1}_0, \ldots, \mu^{\Delta_1}_{\text{EXPL}(\Delta_1)-1}\} \subseteq$ BEXP such that $\mu = \ulcorner\mu^*_0\ldots\mu^*_{\text{EXPL}(\mu)-1}\urcorner$ and for all $i < 2$: $\Delta'_i = \ulcorner\mu^{\Delta'_i}_0\ldots\mu^{\Delta'_i}_{\text{EXPL}(\Delta'_i)-1}\urcorner$ and $\Delta_i = \ulcorner\mu^{\Delta_i}_0\ldots\mu^{\Delta_i}_{\text{EXPL}(\Delta_i)-1}\urcorner$. With Theorem 1-5-(i), we then have that

$$\ulcorner\mu^{\Delta'_0}_0\ldots\mu^{\Delta'_0}_{\text{EXPL}(\Delta'_0)-1}\psi'\mu^{\Delta'_1}_0\ldots\mu^{\Delta'_1}_{\text{EXPL}(\Delta'_1)-1})\urcorner$$
$$=$$
$$\ulcorner\mu^{\Delta_0}_0\ldots\mu^{\Delta_0}_{\text{EXPL}(\Delta_0)-1}\psi\mu^{\Delta_1}_0\ldots\mu^{\Delta_1}_{\text{EXPL}(\Delta_1)-1})\mu^*_0\ldots\mu^*_{\text{EXPL}(\mu)-1}\urcorner.$$

Now, suppose for contradiction that EXPL(Δ'_0) < EXPL(Δ_0). With Theorem 1-5-(iii), it then it holds for all j < EXPL(Δ'_0) that $\mu^{\Delta'_0}_j = \mu^{\Delta_0}_j$. With Postulate 1-2-(i), we then have $\Delta'_0 = \ulcorner\mu^{\Delta'_0}_0\ldots\mu^{\Delta'_0}_{\text{EXPL}(\Delta'_0)-1}\urcorner = \ulcorner\mu^{\Delta_0}_0\ldots\mu^{\Delta_0}_{\text{EXPL}(\Delta'_0)-1}\urcorner$. With Theorem 1-6, we then have that $\ulcorner\Delta'_0\mu^{\Delta_0}_{\text{EXPL}(\Delta'_0)}\ldots\mu^{\Delta_0}_{\text{EXPL}(\Delta_0)-1}\urcorner = \ulcorner\mu^{\Delta_0}_0\ldots\mu^{\Delta_0}_{\text{EXPL}(\Delta'_0)-1}\mu^{\Delta_0}_{\text{EXPL}(\Delta'_0)}\ldots\mu^{\Delta_0}_{\text{EXPL}(\Delta_0)-1}\urcorner = \ulcorner\mu^{\Delta_0}_0\ldots\mu^{\Delta_0}_{\text{EXPL}(\Delta_0)-1}\urcorner = \Delta_0$, which contradicts the I.H. In case of EXPL(Δ_0) < EXPL(Δ'_0), a contradiction follows analogously. Therefore one has that EXPL(Δ'_0) = EXPL(Δ_0). Thus it holds, with Theorem 1-5-(iii), that $\ulcorner\mu^{\Delta'_0}_0\ldots\mu^{\Delta'_0}_{\text{EXPL}(\Delta'_0)-1}\psi'\urcorner = \ulcorner\mu^{\Delta_0}_0\ldots\mu^{\Delta_0}_{\text{EXPL}(\Delta_0)-1}\psi\urcorner$ and thus, with Theorem 1-7-(i), also that $\ulcorner\mu^{\Delta'_1}_0\ldots\mu^{\Delta'_1}_{\text{EXPL}(\Delta'_1)-1})\urcorner = \ulcorner\mu^{\Delta_1}_0\ldots\mu^{\Delta_1}_{\text{EXPL}(\Delta_1)-1})\mu^*_0\ldots\mu^*_{\text{EXPL}(\mu)-1}\urcorner$. As we have just done for Δ'_0, Δ_0, we can show that EXPL(Δ'_1) = EXPL(Δ_1). But then we have, with Theorem 1-5-(iii), that $\Delta'_1 = \Delta_1$ and thus, with Theorem 1-7-(i), that $\ulcorner)\urcorner = \ulcorner)\mu^*_0\ldots\mu^*_{\text{EXPL}(\mu)-1}\urcorner$, which contradicts Postulate 1-2-(ii).

Fourth: Now, suppose $\Delta' \in$ QFORM. Then there are $\Delta^\# \in$ FORM and $\Pi' \in$ QUANT and $\xi' \in$ VAR such that $\Delta' = \ulcorner\Pi'\xi'\Delta^{\#}\urcorner$, and also EXPL($\Delta^\#$) < EXPL($\Delta'$). Suppose $\Delta' = \ulcorner\Delta\mu\urcorner$ and thus $\ulcorner\Delta\mu\urcorner = \ulcorner\Pi'\xi'\Delta^{\#}\urcorner$. Suppose for contradiction $\Delta \in$ AFORM $\cup$ CONFORM. Then there would be $\mu' \in \text{PRED} \cup \{\ulcorner\neg\urcorner, \ulcorner(\urcorner\}$ and $\mu^* \in$ EXP such that $\Delta = \ulcorner\mu'\mu^*\urcorner$. Therefore according to Theorem 1-6 $\ulcorner\Pi'\xi'\Delta^{\#}\urcorner = \ulcorner\Delta\mu\urcorner = \ulcorner\mu'\mu^*\mu\urcorner$ and thus $\Pi' = \mu'$. Thus we would have that $\Pi' \in \text{PRED} \cup \{\ulcorner\neg\urcorner, \ulcorner(\urcorner\}$. Contradiction! Therefore $\Delta \in$ QFORM and there are $\Delta^+ \in$ FORM and $\Pi \in$ QUANT and $\xi \in$ VAR such that $\Delta = \ulcorner\Pi\xi\Delta^{+}\urcorner$. Therefore $\ulcorner\Pi'\xi'\Delta^{\#}\urcorner = \ulcorner\Pi\xi\Delta^{+}\mu\urcorner$. With Theorem 1-7-(iii) and -(i), we then have first $\ulcorner\xi'\Delta^{\#}\urcorner = \ulcorner\xi\Delta^{+}\mu\urcorner$ and then $\Delta^\# = \ulcorner\Delta^{+}\mu\urcorner$, which contradicts the I.H.

Thus $\Delta' = \ulcorner\Delta\mu\urcorner$ leads to a contradiction in all four cases. Therefore $\Delta' \neq \ulcorner\Delta\mu\urcorner$. ■

Theorem 1-10. *Unique readability without sentences (a – unique categories)*

(i) CONST ∩ (PAR ∪ VAR ∪ FTERM ∪ QUANTOR ∪ AFORM ∪ {⌜¬Δ⌝ | Δ ∈ FORM} ∪ {⌜(Δ_0 ψ Δ_1)⌝ | Δ_0, Δ_1 ∈ FORM and ψ ∈ CON\{⌜¬⌝}} ∪ QFORM) = ∅,

(ii) PAR ∩ (CONST ∪ VAR ∪ FTERM ∪ QUANTOR ∪ AFORM ∪ {⌜¬Δ⌝ | Δ ∈ FORM} ∪ {⌜(Δ_0 ψ Δ_1)⌝ | Δ_0, Δ_1 ∈ FORM and ψ ∈ CON\{⌜¬⌝}} ∪ QFORM) = ∅,

(iii) VAR ∩ (CONST ∪ PAR ∪ FTERM ∪ QUANTOR ∪ AFORM ∪ {⌜¬Δ⌝ | Δ ∈ FORM} ∪ {⌜(Δ_0 ψ Δ_1)⌝ | Δ_0, Δ_1 ∈ FORM and ψ ∈ CON\{⌜¬⌝}} ∪ QFORM) = ∅,

(iv) FTERM ∩ (CONST ∪ PAR ∪ VAR ∪ QUANTOR ∪ AFORM ∪ {⌜¬Δ⌝ | Δ ∈ FORM} ∪ {⌜(Δ_0 ψ Δ_1)⌝ | Δ_0, Δ_1 ∈ FORM and ψ ∈ CON\{⌜¬⌝}} ∪ QFORM) = ∅,

(v) QUANTOR ∩ (CONST ∪ PAR ∪ VAR ∪ FTERM ∪ AFORM ∪ {⌜¬Δ⌝ | Δ ∈ FORM} ∪ {⌜(Δ_0 ψ Δ_1)⌝ | Δ_0, Δ_1 ∈ FORM and ψ ∈ CON\{⌜¬⌝}} ∪ QFORM) = ∅,

(vi) AFORM ∩ (CONST ∪ PAR ∪ VAR ∪ FTERM ∪ QUANTOR ∪ {⌜¬Δ⌝ | Δ ∈ FORM} ∪ {⌜(Δ_0 ψ Δ_1)⌝ | Δ_0, Δ_1 ∈ FORM and ψ ∈ CON\{⌜¬⌝}} ∪ QFORM) = ∅,

(vii) {⌜¬Δ⌝ | Δ ∈ FORM} ∩ (CONST ∪ PAR ∪ VAR ∪ FTERM ∪ QUANTOR ∪ AFORM ∪ {⌜(Δ_0 ψ Δ_1)⌝ | Δ_0, Δ_1 ∈ FORM and ψ ∈ CON\{⌜¬⌝}} ∪ QFORM) = ∅,

(viii) {⌜(Δ_0 ψ Δ_1)⌝ | Δ_0, Δ_1 ∈ FORM and ψ ∈ CON\{⌜¬⌝}} ∩ (CONST ∪ PAR ∪ VAR ∪ FTERM ∪ QUANTOR ∪ AFORM ∪ {⌜¬Δ⌝ | Δ ∈ FORM} ∪ QFORM) = ∅, and

(ix) QFORM ∩ (CONST ∪ PAR ∪ VAR ∪ FTERM ∪ QUANTOR ∪ AFORM ∪ {⌜¬Δ⌝ | Δ ∈ FORM} ∪ {⌜(Δ_0 ψ Δ_1)⌝ | Δ_0, Δ_1 ∈ FORM and ψ ∈ CON\{⌜¬⌝}}) = ∅.

Proof: Suppose μ ∈ CONST. According to Postulate 1-1, we then have that μ ∉ PAR ∪ VAR and, according to Definition 1-7, that μ ∉ FTERM. Suppose for contradiction that μ ∈ QUANTOR ∪ AFORM ∪ {⌜¬Δ⌝ | Δ ∈ FORM} ∪ {⌜(Δ_0 ψ Δ_1)⌝ | Δ_0, Δ_1 ∈ FORM and ψ ∈ CON\{⌜¬⌝}} ∪ QFORM. Then, there would be μ' ∈ BEXP and μ* ∈ EXP such that μ = ⌜μ'μ*⌝. This contradicts Postulate 1-2-(ii). Therefore μ ∉ QUANTOR ∪ AFORM ∪ {⌜¬Δ⌝ | Δ ∈ FORM} ∪ {⌜(Δ_0 ψ Δ_1)⌝ | Δ_0, Δ_1 ∈ FORM and ψ ∈ CON\{⌜¬⌝}} ∪ QFORM.

For μ ∈ PAR and μ ∈ VAR, the proof is carried out analogously.

Now, suppose μ ∈ FTERM. According to Definition 1-7, we then have μ ∉ CONST ∪ PAR ∪ VAR and we have μ ∈ TERM. According to Definition 1-6, there are thus φ ∈ FUNC and μ^+ ∈ EXP such that μ = ⌜φμ^+⌝. Suppose for contradiction that μ ∈ QUANTOR ∪ AFORM ∪ {⌜¬Δ⌝ | Δ ∈ FORM} ∪ {⌜(Δ_0 ψ Δ_1)⌝ | Δ_0, Δ_1 ∈ FORM and ψ ∈ CON\{⌜¬⌝}} ∪ QFORM. Then there would be μ' ∈ PRED ∪ QUANT ∪ {'¬', '('} and μ* ∈ EXP such that μ = ⌜μ'μ*⌝. According to Theorem 1-7-(iii), we would then have μ' = φ and thus μ' ∈ FUNC. This contradicts Postulate 1-1. Therefore μ ∉ QUANTOR ∪ AFORM ∪ {⌜¬Δ⌝ | Δ ∈ FORM} ∪ {⌜(Δ_0 ψ Δ_1)⌝ | Δ_0, Δ_1 ∈ FORM and ψ ∈ CON\{⌜¬⌝}} ∪ QFORM.

For $\mu \in$ QUANTOR, $\mu \in$ AFORM, $\mu \in \{\ulcorner\neg\Delta\urcorner \mid \Delta \in$ FORM$\}$, $\mu \in \{\ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner \mid \Delta_0, \Delta_1 \in$ FORM and $\psi \in$ CON$\setminus\{\ulcorner\neg\urcorner\}\}$ and $\mu \in$ QFORM, the proof is carried out analogously. ■

Theorem 1-11. *Unique readability without sentences (b – unique decomposability)*
If $\mu \in$ TERM $\cup$ QUANTOR $\cup$ FORM, then:

(i) $\mu \in$ ATERM or

(ii) $\mu \in$ FTERM and there are $n \in \mathbb{N}\setminus\{0\}$, $\varphi \in$ FUNC and $\{\theta_0, \ldots, \theta_{n-1}\} \subseteq$ TERM such that $\mu = \ulcorner\varphi(\theta_0, \ldots, \theta_{n-1})\urcorner$ and for all $n' \in \mathbb{N}\setminus\{0\}$, $\varphi' \in$ FUNC and $\{\theta'_0, \ldots, \theta'_{n'-1}\} \subseteq$ TERM with $\mu = \ulcorner\varphi'(\theta'_0, \ldots, \theta'_{n'-1})\urcorner$ it holds that $n = n'$ and $\varphi = \varphi'$ and for all $i < n$: $\theta_i = \theta'_i$, or

(iii) $\mu \in$ QUANTOR and there are $\Pi \in$ QUANT and $\xi \in$ VAR such that $\mu = \ulcorner\Pi\xi\urcorner$ and for all $\Pi' \in$ QUANT and $\xi' \in$ VAR with $\mu = \ulcorner\Pi'\xi'\urcorner$ it holds that $\Pi = \Pi'$ and $\xi = \xi'$, or

(iv) $\mu \in$ AFORM and there are $n \in \mathbb{N}\setminus\{0\}$, $\Phi \in$ PRED and $\{\theta_0, \ldots, \theta_{n-1}\} \subseteq$ TERM such that $\mu = \ulcorner\Phi(\theta_0, \ldots, \theta_{n-1})\urcorner$ and for all $n' \in \mathbb{N}\setminus\{0\}$, $\Phi' \in$ PRED and $\{\theta'_0, \ldots, \theta'_{n'-1}\} \subseteq$ TERM with $\mu = \ulcorner\Phi'(\theta'_0, \ldots, \theta'_{n'-1})\urcorner$ it holds that $n = n'$ and $\Phi = \Phi'$ and for all $i < n$: $\theta_i = \theta'_i$, or

(v) $\mu \in \{\ulcorner\neg\Delta\urcorner \mid \Delta \in$ FORM$\}$ and there is $\Delta \in$ FORM such that $\mu = \ulcorner\neg\Delta\urcorner$ and for all $\Delta' \in$ FORM with $\mu = \ulcorner\neg\Delta'\urcorner$ it holds that $\Delta = \Delta'$, or

(vi) $\mu \in \{\ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner \mid \Delta_0, \Delta_1 \in$ FORM and $\psi \in$ CON$\setminus\{\ulcorner\neg\urcorner\}\}$ and there are $\Delta_0, \Delta_1 \in$ FORM and $\psi \in$ CON$\setminus\{\ulcorner\neg\urcorner\}$ such that $\mu = \ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner$ and for all $\Delta'_0, \Delta'_1 \in$ FORM and $\psi' \in$ CON$\setminus\{\ulcorner\neg\urcorner\}$ with $\mu = \ulcorner(\Delta'_0\ \psi'\ \Delta'_1)\urcorner$ it holds that $\Delta_0 = \Delta'_0$ and $\Delta_1 = \Delta'_1$ and $\psi = \psi'$, or

(vii) $\mu \in$ QFORM and there are $\Pi \in$ QUANT, $\xi \in$ VAR and $\Delta \in$ FORM such that $\mu = \ulcorner\Pi\xi\Delta\urcorner$ and for all $\Pi' \in$ QUANT, $\xi' \in$ VAR and $\Delta' \in$ FORM with $\mu = \ulcorner\Pi'\xi'\Delta'\urcorner$ it holds that $\Pi = \Pi'$ and $\xi = \xi'$ and $\Delta = \Delta'$.

Proof: Suppose $\mu \in$ TERM $\cup$ QUANTOR $\cup$ FORM. Therefore $\mu \in$ ATERM $\cup$ FTERM $\cup$ QUANTOR $\cup$ AFORM $\cup$ $\{\ulcorner\neg\Delta\urcorner \mid \Delta \in$ FORM$\}$ $\cup$ $\{\ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner \mid \Delta_0, \Delta_1 \in$ FORM and $\psi \in$ CON$\setminus\{\ulcorner\neg\urcorner\}\}$ $\cup$ QFORM. These *seven* cases will be treated separately. *First*: Suppose $\mu \in$ ATERM. Then (i) is satisfied trivially.

Second: Suppose $\mu \in$ FTERM. According to Definition 1-6 and Definition 1-7, there are then $n \in \mathbb{N}\setminus\{0\}$, $\varphi \in$ FUNC and $\{\theta_0, \ldots, \theta_{n-1}\} \subseteq$ TERM such that $\mu = \ulcorner\varphi(\theta_0, \ldots, \theta_{n-1})\urcorner$. Now, let also $n' \in \mathbb{N}\setminus\{0\}$, $\varphi' \in$ FUNC and $\{\theta'_0, \ldots, \theta'_{n'-1}\} \subseteq$ TERM be such that $\mu = \ulcorner\varphi'(\theta'_0, \ldots, \theta'_{n'-1})\urcorner$. $\varphi = \varphi'$ follows from Theorem 1-7-(iii). With Theorem 1-7-(i), we thus have $\ulcorner\theta_0, \ldots, \theta_{n-1})\urcorner = \ulcorner\theta'_0, \ldots, \theta'_{n'-1})\urcorner$. By induction on i we will now show that for all $i \in \mathbb{N}$: If $i < n$, then $i < n'$ and $\theta_i = \theta'_i$. For this, suppose that the statement holds for all $k < i$. Suppose $i < n$. Suppose $i = 0$. We have that $0 < n'$. We also have that there are $\{\mu_0, \ldots,$

$\mu_{\text{EXPL}(\theta_0)-1}\} \cup \{\mu'_0, \ldots, \mu'_{\text{EXPL}(\theta'_0)-1}\} \subseteq$ BEXP such that $\theta_0 = \ulcorner\mu_0\ldots\mu_{\text{EXPL}(\theta_0)-1}\urcorner$ and $\theta'_0 = \ulcorner\mu'_0\ldots\mu'_{\text{EXPL}(\theta'_0)-1}\urcorner$ and thus, with Theorem 1-6, $\ulcorner\mu_0\ldots\mu_{\text{EXPL}(\theta_0)-1}, \ldots, \theta_{n-1})\urcorner = \ulcorner\mu'_0\ldots\mu'_{\text{EXPL}(\theta'_0)-1}, \ldots, \theta'_{n'-1})\urcorner$. Now, suppose $\text{EXPL}(\theta_0) < \text{EXPL}(\theta'_0)$. With Theorem 1-5-(iii), it would then hold for all $l < \text{EXPL}(\theta_0)$ that $\mu_l = \mu'_l$. With Postulate 1-2-(i), we would thus have $\theta_0 = \ulcorner\mu_0\ldots\mu_{\text{EXPL}(\theta_0)-1}\urcorner = \ulcorner\mu'_0\ldots\mu'_{\text{EXPL}(\theta_0)-1}\urcorner$. But then we would have, with Theorem 1-6, that $\ulcorner\theta_0\mu'_{\text{EXPL}(\theta_0)}\ldots\mu'_{\text{EXPL}(\theta'_0)-1}\urcorner = \ulcorner\mu'_0\ldots\mu'_{\text{EXPL}(\theta_0)-1}\mu'_{\text{EXPL}(\theta_0)}\ldots\mu'_{\text{EXPL}(\theta'_0)-1}\urcorner = \theta'_0$, which contradicts Theorem 1-9-(i). In the same way, a contradiction follows for $\text{EXPL}(\theta'_0) < \text{EXPL}(\theta_0)$. Therefore we have that $\text{EXPL}(\theta_0) = \text{EXPL}(\theta'_0)$ and thus, with Theorem 1-5-(iii), also $\theta_0 = \theta'_0$.

Now, suppose $0 < i$. Then it holds for all $k < i$ that $k < n$. With the I.H., we thus have for all $k < i$ that $k < n'$ and $\theta_k = \theta'_k$. With Theorem 1-5-(iii), we then have that $\ulcorner\theta_0, \ldots, \theta_{i-1}\urcorner = \ulcorner\theta'_0, \ldots, \theta'_{i-1}\urcorner$. We also have that $i\text{-}1 < n'$ and thus that $i \leq n'$. Suppose for contradiction that $i = n'$. Then we would have that $\ulcorner\theta_0, \ldots, \theta_{i-1}\urcorner = \ulcorner\theta'_0, \ldots, \theta'_{n'-1}\urcorner$. With Theorem 1-7-(i), we would then have that $\ulcorner, \theta_i, \ldots, \theta_{n-1})\urcorner = \ulcorner)\urcorner$, which contradicts Postulate 1-2-(ii). Thus we have $i < n'$. Again with Theorem 1-7-(i), we then have that $\ulcorner\theta_i, \ldots, \theta_{n-1})\urcorner = \ulcorner\theta'_i, \ldots, \theta'_{n'-1})\urcorner$. From this, we can derive $\theta_i = \theta'_i$ in the same way as $\theta_0 = \theta'_0$ for $i = 0$. Therefore it holds for all $i < n$ that $i < n'$ and $\theta_i = \theta'_i$. Analogously, we can show that for all $i < n'$ we have that $i < n$ and $\theta'_i = \theta_i$. Taken together, we thus have that $n = n'$ and that for all $i < n$: $\theta_i = \theta'_i$.

Third: Suppose $\mu \in$ QUANTOR. According to Definition 1-8, there are then $\Pi \in$ QUANT and $\xi \in$ VAR such that $\mu = \ulcorner\Pi\xi\urcorner$. Now, let also $\Pi' \in$ QUANT, $\xi' \in$ VAR such that $\mu = \ulcorner\Pi'\xi'\urcorner$. From Theorem 1-7-(iii) and -(i) follows immediately $\Pi = \Pi'$ and $\xi = \xi'$.

Fourth: Suppose $\mu \in$ AFORM. According to Definition 1-10-(i), there are then $n \in \mathbb{N}\setminus\{0\}$, $\Phi \in$ PRED and $\{\theta_0, \ldots, \theta_{n-1}\} \subseteq$ TERM such that $\mu = \ulcorner\Phi(\theta_0, \ldots, \theta_{n-1})\urcorner$. Let now also $n' \in \mathbb{N}\setminus\{0\}$, $\Phi' \in$ PRED and $\{\theta'_0, \ldots, \theta'_{n'-1}\} \subseteq$ TERM such that $\mu = \ulcorner\Phi'(\theta'_0, \ldots, \theta'_{n'-1})\urcorner$. $\Phi = \Phi'$ follows from Theorem 1-7-(iii). With Theorem 1-7-(i), we then get that $\ulcorner\theta_0, \ldots, \theta_{n-1})\urcorner = \ulcorner\theta'_0, \ldots, \theta'_{n'-1})\urcorner$. In the same way as in the second case, we can then show that $n = n'$ and that for all $i < n$: $\theta_i = \theta'_i$.

Fifth: Suppose $\mu \in \{\ulcorner\neg\Delta\urcorner \mid \Delta \in \text{FORM}\}$. Then there is $\Delta \in$ FORM such that $\mu = \ulcorner\neg\Delta\urcorner$. Now, suppose $\Delta' \in$ FORM and $\mu = \ulcorner\neg\Delta'\urcorner$. From Theorem 1-7-(i) follows immediately $\Delta = \Delta'$.

Sixth: Suppose $\mu \in \{\ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner \mid \Delta_0, \Delta_1 \in \text{FORM and } \psi \in \text{CON}\setminus\{\ulcorner\neg\urcorner\}\}$. Then there are $\Delta_0, \Delta_1 \in$ FORM and $\psi \in \text{CON}\setminus\{\ulcorner\neg\urcorner\}$ such that $\mu = \ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner$. Let now also Δ'_0, Δ'_1

$\in$ FORM and $\psi' \in$ CON\{⌜¬⌝} be such that μ = ⌜(Δ'_0 ψ' Δ'_1)⌝. With Theorem 1-7-(i), we then have ⌜Δ_0 ψ Δ_1)⌝ = ⌜Δ'_0 ψ' Δ'_1)⌝. Also, there is $\{\mu_0, \ldots, \mu_{\mathrm{EXPL}(\Delta_0)-1}\} \cup \{\mu'_0, \ldots, \mu'_{\mathrm{EXPL}(\Delta'_0)-1}\} \subseteq$ BEXP such that Δ_0 = ⌜$\mu_0\ldots\mu_{\mathrm{EXPL}(\Delta_0)-1}$⌝ and Δ'_0 = ⌜$\mu'_0\ldots\mu'_{\mathrm{EXPL}(\Delta'_0)-1}$⌝. Suppose for contradiction that EXPL(Δ_0) < EXPL(Δ'_0). WithTheorem 1-5-(iii), we would then have $\mu_i = \mu'_i$ for all i < EXPL(Δ_0). But then we would have, with Postulate 1-2-(i), that Δ_0 = ⌜$\mu_0\ldots\mu_{\mathrm{EXPL}(\Delta_0)-1}$⌝ = ⌜$\mu'_0\ldots\mu'_{\mathrm{EXPL}(\Delta_0)-1}$⌝. With Theorem 1-6, we would then have ⌜$\Delta_0\mu'_{\mathrm{EXPL}(\Delta_0)}\ldots\mu'_{\mathrm{EXPL}(\Delta'_0)-1}$⌝ = ⌜$\mu'_0\ldots\mu'_{\mathrm{EXPL}(\Delta_0)-1}\mu'_{\mathrm{EXPL}(\Delta_0)}\ldots\mu'_{\mathrm{EXPL}(\Delta'_0)-1}$⌝ = ⌜$\mu'_0\ldots\mu'_{\mathrm{EXPL}(\Delta'_0)-1}$⌝ = Δ'_0, which contradicts Theorem 1-9-(ii). Analogously, a contradiction follows from EXPL(Δ'_0) < EXPL(Δ_0). Therefore EXPL(Δ_0) = EXPL(Δ'_0) and thus Δ_0 = ⌜$\mu_0\ldots\mu_{\mathrm{EXPL}(\Delta_0)-1}$⌝ = ⌜$\mu'_0\ldots\mu'_{\mathrm{EXPL}(\Delta'_0)-1}$⌝ = Δ'_0. With Theorem 1-7, it then follows first that ⌜ψ Δ_1)⌝ = ⌜ψ' Δ'_1)⌝, then that $\psi = \psi'$, then that ⌜Δ_1)⌝ = ⌜Δ'_1)⌝ and finally that $\Delta_1 = \Delta'_1$.

Seventh: Suppose $\mu \in$ QFORM. According to Definition 1-10-(iii), there are then $\Pi \in$ QUANT, $\xi \in$ VAR and $\Delta \in$ FORM such that μ = ⌜$\Pi\xi\Delta$⌝. Let now also $\Pi' \in$ QUANT, $\xi' \in$ VAR, $\Delta' \in$ FORM such that μ = ⌜$\Pi'\xi'\Delta'$⌝. From Theorem 1-7-(iii) and -(i) follows immediately $\Pi = \Pi'$ and $\xi = \xi'$ and $\Delta = \Delta'$. ■

With Theorem 1-10 and Theorem 1-11, one can now define functions on the sets TERM, FORM and their union by recursion on the complexity of terms and formulas. The following definitions of the degree of a term and the degree of a formula (Definition 1-11 and Definition 1-12), allow us to prove properties of terms and formulas by induction on the natural numbers more conveniently then this can be done by using EXPL.

Definition 1-11. *Degree of a term*[8] *(TDEG)*

TDEG is a function on TERM and

(i) If $\theta \in$ ATERM, then TDEG(θ) = 0,

(ii) If ⌜$\varphi(\theta_0, \ldots, \theta_{n-1})$⌝ $\in$ FTERM, then
TDEG(⌜$\varphi(\theta_0, \ldots, \theta_{n-1})$⌝) = max({TDEG($\theta_0$), …, TDEG($\theta_{n-1}$)})+1.

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[8] Let 'min(..)' be defined as usual for non-empty subsets of $\mathbb{N}$ and 'max(..)' as usual for non-empty and finite subsets of $\mathbb{N}$. If X is not a non-empty subset of $\mathbb{N}$, let min(X) = 0, and if X is not a non-empty finite subset of $\mathbb{N}$, also let max(X) = 0.

Definition 1-12. *Degree of a formula (FDEG)*

FDEG is a function on FORM and

(i) If $\Delta \in$ AFORM, then FDEG(Δ) = 0,

(ii) If $\ulcorner\neg\Delta\urcorner \in$ CONFORM, then FDEG($\ulcorner\neg\Delta\urcorner$) = FDEG($\Delta$)+1,

(iii) If $\ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner \in$ CONFORM, then
FDEG($\ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner$) = max({FDEG($\Delta_0$), FDEG($\Delta_1$)})+1,

(iv) If $\ulcorner\Pi\xi\Delta\urcorner \in$ QFORM, then FDEG($\ulcorner\Pi\xi\Delta\urcorner$) = FDEG($\Delta$)+1.

We will henceforth use the usual infix notation without parentheses for identity formulas, e.g. $\ulcorner\theta = \theta^*\urcorner$ for $\ulcorner=(\theta, \theta^*)\urcorner$. Furthermore, we will often omit the outermost parentheses, e.g. $\ulcorner A\ \psi\ B\urcorner$ for $\ulcorner(A\ \psi\ B)\urcorner$. With Definition 1-13, we can now characterise the free variables of terms and formulas.

Definition 1-13. *Assignment of the set of variables that occur free in a term* θ *or in a formula* Γ *(FV)*

FV is a function on TERM $\cup$ FORM and

(i) If $\alpha \in$ CONST, then FV(α) = $\emptyset$,

(ii) If $\beta \in$ PAR, then FV(β) = $\emptyset$,

(iii) If $\xi \in$ VAR, then FV(ξ) = {ξ},

(iv) If $\ulcorner\varphi(\theta_0, \ldots, \theta_{n-1})\urcorner \in$ FTERM, then
FV($\ulcorner\varphi(\theta_0, \ldots, \theta_{n-1})\urcorner$) = $\bigcup$\{FV(θ_i) | $i < n$\},

(v) If $\ulcorner\Phi(\theta_0, \ldots, \theta_{n-1})\urcorner \in$ AFORM, then
FV($\ulcorner\Phi(\theta_0, \ldots, \theta_{n-1})\urcorner$) = $\bigcup$\{FV(θ_i) | $i < n$\},

(vi) If $\ulcorner\neg\Delta\urcorner \in$ CONFORM, then FV($\ulcorner\neg\Delta\urcorner$) = FV($\Delta$),

(vii) If $\ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner \in$ CONFORM, then FV($\ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner$) = FV($\Delta_0$) $\cup$ FV(Δ_1), and

(viii) If $\ulcorner\Pi\xi\Delta\urcorner \in$ QFORM and, then FV($\ulcorner\Pi\xi\Delta\urcorner$) = FV($\Delta$)\{$\xi$}.

Definition 1-14. *The set of closed terms (CTERM)*

CTERM = {θ | $\theta \in$ TERM and FV(θ) = $\emptyset$}.

Note that, according to Definition 1-14, parameters are closed terms.

Definition 1-15. *The set of closed formulas (CFORM)*
CFORM = {Δ | Δ ∈ FORM and FV(Δ) = ∅}.

Closed formulas are also called propositions. Note that closed formulas can have parameters among their subexpression (see Definition 1-20). Sentences are now defined as the result of applying a performator to a closed formula.

Definition 1-16. *The set of sentences (SENT; metavariables:* Σ, Σ', Σ*, …*)*
SENT = {⌜ΞΓ⌝ | Ξ ∈ PERF and Γ ∈ CFORM}.

Definition 1-17. *Assumption- and inference-sentences (ASENT and ISENT)*
(i) ASENT = {⌜Suppose Γ⌝ | Γ ∈ CFORM},
(ii) ISENT = {⌜Therefore Γ⌝ | Γ ∈ CFORM}.

Theorem 1-12. *Unique category and unique decomposability for sentences*
If Σ ∈ SENT, then Σ ∉ TERM ∪ QUANTOR ∪ FORM and
(i) Σ ∈ ASENT and Σ ∉ ISENT and there is Γ ∈ CFORM such that Σ = ⌜Suppose Γ⌝ and for all Γ' ∈ CFORM with Σ = ⌜Suppose Γ'⌝ holds: Γ = Γ', or
(ii) Σ ∈ ISENT and Σ ∉ ASENT and there is Γ ∈ CFORM such that Σ = ⌜Therefore Γ⌝ and for all Γ' ∈ CFORM with Σ = ⌜Therefore Γ'⌝ holds: Γ = Γ'.

Proof: Suppose Σ ∈ SENT. Then there are Ξ ∈ PERF and Γ ∈ CFORM such that Σ = ⌜ΞΓ⌝. If Σ ∈ TERM ∪ QUANTOR ∪ FORM, then we would have that Σ ∈ ATERM or Σ ∈ FTERM ∪ QUANTOR ∪ FORM. In the first case, we would have Σ ∈ BEXP, which contradicts Postulate 1-2-(ii). In the second case, there would be μ ∈ FUNC ∪ QUANT ∪ PRED ∪ {⌜¬⌝, ⌜(⌝} and μ' ∈ EXP such that Σ = ⌜μμ'⌝. Thus we would have Ξ = μ and therefore Ξ ∈ FUNC ∪ QUANT ∪ PRED ∪ {⌜¬⌝, ⌜(⌝}, which contradicts Postulate 1-1. Therefore Σ ∉ TERM ∪ QUANTOR ∪ FORM.

If now Σ ∈ SENT, then by Postulate 1-1-(viii) Σ ∈ ASENT or Σ ∈ ISENT. The *two* cases will be treated separately. *First*: Suppose Σ ∈ ASENT. Then there is Γ ∈ CFORM such that Σ = ⌜Suppose Γ⌝. If Σ ∈ ISENT, then there would be Γ* such that Σ = ⌜Therefore Γ*⌝ and thus, according to Theorem 1-7-(iii), ⌜Suppose⌝ = ⌜Therefore⌝. Then {⌜Suppose⌝, ⌜Therefore⌝} would not be a 2-element set, which contradicts Postulate 1-1-(viii). Therefore Σ ∉ ISENT. Now, suppose Γ' ∈ CFORM and Σ = ⌜Suppose Γ'⌝.

Then we have ⌜Suppose Γ⌝ = ⌜Suppose Γ'⌝. With Theorem 1-7-(i), it follows immediately that Γ = Γ'.

Second: Suppose Σ ∈ ISENT. Then there is Γ ∈ CFORM such that Σ = ⌜Therefore Γ⌝. For Σ ∈ ASENT we would again have a contradiction to Postulate 1-1-(viii). Therefore Σ ∉ ASENT. Now, suppose Γ' ∈ CFORM and Σ = ⌜Therefore Γ'⌝. Then we have ⌜Therefore Γ⌝ = ⌜Therefore Γ'⌝. With Theorem 1-7-(i), it follows immediately that Γ = Γ'. ■

With Theorem 1-12, we can now define functions on the set TERM ∪ FORM ∪ SENT by recursion on the complexity of terms, formulas and sentences.

Definition 1-18. *Assignment of the proposition of a sentence (P)*
P = {(⌜ΞΓ⌝, Γ) | Ξ ∈ PERF and Γ ∈ CFORM}.

Note: With Definition 1-16 and Theorem 1-12, it follows immediately that P is a function on SENT. Because of this, we use function notation: P(⌜ΞΓ⌝) = Γ. We now define the set of proper expressions as the union of the set of basic expressions and the grammatical categories.

Definition 1-19. *The set of proper expressions (PEXP)*
PEXP = BEXP ∪ QUANTOR ∪ TERM ∪ FORM ∪ SENT.

Definition 1-20. *The subexpression function (SE)*
SE is a function on PEXP and

(i) If τ ∈ BEXP, then SE(τ) = {τ},

(ii) If ⌜$\varphi(\theta_0, \ldots, \theta_{n-1})$⌝ ∈ FTERM, then
SE(⌜$\varphi(\theta_0, \ldots, \theta_{n-1})$⌝) = {⌜$\varphi(\theta_0, \ldots, \theta_{n-1})$⌝, φ} ∪ $\bigcup$\{SE(θ_i) | $i < n$\},

(iii) If ⌜Πξ⌝ ∈ QUANTOR, then SE(⌜Πξ⌝) = {⌜Πξ⌝, Π, ξ},

(iv) If ⌜$\Phi(\theta_0, \ldots, \theta_{n-1})$⌝ ∈ AFORM, then
SE(⌜$\Phi(\theta_0, \ldots, \theta_{n-1})$⌝) = {⌜$\Phi(\theta_0, \ldots, \theta_{n-1})$⌝, Φ} ∪ $\bigcup$\{SE(θ_i) | $i < n$\},

(v) If ⌜¬Δ⌝ ∈ CONFORM, then SE(⌜¬Δ⌝) = {⌜¬Δ⌝, ⌜¬⌝} ∪ SE(Δ),

(vi) If ⌜(Δ_0 ψ Δ_1)⌝ ∈ CONFORM, then
SE(⌜(Δ_0 ψ Δ_1)⌝) = {⌜(Δ_0 ψ Δ_1)⌝, ψ} ∪ SE(Δ_0) ∪ SE(Δ_1),

(vii) If ⌜ΠξΔ⌝ ∈ QFORM, then
SE(⌜ΠξΔ⌝) = {⌜ΠξΔ⌝} ∪ SE(⌜Πξ⌝) ∪ SE(Δ), and

(viii) If ⌜ΞΔ⌝ ∈ SENT, then SE(⌜ΞΔ⌝) = {⌜ΞΔ⌝, Ξ} ∪ SE(Δ).

Definition 1-21. *The subterm function (ST)*
ST is a function on TERM ∪ FORM ∪ SENT and for all τ ∈ TERM ∪ FORM ∪ SENT: ST(τ) = SE(τ) ∩ TERM.

Definition 1-22. *The subformula function (SF)*
SF is a function on FORM ∪ SENT and for all τ ∈ FORM ∪ SENT: SF(τ) = SE(τ) ∩ FORM.

The following definitions describe the syntax of L insofar as it goes beyond the sentence level. As before, we suppress explicit references to L. Definition 1-23 characterises sentence sequences as finite sequences of inference- and assumption-sentences:

Definition 1-23. *Sentence sequence (metavariables: $\mathfrak{H}$, $\mathfrak{H}'$, $\mathfrak{H}^*$, …)*
$\mathfrak{H}$ is a sentence sequence
iff
$\mathfrak{H}$ is a finite sequence and for all $i \in \mathrm{Dom}(\mathfrak{H})$ holds: $\mathfrak{H}_i \in$ SENT.

Definition 1-24. *The set of sentence sequences (SEQ)*
SEQ = {$\mathfrak{H}$ | $\mathfrak{H}$ is a sentence sequence}.

Definition 1-25. *Conclusion assignment (C)*
C = {($\mathfrak{H}$, Γ) | $\mathfrak{H}$ ∈ SEQ\{∅} and Γ = P($\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-1}$)}.

Note: From this definition it follows directly that C is a function on SEQ\{∅}.

Definition 1-26. *Assignment of the subset of a sequence $\mathfrak{H}$ whose members are the assumption-sentences of $\mathfrak{H}$ (AS)*
AS = {($\mathfrak{H}$, X) | $\mathfrak{H}$ ∈ SEQ and X = {(i, $\mathfrak{H}_i$) | $i \in \mathrm{Dom}(\mathfrak{H})$ and $\mathfrak{H}_i \in$ ASENT}}.

Definition 1-27. *Assignment of the set of assumptions (AP)*
AP = {($\mathfrak{H}$, X) | $\mathfrak{H}$ ∈ SEQ and X = {Γ | There is an $i \in \mathrm{Dom}(\mathrm{AS}(\mathfrak{H}))$ such that Γ = P($\mathfrak{H}_i$)}}.

Definition 1-28. *Assignment of the subset of a sequence $\mathfrak{H}$ whose members are the inference-sentences of $\mathfrak{H}$ (IS)*
IS = {($\mathfrak{H}$, X) | $\mathfrak{H}$ ∈ SEQ and X = {(i, $\mathfrak{H}_i$) | $i \in \mathrm{Dom}(\mathfrak{H})$ and $\mathfrak{H}_i \in$ ISENT}}.

Note: From these definitions it follows directly that AS, AP and IS are functions on SEQ.

Definition 1-29. *Assignment of the set of subterms of the members of a sequence $\mathfrak{H}$ (STSEQ)*
STSEQ = $\{(\mathfrak{H}, X) \mid \mathfrak{H} \in$ SEQ and $X = \bigcup\{$ST$(\mathfrak{H}_i) \mid i \in$ Dom$(\mathfrak{H})\}\}$.

Note: From this definition it follows directly that STSEQ a function on SEQ.

Definition 1-30. *Assignment of the set of subterms of the elements of a set of formulas X (STSF)*
STSF = $\{(X, Y) \mid X \subseteq$ FORM and $Y = \bigcup\{$ST(A) $\mid$ A $\in X\}\}$.

Note: From this definition, it follows directly that STSF is a function on Pot(FORM).

1.2 Substitution

Now the substitution concept is to be established. In this, we restrict the usual substitution concept: Only atomic terms are substituenda and only closed terms are substituentia. This makes it superfluous to rename bound variables in order to avoid variable clashes. The tasks that are fulfilled by free variables in many calculi and usually in model-theory are fulfilled by parameters , which are closed terms (see Definition 1-14), in the Speech Act Calculus as well as in the model-theory developed here. Furthermore, also sentences and sentence sequences are substitution bases and not just terms and formulas (clauses (ix) and (x) of Definition 1-31).

Definition 1-31. *Substitution of closed terms for atomic terms in terms, formulas, sentences and sentence sequences*[9]

Substitution is a 3-ary function on $\{\langle\langle\theta'_0, \ldots, \theta'_{k-1}\rangle, \langle\theta_0, \ldots, \theta_{k-1}\rangle, \mu\rangle \mid k \in \mathbb{N}\setminus\{0\}, \langle\theta'_0, \ldots, \theta'_{k-1}\rangle \in {}^{k}\text{CTERM}, \langle\theta_0, \ldots, \theta_{k-1}\rangle \in {}^{k}\text{ATERM}$ and $\mu \in \text{TERM} \cup \text{FORM} \cup \text{SENT} \cup \text{SEQ}\}$. '[.., .., ..]' is used as substitution operator. Values are assigned as follows:

(i) If $\theta^+ \in \text{ATERM}$ and $\theta^+ = \theta_{k-1}$, then $[\langle\theta'_0, \ldots, \theta'_{k-1}\rangle, \langle\theta_0, \ldots, \theta_{k-1}\rangle, \theta^+] = \theta'_{k-1}$,

(ii) If $\theta^+ \in \text{ATERM}$, $\theta^+ \neq \theta_{k-1}$ and $k = 1$, then $[\langle\theta'_0, \ldots, \theta'_{k-1}\rangle, \langle\theta_0, \ldots, \theta_{k-1}\rangle, \theta^+] = \theta^+$,

(iii) If $\theta^+ \in \text{ATERM}$, $\theta^+ \neq \theta_{k-1}$ and $k \neq 1$, then
$[\langle\theta'_0, \ldots, \theta'_{k-1}\rangle, \langle\theta_0, \ldots, \theta_{k-1}\rangle, \theta^+] = [\langle\theta'_0, \ldots, \theta'_{k-2}\rangle, \langle\theta_0, \ldots, \theta_{k-2}\rangle, \theta^+]$,

(iv) If $\ulcorner\varphi(\theta^*_0, \ldots, \theta^*_{l-1})\urcorner \in \text{FTERM}$, then
$[\langle\theta'_0, \ldots, \theta'_{k-1}\rangle, \langle\theta_0, \ldots, \theta_{k-1}\rangle, \ulcorner\varphi(\theta^*_0, \ldots, \theta^*_{l-1})\urcorner]$
$= \ulcorner\varphi([\langle\theta'_0, \ldots, \theta'_{k-1}\rangle, \langle\theta_0, \ldots, \theta_{k-1}\rangle, \theta^*_0], \ldots, [\langle\theta'_0, \ldots, \theta'_{k-1}\rangle, \langle\theta_0, \ldots, \theta_{k-1}\rangle, \theta^*_{l-1}])\urcorner$,

(v) If $\ulcorner\Phi(\theta_0, \ldots, \theta_{l-1})\urcorner \in \text{AFORM}$, then
$[\langle\theta'_0, \ldots, \theta'_{k-1}\rangle, \langle\theta_0, \ldots, \theta_{k-1}\rangle, \ulcorner\Phi(\theta^*_0, \ldots, \theta^*_{l-1})\urcorner]$
$= \ulcorner\Phi([\langle\theta'_0, \ldots, \theta'_{k-1}\rangle, \langle\theta_0, \ldots, \theta_{k-1}\rangle, \theta^*_0], \ldots, [\langle\theta'_0, \ldots, \theta'_{k-1}\rangle, \langle\theta_0, \ldots, \theta_{k-1}\rangle, \theta^*_{l-1}])\urcorner$,

(vi) If $\ulcorner\neg\Delta\urcorner \in \text{CONFORM}$, then
$[\langle\theta'_0, \ldots, \theta'_{k-1}\rangle, \langle\theta_0, \ldots, \theta_{k-1}\rangle, \ulcorner\neg\Delta\urcorner] = \ulcorner\neg[\langle\theta'_0, \ldots, \theta'_{k-1}\rangle, \langle\theta_0, \ldots, \theta_{k-1}\rangle, \Delta]\urcorner$,

(vii) If $\ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner \in \text{CONFORM}$, then
$[\langle\theta'_0, \ldots, \theta'_{k-1}\rangle, \langle\theta_0, \ldots, \theta_{k-1}\rangle, \ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner]$
$= \ulcorner([\langle\theta'_0, \ldots, \theta'_{k-1}\rangle, \langle\theta_0, \ldots, \theta_{k-1}\rangle, \Delta_0]\ \psi\ [\langle\theta'_0, \ldots, \theta'_{k-1}\rangle, \langle\theta_0, \ldots, \theta_{k-1}\rangle, \Delta_1])\urcorner$,

(viii) If $\ulcorner\Pi\xi\Delta\urcorner \in \text{QFORM}$, then let $\langle i_0, \ldots, i_{s-1}\rangle$ be such that $s = |\{j \mid j < k \text{ and } \theta_j \neq \xi\}|$ and for all $l < s$: $i_l \in \{j \mid j < k \text{ and } \theta_j \neq \xi\}$ and for all $k < l < s$: $i_k < i_l$, and let
$[\langle\theta'_0, \ldots, \theta'_{k-1}\rangle, \langle\theta_0, \ldots, \theta_{k-1}\rangle, \ulcorner\Pi\xi\Delta\urcorner] = \ulcorner\Pi\xi[\langle\theta'_{i_0}, \ldots, \theta'_{i_{s-1}}\rangle, \langle\theta_{i_0}, \ldots, \theta_{i_{s-1}}\rangle, \Delta]\urcorner$, if $|\{j \mid j < k \text{ and } \theta_j \neq \xi\}| \neq 0$, $[\langle\theta'_0, \ldots, \theta'_{k-1}\rangle, \langle\theta_0, \ldots, \theta_{k-1}\rangle, \ulcorner\Pi\xi\Delta\urcorner] = \ulcorner\Pi\xi\Delta\urcorner$ otherwise,

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9 Let ${}^{X}Y = \{f \mid f \in \text{Pot}(X \times Y)$ and f is function on X and $\text{Ran}(f) \subseteq Y\}$ and let $\langle a_0, \ldots, a_{k-1}\rangle = \{(i, a_i) \mid i < k\}$. In the following we will designate 1-tuples by their values if we write down substitution results. So, for example, $[\theta'_0, \theta_0, \Delta]$ for $[\langle\theta'_0\rangle, \langle\theta_0\rangle, \Delta]$.

(ix) If $\ulcorner\Xi\Delta\urcorner \in$ SENT, then
$[\langle\theta'_0, \ldots, \theta'_{k\text{-}1}\rangle, \langle\theta_0, \ldots, \theta_{k\text{-}1}\rangle, \ulcorner\Xi\Delta\urcorner] = \ulcorner\Xi[\langle\theta'_0, \ldots, \theta'_{k\text{-}1}\rangle, \langle\theta_0, \ldots, \theta_{k\text{-}1}\rangle, \Delta]\urcorner$, and

(x) If $\mathfrak{H} \in$ SEQ, then $[\langle\theta'_0, \ldots, \theta'_{k\text{-}1}\rangle, \langle\theta_0, \ldots, \theta_{k\text{-}1}\rangle, \mathfrak{H}]$
$= \{(j, [\langle\theta'_0, \ldots, \theta'_{k\text{-}1}\rangle, \langle\theta_0, \ldots, \theta_{k\text{-}1}\rangle, \mathfrak{H}_j]) \mid j \in \text{Dom}(\mathfrak{H})\}$.

Clause (viii) regulates the substitution in quantificational formulas. In this case, the substituion is to be carried out for and only for those members of the substituendum sequence that are not identical to the variable bound by the respective quantifier (if such members exist). Accordingly, the desired members of the substituendum sequence and the corresponding members of the substituens sequence have to be singled out. This is achieved by the (in each case uniquely determined) number sequence $\langle i_0, \ldots, i_{s\text{-}1}\rangle$, which picks exactly those indices whose values in the substituendum sequence are different from the bound variable. The new substituendum resp. substituens sequences, which have the desired properties, are then simply the result of the composition of the original substituendum resp. substituens sequences with $\langle i_0, \ldots, i_{s\text{-}1}\rangle$. If, however, all members of the substituendum sequence are identical to the bound variable, then the substitution result is to be identical to the substitution basis, i.e. the respective quantificational formula.

Now, some theorems are to be established which are needed for the meta-theory of the Speech Act Calculus – especially from ch. 4 onwards. We recommend that more impatient readers skip these theorems for now and return here if the need arises. The first theorem eases proofs by induction on the *degree* of a formula. It is proved by induction on the *complexity* of a formula.

Theorem 1-13. *Conservation of the degree of a formula as substitution basis*
If $\theta \in$ CTERM, $\theta' \in$ ATERM and $\Delta \in$ FORM, then FDEG(Δ) = FDEG($[\theta, \theta', \Delta]$).

Proof: Suppose $\theta \in$ CTERM, $\theta' \in$ ATERM and $\Delta \in$ FORM. The proof is carried out by induction on the complexity of Δ. Suppose $\Delta = \ulcorner\Phi(\theta_0, \ldots, \theta_{n\text{-}1})\urcorner \in$ AFORM. According to Definition 1-12, we then have FDEG(Δ) = 0. Then we have that $[\theta, \theta', \Delta] = [\theta, \theta', \ulcorner\Phi(\theta_0, \ldots, \theta_{n\text{-}1})\urcorner] = \ulcorner\Phi([\theta, \theta', \theta_0], \ldots, [\theta, \theta', \theta_{n\text{-}1}])\urcorner \in$ AFORM. Therefore also FDEG($[\theta, \theta', \Delta]$) = 0. Suppose the statement holds for $\Delta_0, \Delta_1 \in$ FORM. That is: FDEG(Δ_0) = FDEG($[\theta, \theta', \Delta_0]$) and FDEG($\Delta_1$) = FDEG($[\theta, \theta', \Delta_1]$).

Ad CONFORM: Now, suppose $\Delta = \ulcorner\neg\Delta_0\urcorner$. Then we have that FDEG(Δ) = FDEG($\ulcorner\neg\Delta_0\urcorner$) = FDEG($\Delta_0$)+1 = FDEG($[\theta, \theta', \Delta_0]$)+1 = FDEG($\ulcorner\neg[\theta, \theta', \Delta_0]\urcorner$) =

$FDEG([\theta, \theta', \ulcorner\neg\Delta_0\urcorner]) = FDEG([\theta, \theta', \Delta])$. Now, suppose $\Delta = \ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner$ for some $\psi \in CON\backslash\{\ulcorner\neg\urcorner\}$. Then we have that $FDEG(\Delta) = FDEG(\ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner) = \max(\{FDEG(\Delta_0), FDEG(\Delta_1)\})+1 = \max(\{FDEG([\theta, \theta', \Delta_0]), FDEG([\theta, \theta', \Delta_1])\})+1 = FDEG(\ulcorner([\theta, \theta', \Delta_0]\ \psi\ [\theta, \theta', \Delta_1])\urcorner) = FDEG([\theta, \theta', \ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner]) = FDEG([\theta, \theta', \Delta])$.

Ad QFORM: Now, suppose $\Delta = \ulcorner\Pi\xi\Delta_0\urcorner$. First, let $\xi \neq \theta'$. Then we have that $FDEG(\Delta) = FDEG(\ulcorner\Pi\xi\Delta_0\urcorner) = FDEG(\Delta_0)+1 = FDEG([\theta, \theta', \Delta_0])+1 = FDEG(\ulcorner\Pi\xi[\theta, \theta', \Delta_0]\urcorner) = FDEG([\theta, \theta', \ulcorner\Pi\xi\Delta_0\urcorner]) = FDEG([\theta, \theta', \Delta])$. Now, suppose $\xi = \theta'$. Then we have that $FDEG(\Delta) = FDEG(\ulcorner\Pi\xi\Delta_0\urcorner) = FDEG([\theta, \theta', \ulcorner\Pi\xi\Delta_0\urcorner]) = FDEG([\theta, \theta', \Delta])$. ■

Theorem 1-14. *For all substituenda and substitution bases it holds that either all closed terms are subterms of the respective substitution result or that the respective substitution result is identical to the respective substitution basis for all closed terms*

If $\theta' \in ATERM$, $\theta^* \in TERM$, $\Delta \in FORM$, then:

(i) $\theta \in ST([\theta, \theta', \theta^*])$ for all $\theta \in CTERM$ or $[\theta, \theta', \theta^*] = \theta^*$ for all $\theta \in CTERM$, and

(ii) $\theta \in ST([\theta, \theta', \Delta])$ for all $\theta \in CTERM$ or $[\theta, \theta', \Delta] = \Delta$ for all $\theta \in CTERM$.

Proof: Suppose $\theta' \in ATERM$, $\theta^* \in TERM$, $\Delta \in FORM$. *Ad (i)*: The proof is carried out by induction on the complexity of θ^*. Suppose $\theta^* \in ATERM$. If $\theta' = \theta^*$, then we have that $[\theta, \theta', \theta^*] = \theta$ and thus that $\theta \in ST([\theta, \theta', \theta^*])$ for all $\theta \in CTERM$. If $\theta' \neq \theta^*$, then we have that $[\theta, \theta', \theta^*] = \theta^*$ for all $\theta \in CTERM$. Suppose the statement holds for $\theta^*_0, \ldots, \theta^*_{r-1} \in TERM$ and let $\theta^* = \ulcorner\varphi(\theta^*_0, \ldots, \theta^*_{r-1})\urcorner \in FTERM$. Then we have that $[\theta, \theta', \theta^*] = [\theta, \theta', \ulcorner\varphi(\theta^*_0, \ldots, \theta^*_{r-1})\urcorner] = \ulcorner\varphi([\theta, \theta', \theta^*_0], \ldots, [\theta, \theta', \theta^*_{r-1}])\urcorner$ for all $\theta \in CTERM$. According to the I.H., we have that for all $i < r$: $\theta \in ST([\theta, \theta', \theta^*_i])$ for all $\theta \in CTERM$ or $[\theta, \theta', \theta^*_i] = \theta^*_i$ for all $\theta \in CTERM$. Suppose there is an $i < r$ such that $\theta \in ST([\theta, \theta', \theta^*_i])$ for all $\theta \in CTERM$. Then we have that $\theta \in ST(\ulcorner\varphi([\theta, \theta', \theta^*_0], \ldots, [\theta, \theta', \theta^*_{r-1}])\urcorner) = ST([\theta, \theta', \theta^*])$ for all $\theta \in CTERM$. Suppose there is no $i < r$ such that $\theta \in ST([\theta, \theta', \theta^*_i])$ for all $\theta \in CTERM$. According to the I.H., we then have that $[\theta, \theta', \theta^*_i] = \theta^*_i$ for all $\theta \in CTERM$ and all $i < r$. Therefore $[\theta, \theta', \theta^*] = \ulcorner\varphi([\theta, \theta', \theta^*_0], \ldots, [\theta, \theta', \theta^*_{r-1}])\urcorner = \ulcorner\varphi(\theta^*_0, \ldots, \theta^*_{r-1})\urcorner = \theta^*$ for all $\theta \in CTERM$.

Ad (ii): Suppose $\Delta \in FORM$. The proof is carried out by induction on the complexity of Δ. Suppose $\Delta = \ulcorner\Phi(\theta_0, \ldots, \theta_{r-1})\urcorner \in AFORM$. This case is proved in the same way as the FTERM-case by applying (i).

Suppose the statement holds for $\Delta_0, \Delta_1 \in FORM$ and let $\Delta = \ulcorner\neg\Delta_0\urcorner \in CONFORM$. Then we have that $[\theta, \theta', \Delta] = [\theta, \theta', \ulcorner\neg\Delta_0\urcorner] = \ulcorner\neg[\theta, \theta', \Delta_0]\urcorner$ for all $\theta \in CTERM$. According to the I.H., we have that $\theta \in ST([\theta, \theta', \Delta_0])$ for all $\theta \in CTERM$ or $[\theta, \theta', \Delta_0] = \Delta_0$ for all $\theta \in CTERM$. In the first case, we thus have that $\theta \in ST(\ulcorner\neg[\theta, \theta', \Delta_0]\urcorner) = ST([\theta, \theta', \Delta])$ for all $\theta \in CTERM$. In the second case, we have that $[\theta, \theta', \Delta] = \ulcorner\neg[\theta, \theta', \Delta_0]\urcorner = \ulcorner\neg\Delta_0\urcorner = \Delta$ for all $\theta \in CTERM$. Suppose $\Delta = \ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner$. This case is proved in the same way as the negation-case.

Suppose $\Delta = \ulcorner\Pi\xi\Delta_0\urcorner$. First, suppose $\xi = \theta'$. Then we have that $[\theta, \theta', \Delta] = [\theta, \theta', \ulcorner\Pi\xi\Delta_0\urcorner] = \ulcorner\Pi\xi\Delta_0\urcorner = \Delta$ for all $\theta \in CTERM$. Now, suppose $\xi \neq \theta'$. Then we have that $[\theta, \theta', \Delta] = [\theta, \theta', \ulcorner\Pi\xi\Delta_0\urcorner] = \ulcorner\Pi\xi[\theta, \theta', \Delta_0]\urcorner$ for all $\theta \in CTERM$. According to the I.H., we then have that $\theta \in ST([\theta, \theta', \Delta_0])$ for all $\theta \in CTERM$ or $[\theta, \theta', \Delta_0] = \Delta_0$ for all $\theta \in CTERM$. In the first case, we thus have that $\theta \in ST(\ulcorner\Pi\xi[\theta, \theta', \Delta_0]\urcorner) = ST([\theta, \theta', \Delta])$ for all $\theta \in CTERM$. In the second case, we have that $[\theta, \theta', \Delta] = \ulcorner\Pi\xi[\theta, \theta', \Delta_0]\urcorner = \ulcorner\Pi\xi\Delta_0\urcorner = \Delta$ for all $\theta \in CTERM$. ■

Theorem 1-15. *Bases for the substitution of closed terms in terms*
If $\theta \in TERM$, $k \in \mathbb{N}\backslash\{0\}$, $\{\theta_0, \ldots, \theta_{k-1}\} \subseteq CTERM$ and $\{\xi_0, \ldots, \xi_{k-1}\} \subseteq VAR\backslash ST(\theta)$, where $\xi_i \neq \xi_j$ for all $i, j < k$ with $i \neq j$, then there is a $\theta^+ \in TERM$, where $FV(\theta^+) \subseteq \{\xi_0, \ldots, \xi_{k-1}\} \cup FV(\theta)$ and $ST(\theta^+) \cap \{\theta_0, \ldots, \theta_{k-1}\} = \emptyset$ such that $\theta = [\langle\theta_0, \ldots, \theta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \theta^+]$.

Proof: By induction on the complexity of θ. Suppose $\theta \in ATERM$. Now, suppose $k \in \mathbb{N}\backslash\{0\}$, $\{\theta_0, \ldots, \theta_{k-1}\} \subseteq CTERM$ and $\{\xi_0, \ldots, \xi_{k-1}\} \subseteq VAR\backslash ST(\theta)$, where $\xi_i \neq \xi_j$ for all $i, j < k$ with $i \neq j$. Then we have that $\theta \in CONST \cup PAR \cup VAR$. First, suppose $\theta \in PAR \cup CONST$. Then there is no $i < k$ such that $\theta = \theta_i$, or there is an $i < k$ such that $\theta = \theta_i$. In the *first case*, it follows that $\theta = [\langle\theta_0, \ldots, \theta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \theta]$ and we have that $FV(\theta) \subseteq \{\xi_0, \ldots, \xi_{k-1}\} \cup FV(\theta)$ and $ST(\theta) \cap \{\theta_0, \ldots, \theta_{k-1}\} = \emptyset$. In the *second case*, there is an $i < k$ such that $\theta = [\langle\theta_0, \ldots, \theta_i\rangle, \langle\xi_0, \ldots, \xi_i\rangle, \xi_i]$. Because of $\xi_i \neq \xi_j$ for all $i, j < k$ with $i \neq j$, we then also have that $\theta = [\langle\theta_0, \ldots, \theta_i\rangle, \langle\xi_0, \ldots, \xi_i\rangle, \xi_i] = [\langle\theta_0, \ldots, \theta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \xi_i]$ and we have that $FV(\xi_i) \subseteq \{\xi_0, \ldots, \xi_{k-1}\} \cup FV(\theta)$ and $ST(\xi_i) \cap \{\theta_0, \ldots, \theta_{k-1}\} = \emptyset$. Now, suppose $\theta \in VAR$. Because of $\{\xi_0, \ldots, \xi_{k-1}\} \subseteq VAR\backslash ST(\theta)$, we then have that $\theta = [\langle\theta_0, \ldots, \theta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \theta]$ and $FV(\theta) \subseteq \{\xi_0, \ldots, \xi_{k-1}\} \cup FV(\theta)$ and because of $ST(\theta) \cap \{\theta_0, \ldots, \theta_{k-1}\} \subseteq VAR \cap CTERM = \emptyset$ we also have that $ST(\theta) \cap \{\theta_0, \ldots, \theta_{k-1}\} = \emptyset$.

Suppose the statement holds for $\theta'_0, \ldots, \theta'_{r-1} \in TERM$ and let $\theta = \ulcorner\varphi(\theta'_0, \ldots \theta'_{r-1})\urcorner \in FTERM$. Now, suppose $k \in \mathbb{N}\backslash\{0\}$, $\{\theta_0, \ldots, \theta_{k-1}\} \subseteq CTERM$ and $\{\xi_0, \ldots, \xi_{k-1}\} \subseteq VAR\backslash ST(\theta)$, where $\xi_i \neq \xi_j$ for all $i, j < k$ with $i \neq j$. With $\bigcup\{ST(\theta'_i) \mid i < r\} \subseteq ST(\theta)$, it then holds for all $i < r$ that $\{\xi_0, \ldots, \xi_{k-1}\} \subseteq VAR\backslash ST(\theta'_i)$. According to the I.H., we then have that for every θ'_i ($i < r$) there is a $\theta^+_i \in$ TERM such that $\theta'_i = [\langle\theta_0, \ldots, \theta_{k-1}\rangle, \langle\xi_1, \ldots, \xi_{k-1}\rangle, \theta^+_i]$ and $\mathrm{FV}(\theta^+_i) \subseteq \{\xi_0, \ldots, \xi_{k-1}\} \cup \mathrm{FV}(\theta'_i)$ and $\mathrm{ST}(\theta^+_i) \cap \{\theta_0, \ldots, \theta_{k-1}\} = \emptyset$. Then there is no $i < k$ such that $\ulcorner\varphi(\theta'_0, \ldots \theta'_{r-1})\urcorner = \theta_i$, or there is an $i < k$ such that $\ulcorner\varphi(\theta'_0, \ldots \theta'_{r-1})\urcorner = \theta_i$. In the *first case*, we have that $\ulcorner\varphi(\theta'_0, \ldots \theta'_{r-1})\urcorner = \ulcorner\varphi([\langle\theta_0, \ldots, \theta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \theta^+_0], \ldots, [\langle\theta_0, \ldots, \theta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \theta^+_{r-1}])\urcorner = [\langle\theta_0, \ldots, \theta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \ulcorner\varphi(\theta^+_0, \ldots, \theta^+_{r-1})\urcorner]$. We also have that $\mathrm{FV}(\ulcorner\varphi(\theta^+_0, \ldots, \theta^+_{r-1}\rangle\urcorner) = \bigcup\{\mathrm{FV}(\theta^+_i) \mid i < r\}$ and hence, with the I.H., that $\mathrm{FV}(\ulcorner\varphi(\theta^+_0, \ldots, \theta^+_{r-1})\urcorner) \subseteq \bigcup\{\mathrm{FV}(\theta'_i) \mid i < r\} \cup \{\xi_0, \ldots, \xi_{k-1}\} = \mathrm{FV}(\ulcorner\varphi(\theta'_0, \ldots, \theta'_{r-1})\urcorner) \cup \{\xi_0, \ldots, \xi_{k-1}\}$. According to the case assumption and the I.H., we also have that $\mathrm{ST}(\ulcorner\varphi(\theta^+_0, \ldots, \theta^+_{r-1})\urcorner) \cap \{\theta_0, \ldots, \theta_{k-1}\} = (\{\ulcorner\varphi(\theta^+_0, \ldots, \theta^+_{r-1})\urcorner\} \cup \bigcup\{\mathrm{ST}(\theta^+_i) \mid i < r\}) \cap \{\theta_0, \ldots, \theta_{k-1}\} = (\{\ulcorner\varphi(\theta^+_0, \ldots, \theta^+_{r-1})\urcorner\} \cap \{\theta_0, \ldots, \theta_{k-1}\}) \cup (\bigcup\{\mathrm{ST}(\theta^+_i) \mid i < r\} \cap \{\theta_0, \ldots, \theta_{k-1}\}) = \emptyset \cup \bigcup\{\mathrm{ST}(\theta^+_i) \cap \{\theta_0, \ldots, \theta_{k-1}\} \mid i < r\} = \emptyset$. In the *second case* there is an $i < k$ such that $\ulcorner\varphi(\theta'_0, \ldots \theta'_{r-1})\urcorner = [\langle\theta_0, \ldots, \theta_i\rangle, \langle\xi_0, \ldots, \xi_i\rangle, \xi_i]$. Because of $\xi_i \neq \xi_j$ for all $i, j < k$ with $i \neq j$, we then also have that $\ulcorner\varphi(\theta'_0, \ldots \theta'_{r-1})\urcorner = [\langle\theta_0, \ldots, \theta_i\rangle, \langle\xi_0, \ldots, \xi_i\rangle, \xi_i] = [\langle\theta_0, \ldots, \theta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \xi_i]$ and $\mathrm{FV}(\xi_i) \subseteq \{\xi_0, \ldots, \xi_{k-1}\} \cup \mathrm{FV}(\ulcorner\varphi(\theta'_0, \ldots \theta'_{r-1})\urcorner)$ and because of $\xi_i \notin$ CTERM also $\mathrm{ST}(\xi_i) \cap \{\theta_0, \ldots, \theta_{k-1}\} = \emptyset$. ■

Theorem 1-16. *Bases for the substitution of closed terms in formulas*
If $\Delta \in$ FORM, $k \in \mathbb{N}\backslash\{0\}$,$\{\theta_0, \ldots, \theta_{k-1}\} \subseteq$ CTERM and $\{\xi_0, \ldots, \xi_{k-1}\} \subseteq \mathrm{VAR}\backslash\mathrm{ST}(\Delta)$, where $\xi_i \neq \xi_j$ for all $i, j < k$ with $i \neq j$, then there is a $\Delta^+ \in$ FORM, where $\mathrm{FV}(\Delta^+) \subseteq \{\xi_0, \ldots, \xi_{k-1}\} \cup \mathrm{FV}(\Delta)$ and $\mathrm{ST}(\Delta^+) \cap \{\theta_0, \ldots, \theta_{k-1}\} = \emptyset$ such that $\Delta = [\langle\theta_0, \ldots, \theta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta^+]$.

Proof: By induction on the complexity of Δ. Suppose $\Delta = \ulcorner\Phi(\theta'_0, \ldots \theta'_{r-1})\urcorner \in$ AFORM. Now, suppose $k \in \mathbb{N}\backslash\{0\}$, $\{\theta_0, \ldots, \theta_{k-1}\} \subseteq$ CTERM and $\{\xi_0, \ldots, \xi_{k-1}\} \subseteq \mathrm{VAR}\backslash\mathrm{ST}(\ulcorner\Phi(\theta'_0, \ldots \theta'_{r-1})\urcorner)$, where $\xi_i \neq \xi_j$ for all $i, j < k$ with $i \neq j$. With $\bigcup\{\mathrm{ST}(\theta'_i) \mid i < r\} = \mathrm{ST}(\ulcorner\Phi(\theta'_0, \ldots \theta'_{r-1})\urcorner)$, it then holds for all $i < r$ that $\{\xi_0, \ldots, \xi_{k-1}\} \subseteq \mathrm{VAR}\backslash\mathrm{ST}(\theta'_i)$. According to Theorem 1-15, we then have that for every θ'_i ($i < r$) there is a $\theta^+_i \in$ TERM such that $\theta'_i = [\langle\theta_0, \ldots, \theta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \theta^+_i]$ and $\mathrm{FV}(\theta^+_i) \subseteq \{\xi_0, \ldots, \xi_{k-1}\} \cup \mathrm{FV}(\theta'_i)$ and $\mathrm{ST}(\theta^+_i) \cap \{\theta_0, \ldots, \theta_{k-1}\} = \emptyset$. Then we also have that $\ulcorner\Phi(\theta'_0, \ldots \theta'_{r-1})\urcorner = \ulcorner\Phi([\langle\theta_0, \ldots, \theta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \theta^+_0], \ldots, [\langle\theta_0, \ldots, \theta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \theta^+_{r-1}])\urcorner = [\langle\theta_0, \ldots, \theta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \ulcorner\Phi(\theta^+_0, \ldots, \theta^+_{r-1})\urcorner]$. We also have that $\mathrm{FV}(\ulcorner\Phi(\theta^+_0, \ldots, \theta^+_{r-1})\urcorner) = \bigcup\{\mathrm{FV}(\theta^+_i) \mid i < r\}$ and thus $\mathrm{FV}(\ulcorner\Phi(\theta^+_0, \ldots, \theta^+_{r-1})\urcorner) \subseteq \bigcup\{\mathrm{FV}(\theta'_i) \mid i < r\} \cup \{\xi_0, \ldots, \xi_{k-1}\} = \mathrm{FV}(\ulcorner\Phi(\theta'_0, \ldots, \theta'_{r-1})\urcorner) \cup \{\xi_0, \ldots, \xi_{k-1}\}$. We then also have that $\mathrm{ST}(\ulcorner\Phi(\theta^+_0, \ldots, \theta^+_{r-1})\urcorner) \cap \{\theta_0, \ldots, \theta_{k-1}\} = \bigcup\{\mathrm{ST}(\theta^+_i) \mid i < r\} \cap \{\theta_0, \ldots, \theta_{k-1}\} = \bigcup\{\mathrm{ST}(\theta^+_i) \cap \{\theta_0, \ldots, \theta_{k-1}\} \mid i < r\} = \emptyset$.

Now, suppose that the statement holds for Δ_0, $\Delta_1 \in$ FORM and let $\Delta = \ulcorner\neg\Delta_0\urcorner \in$ CONFORM. Now, suppose $k \in \mathbb{N}\backslash\{0\}$, $\{\theta_0, \ldots, \theta_{k\text{-}1}\} \subseteq$ CTERM and $\{\xi_0, \ldots, \xi_{k\text{-}1}\} \subseteq$ VAR\ST($\ulcorner\neg\Delta_0\urcorner$), where $\xi_i \neq \xi_j$ for all i, $j < k$ with $i \neq j$. With ST(Δ_0) = ST($\ulcorner\neg\Delta_0\urcorner$), we then have $\{\xi_0, \ldots, \xi_{k\text{-}1}\} \subseteq$ VAR\ST(Δ_0). According to the I.H. for Δ_0, there is then a $\Delta^+_0 \in$ FORM such that $\Delta_0 = [\langle\theta_0, \ldots, \theta_{k\text{-}1}\rangle, \langle\xi_0, \ldots, \xi_{k\text{-}1}\rangle, \Delta^+_0]$ and FV(Δ^+_0) $\subseteq$ FV(Δ_0) $\cup$ $\{\xi_0, \ldots, \xi_{k\text{-}1}\}$ and ST(Δ^+_0) $\cap$ $\{\theta_0, \ldots, \theta_{k\text{-}1}\} = \emptyset$. Then we also have that $\ulcorner\neg\Delta_0\urcorner = \ulcorner\neg[\langle\theta_0, \ldots, \theta_{k\text{-}1}\rangle, \langle\xi_0, \ldots, \xi_{k\text{-}1}\rangle, \Delta^+_0]\urcorner = [\langle\theta_0, \ldots, \theta_{k\text{-}1}\rangle, \langle\xi_1, \ldots, \xi_{k\text{-}1}\rangle, \ulcorner\neg\Delta^+_0\urcorner]$. Furthermore, we have that FV($\ulcorner\neg\Delta^+_0\urcorner$) = FV($\Delta^+_0$) and thus, with the I.H., that FV($\ulcorner\neg\Delta^+_0\urcorner$) $\subseteq$ FV(Δ_0) $\cup$ $\{\xi_0, \ldots, \xi_{k\text{-}1}\}$ = FV($\ulcorner\neg\Delta_0\urcorner$) $\cup$ $\{\xi_0, \ldots, \xi_{k\text{-}1}\}$. According to the I.H., we also have that ST($\ulcorner\neg\Delta^+_0\urcorner$) $\cap$ $\{\theta_0, \ldots, \theta_{k\text{-}1}\}$ = ST(Δ^+_0) $\cap$ $\{\theta_0, \ldots, \theta_{k\text{-}1}\} = \emptyset$.

Now, let $\Delta = \ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner \in$ CONFORM. Now, suppose $k \in \mathbb{N}\backslash\{0\}$, $\{\theta_0, \ldots, \theta_{k\text{-}1}\} \subseteq$ CTERM and $\{\xi_0, \ldots, \xi_{k\text{-}1}\} \subseteq$ VAR\ST($\ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner$), where $\xi_i \neq \xi_j$ for all i, $j < k$ with $i \neq j$. With ST(Δ_0) $\cup$ ST(Δ_1) = ST($\ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner$), we then have $\{\xi_0, \ldots, \xi_{k\text{-}1}\} \subseteq$ VAR\(ST(Δ_0) $\cup$ ST(Δ_1)). According to the I.H. for Δ_0, Δ_1, there are then Δ^+_0, $\Delta^+_1 \in$ FORM such that for $l < 2$: $\Delta_l = [\langle\theta_0, \ldots, \theta_{k\text{-}1}\rangle, \langle\xi_0, \ldots, \xi_{k\text{-}1}\rangle, \Delta^+_l]$ and FV(Δ^+_l) $\subseteq$ $\{\xi_0, \ldots, \xi_{k\text{-}1}\}$ $\cup$ FV(Δ_l) and ST(Δ^+_l) $\cap$ $\{\theta_0, \ldots, \theta_{k\text{-}1}\} = \emptyset$. We then have that $\ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner = \ulcorner([\langle\theta_0, \ldots, \theta_{k\text{-}1}\rangle, \langle\xi_0, \ldots, \xi_{k\text{-}1}\rangle, \Delta^+_0]\ \psi\ [\langle\theta_0, \ldots, \theta_{k\text{-}1}\rangle, \langle\xi_1, \ldots, \xi_{k\text{-}1}\rangle, \Delta^+_1])\urcorner = [\langle\theta_0, \ldots, \theta_{k\text{-}1}\rangle, \langle\xi_0, \ldots, \xi_{k\text{-}1}\rangle, \ulcorner(\Delta^+_0\ \psi\ \Delta^+_1)\urcorner]$. Also, we have that FV($\ulcorner(\Delta^+_0\ \psi\ \Delta^+_1)\urcorner$) = FV($\Delta^+_0$) $\cup$ FV(Δ^+_1) and thus FV($\ulcorner(\Delta^+_0\ \psi\ \Delta^+_1)\urcorner$) $\subseteq$ FV(Δ_0) $\cup$ FV(Δ_1) $\cup$ $\{\xi_0, \ldots, \xi_{k\text{-}1}\}$ = FV($\ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner$) $\cup$ $\{\xi_0, \ldots, \xi_{k\text{-}1}\}$. We also have that ST($\ulcorner(\Delta^+_0\ \psi\ \Delta^+_1)\urcorner$) $\cap$ $\{\theta_0, \ldots, \theta_{k\text{-}1}\}$ = (ST(Δ^+_0) $\cap$ $\{\theta_0, \ldots, \theta_{k\text{-}1}\}$) $\cup$ (ST(Δ^+_1) $\cap$ $\{\theta_0, \ldots, \theta_{k\text{-}1}\}$) = $\emptyset$.

Now, let $\Delta = \ulcorner\Pi\zeta\Delta_0\urcorner \in$ QFORM and suppose $k \in \mathbb{N}\backslash\{0\}$, $\{\theta_0, \ldots, \theta_{k\text{-}1}\} \subseteq$ CTERM and $\{\xi_0, \ldots, \xi_{k\text{-}1}\} \subseteq$ VAR\ST($\ulcorner\Pi\zeta\Delta_0\urcorner$), where $\xi_i \neq \xi_j$ for all i, $j < k$ with $i \neq j$. Then, we have in particular $\zeta \notin \{\xi_0, \ldots, \xi_{k\text{-}1}\}$. With ST($\Delta_0$) $\subseteq$ ST($\ulcorner\Pi\zeta\Delta_0\urcorner$), we have that $\{\xi_0, \ldots, \xi_{k\text{-}1}\} \subseteq$ VAR\ST(Δ_0). According to the I.H. for Δ_0, there is then a $\Delta^+_0 \in$ FORM such that $\Delta_0 = [\langle\theta_0, \ldots, \theta_{k\text{-}1}\rangle, \langle\xi_0, \ldots, \xi_{k\text{-}1}\rangle, \Delta^+_0]$ and FV(Δ^+_0) $\subseteq$ $\{\xi_0, \ldots, \xi_{k\text{-}1}\}$ $\cup$ FV(Δ_0) and ST(Δ^+_0) $\cap$ $\{\theta_0, \ldots, \theta_{k\text{-}1}\} = \emptyset$. Since $\zeta \notin \{\xi_0, \ldots, \xi_{k\text{-}1}\}$, we then have $\ulcorner\Pi\zeta\Delta_0\urcorner = \ulcorner\Pi\zeta[\langle\theta_0, \ldots, \theta_{k\text{-}1}\rangle, \langle\xi_0, \ldots, \xi_{k\text{-}1}\rangle, \Delta^+_0]\urcorner = [\langle\theta_0, \ldots, \theta_{k\text{-}1}\rangle, \langle\xi_0, \ldots, \xi_{k\text{-}1}\rangle, \ulcorner\Pi\zeta\Delta^+_0\urcorner]$. We then have FV($\ulcorner\Pi\zeta\Delta^+_0\urcorner$) = FV($\Delta^+_0$)\{$\zeta$} $\subseteq$ (FV(Δ_0)\{ζ}) $\cup$ $\{\xi_0, \ldots, \xi_{k\text{-}1}\}$ = FV($\ulcorner\Pi\zeta\Delta_0\urcorner$) $\cup$ $\{\xi_0, \ldots, \xi_{k\text{-}1}\}$. With VAR $\cap$ CTERM = $\emptyset$ we then also have ST($\ulcorner\Pi\zeta\Delta^+_0\urcorner$) $\cap$ $\{\theta_0, \ldots, \theta_{k\text{-}1}\}$ = (ST(Δ^+_0) $\cup$ $\{\zeta\}$) $\cap$ $\{\theta_0, \ldots, \theta_{k\text{-}1}\} = \emptyset$. ■

Theorem 1-17. *Alternative bases for the substitution of closed terms for variables in terms*
If {ξ, ζ} ∪ X ⊆ VAR, where {ξ, ζ} ∩ X = ∅, and θ ∈ TERM, where FV(θ) ⊆ {ξ} ∪ X, then there is a θ* ∈ TERM, where FV(θ*) ⊆ {ζ} ∪ X, such that for all θ' ∈ CTERM it holds that [θ', ξ, θ] = [θ', ζ, θ*].

Proof: Suppose {ξ, ζ} ∪ X ⊆ VAR, where {ξ, ζ} ∩ X = ∅, and θ ∈ TERM, where FV(θ) ⊆ {ξ} ∪ X. For ξ = ζ, the statement follows immediately with θ* = θ. Now, suppose ξ ≠ ζ. The proof is now carried out by induction on the complexity of θ. Suppose θ ∈ CONST ∪ PAR. Then it holds with θ* = θ that FV(θ*) = ∅ ⊆ {ζ} ∪ X and that for all θ' ∈ CTERM: [θ', ξ, θ] = [θ', ζ, θ*]. Now, suppose θ ∈ VAR. Suppose θ = ξ. Then it holds with θ* = ζ that FV(θ*) ⊆ {ζ} ∪ X and that for all θ' ∈ CTERM: [θ', ξ, θ] = θ' = [θ', ζ, θ*]. Suppose θ ≠ ξ. Then we have θ ∈ X and thus θ ∉ {ξ, ζ}. Then it holds with θ* = θ that FV(θ*) = {θ} ⊆ {ζ} ∪ X and that for all θ' ∈ CTERM: [θ', ξ, θ] = θ = θ* = [θ', ζ, θ*].

Now, suppose the statement holds for θ_0, …, θ_{r-1} ∈ TERM and suppose θ = ⌜φ(θ_0, … θ_{r-1})⌝ ∈ FTERM. Then we have for all $i < r$: FV(θ_i) ⊆ {ξ} ∪ X. According to the I.H., we then have that for all $i < r$ there is a θ^*_i ∈ TERM, with FV(θ^*_i) ⊆ {ζ} ∪ X, such that for all θ' ∈ CTERM it holds that [θ', ξ, θ_i] = [θ', ζ, θ^*_i]. With θ* = ⌜φ(θ^*_0, … θ^*_{r-1})⌝ it then holds that FV(θ*) ⊆ {ζ} ∪ X and that for all θ' ∈ CTERM: [θ', ξ, θ] = [θ', ξ, ⌜φ(θ_0, … θ_{r-1})⌝] = ⌜φ([θ', ξ, θ_0], … [θ', ξ, θ_{r-1}])⌝ = ⌜φ([θ', ζ, θ^*_0], … [θ', ζ, θ^*_{r-1}])⌝ = [θ', ζ, ⌜φ(θ^*_0, … θ^*_{r-1})⌝] = [θ', ζ, θ*]. ■

Theorem 1-18. *Alternative bases for the substitution of closed terms for variables in formulas*
If {ξ, ζ} ∪ X ⊆ VAR, where {ξ, ζ} ∩ X = ∅, and Δ ∈ FORM, where FV(Δ) ⊆ {ξ} ∪ X and ζ ∉ ST(Δ), then there is a Δ* ∈ FORM, where FV(Δ*) ⊆ {ζ} ∪ X, such that for all θ' ∈ CTERM it holds that [θ', ξ, Δ] = [θ', ζ, Δ*].

Proof: The proof is carried out by induction on the complexity of Δ. Suppose Δ = ⌜Φ(θ_0, … θ_{r-1})⌝ ∈ AFORM. Let {ξ, ζ} ∪ X ⊆ VAR, where {ξ, ζ} ∩ X = ∅, and FV(Δ) ⊆ {ξ} ∪ X and ζ ∉ ST(Δ). Then we have for all $i < r$: FV(θ_i) ⊆ {ξ} ∪ X. According to Theorem 1-17, there is then for all $i < r$ a θ^*_i ∈ TERM, where FV(θ^*_i) ⊆ {ζ} ∪ X such that for all θ' ∈ CTERM holds: [θ', ξ, θ_i] = [θ', ζ, θ^*_i]. Then it holds with Δ* = ⌜Φ(θ^*_0, … θ^*_{r-1})⌝ that FV(Δ*) ⊆ {ζ} ∪ X and that for all θ' ∈ CTERM holds: [θ', ξ, ⌜Φ(θ_0, … θ_{r-1})⌝] = ⌜Φ([θ',

$\xi, \theta_0], \ldots [\theta', \xi, \theta_{r\text{-}1}])^\urcorner = {}^\ulcorner\Phi([\theta', \zeta, \theta^*_0], \ldots [\theta', \zeta, \theta^*_{r\text{-}1}])^\urcorner = [\theta', \zeta, {}^\ulcorner\Phi(\theta^*_0, \ldots \theta^*_{r\text{-}1})^\urcorner] = [\theta', \zeta, \Delta^*]$.

Now, suppose the statement holds for $\Delta_0, \Delta_1 \in$ FORM and let $\Delta \in$ CONFORM. Let $\{\xi, \zeta\} \cup X \subseteq$ VAR, where $\{\xi, \zeta\} \cap X = \emptyset$, and FV$(\Delta) \subseteq \{\xi\} \cup X$ and $\zeta \notin$ ST(Δ). First, suppose $\Delta = {}^\ulcorner\neg\Delta_0{}^\urcorner$. Then we have FV$(\Delta_0)$ = FV$(\Delta) \subseteq \{\xi\} \cup X$ and $\zeta \notin$ ST(Δ_0). According to the I.H., we have a $\Delta^*_0 \in$ FORM, where FV$(\Delta^*_0) \subseteq \{\zeta\} \cup X$, such that for all $\theta' \in$ CTERM holds: $[\theta', \xi, \Delta_0] = [\theta', \zeta, \Delta^*_0]$. With $\Delta^* = {}^\ulcorner\neg\Delta^*_0{}^\urcorner$, it then holds that FV$(\Delta^*) \subseteq \{\zeta\} \cup X$ and that for all $\theta' \in$ CTERM: $[\theta', \xi, {}^\ulcorner\neg\Delta_0{}^\urcorner] = {}^\ulcorner\neg[\theta', \xi, \Delta_0]^\urcorner = {}^\ulcorner\neg[\theta', \zeta, \Delta^*_0]^\urcorner = [\theta', \zeta, {}^\ulcorner\neg\Delta^*_0{}^\urcorner] = [\theta', \zeta, \Delta^*]$.

Now, suppose $\Delta = {}^\ulcorner(\Delta_0\ \psi\ \Delta_1)^\urcorner \in$ CONFORM. Then we have FV$(\Delta_0) \subseteq$ FV$(\Delta) \subseteq \{\xi\} \cup X$ and $\zeta \notin$ ST(Δ_0) and FV$(\Delta_1) \subseteq$ FV$(\Delta) \subseteq \{\xi\} \cup X$ and $\zeta \notin$ ST(Δ_1). According to the I.H., there are then $\Delta^*_0, \Delta^*_1 \in$ FORM, where FV$(\Delta^*_0) \subseteq \{\zeta\} \cup X$ and FV$(\Delta^*_1) \subseteq \{\zeta\} \cup X$, such that for all $\theta' \in$ CTERM holds: $[\theta', \xi, \Delta_0] = [\theta', \zeta, \Delta^*_0]$ and $[\theta', \xi, \Delta_1] = [\theta', \zeta, \Delta^*_1]$. With $\Delta^* = {}^\ulcorner(\Delta^*_0\ \psi\ \Delta^*_1)^\urcorner$, it then holds that FV$(\Delta^*) \subseteq \{\zeta\} \cup X$ and that for all $\theta' \in$ CTERM: $[\theta', \xi, {}^\ulcorner(\Delta_0\ \psi\ \Delta_1)^\urcorner] = {}^\ulcorner([\theta', \xi, \Delta_0]\ \psi\ [\theta', \xi, \Delta_1])^\urcorner = {}^\ulcorner([\theta', \zeta, \Delta^*_0]\ \psi\ [\theta', \zeta, \Delta^*_1])^\urcorner = [\theta', \zeta, {}^\ulcorner(\Delta^*_0\ \psi\ \Delta^*_1)^\urcorner] = [\theta', \zeta, \Delta^*]$.

Now, suppose $\Delta = {}^\ulcorner\Pi\xi'\Delta_0{}^\urcorner \in$ QFORM. Let $\{\xi, \zeta\} \cup X \subseteq$ VAR, where $\{\xi, \zeta\} \cap X = \emptyset$, and FV$(\Delta) \subseteq \{\xi\} \cup X$ and $\zeta \notin$ ST(Δ). Then we have in particular $\zeta \neq \xi'$. First, suppose $\xi = \xi'$. Then we have $[\theta', \xi, {}^\ulcorner\Pi\xi'\Delta_0{}^\urcorner] = {}^\ulcorner\Pi\xi'\Delta_0{}^\urcorner$ for all $\theta' \in$ CTERM and FV$(\Delta) \subseteq X$. Let $\Delta^* = \Delta = {}^\ulcorner\Pi\xi'\Delta_0{}^\urcorner$. Since $\zeta \notin$ ST(Δ), we also have $[\theta', \zeta, {}^\ulcorner\Pi\xi'\Delta_0{}^\urcorner] = {}^\ulcorner\Pi\xi'\Delta_0{}^\urcorner$ for all $\theta' \in$ CTERM and FV(Δ^*) = FV$(\Delta) \subseteq X \subseteq \{\zeta\} \cup X$. Now, suppose $\xi \neq \xi'$. Then we have FV$(\Delta_0) \subseteq$ FV$(\Delta) \cup \{\xi'\} \subseteq \{\xi\} \cup X \cup \{\xi'\}$ and $\zeta \notin$ ST(Δ_0). According to the I.H., there is then $\Delta^*_0 \in$ FORM, where FV$(\Delta^*_0) \subseteq \{\zeta\} \cup X \cup \{\xi'\}$, such that for all $\theta' \in$ CTERM it holds that $[\theta', \xi, \Delta_0] = [\theta', \zeta, \Delta^*_0]$. With $\Delta^* = {}^\ulcorner\Pi\xi'\Delta^*_0{}^\urcorner$, it then holds that FV$(\Delta^*) \subseteq \{\zeta\} \cup X$ and that for all $\theta' \in$ CTERM it holds that $[\theta', \xi, {}^\ulcorner\Pi\xi'\Delta_0{}^\urcorner] = {}^\ulcorner\Pi\xi'[\theta', \xi, \Delta_0]^\urcorner = {}^\ulcorner\Pi\xi'[\theta', \zeta, \Delta^*_0]^\urcorner = [\theta', \zeta, {}^\ulcorner\Pi\xi'\Delta^*_0{}^\urcorner] = [\theta', \zeta, \Delta^*]$. ■

Theorem 1-19. *Unique substitution bases (a) for terms*
If θ, $\theta^+ \in$ TERM, $\theta^* \in$ CTERM\(ST(θ) $\cup$ ST(θ^+)) and $\theta^\S \in$ ATERM and if $[\theta^*, \theta^\S, \theta] = [\theta^*, \theta^\S, \theta^+]$, then $\theta = \theta^+$.

Proof: By induction on the complexity of θ. Suppose $\theta \in$ ATERM. Now, suppose $\theta^+ \in$ TERM, $\theta^* \in$ CTERM\(ST(θ) $\cup$ ST(θ^+)) and $\theta^\S \in$ ATERM and suppose $[\theta^*, \theta^\S, \theta] = [\theta^*, \theta^\S, \theta^+]$. Now, suppose $\theta^\S = \theta$. Then we have $[\theta^*, \theta^\S, \theta] = \theta^*$. Then we also have $\theta^* = [\theta^*, \theta, \theta^+]$. Since, according to the hypothesis, $\theta^* \notin$ ST(θ^+) and thus $\theta^+ \neq \theta^*$, we then have $\theta = \theta^+$. Now, suppose $\theta^\S \neq \theta$. Then we have $[\theta^*, \theta^\S, \theta] = \theta$. Then we have $\theta = [\theta^*, \theta^\S, \theta^+]$. Because of $\theta^* \notin$ ST(θ) and Theorem 1-14-(i), we then also have $\theta = \theta^+$.

Now, suppose the statement holds for $\{\theta_0, \ldots, \theta_{r-1}\} \subseteq$ TERM and let $\ulcorner\varphi(\theta_0, \ldots \theta_{r-1})\urcorner \in$ FTERM. Now, suppose $\theta^+ \in$ TERM, $\theta^* \in$ CTERM\(ST($\ulcorner\varphi(\theta_0, \ldots \theta_{r-1})\urcorner$) $\cup$ ST(θ^+)) and $\theta^\S \in$ ATERM and suppose $[\theta^*, \theta^\S, \ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner] = [\theta^*, \theta^\S, \theta^+]$. Therefore $[\theta^*, \theta^\S, \theta^+] = \ulcorner\varphi([\theta^*, \theta^\S, \theta_0], \ldots, [\theta^*, \theta^\S, \theta_{r-1}])\urcorner \in$ FTERM. Suppose for contradiction that $\theta^+ \in$ ATERM. We have $\theta^\S \neq \theta^+$ or $\theta^\S = \theta^+$. Suppose $\theta^\S \neq \theta^+$. Then we have $\theta^+ = [\theta^*, \theta^\S, \theta^+] = \ulcorner\varphi([\theta^*, \theta^\S, \theta_0], \ldots, [\theta^*, \theta^\S, \theta_{r-1}])\urcorner \in$ FTERM. Contradiction! Suppose $\theta^\S = \theta^+$. Then we have $\theta^* = [\theta^*, \theta^\S, \theta^+] = \ulcorner\varphi([\theta^*, \theta^\S, \theta_0], \ldots, [\theta^*, \theta^\S, \theta_{r-1}])\urcorner$. With Theorem 1-14-(i), it then follows that for all $i < r$: $[\theta^*, \theta^\S, \theta_i] = \theta_i$ or there is an $i < r$ such that $\theta^* \in$ ST($[\theta^*, \theta^\S, \theta_i]$). If $[\theta^*, \theta^\S, \theta_i] = \theta_i$ for all $i < r$, then $\theta^* = \ulcorner\varphi([\theta^*, \theta^\S, \theta_0], \ldots, [\theta^*, \theta^\S, \theta_{r-1}])\urcorner = \ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner$ and thus, in contradiction to the hypothesis, $\theta^* \in$ ST($\ulcorner\varphi(\theta_0, \ldots \theta_{r-1})\urcorner$). If, on the other hand, there was an $i < r$ such that $\theta^* \in$ ST($[\theta^*, \theta^\S, \theta_i]$), then θ^* would be a proper subterm of $\ulcorner\varphi([\theta^*, \theta^\S, \theta_0], \ldots, [\theta^*, \theta^\S, \theta_{r-1}])\urcorner$ and therefore a proper subterm of itself, which contradicts Theorem 1-8. Therefore $\theta^+ \notin$ ATERM, but $\theta^+ \in$ FTERM. Therefore there are $\{\theta'_0, \ldots, \theta'_{k-1}\} \subseteq$ TERM and $\varphi' \in$ FUNC such that $\theta^+ = \ulcorner\varphi'(\theta'_0, \ldots, \theta'_{k-1})\urcorner$. Thus we have $\ulcorner\varphi'([\theta^*, \theta^\S, \theta'_0], \ldots, [\theta^*, \theta^\S, \theta'_{k-1}])\urcorner = [\theta^*, \theta^\S, \ulcorner\varphi'(\theta'_0, \ldots, \theta'_{k-1})\urcorner] = [\theta^*, \theta^\S, \theta^+] = \ulcorner\varphi([\theta^*, \theta^\S, \theta_0], \ldots, [\theta^*, \theta^\S, \theta_{r-1}])\urcorner$. With Theorem 1-11-(ii), it then follows that $k = r$ and $\varphi' = \varphi$ and $[\theta^*, \theta^\S, \theta_i] = [\theta^*, \theta^\S, \theta'_i]$ for all $i < r$. With the I.H., it follows that $\theta_i = \theta'_i$ for all $i < r$. Thus we then have $\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner = \ulcorner\varphi'(\theta'_0, \ldots, \theta'_{k-1})\urcorner = \theta^+$. ■

Theorem 1-20. *Unique substitution bases (a) for formulas*
If $\Delta, \Delta^+ \in$ FORM, $\theta^* \in$ CTERM$\backslash$(ST(Δ) $\cup$ ST(Δ^+)) and $\theta^§ \in$ ATERM and if $[\theta^*, \theta^§, \Delta] = [\theta^*, \theta^§, \Delta^+]$, then $\Delta = \Delta^+$.

Proof: Suppose $\Delta, \Delta^+ \in$ FORM, $\theta^* \in$ CTERM$\backslash$(ST(Δ) $\cup$ ST(Δ^+)) and $\theta^§ \in$ ATERM and $[\theta^*, \theta^§, \Delta] = [\theta^*, \theta^§, \Delta^+]$. In the same way as we did in the inductive step of the preceding proof for functional terms, one can show for all formulas that substitution bases (Δ and Δ^+) belong to the same category and have the same main operator (predicate, connective or quantifier) as the respective substitution results ($[\theta^*, \theta^§, \Delta]$ and $[\theta^*, \theta^§, \Delta^+]$). The proof is carried out by induction on the complexity of Δ. Suppose $\Delta = \ulcorner\Phi(\theta_0, \ldots \theta_{r-1})\urcorner \in$ AFORM. Then we also have $[\theta^*, \theta^§, \Delta] = \ulcorner\Phi([\theta^*, \theta^§, \theta_0], \ldots, [\theta^*, \theta^§, \theta_{r-1}])\urcorner \in$ AFORM and there are $\{\theta'_0, \ldots, \theta'_{r-1}\} \subseteq$ TERM with $\ulcorner\Phi(\theta'_0, \ldots \theta'_{r-1})\urcorner = \Delta^+$. Therefore also $\ulcorner\Phi([\theta^*, \theta^§, \theta_0], \ldots, [\theta^*, \theta^§, \theta_{r-1}])\urcorner = [\theta^*, \theta^§, \Delta] = [\theta^*, \theta^§, \Delta^+] = [\theta^*, \theta^§, \ulcorner\Phi(\theta'_0, \ldots \theta'_{r-1})\urcorner] = \ulcorner\Phi([\theta^*, \theta^§, \theta'_0], \ldots, [\theta^*, \theta^§, \theta'_{r-1}])\urcorner \in$ AFORM. With Theorem 1-11-(iv), it then follows that $[\theta^*, \theta^§, \theta_i] = [\theta^*, \theta^§, \theta'_i]$ for all $i < r$. With Theorem 1-19, it then follows that $\theta_i = \theta'_i$ for all $i < r$. Thus we have $\ulcorner\Phi(\theta_0, \ldots \theta_{r-1})\urcorner = \ulcorner\Phi(\theta'_0, \ldots \theta'_{r-1})\urcorner = \Delta^+$.

Now, suppose the statement holds for $\Delta_0, \Delta_1 \in$ FORM and let $\Delta = \ulcorner\neg\Delta_0\urcorner \in$ CONFORM. Then we also have $[\theta^*, \theta^§, \Delta] = \ulcorner\neg[\theta^*, \theta^§, \Delta_0]\urcorner \in$ CONFORM and there is $\Delta'_0 \in$ FORM with $\ulcorner\neg\Delta'_0\urcorner = \Delta^+$. Therefore also $\ulcorner\neg[\theta^*, \theta^§, \Delta_0]\urcorner = [\theta^*, \theta^§, \Delta] = [\theta^*, \theta^§, \Delta^+] = [\theta^*, \theta^§, \ulcorner\neg\Delta'_0\urcorner] = \ulcorner\neg[\theta^*, \theta^§, \Delta'_0]\urcorner \in$ CONFORM. With Theorem 1-11-(v), it then follows that $[\theta^*, \theta^§, \Delta_0] = [\theta^*, \theta^§, \Delta'_0]$. With the I.H., it follows that $\Delta_0 = \Delta'_0$ and thus $\Delta = \ulcorner\neg\Delta_0\urcorner = \ulcorner\neg\Delta'_0\urcorner = \Delta^+$. Suppose $\Delta = \ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner \in$ CONFORM. Then we also have $[\theta^*, \theta^§, \Delta] = \ulcorner([\theta^*, \theta^§, \Delta_0]\ \psi\ [\theta^*, \theta^§, \Delta_1])\urcorner \in$ CONFORM and there are $\Delta'_0, \Delta'_1 \in$ FORM with $\ulcorner(\Delta'_0\ \psi\ \Delta'_1)\urcorner = \Delta^+$. Therefore also $\ulcorner([\theta^*, \theta^§, \Delta_0]\ \psi\ [\theta^*, \theta^§, \Delta_1])\urcorner = [\theta^*, \theta^§, \Delta] = [\theta^*, \theta^§, \Delta^+] = [\theta^*, \theta^§, \ulcorner(\Delta'_0\ \psi\ \Delta'_1)\urcorner] = \ulcorner([\theta^*, \theta^§, \Delta'_0]\ \psi\ [\theta^*, \theta^§, \Delta'_1])\urcorner \in$ CONFORM. With Theorem 1-11-(vi), it then follows that $[\theta^*, \theta^§, \Delta_0] = [\theta^*, \theta^§, \Delta'_0]$ and $[\theta^*, \theta^§, \Delta_1] = [\theta^*, \theta^§, \Delta'_1]$. With the I.H., it follows that $\Delta_0 = \Delta'_0$ and $\Delta_1 = \Delta'_1$ and thus that $\Delta = \ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner = \ulcorner(\Delta'_0\ \psi\ \Delta'_1)\urcorner = \Delta^+$.

Suppose $\Delta = \ulcorner\Pi\xi\Delta_0\urcorner \in$ QFORM. Then we also have $[\theta^*, \theta^§, \Delta] \in$ QFORM and there is $\Delta'_0 \in$ FORM with $\ulcorner\Pi\xi\Delta'_0\urcorner = \Delta^+$. Suppose $\xi = \theta^§$. Then we have $\Delta = \ulcorner\Pi\xi\Delta_0\urcorner = [\theta^*, \theta^§, \ulcorner\Pi\xi\Delta_0\urcorner] = [\theta^*, \theta^§, \Delta] = [\theta^*, \theta^§, \Delta^+] = [\theta^*, \theta^§, \ulcorner\Pi\xi\Delta'_0\urcorner] = \ulcorner\Pi\xi\Delta'_0\urcorner = \Delta^+$. Suppose $\xi \neq \theta^§$. Then we have $\ulcorner\Pi\xi[\theta^*, \theta^§, \Delta_0]\urcorner = [\theta^*, \theta^§, \Delta] = [\theta^*, \theta^§, \Delta^+] = [\theta^*, \theta^§, \ulcorner\Pi\xi\Delta'_0\urcorner] = \ulcorner\Pi\xi[\theta^*, \theta^§, \Delta'_0]\urcorner \in$ QFORM. With Theorem 1-11-(vii), it then follows that $[\theta^*, \theta^§, \Delta_0] = [\theta^*, \theta^§, \Delta'_0]$. With the I.H., it follows that $\Delta_0 = \Delta'_0$ and thus that $\Delta = \ulcorner\Pi\xi\Delta_0\urcorner = \ulcorner\Pi\xi\Delta'_0\urcorner = \Delta^+$. ■

Theorem 1-21. *Unique substitution bases (a) for sentences*
If $\Sigma, \Sigma^+ \in$ SENT, $\theta^* \in$ CTERM\(ST(Σ) $\cup$ ST(Σ^+)) and $\theta^\S \in$ ATERM and if $[\theta^*, \theta^\S, \Sigma] = [\theta^*, \theta^\S, \Sigma^+]$, then $\Sigma = \Sigma^+$.

Proof: The theorem is proved analogously to the negation-case in the proof of Theorem 1-20 by applying Theorem 1-20 and Theorem 1-12. ■

Theorem 1-22. *Unique substitution bases (b) for terms*
If $\theta, \theta^+ \in$ TERM, $\theta^* \in$ CTERM\(ST(θ) $\cup$ ST(θ^+)), $\xi \in$ VAR, $\beta \in$ PAR and $[\theta^*, \xi, \theta] = [\theta^*, \beta, \theta^+]$, then $\theta^+ = [\beta, \xi, \theta]$.

Proof: By induction on the complexity of θ. Suppose $\theta \in$ ATERM. Now, suppose $\theta^+ \in$ TERM, $\theta^* \in$ CTERM\(ST(θ) $\cup$ ST(θ^+)), $\xi \in$ VAR, $\beta \in$ PAR and $[\theta^*, \xi, \theta] = [\theta^*, \beta, \theta^+]$. Then we have $\theta \in$ CONST $\cup$ PAR $\cup$ VAR. Now, suppose $\theta \in$ CONST. Then we have $[\theta^*, \xi, \theta] = \theta$. Then we have $\theta = [\theta^*, \beta, \theta^+]$. Because of $\theta^* \notin$ ST(θ) and Theorem 1-14-(i), we then have that $\theta = \theta^+$ and because of $\theta \neq \xi$ we have $\theta^+ = \theta = [\beta, \xi, \theta]$. Now, suppose $\theta \in$ PAR. Then we have $[\theta^*, \xi, \theta] = \theta$. Then we have $\theta = [\theta^*, \beta, \theta^+]$. Because of $\theta^* \notin$ ST(θ) and Theorem 1-14-(i), we then have again $\theta = \theta^+$ and because of $\xi \neq \theta$: $\theta^+ = \theta = [\beta, \xi, \theta]$. Now, suppose $\theta \in$ VAR. Suppose $\theta = \xi$. Then we have $[\theta^*, \xi, \theta] = \theta^*$. Then we have $\theta^* = [\theta^*, \beta, \theta^+]$. Because of $\theta^* \neq \theta^+$, we then have $\beta \in$ ST(θ^+). Thus we have $\theta^* \in$ ST($[\theta^*, \beta, \theta^+]$). If $\theta^+ \neq \beta$, we would have, with $\theta^* = [\theta^*, \beta, \theta^+]$, that θ^* is a proper subterm of itself, which contradicts Theorem 1-8. Therefore we have $\theta^+ = \beta = [\beta, \xi, \theta]$. Now, suppose $\theta \neq \xi$. Then we have $\theta = [\theta^*, \xi, \theta]$. Then we have $\theta = [\theta^*, \beta, \theta^+]$. Because of $\theta^* \notin$ ST(θ) and Theorem 1-14-(i), we then have $\theta = \theta^+$ and, because of $\theta \neq \xi$, we thus have $\theta^+ = \theta = [\beta, \xi, \theta]$.

Now, suppose the statement holds for $\{\theta_0, \ldots, \theta_{r-1}\} \subseteq$ TERM and suppose $\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner \in$ FTERM. Now, suppose $\theta^+ \in$ TERM, $\theta^* \in$ TERM\(ST($\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner$) $\cup$ ST(θ^+)), $\xi \in$ VAR, $\beta \in$ PAR and $[\theta^*, \xi, \ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner] = [\theta^*, \beta, \theta^+]$. Therefore $[\theta^*, \beta, \theta^+] = \ulcorner\varphi([\theta^*, \xi, \theta_0], \ldots, [\theta^*, \xi, \theta_{r-1}])\urcorner \in$ FTERM. Suppose for contradiction that $\theta^+ \in$ ATERM. We have $\beta \neq \theta^+$ or $\beta = \theta^+$. Suppose $\beta \neq \theta^+$. Then we have $\theta^+ = [\theta^*, \beta, \theta^+] = \ulcorner\varphi([\theta^*, \xi, \theta_0], \ldots, [\theta^*, \xi, \theta_{r-1}])\urcorner \in$ FTERM. Contradiction! Suppose $\beta = \theta^+$. Then we have $\theta^* = [\theta^*, \beta, \theta^+] = \ulcorner\varphi([\theta^*, \xi, \theta_0], \ldots, [\theta^*, \xi, \theta_{r-1}])\urcorner$. With Theorem 1-14-(i), it then follows that for all $i < r$: $[\theta^*, \xi, \theta_i] = \theta_i$ or there is an $i < r$ such that $\theta^* \in$ ST($[\theta^*, \xi, \theta_i]$). If $[\theta^*, \xi, \theta_i] = \theta_i$ for all $i < r$, then we would have $\theta^* = \ulcorner\varphi([\theta^*, \xi, \theta_0], \ldots, [\theta^*, \xi, \theta_{r-1}])\urcorner = \ulcorner\varphi(\theta_0, \ldots,$

$\theta_{r\text{-}1})\urcorner$ and thus $\theta^* \in \mathrm{ST}(\ulcorner\varphi(\theta_0, \ldots \theta_{r\text{-}1})\urcorner)$, which contradicts the hypothesis. If, on the other hand, there was an $i < r$ such that $\theta^* \in \mathrm{ST}([\theta^*, \xi, \theta_i])$, then θ^* would be a proper subterm of $\ulcorner\varphi([\theta^*, \xi, \theta_0], \ldots, [\theta^*, \xi, \theta_{r\text{-}1}])\urcorner$ and therefore a proper subterm of itself, which contradicts Theorem 1-8. Therefore $\theta^+ \notin \mathrm{ATERM}$, but $\theta^+ \in \mathrm{FTERM}$. Therefore there are $\{\theta'_0, \ldots, \theta'_{k\text{-}1}\} \subseteq \mathrm{TERM}$ and $\varphi' \in \mathrm{FUNC}$ such that $\theta^+ = \ulcorner\varphi'(\theta'_0, \ldots, \theta'_{k\text{-}1})\urcorner$. Thus we have $\ulcorner\varphi'([\theta^*, \beta, \theta'_0], \ldots, [\theta^*, \beta, \theta'_{k\text{-}1}])\urcorner = [\theta^*, \beta, \ulcorner\varphi'(\theta'_0, \ldots, \theta'_{k\text{-}1})\urcorner] = [\theta^*, \beta, \theta^+] = \ulcorner\varphi([\theta^*, \xi, \theta_0], \ldots, [\theta^*, \xi, \theta_{r\text{-}1}])\urcorner$. With Theorem 1-11-(ii), it then follows that $k = r$ and $\varphi' = \varphi$ and $[\theta^*, \beta, \theta'_i] = [\theta^*, \xi, \theta_i]$ for all $i < r$. With the I.H., it follows that $\theta'_i = [\beta, \xi, \theta_i]$ for all $i < r$. Thus we have $\theta^+ = \ulcorner\varphi'(\theta'_0, \ldots, \theta'_{k\text{-}1})\urcorner = \ulcorner\varphi([\beta, \xi, \theta_0], \ldots, [\beta, \xi, \theta_{r\text{-}1}])\urcorner = [\beta, \xi, \ulcorner\varphi(\theta_0, \ldots, \theta_{r\text{-}1})\urcorner]$. ■

Theorem 1-23. *Unique substitution bases (b) for formulas*
If $\Delta, \Delta^+ \in \mathrm{FORM}$, $\theta^* \in \mathrm{TERM}\backslash(\mathrm{ST}(\Delta) \cup \mathrm{ST}(\Delta^+))$, $\xi \in \mathrm{VAR}$, $\beta \in \mathrm{PAR}$ and $[\theta^*, \xi, \Delta] = [\theta^*, \beta, \Delta^+]$, then $\Delta^+ = [\beta, \xi, \Delta]$.

Proof: Let $\Delta, \Delta^+ \in \mathrm{FORM}$, $\theta^* \in \mathrm{CTERM}\backslash(\mathrm{ST}(\Delta) \cup \mathrm{ST}(\Delta^+))$ and $\xi \in \mathrm{VAR}$, $\beta \in \mathrm{PAR}$ and $[\theta^*, \xi, \Delta] = [\theta^*, \beta, \Delta^+]$. In the same way as we did in the inductive step of the preceding proof for functional terms, one can show for all formulas that substitution bases (Δ and Δ^+) belong to the same category and have the same main operator (predicate, connective or quantifier) as the respective substitution results ($[\theta^*, \xi, \Delta]$ and $[\theta^*, \beta, \Delta^+]$). The proof is carried out by induction on the complexity of Δ. Suppose $\Delta = \ulcorner\Phi(\theta_0, \ldots \theta_{r\text{-}1})\urcorner \in \mathrm{AFORM}$. Then we also have $[\theta^*, \xi, \Delta] = \ulcorner\Phi([\theta^*, \xi, \theta_0], \ldots, [\theta^*, \xi, \theta_{r\text{-}1}])\urcorner \in \mathrm{AFORM}$ and there are $\{\theta'_0, \ldots, \theta'_{r\text{-}1}\} \subseteq \mathrm{TERM}$ with $\ulcorner\Phi(\theta'_0, \ldots, \theta'_{r\text{-}1})\urcorner = \Delta^+$. Therefore we also have $\ulcorner\Phi([\theta^*, \xi, \theta_0], \ldots, [\theta^*, \xi, \theta_{r\text{-}1}])\urcorner = [\theta^*, \xi, \Delta] = [\theta^*, \beta, \Delta^+] = [\theta^*, \beta, \ulcorner\Phi(\theta'_0, \ldots, \theta'_{r\text{-}1})\urcorner] = \ulcorner\Phi([\theta^*, \beta, \theta'_0], \ldots, [\theta^*, \beta, \theta'_{r\text{-}1}])\urcorner \in \mathrm{AFORM}$. With Theorem 1-11-(iv), it then follows that $[\theta^*, \xi, \theta_i] = [\theta^*, \beta, \theta'_i]$ for all $i < r$. With Theorem 1-22, it follows that $\theta'_i = [\beta, \xi, \theta_i]$ for all $i < r$. Thus we then have $\Delta^+ = \ulcorner\Phi(\theta'_0, \ldots \theta'_{r\text{-}1})\urcorner = \ulcorner\Phi([\beta, \xi, \theta_0], \ldots, [\beta, \xi, \theta_{r\text{-}1}])\urcorner = [\beta, \xi, \ulcorner\Phi(\theta_0, \ldots, \theta_{r\text{-}1})\urcorner] = [\beta, \xi, \Delta]$.

Now, suppose the statement holds for $\Delta_0, \Delta_1 \in \mathrm{FORM}$ and let $\Delta = \ulcorner\neg\Delta_0\urcorner \in \mathrm{CONFORM}$. Then we also have $[\theta^*, \xi, \Delta] = \ulcorner\neg[\theta^*, \xi, \Delta_0]\urcorner \in \mathrm{CONFORM}$ and there is $\Delta'_0 \in \mathrm{FORM}$ with $\ulcorner\neg\Delta'_0\urcorner = \Delta^+$. Therefore we also have $\ulcorner\neg[\theta^*, \xi, \Delta_0]\urcorner = [\theta^*, \beta, \Delta^+] = [\theta^*, \beta, \ulcorner\neg\Delta'_0\urcorner] = \ulcorner\neg[\theta^*, \beta, \Delta'_0]\urcorner \in \mathrm{CONFORM}$. With Theorem 1-11-(v), it then follows that $[\theta^*, \xi, \Delta_0] = [\theta^*, \beta, \Delta'_0]$. With the I.H., it follows that $\Delta'_0 = [\beta, \xi, \Delta_0]$ and thus that $\Delta^+ = \ulcorner\neg\Delta'_0\urcorner = \ulcorner\neg[\beta, \xi, \Delta_0]\urcorner = [\beta, \xi, \ulcorner\neg\Delta_0\urcorner] = [\beta, \xi, \Delta]$. Suppose $\Delta = \ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner \in \mathrm{CONFORM}$. Then we also have $[\theta^*, \xi, \Delta] = \ulcorner([\theta^*, \xi, \Delta_0]\ \psi\ [\theta^*, \xi, \Delta_1])\urcorner \in \mathrm{CONFORM}$

and there are Δ'_0, $\Delta'_1 \in$ FORM with $\ulcorner(\Delta'_0\ \psi\ \Delta'_1)\urcorner = \Delta^+$. Therefore we also have $\ulcorner([\theta^*, \xi, \Delta_0]\ \psi\ [\theta^*, \xi, \Delta_1])\urcorner = [\theta^*, \beta, \Delta^+] = [\theta^*, \beta, \ulcorner(\Delta'_0\ \psi\ \Delta'_1)\urcorner] = \ulcorner([\theta^*, \beta, \Delta'_0]\ \psi\ [\theta^*, \beta, \Delta'_1])\urcorner \in$ CONFORM. With Theorem 1-11-(vi), it then follows that $[\theta^*, \xi, \Delta_0] = [\theta^*, \beta, \Delta'_0]$ and $[\theta^*, \xi, \Delta_1] = [\theta^*, \beta, \Delta'_1]$. With the I.H., it follows that $\Delta'_0 = [\beta, \xi, \Delta_0]$ and $\Delta'_1 = [\beta, \xi, \Delta_1]$ and thus we have $\Delta^+ = \ulcorner(\Delta'_0\ \psi\ \Delta'_1)\urcorner = \ulcorner([\beta, \xi, \Delta_0]\ \psi\ [\beta, \xi, \Delta_1])\urcorner = [\beta, \xi, \ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner] = [\beta, \xi, \Delta]$.

Suppose $\Delta = \ulcorner\Pi\xi'\Delta_0\urcorner \in$ QFORM. Suppose $\xi' = \xi$. Then we have $\Delta = \ulcorner\Pi\xi'\Delta_0\urcorner = [\theta^*, \xi, \ulcorner\Pi\xi'\Delta_0\urcorner] = [\theta^*, \xi, \Delta] = [\theta^*, \beta, \Delta^+]$. With Theorem 1-14-(ii), we then have $\theta^* \in$ ST($[\theta^*, \beta, \Delta^+]$) = ST($\Delta$) or $[\theta^*, \beta, \Delta^+] = \Delta^+$. This first case is excluded by the hypothesis. In the second case, we have that $\Delta^+ = \ulcorner\Pi\xi'\Delta_0\urcorner = [\beta, \xi, \ulcorner\Pi\xi'\Delta_0\urcorner] = [\beta, \xi, \Delta]$. Suppose $\xi' \neq \xi$. Then we have $[\theta^*, \xi, \Delta] = \ulcorner\Pi\xi'[\theta^*, \xi, \Delta_0]\urcorner \in$ QFORM and there is $\Delta'_0 \in$ FORM with $\ulcorner\Pi\xi'\Delta'_0\urcorner = \Delta^+$. Therefore we also have $\ulcorner\Pi\xi'[\theta^*, \xi, \Delta_0]\urcorner = [\theta^*, \beta, \Delta^+] = [\theta^*, \beta, \ulcorner\Pi\xi'\Delta'_0\urcorner] = \ulcorner\Pi\xi'[\theta^*, \beta, \Delta'_0]\urcorner \in$ QFORM. With Theorem 1-11-(vii), it then follows that $[\theta^*, \xi, \Delta_0] = [\theta^*, \beta, \Delta'_0]$. With the I.H., it follows that $\Delta'_0 = [\beta, \xi, \Delta_0]$ and thus $\Delta^+ = \ulcorner\Pi\xi'\Delta'_0\urcorner = \ulcorner\Pi\xi'[\beta, \xi, \Delta_0]\urcorner = [\beta, \xi, \ulcorner\Pi\xi'\Delta_0\urcorner] = [\beta, \xi, \Delta]$. ■

Theorem 1-24. *Cancellation of parameters in substitution results*

If $\theta \in$ TERM, $\Delta \in$ FORM, $\Sigma \in$ SENT, $\theta^* \in$ CTERM, $\beta \in$ PAR\(ST(θ) ∪ ST(Δ) ∪ ST(Σ)) and $\theta^+ \in$ ATERM, then:

(i) $[\theta^*, \theta^+, \theta] = [\theta^*, \beta, [\beta, \theta^+, \theta]]$,

(ii) $[\theta^*, \theta^+, \Delta] = [\theta^*, \beta, [\beta, \theta^+, \Delta]]$, and

(iii) $[\theta^*, \theta^+, \Sigma] = [\theta^*, \beta, [\beta, \theta^+, \Sigma]]$.

Proof: Let $\theta \in$ TERM, $\Delta \in$ FORM, $\Sigma \in$ SENT, $\theta^* \in$ CTERM, $\beta \in$ PAR\(ST(θ) ∪ ST(Δ) ∪ ST(Σ)) and $\theta^+ \in$ ATERM. *Ad (i)*: The proof is carried out by induction on the complexity of θ. Suppose $\theta \in$ ATERM. Then we have $\theta = \theta^+$ or $\theta \neq \theta^+$. First, suppose $\theta = \theta^+$. Then we have $[\beta, \theta^+, \theta] = \beta$ and $[\theta^*, \theta^+, \theta] = \theta^*$. Then we have $[\theta^*, \theta^+, \theta] = \theta^* = [\theta^*, \beta, \beta] = [\theta^*, \beta, [\beta, \theta^+, \theta]]$. Now, suppose $\theta \neq \theta^+$. Then we have $[\beta, \theta^+, \theta] = \theta$ and $[\theta^*, \theta^+, \theta] = \theta$. Because of $\beta \notin$ ST(θ), we have $\beta \neq \theta$ and thus $\theta = [\theta^*, \beta, \theta]$. Therefore we have $[\theta^*, \theta^+, \theta] = \theta = [\theta^*, \beta, \theta] = [\theta^*, \beta, [\beta, \theta^+, \theta]]$.

Now, suppose the statement holds for $\{\theta_0, \ldots, \theta_{r-1}\} \subseteq$ TERM and suppose $\theta = \ulcorner\varphi(\theta_0, \ldots \theta_{r-1})\urcorner \in$ FTERM. Because of $\beta \notin$ ST(θ), we also have that $\beta \notin$ ST(θ_i) for all $i < r$. With the I.H., it then holds that $[\theta^*, \theta^+, \theta_i] = [\theta^*, \beta, [\beta, \theta^+, \theta_i]]$ for all $i < r$. Then we have $[\theta^*,$

$\theta^+, \ulcorner\varphi(\theta_0, \ldots \theta_{r\text{-}1})\urcorner] = \ulcorner\varphi([\theta^*, \theta^+, \theta_0], \ldots, [\theta^*, \theta^+, \theta_{r\text{-}1}])\urcorner = \ulcorner\varphi([\theta^*, \beta, [\beta, \theta^+, \theta_0]], \ldots, [\theta^*, \beta, [\beta, \theta^+, \theta_{r\text{-}1}]])\urcorner = [\theta^*, \beta, \ulcorner\varphi([\beta, \theta^+, \theta_0], \ldots, [\beta, \theta^+, \theta_{r\text{-}1}])\urcorner] = [\theta^*, \beta, [\beta, \theta^+, \ulcorner\varphi(\theta_0, \ldots \theta_{r\text{-}1})\urcorner]]$.

Ad (ii): The proof is carried out by induction on the complexity of Δ. Suppose $\Delta = \ulcorner\Phi(\theta_0, \ldots \theta_{r\text{-}1})\urcorner \in$ AFORM. Then we have $\beta \notin \mathrm{ST}(\theta_i)$ for all $i < r$ and $[\theta^*, \theta^+, \Delta] = [\theta^*, \theta^+, \ulcorner\Phi(\theta_0, \ldots \theta_{r\text{-}1})\urcorner] = \ulcorner\Phi([\theta^*, \theta^+, \theta_0], \ldots [\theta^*, \theta^+, \theta_{r\text{-}1}])\urcorner$. With (i), it holds that $[\theta^*, \theta^+, \theta_i] = [\theta^*, \beta, [\beta, \theta^+, \theta_i]]$ for all $i < r$. Therefore we have $[\theta^*, \theta^+, \Delta] = \ulcorner\Phi([\theta^*, \beta, [\beta, \theta^+, \theta_0]], \ldots, [\theta^*, \beta, [\beta, \theta^+, \theta_{r\text{-}1}]])\urcorner = [\theta^*, \beta, \ulcorner\Phi([\beta, \theta^+, \theta_0], \ldots, [\beta, \theta^+, \theta_{r\text{-}1}])\urcorner = [\theta^*, \beta, [\beta, \theta^+, \ulcorner\Phi(\theta_0, \ldots \theta_{r\text{-}1})\urcorner]] = [\theta^*, \beta, [\beta, \theta^+, \Delta]]$.

Now, suppose the statement holds for $\Delta_0, \Delta_1 \in$ FORM. First, let $\Delta = \ulcorner\neg\Delta_0\urcorner \in$ CONFORM. Then we have $\beta \notin \mathrm{ST}(\Delta_0)$ and $[\theta^*, \theta^+, \Delta] = [\theta^*, \theta^+, \ulcorner\neg\Delta_0\urcorner] = \ulcorner\neg[\theta^*, \theta^+, \Delta_0]\urcorner$. With the I.H., it holds that $[\theta^*, \theta^+, \Delta_0] = [\theta^*, \beta, [\beta, \theta^+, \Delta_0]]$. Therefore $[\theta^*, \theta^+, \Delta] = \ulcorner\neg[\theta^*, \beta, [\beta, \theta^+, \Delta_0]]\urcorner = [\theta^*, \beta, [\beta, \theta^+, \ulcorner\neg\Delta_0\urcorner]] = [\theta^*, \beta, [\beta, \theta^+, \Delta]]$. Suppose $\Delta = \ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner \in$ CONFORM. This case is proved analogously to the negation-case.

Suppose $\Delta = \ulcorner\Pi\xi\Delta_0\urcorner \in$ QFORM. Suppose $\xi = \theta^+$. Then we have $[\theta^*, \theta^+, \Delta] = [\theta^*, \theta^+, \ulcorner\Pi\xi\Delta_0\urcorner] = \ulcorner\Pi\xi\Delta_0\urcorner = [\beta, \theta^+, \ulcorner\Pi\xi\Delta_0\urcorner] = [\beta, \theta^+, \Delta]$. Then we have $\beta \notin \mathrm{ST}([\beta, \theta^+, \Delta]) = \mathrm{ST}(\Delta)$. Therefore $[\theta^*, \theta^+, \Delta] = [\beta, \theta^+, \Delta] = [\theta^*, \beta, [\beta, \theta^+, \Delta]]$. Suppose $\xi \neq \theta^+$. This case is proved analogously to the negation-case.

Ad (iii): This case is proved analogously to the negation-case. ■

Theorem 1-25. *A sufficient condition for the commutativity of a substitution in terms and formulas*

If $\theta^*_0, \theta^*_1 \in$ CTERM, $\theta_0, \theta_1 \in$ ATERM, $\theta_0 \neq \theta_1$, $\theta_1 \notin \mathrm{ST}(\theta^*_0)$ and $\theta_0 \notin \mathrm{ST}(\theta^*_1)$, then:

(i) If $\theta^+ \in$ TERM, then $[\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \theta^+]] = [\theta^*_0, \theta_0, [\theta^*_1, \theta_1, \theta^+]]$, and

(ii) If $\Delta \in$ FORM, then $[\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \Delta]] = [\theta^*_0, \theta_0, [\theta^*_1, \theta_1, \Delta]]$.

Proof: Let $\theta^*_0, \theta^*_1 \in$ CTERM, $\theta_0, \theta_1 \in$ ATERM, $\theta_0 \neq \theta_1$, $\theta_1 \notin \mathrm{ST}(\theta^*_0)$ and $\theta_0 \notin \mathrm{ST}(\theta^*_1)$. *Ad (i)*: Suppose $\theta^+ \in$ TERM. The proof is carried out by induction on the complexity of θ^+. Suppose $\theta^+ \in$ ATERM. Suppose $\theta^+ = \theta_0$. Then we have $\theta^+ \neq \theta_1$ and $[\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \theta^+]] = [\theta^*_1, \theta_1, \theta^*_0]$. Because of $\theta_1 \notin \mathrm{ST}(\theta^*_0)$, we have $[\theta^*_1, \theta_1, \theta^*_0] = \theta^*_0$. On the other hand, we have $[\theta^*_0, \theta_0, [\theta^*_1, \theta_1, \theta^+]] = [\theta^*_0, \theta_0, \theta^+] = \theta^*_0$. Therefore $[\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \theta^+]] = [\theta^*_0, \theta_0, [\theta^*_1, \theta_1, \theta^+]]$. Now, suppose $\theta^+ \neq \theta_0$. Suppose $\theta^+ = \theta_1$. Then we have $[\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \theta^+]] = [\theta^*_1, \theta_1, \theta^+] = \theta^*_1$. Because of $\theta_0 \notin \mathrm{ST}(\theta^*_1)$, we have $[\theta^*_0, \theta_0, \theta^*_1] = \theta^*_1$. Thus we have $[\theta^*_0, \theta_0, [\theta^*_1, \theta_1, \theta^+]] = [\theta^*_0, \theta_0, \theta^*_1] = \theta^*_1$. Therefore $[\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \theta^+]] = [\theta^*_0, \theta_0, [\theta^*_1, \theta_1, \theta^+]]$. Suppose $\theta^+ \neq \theta_1$. Then we have $[\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \theta^+]] = [\theta^*_1, \theta_1,$

$\theta^+] = \theta^+$ and $[\theta^*_0, \theta_0, [\theta^*_1, \theta_1, \theta^+]] = [\theta^*_0, \theta_0, \theta^+] = \theta^+$. Therefore we have again that $[\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \theta^+]] = [\theta^*_0, \theta_0, [\theta^*_1, \theta_1, \theta^+]]$.

Now, suppose the statement holds for $\{\theta'_0, \ldots, \theta'_{r\text{-}1}\} \subseteq$ TERM and suppose $\theta^+ = \ulcorner\varphi(\theta'_0, \ldots, \theta'_{r\text{-}1})\urcorner \in$ FTERM. Then we have $[\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \theta^+]] = [\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \ulcorner\varphi(\theta'_0, \ldots, \theta'_{r\text{-}1})\urcorner]] = \ulcorner\varphi([\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \theta'_0]], \ldots, [\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \theta'_{r\text{-}1}]])\urcorner$. With the I.H., it holds that $[\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \theta'_i]] = [\theta^*_0, \theta_0, [\theta^*_1, \theta_1, \theta'_i]]$ for all $i < r$. Therefore we have $[\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \theta^+]] = \ulcorner\varphi([\theta^*_0, \theta_0, [\theta^*_1, \theta_1, \theta'_0]], \ldots, [\theta^*_0, \theta_0, [\theta^*_1, \theta_1, \theta'_{r\text{-}1}]])\urcorner = [\theta^*_0, \theta_0, [\theta^*_1, \theta_1, \ulcorner\varphi(\theta'_0, \ldots \theta'_{r\text{-}1})\urcorner]] = [\theta^*_0, \theta_0, [\theta^*_1, \theta_1, \theta^+]]$.

Ad (ii): Suppose $\Delta \in$ FORM. The proof is carried out by induction on the complexity of Δ. Suppose $\Delta = \ulcorner\Phi(\theta'_0, \ldots \theta'_{r\text{-}1})\urcorner \in$ AFORM. Then we have $[\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \Delta]] = [\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \ulcorner\Phi(\theta'_0, \ldots, \theta'_{r\text{-}1})\urcorner]] = \ulcorner\Phi([\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \theta'_0]], \ldots, [\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \theta'_{r\text{-}1}]])\urcorner$. With (i), we have that $[\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \theta'_i]] = [\theta^*_0, \theta_0, [\theta^*_1, \theta_1, \theta'_i]]$ for all $i < r$. Therefore we have $[\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \Delta]] = \ulcorner\Phi([\theta^*_0, \theta_0, [\theta^*_1, \theta_1, \theta'_0]], \ldots, [\theta^*_0, \theta_0, [\theta^*_1, \theta_1, \theta'_{r\text{-}1}]])\urcorner = [\theta^*_0, \theta_0, [\theta^*_1, \theta_1, \ulcorner\Phi(\theta'_0, \ldots \theta'_{r\text{-}1})\urcorner]] = [\theta^*_0, \theta_0, [\theta^*_1, \theta_1, \Delta]]$.

Now, suppose the statement holds for $\Delta_0, \Delta_1 \in$ FORM and suppose $\Delta = \ulcorner\neg\Delta_0\urcorner \in$ CONFORM. Then we have $[\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \Delta]] = [\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \ulcorner\neg\Delta_0\urcorner]] = \ulcorner\neg[\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \Delta_0]]\urcorner$. With the I.H., it holds that $[\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \Delta_0]] = [\theta^*_0, \theta_0, [\theta^*_1, \theta_1, \Delta_0]]$. Therefore we have $[\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \Delta]] = \ulcorner\neg[\theta^*_0, \theta_0, [\theta^*_1, \theta_1, \Delta_0]]\urcorner = [\theta^*_0, \theta_0, [\theta^*_1, \theta_1, \ulcorner\neg\Delta_0\urcorner]] = [\theta^*_0, \theta_0, [\theta^*_1, \theta_1, \Delta]]$. Suppose $\Delta = \ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner \in$ CONFORM. This case is proved analogously to the negation-case.

Suppose $\Delta = \ulcorner\Pi\xi\Delta_0\urcorner \in$ QFORM. Suppose $\xi = \theta_0$. Then we have $\xi \neq \theta_1$ and $[\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \Delta]] = [\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \ulcorner\Pi\xi\Delta_0\urcorner]] = [\theta^*_1, \theta_1, \ulcorner\Pi\xi\Delta_0\urcorner] = \ulcorner\Pi\xi[\theta^*_1, \theta_1, \Delta_0]\urcorner = [\theta^*_0, \theta_0, \ulcorner\Pi\xi[\theta^*_1, \theta_1, \Delta_0]\urcorner] = [\theta^*_0, \theta_0, [\theta^*_1, \theta_1, \ulcorner\Pi\xi\Delta_0\urcorner]] = [\theta^*_0, \theta_0, [\theta^*_1, \theta_1, \Delta]]$. Suppose $\xi = \theta_1$. Then we have $\xi \neq \theta_0$ and $[\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \Delta]] = [\theta^*_1, \theta_1, [\theta^*_0, \theta_0, \ulcorner\Pi\xi\Delta_0\urcorner]] = [\theta^*_1, \theta_1, \ulcorner\Pi\xi[\theta^*_0, \theta_0, \Delta_0]\urcorner] = \ulcorner\Pi\xi[\theta^*_0, \theta_0, \Delta_0]\urcorner = [\theta^*_0, \theta_0, \ulcorner\Pi\xi\Delta_0\urcorner] = [\theta^*_0, \theta_0, [\theta^*_1, \theta_1, \ulcorner\Pi\xi\Delta_0\urcorner]] = [\theta^*_0, \theta_0, [\theta^*_1, \theta_1, \Delta]]$. Suppose $\theta_0 \neq \xi \neq \theta_1$. This case is proved analogously to the negation-case. ■

Theorem 1-26. *Substitution in substitution results*
If $\zeta \in$ VAR, $\theta', \theta* \in$ CTERM and $\theta^+ \in$ CONST $\cup$ PAR, then:

(i) If $\theta \in$ TERM, then $[\theta', \theta^+, [\theta*, \zeta, \theta]] = [[\theta', \theta^+, \theta*], \zeta, [\theta', \theta^+, \theta]]$, and

(ii) If $\Delta \in$ FORM, then $[\theta', \theta^+, [\theta*, \zeta, \Delta]] = [[\theta', \theta^+, \theta*], \zeta, [\theta', \theta^+, \Delta]]$.

Proof: Suppose $\zeta \in$ VAR, $\theta', \theta* \in$ CTERM and $\theta^+ \in$ CONST $\cup$ PAR. *Ad (i)*: Suppose $\theta \in$ TERM. The proof is carried out by induction on the complexity of θ. Suppose $\theta \in$ ATERM. First, suppose $\theta \in$ CONST $\cup$ PAR. Suppose $\theta = \theta^+$. Then we have $[\theta', \theta^+, [\theta*, \zeta, \theta]] = [\theta', \theta^+, \theta] = \theta'$. We have $\zeta \notin$ ST(θ') $\in$ CTERM and thus $[\theta', \theta^+, [\theta*, \zeta, \theta]] = \theta' = [[\theta', \theta^+, \theta*], \zeta, \theta'] = [[\theta', \theta^+, \theta*], \zeta, [\theta', \theta^+, \theta]]$. Suppose $\theta \neq \theta^+$. Then we have $[\theta', \theta^+, [\theta*, \zeta, \theta]] = [\theta', \theta^+, \theta] = \theta = [[\theta', \theta^+, \theta*], \zeta, \theta] = [[\theta', \theta^+, \theta*], \zeta, [\theta', \theta^+, \theta]]$. Now, suppose $\theta \in$ VAR. Suppose $\theta = \zeta$. Then we have $[\theta', \theta^+, [\theta*, \zeta, \theta]] = [\theta', \theta^+, \theta*] = [[\theta', \theta^+, \theta*], \zeta, \theta] = [[\theta', \theta^+, \theta*], \zeta, [\theta', \theta^+, \theta]]$. Suppose $\theta \neq \zeta$. Then we have $[\theta', \theta^+, [\theta*, \zeta, \theta]] = [\theta', \theta^+, \theta] = \theta = [[\theta', \theta^+, \theta*], \zeta, \theta] = [[\theta', \theta^+, \theta*], \zeta, [\theta', \theta^+, \theta]]$.

Now, suppose the statement holds for $\{\theta_0, \ldots, \theta_{r-1}\} \subseteq$ TERM and suppose $\theta = \ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner \in$ FTERM. Then we have $[\theta', \theta^+, [\theta*, \zeta, \theta]] = [\theta', \theta^+, [\theta*, \zeta, \ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner]] = \ulcorner\varphi([\theta', \theta^+, [\theta*, \zeta, \theta_0]], \ldots, [\theta', \theta^+, [\theta*, \zeta, \theta_{r-1}]])\urcorner$. With the I.H., it holds that $[\theta', \theta^+, [\theta*, \zeta, \theta_i]] = [[\theta', \theta^+, \theta*], \zeta, [\theta', \theta^+, \theta_i]]$ for all $i < r$. Therefore we have $[\theta', \theta^+, [\theta*, \zeta, \theta]] = \ulcorner\varphi([[\theta', \theta^+, \theta*], \zeta, [\theta', \theta^+, \theta_0]], \ldots, [[\theta', \theta^+, \theta*], \zeta, [\theta', \theta^+, \theta_{r-1}]])\urcorner = [[\theta', \theta^+, \theta*], \zeta, [\theta', \theta^+, \ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner]] = [[\theta', \theta^+, \theta*], \zeta, [\theta', \theta^+, \theta]]$.

Ad (ii): Suppose $\Delta \in$ FORM. The proof is carried out by induction on the complexity of Δ. Suppose $\Delta = \ulcorner\Phi(\theta_0, \ldots \theta_{r-1})\urcorner \in$ AFORM. This case is proved analogously to the FTERM-case by applying (i).

Now, suppose the statement holds for $\Delta_0, \Delta_1 \in$ FORM and suppose $\Delta = \ulcorner\neg\Delta_0\urcorner \in$ CONFORM. Then we have $[\theta', \theta^+, [\theta*, \zeta, \Delta]] = [\theta', \theta^+, [\theta*, \zeta, \ulcorner\neg\Delta_0\urcorner]] = \ulcorner\neg[\theta', \theta^+, [\theta*, \zeta, \Delta_0]]\urcorner$. With the I.H., it holds that $[\theta', \theta^+, [\theta*, \zeta, \Delta_0]] = [[\theta', \theta^+, \theta*], \zeta, [\theta', \theta^+, \Delta_0]]$. Therefore $[\theta', \theta^+, [\theta*, \zeta, \Delta]] = \ulcorner\neg[[\theta', \theta^+, \theta*], \zeta, [\theta', \theta^+, \Delta_0]]\urcorner = [[\theta', \theta^+, \theta*], \zeta, [\theta', \theta^+, \ulcorner\neg\Delta_0\urcorner]] = [[\theta', \theta^+, \theta*], \zeta, [\theta', \theta^+, \Delta]]$. Suppose $\Delta = \ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner \in$ CONFORM. This case is proved analogously to the negation-case.

Suppose $\Delta = \ulcorner\Pi\xi\Delta_0\urcorner \in$ QFORM. Suppose $\xi = \zeta$. Then we have $[\theta', \theta^+, [\theta*, \zeta, \Delta]] = [\theta', \theta^+, [\theta*, \zeta, \ulcorner\Pi\xi\Delta_0\urcorner]] = [\theta', \theta^+, \ulcorner\Pi\xi\Delta_0\urcorner] = \ulcorner\Pi\xi[\theta', \theta^+, \Delta_0]\urcorner = [[\theta', \theta^+, \theta*], \zeta, \ulcorner\Pi\xi[\theta', \theta^+, \Delta_0]\urcorner] = [[\theta', \theta^+, \theta*], \zeta, [\theta', \theta^+, \ulcorner\Pi\xi\Delta_0\urcorner]] = [[\theta', \theta^+, \theta*], \zeta, [\theta', \theta^+, \Delta]]$. Suppose $\xi \neq \zeta$. This case is proved analogously to the negation-case. ■

Theorem 1-27. *Multiple substitution of new and pairwise different parameters for pairwise different parameters in terms, formulas, sentences and sequences*

If $\theta \in$ TERM, $\Delta \in$ FORM, $\Sigma \in$ SENT, $\mathfrak{H} \in$ SEQ, $k \in \mathbb{N}\backslash\{0\}$ and $\{\beta^*_0, \ldots, \beta^*_k\} \subseteq$ PAR$\backslash$(ST(θ) $\cup$ ST(Δ) $\cup$ ST(Σ) $\cup$ STSEQ($\mathfrak{H}$)) and $\{\beta_0, \ldots, \beta_k\} \subseteq$ PAR$\backslash\{\beta^*_0, \ldots, \beta^*_k\}$, where $\beta^*_i \neq \beta^*_j$ and $\beta_i \neq \beta_j$ for all $i, j < k+1$ with $i \neq j$, then:

(i) $[\beta^*_k, \beta_k, [\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \theta]] = [\langle\beta^*_0, \ldots, \beta^*_k\rangle, \langle\beta_0, \ldots, \beta_k\rangle, \theta]$,

(ii) $[\beta^*_k, \beta_k, [\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \Delta]] = [\langle\beta^*_0, \ldots, \beta^*_k\rangle, \langle\beta_0, \ldots, \beta_k\rangle, \Delta]$,

(iii) $[\beta^*_k, \beta_k, [\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \Sigma]] = [\langle\beta^*_0, \ldots, \beta^*_k\rangle, \langle\beta_0, \ldots, \beta_k\rangle, \Sigma]$, and

(iv) $[\beta^*_k, \beta_k, [\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \mathfrak{H}]] = [\langle\beta^*_0, \ldots, \beta^*_k\rangle, \langle\beta_0, \ldots, \beta_k\rangle, \mathfrak{H}]$.

Proof: Suppose $\theta \in$ TERM, $\Delta \in$ FORM, $\Sigma \in$ SENT, $\mathfrak{H} \in$ SEQ, $k \in \mathbb{N}\backslash\{0\}$ and $\{\beta^*_0, \ldots, \beta^*_k\} \subseteq$ PAR$\backslash$(ST(θ) $\cup$ ST(Δ)) and $\{\beta_0, \ldots, \beta_k\} \subseteq$ PAR$\backslash\{\beta^*_0, \ldots, \beta^*_k\}$, where $\beta^*_i \neq \beta^*_j$ and $\beta_i \neq \beta_j$ for all $i, j < k+1$ with $i \neq j$. *Ad (i)*: The proof is carried out by induction on the complexity of θ. Suppose $\theta \in$ ATERM. Then we have $\theta \in$ CONST $\cup$ PAR $\cup$ VAR. Now, suppose $\theta \in$ CONST $\cup$ VAR $\cup$ (PAR$\backslash\{\beta_0, \ldots, \beta_k\}$). Then we have $\theta = [\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \theta]$ and we have $\theta = [\langle\beta^*_0, \ldots, \beta^*_k\rangle, \langle\beta_0, \ldots, \beta_k\rangle, \theta]$ and thus $[\beta^*_k, \beta_k, [\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \theta]] = [\beta^*_k, \beta_k, \theta] = \theta = [\langle\beta^*_0, \ldots, \beta^*_k\rangle, \langle\beta_0, \ldots, \beta_k\rangle, \theta]$.

Now, suppose $\theta \in \{\beta_0, \ldots, \beta_k\}$. Then we have $\theta = \beta_i$ for an $i < k+1$. According to the hypothesis, we then have that for all $j < k+1$ with $j \neq i$ it holds that $\theta \neq \beta_j$. Thus we have $[\langle\beta^*_0, \ldots, \beta^*_k\rangle, \langle\beta_0, \ldots, \beta_k\rangle, \theta] = \beta^*_i$. Now, suppose $i < k$. Then we have $[\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \theta] = \beta^*_i$ and thus $[\beta^*_k, \beta_k, [\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \theta]] = [\beta^*_k, \beta_k, \beta^*_i]$. By hypothesis, we have that $\beta_k \neq \beta^*_i$ and thus that $[\beta^*_k, \beta_k, \beta^*_i] = \beta^*_i$. Now, suppose $i = k$. Then we have $[\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \theta] = \theta = \beta_k$ and hence $[\beta^*_k, \beta_k, [\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \theta]] = [\beta^*_k, \beta_k, \beta_k] = \beta^*_k = \beta^*_i$.

Now, suppose the statement holds for $\{\theta_0, \ldots, \theta_{r-1}\} \subseteq$ TERM and suppose $\theta = \ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner \in$ FTERM. Then we have $[\beta^*_k, \beta_k, [\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \theta]] = [\beta^*_k, \beta_k, [\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner]] = \ulcorner\varphi([\beta^*_k, \beta_k, [\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \theta_0]], \ldots, [\beta^*_k, \beta_k, [\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \theta_{r-1}]])\urcorner$. With the I.H., it holds that $[\beta^*_k, \beta_k, [\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \theta_i]] = [\langle\beta^*_0, \ldots, \beta^*_k\rangle, \langle\beta_0, \ldots, \beta_k\rangle, \theta_i]$ for all $i < r$. Therefore we have $[\beta^*_k, \beta_k, [\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \theta]] = \ulcorner\varphi([\langle\beta^*_0, \ldots, \beta^*_k\rangle, \langle\beta_0, \ldots, \beta_k\rangle, \theta_0], \ldots, [\langle\beta^*_0, \ldots, \beta^*_k\rangle, \langle\beta_0, \ldots, \beta_k\rangle, \theta_{r-1}])\urcorner = [\langle\beta^*_0, \ldots, \beta^*_k\rangle, \langle\beta_0, \ldots, \beta_k\rangle, \ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner] = [\langle\beta^*_0, \ldots, \beta^*_k\rangle, \langle\beta_0, \ldots, \beta_k\rangle, \theta]$.

Ad (ii): The proof is carried out by induction on the complexity of Δ. Suppose $\Delta = \ulcorner\Phi(\theta_0, \ldots \theta_{r-1})\urcorner \in$ AFORM. This case is proved analogously to the FTERM-case by applying (i).

Now, suppose the statement holds for Δ_0, $\Delta_1 \in$ FORM and suppose $\Delta = \ulcorner\neg\Delta_0\urcorner \in$ CONFORM. Then we have $[\beta^*_k, \beta_k, [\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \Delta]] = [\beta^*_k, \beta_k, [\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \ulcorner\neg\Delta_0\urcorner]] = \ulcorner\neg[\beta^*_k, \beta_k, [\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \Delta_0]]\urcorner$. With the I.H., it holds that $[\beta^*_k, \beta_k, [\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \Delta_0]] = [\langle\beta^*_0, \ldots, \beta^*_k\rangle, \langle\beta_0, \ldots, \beta_k\rangle, \Delta_0]$. Therefore we have $[\beta^*_k, \beta_k, [\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \Delta]] = \ulcorner\neg[\langle\beta^*_0, \ldots, \beta^*_k\rangle, \langle\beta_0, \ldots, \beta_k\rangle, \Delta_0]\urcorner = [\langle\beta^*_0, \ldots, \beta^*_k\rangle, \langle\beta_0, \ldots, \beta_k\rangle, \ulcorner\neg\Delta_0\urcorner] = [\langle\beta^*_0, \ldots, \beta^*_k\rangle, \langle\beta_0, \ldots, \beta_k\rangle, \Delta]$. Suppose $\Delta = \ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner \in$ CONFORM. This case is proved analogously to the negation-case. Suppose $\Delta = \ulcorner\Pi\xi\Delta_0\urcorner \in$ QFORM. This case is also proved analogously to the negation-case.

Ad (iii) and (iv): (iii) follows analogously to the negation-case by applying (ii), and (iv) follows analogously to the FTERM-case by applying (iii). ■

Note: For sets of formulas, a theorem that is analogous to Theorem 1-27 can be proved.

Theorem 1-28. *Multiple substitution of closed terms for pairwise different variables in terms and formulas (a)*

If $k \in \mathbb{N}\setminus\{0\}$, $\{\theta^*_0, \ldots, \theta^*_k\} \subseteq$ CTERM and $\{\xi_0, \ldots, \xi_k\} \subseteq$ VAR, where $\xi_i \neq \xi_j$ for all i, $j < k+1$ with $i \neq j$, then:

(i) If $\theta \in$ TERM, then
$$[\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \theta]] = [\langle\theta^*_0, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \theta], \text{ and}$$

(ii) If $\Delta \in$ FORM, then
$$[\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta]] = [\langle\theta^*_0, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \Delta].$$

Proof: Let $k \in \mathbb{N}\setminus\{0\}$, $\{\theta^*_0, \ldots, \theta^*_k\} \subseteq$ CTERM and $\{\xi_0, \ldots, \xi_k\} \subseteq$ VAR, where $\xi_i \neq \xi_j$ for all i, $j < k+1$ with $i \neq j$. *Ad (i)*: Suppose $\theta \in$ TERM. The proof is carried out by induction on the complexity of θ. Suppose $\theta \in$ ATERM. Suppose $\xi_i \neq \theta$ for all $i < k+1$. Then we have $[\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \theta]] = [\theta^*_k, \xi_k, \theta] = \theta = [\langle\theta^*_0, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \theta]$. Suppose $\xi_i = \theta$ for an $i < k$. Then we have $\xi_j \neq \theta$ for all $i < j < k+1$. Then we have $[\langle\theta^*_0, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \theta] = [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \theta] = [\langle\theta^*_0, \ldots, \theta^*_i\rangle, \langle\xi_0, \ldots, \xi_i\rangle, \theta] = \theta^*_i \in$ CTERM. Therefore $[\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \theta]] = [\theta^*_k, \xi_k, \theta^*_i] = \theta^*_i = [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \theta] = [\langle\theta^*_0, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \theta]$. Suppose $\xi_k = \theta$. Then we have $\xi_i \neq \theta$ for all $i < k$ and $[\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \theta] = \theta$. Therefore $[\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \theta]] = [\theta^*_k, \xi_k, \theta] = \theta^*_k = [\langle\theta^*_0, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \theta]$.

Now, suppose the statement holds for $\{\theta_0, \ldots, \theta_{r-1}\} \subseteq$ TERM and suppose $\theta = \ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner \in$ FTERM. Then we have $[\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \theta]] = [\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner]] = \ulcorner\varphi([\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \theta_0]], \ldots, [\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \theta_{r-1}]])\urcorner$. With the I.H., it holds that $[\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \theta_i]] = [\langle\theta^*_0, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \theta_i]$ for all $i < r$. Therefore we have $[\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \theta]] = \ulcorner\varphi([\langle\theta^*_0, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \theta_0], \ldots, [\langle\theta^*_0, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \theta_{r-1}])\urcorner = [\langle\theta^*_0, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner] = [\langle\theta^*_0, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \theta]$.

Ad (ii): Suppose $\Delta \in$ FORM. The proof is carried out by induction on the complexity of Δ. Suppose $\Delta = \ulcorner\Phi(\theta_0, \ldots\ \theta_{r-1})\urcorner \in$ AFORM. This case is proved analogously to the FTERM-case by applying (i).

Now, suppose the theorem holds for $\Delta_0, \Delta_1 \in$ FORM. Suppose $\Delta = \ulcorner\neg\Delta_0\urcorner \in$ CONFORM. Then we have $[\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta]] = [\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \ulcorner\neg\Delta_0\urcorner]] = \ulcorner\neg[\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta_0]]\urcorner$. With the I.H., it holds that $[\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta_0]] = [\langle\theta^*_0, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \Delta_0]$. Therefore $[\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta]] = \ulcorner\neg[\langle\theta^*_0, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \Delta_0]\urcorner = [\langle\theta^*_0, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \ulcorner\neg\Delta_0\urcorner] = [\langle\theta^*_0, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \Delta]$. Suppose $\Delta = \ulcorner(\Delta_0\ \psi\ \Delta_1)\urcorner \in$ CONFORM. This case is proved analogously to the negation-case.

Suppose $\Delta = \ulcorner\Pi\zeta\Delta_0\urcorner \in$ QFORM. Suppose $\xi_i = \zeta$ for one $i < k$. Then we have $\xi_j \neq \zeta$ for all $j < k$+1 with $i \neq j$. Then we have $[\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta]] = [\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \ulcorner\Pi\zeta\Delta_0\urcorner]] = [\theta^*_k, \xi_k, \ulcorner\Pi\zeta[\langle\theta^*_0, \ldots, \theta^*_{i-1}, \theta^*_{i+1}, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_{k-1}\rangle, \Delta_0]\urcorner] = \ulcorner\Pi\zeta[\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{i-1}, \theta^*_{i+1}, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_{k-1}\rangle, \Delta_0]]\urcorner$. With the I.H., it holds that $[\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{i-1}, \theta^*_{i+1}, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_{k-1}\rangle, \Delta_0]] = [\langle\theta^*_0, \ldots, \theta^*_{i-1}, \theta^*_{i+1}, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_k\rangle, \Delta_0]$. Therefore we have $[\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta]] = \ulcorner\Pi\zeta[\langle\theta^*_0, \ldots, \theta^*_{i-1}, \theta^*_{i+1}, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_k\rangle, \Delta_0]\urcorner = [\langle\theta^*_0, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \ulcorner\Pi\zeta\Delta_0\urcorner] = [\langle\theta^*_0, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \Delta]$. Suppose $\xi_k = \zeta$. Then we have $\xi_i \neq \zeta$ for all $i < k$ and $[\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta]] = [\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \ulcorner\Pi\zeta\Delta_0\urcorner]] = [\theta^*_k, \xi_k, \ulcorner\Pi\zeta[\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta_0]\urcorner] = \ulcorner\Pi\zeta[\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta_0]\urcorner = [\langle\theta^*_0, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \ulcorner\Pi\zeta\Delta_0\urcorner] = [\langle\theta^*_0, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \Delta]$.

Suppose $\xi_i \neq \zeta$ for all $i < k$+1. Then we have $[\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta]] = [\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \ulcorner\Pi\zeta\Delta_0\urcorner]] = [\theta^*_k, \xi_k, \ulcorner\Pi\zeta[\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta_0]\urcorner] = \ulcorner\Pi\zeta[\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta_0]]\urcorner$. With the I.H., it holds that

$[\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta_0]] = [\langle\theta^*_0, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \Delta_0]$. Therefore $[\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta]] = \ulcorner\Pi\zeta[\langle\theta^*_0, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \Delta_0]\urcorner = [\langle\theta^*_0, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \ulcorner\Pi\zeta\Delta_0\urcorner] = [\langle\theta^*_0, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \Delta]$. ■

Theorem 1-29. *Multiple substitution of closed terms for pairwise different variables in terms and formulas (b)*
If $k \in \mathbb{N}\setminus\{0\}$, $\{\theta^*_0, \ldots, \theta^*_k\} \subseteq$ CTERM and $\{\xi_0, \ldots, \xi_k\} \subseteq$ VAR, where $\xi_i \neq \xi_j$ for all $i, j < k+1$ with $i \neq j$, then:

(i) If $\theta \in$ TERM, then
$$[\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, [\theta^*_k, \xi_k, \theta]] = [\langle\theta^*_0, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \theta], \text{ and}$$

(ii) If $\Delta \in$ FORM, then
$$[\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, [\theta^*_k, \xi_k, \Delta]] = [\langle\theta^*_0, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \Delta].$$

Proof: Suppose $k \in \mathbb{N}\setminus\{0\}$, $\{\theta^*_0, \ldots, \theta^*_k\} \subseteq$ CTERM and $\{\xi_0, \ldots, \xi_k\} \subseteq$ VAR, where $\xi_i \neq \xi_j$ for all $i, j < k+1$ with $i \neq j$. *Ad (i)*: Suppose $\theta \in$ TERM. The proof is carried out by induction on k. Suppose $k = 1$. With Theorem 1-25-(i) and Theorem 1-28-(i), we then have $[\theta^*_0, \xi_0, [\theta^*_1, \xi_1, \theta]] = [\theta^*_1, \xi_1, [\theta^*_0, \xi_0, \theta]] = [\langle\theta^*_0, \theta^*_1\rangle, \langle\xi_0, \xi_1\rangle, \theta]$. Now, suppose $1 < k$. Applying the I.H., Theorem 1-25-(i), the I.H., Theorem 1-28-(i), the I.H. and Theorem 1-28-(i) (in this order) yields $[\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, [\theta^*_k, \xi_k, \theta]] = [\langle\theta^*_0, \ldots, \theta^*_{k-2}\rangle, \langle\xi_0, \ldots, \xi_{k-2}\rangle, [\theta^*_{k-1}, \xi_{k-1}, [\theta^*_k, \xi_k, \theta]]] = [\langle\theta^*_0, \ldots, \theta^*_{k-2}\rangle, \langle\xi_0, \ldots, \xi_{k-2}\rangle, [\theta^*_k, \xi_k, [\theta^*_{k-1}, \xi_{k-1}, \theta]]] = [\langle\theta^*_0, \ldots, \theta^*_{k-2}, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_{k-2}, \xi_k\rangle, [\theta^*_{k-1}, \xi_{k-1}, \theta]] = [\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-2}\rangle, \langle\xi_0, \ldots, \xi_{k-2}\rangle, [\theta^*_{k-1}, \xi_{k-1}, \theta]]] = [\theta^*_k, \xi_k, [\langle\theta^*_0, \ldots, \theta^*_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \theta]] = [\langle\theta^*_0, \ldots, \theta^*_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \theta]$.

(ii) follows analogously from Theorem 1-25-(ii) and Theorem 1-28-(ii). ■

2 The Availability of Propositions

In this chapter, the availability concepts that are needed for the calculus are established. Our course of action can be sketched as follows: First, preliminary concepts concerning segments and segment sequences are to be established, where a segment in a sentence sequence $\mathfrak{H}$ will be a non-empty, uninterrupted subset of $\mathfrak{H}$ (2.1). Second, closed segments will be characterised as certain CdI-, NI- and RA-like segments, i.e. certain segments of the kinds that are connected to inferences by conditional introduction (CdI), negation introduction (NI) and particular-quantifier elimination (PE) (2.2). The availability concepts themselves will then be established with recourse to closed segments. This will be done in such a way that exactly those propositions are available in a sentence sequence at a position that do not lie within a proper initial segment of a closed segment in this sentence sequence at this position (2.3). With the theorems that are established in this chapter, we can later show that CdI, NI and PE and only CdI, NI and PE can discharge assumptions.

2.1 Segments and Segment Sequences

First, segments in a non-empty sequence $\mathfrak{H}$ will be characterised as non-empty and uninterrupted subsets of $\mathfrak{H}$. Second, some theorems on segments will be proved. Then, some concepts and theorems concerning segment sequences for sentence sequences will be established, where a segment sequence for a sentence sequence $\mathfrak{H}$ is a finite sequence that enumerates disjunct segments in $\mathfrak{H}$. Then, AS-comprising segment sequences for segments in sentence sequences will be defined with recourse to segment sequences. An AS-comprising segment sequence for a segment $\mathfrak{A}$ in $\mathfrak{H}$ will be a segment sequence for $\mathfrak{H}$ for which it holds that all values of the sequence are disjunct subsegments of $\mathfrak{A}$ and that all assumption-sentences in $\mathfrak{A}$ lie in one of the values of the sequence. These AS-comprising segment sequences will later play a crucial role in the inductive generation of closed segments. The end of the chapter contains the proofs of theorems about AS-comprising segment sequences that are needed for the establishment of closed segments and of theorems on these. We start with the segment definition:

Definition 2-1. *Segment in a sequence (metavariables:* $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{A}'$, $\mathfrak{B}'$, $\mathfrak{C}'$, $\mathfrak{A}^*$, $\mathfrak{B}^*$, $\mathfrak{C}^*$, …*)*
$\mathfrak{A}$ <u>is a segment in</u> $\mathfrak{H}$
iff
$\mathfrak{H} \in \text{SEQ}$, $\mathfrak{A} \neq \emptyset$, $\mathfrak{A} \subseteq \mathfrak{H}$ and $\mathfrak{A} = \{(i, \mathfrak{H}_i) \mid \min(\text{Dom}(\mathfrak{A})) \leq i \leq \max(\text{Dom}(\mathfrak{A}))\}$.

Definition 2-2. *Assignment of the set of segments of* $\mathfrak{H}$ *(SG)*
$\text{SG} = \{(\mathfrak{H}, X) \mid \mathfrak{H} \in \text{SEQ} \text{ and } X = \{\mathfrak{A} \mid \mathfrak{A} \text{ is a segment in } \mathfrak{H}\}\}$.

Definition 2-3, Definition 2-4 and Definition 2-5 introduce some useful expressions.

Definition 2-3. *Segment*
$\mathfrak{A}$ <u>is a segment</u> iff there is an $\mathfrak{H}$ such that $\mathfrak{A}$ is a segment in $\mathfrak{H}$.

Definition 2-4. *Subsegment*
$\mathfrak{A}$ <u>is a subsegment of</u> $\mathfrak{A}'$ iff $\mathfrak{A}$, $\mathfrak{A}'$ are segments and $\mathfrak{A} \subseteq \mathfrak{A}'$.

Definition 2-5. *Proper subsegment*
$\mathfrak{A}$ <u>is a proper subsegment of</u> $\mathfrak{A}'$ iff $\mathfrak{A}$ is a subsegment of $\mathfrak{A}'$ and $\mathfrak{A} \neq \mathfrak{A}'$.

Theorem 2-1. *A sentence sequence* $\mathfrak{H}$ *is non-empty if and only if* SG($\mathfrak{H}$) *is non-empty*
If $\mathfrak{H} \in \text{SEQ}$, then: $\mathfrak{H} \neq \emptyset$ iff $\text{SG}(\mathfrak{H}) \neq \emptyset$.

Proof: Suppose $\mathfrak{H} \in \text{SEQ}$. Suppose $\mathfrak{H} \neq \emptyset$. Then $\mathfrak{H}$ is a segment in $\mathfrak{H}$ and thus $\mathfrak{H} \in \text{SG}(\mathfrak{H})$. Now, suppose $\text{SG}(\mathfrak{H}) \neq \emptyset$. Then there is an $\mathfrak{A}$ such that $\mathfrak{A}$ is a segment in $\mathfrak{H}$. Then we have $\mathfrak{A} \neq \emptyset$ and $\mathfrak{A} \subseteq \mathfrak{H}$ and thus $\mathfrak{H} \neq \emptyset$. ■

Theorem 2-2. *The segment predicate is monotone relative to inclusion between sequences*
If $\mathfrak{H}, \mathfrak{H}' \in \text{SEQ}$, $\mathfrak{H} \subseteq \mathfrak{H}'$ and $\mathfrak{A}$ is a segment in $\mathfrak{H}$, then $\mathfrak{A}$ is a segment in $\mathfrak{H}'$.

Proof: Suppose $\mathfrak{H}, \mathfrak{H}' \in \text{SEQ}$, $\mathfrak{H} \subseteq \mathfrak{H}'$ and $\mathfrak{A}$ is a segment in $\mathfrak{H}$. Then we have $\mathfrak{A} \neq \emptyset$ and $\mathfrak{A} \subseteq \mathfrak{H} \subseteq \mathfrak{H}'$. Moreover, we have $\mathfrak{H} = \mathfrak{H}' \upharpoonright \text{Dom}(\mathfrak{H})$. Thus we have

$$\mathfrak{A} = \{(i, \mathfrak{H}_i) \mid \min(\text{Dom}(\mathfrak{A})) \leq i \leq \max(\text{Dom}(\mathfrak{A}))\}$$
$$=$$
$$\{(i, \mathfrak{H}'_i) \mid \min(\text{Dom}(\mathfrak{A})) \leq i \leq \max(\text{Dom}(\mathfrak{A}))\}$$

and hence we have that $\mathfrak{A}$ is a segment in $\mathfrak{H}'$. ■

Remark 2-1. *All of the segment predicates defined in the following are monotone relative to inclusion between sequences. The respective instances of this result are used in the further account without being proven individually*
If F is one of the segment predicates defined in the following, then: If $\mathfrak{H}$, $\mathfrak{H}' \in$ SEQ, $\mathfrak{H} \subseteq \mathfrak{H}'$ and $\mathfrak{A}$ is an F-segment in $\mathfrak{H}$, then $\mathfrak{A}$ is an F-segment in $\mathfrak{H}'$.

Comment: All following definitions of segment predicates have one of the following two forms:

$\mathfrak{A}$ is an F-segment in $\mathfrak{H}$ iff $\mathfrak{H} \in$ SEQ, $\mathfrak{A} \in$ SG($\mathfrak{H}$) and $H(\mathfrak{A}, \mathfrak{H})$.

or

$\mathfrak{A}$ is an F-segment in $\mathfrak{H}$ iff $\mathfrak{A}$ is a segment in $\mathfrak{H}$ and $H(\mathfrak{A}, \mathfrak{H})$.

In each case, H is the variable part of the definiens, which distinguishes the different definitions. For H it holds in each case that if $\mathfrak{H}$, $\mathfrak{H}' \in$ SEQ, $\mathfrak{H} \subseteq \mathfrak{H}'$ and $\mathfrak{A} \in$ SG($\mathfrak{H}$) (or, equivalently: $\mathfrak{A}$ is a segment in $\mathfrak{H}$) and $H(\mathfrak{A}, \mathfrak{H})$, then $H(\mathfrak{A}, \mathfrak{H}')$. With Theorem 2-2 and the respective definition it then follows in each case that if $\mathfrak{H}$, $\mathfrak{H}'$ are sequences, $\mathfrak{H} \subseteq \mathfrak{H}'$ and $\mathfrak{A}$ is an F-segment in $\mathfrak{H}$, then $\mathfrak{A}$ is an F-segment in $\mathfrak{H}'$.

From this, it also follows that if $\mathfrak{H}$, $\mathfrak{H}'$ are sequences and $\mathfrak{A}$ is an F-segment in $\mathfrak{H}$, then $\mathfrak{A}$ is also an F-segment in $\mathfrak{H}^\frown\mathfrak{H}'$.[10] Note, however, that for many of the sequence predicates defined in the following, it does not hold that if $\mathfrak{H}$, $\mathfrak{H}'$ are sequences, and $\mathfrak{A}$ is an F-segment in $\mathfrak{H}$, then $\mathfrak{A}$ is also an F-segment in $\mathfrak{H}'^\frown\mathfrak{H}$. ■

Theorem 2-3. *Segments in restrictions*[11]
If $\mathfrak{H} \in$ SEQ, then: $\mathfrak{A}$ is a segment in $\mathfrak{H}$ iff $\mathfrak{A}$ is a segment in $\mathfrak{H}\restriction\max(\mathrm{Dom}(\mathfrak{A}))+1$.

Proof: Suppose $\mathfrak{H} \in$ SEQ. (*L-R*): Suppose $\mathfrak{A}$ is a segment in $\mathfrak{H}$. Then we have $\mathfrak{A} \neq \emptyset$, $\mathfrak{A} \subseteq \mathfrak{H}$ and thus: $\mathfrak{H}\restriction\max(\mathrm{Dom}(\mathfrak{A}))+1 \in$ SEQ. We also have that $\mathfrak{A} \subseteq \mathfrak{H}\restriction\max(\mathrm{Dom}(\mathfrak{A}))+1 \subseteq \mathfrak{H}$ and hence that $\mathfrak{H}\restriction\max(\mathrm{Dom}(\mathfrak{A}))+1 \in$ SEQ$\setminus\{\emptyset\}$ and also that

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10 Let $f^\frown g = f \cup \{(\mathrm{Dom}(f)+i, g_i) \mid i \in \mathrm{Dom}(g)\}$ if f is a finite sequence and g is a sequence, else $f^\frown g = \emptyset$. We omit parentheses and assume that they are nested from left to right, i.e., $\ulcorner a_0{}^\frown a_1{}^\frown a_2{}^\frown \ldots{}^\frown a_{n\text{-}1}\urcorner = \ulcorner(\ldots((a_0{}^\frown a_1)^\frown a_2)^\frown \ldots)^\frown a_{n\text{-}1})\urcorner$.

11 Let $R\restriction X = \{(a, b) \mid (a, b) \in R \text{ and } a \in X\}$.

$\mathfrak{A} = \{(i, \mathfrak{H}_i) \mid \min(\mathrm{Dom}(\mathfrak{A})) \leq i \leq \max(\mathrm{Dom}(\mathfrak{A}))\}$
$=$
$\{(i, (\mathfrak{H}{\upharpoonright}\max(\mathrm{Dom}(\mathfrak{A}))+1)_i) \mid \min(\mathrm{Dom}(\mathfrak{A})) \leq i \leq \max(\mathrm{Dom}(\mathfrak{A}))\}$.

Thus, $\mathfrak{A}$ is a segment in $\mathfrak{H}{\upharpoonright}\max(\mathrm{Dom}(\mathfrak{A}))+1$. (*R-L*): Suppose $\mathfrak{A}$ is a segment in $\mathfrak{H}{\upharpoonright}\max(\mathrm{Dom}(\mathfrak{A}))+1$. Then we have $\mathfrak{H}{\upharpoonright}\max(\mathrm{Dom}(\mathfrak{A}))+1 \in$ SEQ. According to the initial assumption, we also have $\mathfrak{H} \in$ SEQ. With $\mathfrak{H}{\upharpoonright}\max(\mathrm{Dom}(\mathfrak{A}))+1 \subseteq \mathfrak{H}$ and Theorem 2-2, we then have that $\mathfrak{A}$ is a segment in $\mathfrak{H}$. ■

Remark 2-2. *F-segments in restrictions*
If F is one of the segment predicates defined in the following, then: If $\mathfrak{H} \in$ SEQ, then $\mathfrak{A}$ is an F-segment in $\mathfrak{H}$ iff $\mathfrak{A}$ is an F-segment in $\mathfrak{H}{\upharpoonright}\max(\mathrm{Dom}(\mathfrak{A}))+1$.

Comment: All of the following definitions of segment predicates have one of the two forms noted in Remark 2-1, where for H it holds that if $\mathfrak{H} \in$ SEQ, $\mathfrak{A} \in$ SG($\mathfrak{H}$) (or, equivalently: $\mathfrak{A}$ is a segment in $\mathfrak{H}$) and $H(\mathfrak{A}, \mathfrak{H})$, then $H(\mathfrak{A}, \mathfrak{H}{\upharpoonright}\max(\mathrm{Dom}(\mathfrak{A}))+1)$. The reason for this is in each case that the respective definientia only refer to conditions in $\mathfrak{H}{\upharpoonright}\max(\mathrm{Dom}(\mathfrak{A}))+1$. With Theorem 2-3 and the respective definitions it thus follows in each case that if $\mathfrak{H}$ is a sentence sequence and $\mathfrak{A}$ is an F-segment in $\mathfrak{H}$ ist, then $\mathfrak{A}$ is an F-segment in $\mathfrak{H}{\upharpoonright}\max(\mathrm{Dom}(\mathfrak{A}))+1$. For the right-left-direction see Remark 2-1. ■

Theorem 2-4. *Segments with identical beginning and end are identical*
If $\mathfrak{H} \in$ SEQ, $\mathfrak{A}, \mathfrak{A}' \in$ SG($\mathfrak{H}$), $\min(\mathrm{Dom}(\mathfrak{A})) = \min(\mathrm{Dom}(\mathfrak{A}'))$ and $\max(\mathrm{Dom}(\mathfrak{A})) = \max(\mathrm{Dom}(\mathfrak{A}'))$, then $\mathfrak{A} = \mathfrak{A}'$.

Proof: Suppose $\mathfrak{H} \in$ SEQ, $\mathfrak{A}, \mathfrak{A}' \in$ SG($\mathfrak{H}$), $\min(\mathrm{Dom}(\mathfrak{A})) = \min(\mathrm{Dom}(\mathfrak{A}'))$ and $\max(\mathrm{Dom}(\mathfrak{A})) = \max(\mathrm{Dom}(\mathfrak{A}'))$. Then we have for all $(i, \mathfrak{H}_i)$: $(i, \mathfrak{H}_i) \in \mathfrak{A}$ iff $\min(\mathrm{Dom}(\mathfrak{A})) \leq i \leq \max(\mathrm{Dom}(\mathfrak{A}))$ iff $\min(\mathrm{Dom}(\mathfrak{A}')) \leq i \leq \max(\mathrm{Dom}(\mathfrak{A}'))$ iff $(i, \mathfrak{H}_i) \in \mathfrak{A}'$. ■

Theorem 2-5. *Inclusion between segments*
If $\mathfrak{H} \in$ SEQ and $\mathfrak{A}, \mathfrak{A}' \in$ SG($\mathfrak{H}$), then:
(i) $\min(\mathrm{Dom}(\mathfrak{A})) \leq \min(\mathrm{Dom}(\mathfrak{A}'))$ and $\max(\mathrm{Dom}(\mathfrak{A}')) \leq \max(\mathrm{Dom}(\mathfrak{A}))$ iff $\mathfrak{A}' \subseteq \mathfrak{A}$, and
(ii) If $\min(\mathrm{Dom}(\mathfrak{A})) = \min(\mathrm{Dom}(\mathfrak{A}'))$, then $\mathfrak{A} \subseteq \mathfrak{A}'$ or $\mathfrak{A}' \subseteq \mathfrak{A}$.

Proof: Suppose $\mathfrak{H} \in$ SEQ and $\mathfrak{A}, \mathfrak{A}' \in$ SG($\mathfrak{H}$). Then we have

$\mathfrak{A} = \{(l, \mathfrak{H}_l) \mid \min(\mathrm{Dom}(\mathfrak{A})) \leq l \leq \max(\mathrm{Dom}(\mathfrak{A}))\}$

and
$\mathfrak{A}' = \{(l, \mathfrak{H}_l) \mid \min(\mathrm{Dom}(\mathfrak{A}')) \leq l \leq \max(\mathrm{Dom}(\mathfrak{A}'))\}$.

Ad (i): Suppose $\min(\mathrm{Dom}(\mathfrak{A})) \leq \min(\mathrm{Dom}(\mathfrak{A}'))$ and $\max(\mathrm{Dom}(\mathfrak{A}')) \leq \max(\mathrm{Dom}(\mathfrak{A}))$. Suppose $(l, \mathfrak{H}_l) \in \mathfrak{A}'$. Then we have $\min(\mathrm{Dom}(\mathfrak{A}')) \leq l \leq \max(\mathrm{Dom}(\mathfrak{A}'))$ and thus according to the hypothesis $\min(\mathrm{Dom}(\mathfrak{A})) \leq \min(\mathrm{Dom}(\mathfrak{A}')) \leq l \leq \max(\mathrm{Dom}(\mathfrak{A}')) \leq \max(\mathrm{Dom}(\mathfrak{A}))$. Therefore we have $(l, \mathfrak{H}_l) \in \mathfrak{A}$.

Now, suppose $\mathfrak{A}' \subseteq \mathfrak{A}$. Then we have that $\min(\mathrm{Dom}(\mathfrak{A}')), \max(\mathrm{Dom}(\mathfrak{A}')) \in \mathrm{Dom}(\mathfrak{A})$ and hence $\min(\mathrm{Dom}(\mathfrak{A})) \leq \min(\mathrm{Dom}(\mathfrak{A}'))$ and $\max(\mathrm{Dom}(\mathfrak{A}')) \leq \max(\mathrm{Dom}(\mathfrak{A}))$.

Ad (ii): Suppose $\min(\mathrm{Dom}(\mathfrak{A})) = \min(\mathrm{Dom}(\mathfrak{A}'))$. Then we have $\max(\mathrm{Dom}(\mathfrak{A})) \leq \max(\mathrm{Dom}(\mathfrak{A}'))$ or $\max(\mathrm{Dom}(\mathfrak{A}')) \leq \max(\mathrm{Dom}(\mathfrak{A}))$. In the first case, it follows with (i) that $\mathfrak{A} \subseteq \mathfrak{A}'$. In the second case, it follows with (i) that $\mathfrak{A}' \subseteq \mathfrak{A}$. ■

Theorem 2-6. *Non-empty restrictions of segments are segments*
If $\mathfrak{H} \in \mathrm{SEQ}$ and $\mathfrak{A} \in \mathrm{SG}(\mathfrak{H})$, then for all $k \in \mathrm{Dom}(\mathfrak{A})$: $\mathfrak{A}\restriction k+1 \in \mathrm{SG}(\mathfrak{H})$.

Proof: Suppose $\mathfrak{H} \in \mathrm{SEQ}$ and $\mathfrak{A} \in \mathrm{SG}(\mathfrak{H})$ and suppose $k \in \mathrm{Dom}(\mathfrak{A})$. Then we have that $\min(\mathrm{Dom}(\mathfrak{A})) < k+1 \leq \max(\mathrm{Dom}(\mathfrak{A}))+1$. Thus we have that $\mathfrak{A}\restriction k+1 = \{(i, \mathfrak{H}_i) \mid \min(\mathrm{Dom}(\mathfrak{A})) \leq i \leq \max(\mathrm{Dom}(\mathfrak{A}))\}\restriction k+1 = \{(i, \mathfrak{H}_i) \mid \min(\mathrm{Dom}(\mathfrak{A})) \leq i \leq k\} = \{(i, \mathfrak{H}_i) \mid \min(\mathrm{Dom}(\mathfrak{A}\restriction k+1)) \leq i \leq \max(\mathrm{Dom}(\mathfrak{A}\restriction k+1))\}$ and also that $\mathfrak{A}\restriction k+1 \subseteq \mathfrak{A} \subseteq \mathfrak{H}$. We also have $k \in \mathrm{Dom}(\mathfrak{A}\restriction k+1)$ and thus that $\mathfrak{A}\restriction k+1 \neq \emptyset$. Hence we have $\mathfrak{A}\restriction k+1 \in \mathrm{SG}(\mathfrak{H})$. ■

Theorem 2-7. *Restrictions of segments that are segments themselves have the same beginning as the restricted segment*
If $\mathfrak{A}$ is a segment in $\mathfrak{H}$, then for all $k \in \mathrm{Dom}(\mathfrak{A})$: If $\mathfrak{A}\restriction k$ is a segment in $\mathfrak{H}$, then $\min(\mathrm{Dom}(\mathfrak{A}\restriction k)) = \min(\mathrm{Dom}(\mathfrak{A}))$.

Proof: Suppose $\mathfrak{A}$ is a segment in $\mathfrak{H}$. Now, suppose $k \in \mathrm{Dom}(\mathfrak{A})$ and suppose $\mathfrak{A}\restriction k$ is a segment in $\mathfrak{H}$ and hence $\mathfrak{A}\restriction k \neq \emptyset$. Then we have $\mathfrak{A}\restriction k = \{(i, \mathfrak{H}_i) \mid \min(\mathrm{Dom}(\mathfrak{A})) \leq i \leq \max(\mathrm{Dom}(\mathfrak{A}))\}\restriction k = \{(i, \mathfrak{H}_i) \mid \min(\mathrm{Dom}(\mathfrak{A})) \leq i \leq k\text{-}1\}$ and hence with $\mathfrak{A}\restriction k \neq \emptyset$ that $\min(\mathrm{Dom}(\mathfrak{A}\restriction k)) = \min(\mathrm{Dom}(\mathfrak{A}))$. ■

Theorem 2-8. *Two segments are disjunct if and only if one of them lies before the other*
If $\mathfrak{H} \in \mathrm{SEQ}$ and $\mathfrak{A}, \mathfrak{A}' \in \mathrm{SG}(\mathfrak{H})$, then:
$\mathfrak{A} \cap \mathfrak{A}' = \emptyset$
iff

(i) $\min(\mathrm{Dom}(\mathfrak{A})) < \min(\mathrm{Dom}(\mathfrak{A}'))$ and $\max(\mathrm{Dom}(\mathfrak{A})) < \min(\mathrm{Dom}(\mathfrak{A}'))$, or

or

(ii) $\min(\mathrm{Dom}(\mathfrak{A}')) < \min(\mathrm{Dom}(\mathfrak{A}))$ and $\max(\mathrm{Dom}(\mathfrak{A}')) < \min(\mathrm{Dom}(\mathfrak{A}))$.

Proof: Suppose $\mathfrak{H} \in \mathrm{SEQ}$ and $\mathfrak{A}, \mathfrak{A}' \in \mathrm{SG}(\mathfrak{H})$. (*L-R*): Suppose $\mathfrak{A} \cap \mathfrak{A}' = \emptyset$. Then we have

$\min(\mathrm{Dom}(\mathfrak{A})) < \min(\mathrm{Dom}(\mathfrak{A}'))$
or
$\min(\mathrm{Dom}(\mathfrak{A})) = \min(\mathrm{Dom}(\mathfrak{A}'))$
or
$\min(\mathrm{Dom}(\mathfrak{A}')) < \min(\mathrm{Dom}(\mathfrak{A}))$.

The second case, i.e. $\min(\mathrm{Dom}(\mathfrak{A})) = \min(\mathrm{Dom}(\mathfrak{A}'))$, is impossible because otherwise we would have that $(\min(\mathrm{Dom}(\mathfrak{A})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}))}) \in \mathfrak{A}$ and $(\min(\mathrm{Dom}(\mathfrak{A})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}))}) \in \mathfrak{A}'$ and thus that $\mathfrak{A} \cap \mathfrak{A}' \neq \emptyset$.

Suppose $\min(\mathrm{Dom}(\mathfrak{A})) < \min(\mathrm{Dom}(\mathfrak{A}'))$. If $\min(\mathrm{Dom}(\mathfrak{A}')) \leq \max(\mathrm{Dom}(\mathfrak{A}))$, then we would have $(\min(\mathrm{Dom}(\mathfrak{A}')), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}'))}) \in \mathfrak{A}$ and $(\min(\mathrm{Dom}(\mathfrak{A}')), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}'))}) \in \mathfrak{A}'$. Thus we would have $\mathfrak{A} \cap \mathfrak{A}' \neq \emptyset$, which contradicts the hypothesis. In the first case, we thus have $\min(\mathrm{Dom}(\mathfrak{A})) < \min(\mathrm{Dom}(\mathfrak{A}'))$ and $\max(\mathrm{Dom}(\mathfrak{A})) < \min(\mathrm{Dom}(\mathfrak{A}'))$.

Suppose $\min(\mathrm{Dom}(\mathfrak{A}')) < \min(\mathrm{Dom}(\mathfrak{A}))$. If $\min(\mathrm{Dom}(\mathfrak{A})) \leq \max(\mathrm{Dom}(\mathfrak{A}'))$, then we would have $(\min(\mathrm{Dom}(\mathfrak{A})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}))}) \in \mathfrak{A}'$ and $(\min(\mathrm{Dom}(\mathfrak{A})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}))}) \in \mathfrak{A}$. Thus we would again have $\mathfrak{A} \cap \mathfrak{A}' \neq \emptyset$. In the third case, we thus have $\min(\mathrm{Dom}(\mathfrak{A}')) < \min(\mathrm{Dom}(\mathfrak{A}))$ and $\max(\mathrm{Dom}(\mathfrak{A}')) < \min(\mathrm{Dom}(\mathfrak{A}))$.

(*R-L*): Now, suppose $\min(\mathrm{Dom}(\mathfrak{A})) < \min(\mathrm{Dom}(\mathfrak{A}'))$ and $\max(\mathrm{Dom}(\mathfrak{A})) < \min(\mathrm{Dom}(\mathfrak{A}'))$ or $\min(\mathrm{Dom}(\mathfrak{A}')) < \min(\mathrm{Dom}(\mathfrak{A}))$ and $\max(\mathrm{Dom}(\mathfrak{A}')) < \min(\mathrm{Dom}(\mathfrak{A}))$. Now, suppose for contradiction that $\mathfrak{A} \cap \mathfrak{A}' \neq \emptyset$. Then there would be an i such that $(i, \mathfrak{H}_i) \in \mathfrak{A} \cap \mathfrak{A}'$. Then we would have $\min(\mathrm{Dom}(\mathfrak{A})) \leq i \leq \max(\mathrm{Dom}(\mathfrak{A}))$ and $\min(\mathrm{Dom}(\mathfrak{A}')) \leq i \leq \max(\mathrm{Dom}(\mathfrak{A}'))$. Thus we would have $\min(\mathrm{Dom}(\mathfrak{A}')) < \min(\mathrm{Dom}(\mathfrak{A}'))$ or $\min(\mathrm{Dom}(\mathfrak{A})) < \min(\mathrm{Dom}(\mathfrak{A}))$. Contradiction! Therefore we have $\mathfrak{A} \cap \mathfrak{A}' = \emptyset$. ■

Theorem 2-9. *Two segments have a common element if and only if the beginning of one of them lies within the other*

If $\mathfrak{H} \in$ SEQ and $\mathfrak{A}, \mathfrak{A}' \in$ SG($\mathfrak{H}$), then:
$\mathfrak{A} \cap \mathfrak{A}' \neq \emptyset$
iff

(i) min(Dom($\mathfrak{A}$)) $\in$ Dom($\mathfrak{A}'$) or

or

(ii) min(Dom($\mathfrak{A}'$)) $\in$ Dom($\mathfrak{A}$).

Proof: Suppose $\mathfrak{H} \in$ SEQ and $\mathfrak{A}, \mathfrak{A}' \in$ SG($\mathfrak{H}$). (*L-R*): Suppose $\mathfrak{A} \cap \mathfrak{A}' \neq \emptyset$. Then there is an $i \in$ Dom($\mathfrak{H}$) such that $(i, \mathfrak{H}_i) \in \mathfrak{A} \cap \mathfrak{A}'$. Then we have

min(Dom($\mathfrak{A}$)) $\leq i \leq$ max(Dom($\mathfrak{A}$)) and min(Dom($\mathfrak{A}'$)) $\leq i \leq$ max(Dom($\mathfrak{A}'$))
and
min(Dom($\mathfrak{A}'$)) $\leq$ min(Dom($\mathfrak{A}$)) or min(Dom($\mathfrak{A}$)) $\leq$ min(Dom($\mathfrak{A}'$)).

Thus we then have

min(Dom($\mathfrak{A}'$)) $\leq$ min(Dom($\mathfrak{A}$)) $\leq i \leq$ max(Dom($\mathfrak{A}'$))
or
min(Dom($\mathfrak{A}$)) $\leq$ min(Dom($\mathfrak{A}'$)) $\leq i \leq$ max(Dom($\mathfrak{A}$)).

Thus we have eventually that

min(Dom($\mathfrak{A}$)) $\in$ Dom($\mathfrak{A}'$) or min(Dom($\mathfrak{A}'$)) $\in$ Dom($\mathfrak{A}$).

(R-L): If min(Dom($\mathfrak{A}$)) $\in$ Dom($\mathfrak{A}'$) or min(Dom($\mathfrak{A}'$)) $\in$ Dom($\mathfrak{A}$), then we have (min(Dom($\mathfrak{A}$)), $\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}))}$) $\in \mathfrak{A} \cap \mathfrak{A}'$ or (min(Dom($\mathfrak{A}'$)), $\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}'))}$) $\in \mathfrak{A} \cap \mathfrak{A}'$ and thus in both cases $\mathfrak{A} \cap \mathfrak{A}' \neq \emptyset$. ■

Definition 2-6. *Suitable sequences of natural numbers for subsets of sentence sequences*

g is a suitable sequence of natural numbers for $\mathfrak{A}$
iff
There is an $\mathfrak{H} \in$ SEQ such that $\mathfrak{A} \subseteq \mathfrak{H}$ and g is a strictly monotone increasing sequence of natural numbers with Ran(g) = Dom($\mathfrak{A}$).

The immediate purpose of the definition is to enable us to enumerate the elements (of the domain) of a subset of a sequence in a way that preserves their natural order. Moreover, suitable sequences can be used to turn segments of sequences into sequences by composing the respective segments with a suitable sequence of natural numbers. Such a procedure could be considered as an inverse operation to the concatenation of sequences.

Theorem 2-10. *Existence of suitable sequences of natural numbers*
If $\mathfrak{H} \in$ SEQ and $\mathfrak{A} \subseteq \mathfrak{H}$, then there is a g such that g is a suitable sequence of natural numbers for $\mathfrak{A}$.

Proof: Suppose $\mathfrak{H} \in$ SEQ and $\mathfrak{A} \subseteq \mathfrak{H}$. The proof is carried out by induction on $|\mathfrak{A}|$. Suppose $|\mathfrak{A}| = 0$. Let $g = \emptyset$. Then g is trivially a strictly monotone increasing sequence of natural numbers with Ran(g) = Dom($\mathfrak{A}$). Now, suppose $|\mathfrak{A}| = k+1$. Then we have $k = 0$ or $k > 0$. In the first case, {(0, max(Dom($\mathfrak{A}$)))} is a suitable sequence of natural numbers for $\mathfrak{A}$. Now, suppose $k > 0$. Since $\mathfrak{A}$ is a finite function, we have that $|\mathfrak{A}\setminus\{(\max(\mathrm{Dom}(\mathfrak{A})), \mathfrak{A}_{\max(\mathrm{Dom}(\mathfrak{A}))})\}| = k$. Furthermore, we have $\mathfrak{A}\setminus\{(\max(\mathrm{Dom}(\mathfrak{A})), \mathfrak{A}_{\max(\mathrm{Dom}(\mathfrak{A}))})\} \subseteq \mathfrak{H}$. According to the I.H., we thus have a g such that g is a suitable sequence of natural numbers for $\mathfrak{A}\setminus\{(\max(\mathrm{Dom}(\mathfrak{A})), \mathfrak{A}_{\max(\mathrm{Dom}(\mathfrak{A}))})\}$. Now, let $g' = g \cup \{(\mathrm{Dom}(g), \max(\mathrm{Dom}(\mathfrak{A})))\}$. Obviously it holds that Ran(g') = Dom($\mathfrak{A}$). Because of

$$g(\max(\mathrm{Dom}(g))) = \max(\mathrm{Ran}(g)) = \max(\mathrm{Dom}(\mathfrak{A}\setminus\{(\max(\mathrm{Dom}(\mathfrak{A})), \mathfrak{A}_{\max(\mathrm{Dom}(\mathfrak{A}))})\}))$$
$$< \max(\mathrm{Dom}(\mathfrak{A})) = \max(\mathrm{Ran}(g')) = g'(\mathrm{Dom}(g)) = g'(\max(\mathrm{Dom}(g'))),$$

the strict monotony of g carries over to g'. Therefore we have that g' is a suitable sequence of natural numbers for $\mathfrak{A}$. ■

Theorem 2-11. *Bijectivity of suitable sequences of natural numbers*
If $\mathfrak{H} \in$ SEQ, $\mathfrak{A} \subseteq \mathfrak{H}$, and g is a suitable sequence of natural numbers for $\mathfrak{A}$, then g is a bijection between Dom(g) and Dom($\mathfrak{A}$).

Proof: Suppose $\mathfrak{H} \in$ SEQ, $\mathfrak{A} \subseteq \mathfrak{H}$ and suppose g is a suitable sequence of natural numbers for $\mathfrak{A}$. Then we have Ran(g) = Dom($\mathfrak{A}$) and hence that g is a surjection of Dom(g) onto Dom($\mathfrak{A}$). Furthermore, because g is a strictly monotone sequence of natural numbers, we have that g is an injection of Dom(g) into Dom($\mathfrak{A}$). Hence g is a bijection between Dom(g) and Dom($\mathfrak{A}$). ■

Theorem 2-12. *Uniqueness of suitable sequences of natural numbers*
If $\mathfrak{H} \in$ SEQ, $\mathfrak{A} \subseteq \mathfrak{H}$, and g, g' are suitable sequences of natural numbers for $\mathfrak{A}$, then: $g = g'$.

Proof: Suppose $\mathfrak{H} \in$ SEQ, $\mathfrak{A} \subseteq \mathfrak{H}$ and suppose g, g' are suitable sequences of natural numbers for $\mathfrak{A}$. Then we have Ran(g) = Dom($\mathfrak{A}$) = Ran(g'). With Theorem 2-11, we also have that Dom(g) = |Ran(g)| = |Ran(g')| = Dom(g'). Now, it holds that strictly monotone increasing sequences of natural numbers with identical domains and identical ranges are identical. Therefore we have $g = g'$. ■

Theorem 2-13. *Non-recursive characterisation of the suitable sequence for a segment*
If $\mathfrak{A}$ is a segment in $\mathfrak{H}$, then $\{(l, \min(\text{Dom}(\mathfrak{A}))+l) \mid l < |\text{Dom}(\mathfrak{A})|\}$ is a suitable sequence of natural numbers for $\mathfrak{A}$.

Proof: Suppose $\mathfrak{H} \in$ SEQ and $\mathfrak{A}$ is a segment in $\mathfrak{H}$. Then we have $\mathfrak{A} \neq \emptyset$. The proof is carried out by induction on |Dom($\mathfrak{A}$)|. Suppose |Dom($\mathfrak{A}$)| = 1. Then we have Dom($\mathfrak{A}$) = $\{\min(\text{Dom}(\mathfrak{A}))\}$ and $\{(0, \min(\text{Dom}(\mathfrak{A})))\}$ is a suitable sequence of natural numbers for $\mathfrak{A}$ and $\{(0, \min(\text{Dom}(\mathfrak{A})))\} = \{(l, \min(\text{Dom}(\mathfrak{A}))+l) \mid l < 1\} = \{(l, \min(\text{Dom}(\mathfrak{A}))+l) \mid l < |\text{Dom}(\mathfrak{A})|\}$.

Now, suppose the statement holds for $k \geq 1$ and suppose |Dom($\mathfrak{A}$)| = k+1. Since $\mathfrak{A}$ is a finite function, we have that $|\mathfrak{A}\backslash\{(\max(\text{Dom}(\mathfrak{A})), \mathfrak{A}_{\max(\text{Dom}(\mathfrak{A}))})\}| = k$. Furthermore, we have that $\mathfrak{A}^* = \mathfrak{A}\backslash\{(\max(\text{Dom}(\mathfrak{A})), \mathfrak{A}_{\max(\text{Dom}(\mathfrak{A}))})\}$ is a segment in $\mathfrak{H}$. According to the I.H., we therefore have that $g = \{(l, \min(\text{Dom}(\mathfrak{A}^*))+l) \mid l < |\text{Dom}(\mathfrak{A}^*)|\} = \{(l, \min(\text{Dom}(\mathfrak{A}))+l) \mid l < |\text{Dom}(\mathfrak{A})|-1\}$ is a suitable sequences of natural numbers for $\mathfrak{A}^*$. Let $g' = g \cup \{(|\text{Dom}(\mathfrak{A})|-1, \max(\text{Dom}(\mathfrak{A})))\}$. Then we have Ran($g'$) = Dom($\mathfrak{A}^*$) $\cup$ $\{\max(\text{Dom}(\mathfrak{A}))\}$ = Dom($\mathfrak{A}$) and we have Dom(g') = Dom(g) $\cup$ $\{\text{Dom}(g)\}$ = Dom(g)+1 = |Dom($\mathfrak{A}^*$)|+1 = |Dom($\mathfrak{A}$)|. Since $\mathfrak{A}$ is a segment in $\mathfrak{H}$, it also holds that max(Dom($\mathfrak{A}^*$))+1 = max(Dom($\mathfrak{A}$)). Thus we have $g'(|\text{Dom}(\mathfrak{A})|-1) = \max(\text{Dom}(\mathfrak{A}^*))+1 = g(|\text{Dom}(\mathfrak{A})|-2)+1 = (\min(\text{Dom}(\mathfrak{A}^*))+|\text{Dom}(\mathfrak{A})|-2)+1 = (\min(\text{Dom}(\mathfrak{A}))+|\text{Dom}(\mathfrak{A})|-2)+1 = \min(\text{Dom}(\mathfrak{A}))+|\text{Dom}(\mathfrak{A})|-1$. Hence we then have $g' = \{(l, \min(\text{Dom}(\mathfrak{A}))+l) \mid l < |\text{Dom}(\mathfrak{A})|-1\} \cup \{(|\text{Dom}(\mathfrak{A})|-1, \min(\text{Dom}(\mathfrak{A}))+|\text{Dom}(\mathfrak{A})|-1)\} = \{(l, \min(\text{Dom}(\mathfrak{A}))+l) \mid l < |\text{Dom}(\mathfrak{A})|\}$. Thus we have that g' is also a strictly monotone increasing sequence of natural numbers and hence we have that g' is a suitable sequence of natural numbers for $\mathfrak{A}$. ■

Definition 2-7. *Segment sequences for sentence sequences*

G <u>is a segment sequence for</u> $\mathfrak{H}$
iff
$\mathfrak{H} \in$ SEQ and G is a sequence with $\mathrm{Ran}(G) \subseteq \mathrm{SG}(\mathfrak{H})$ and for all $i, j \in \mathrm{Dom}(G)$: If $i < j$, then $\min(\mathrm{Dom}(G(i))) < \min(\mathrm{Dom}(G(j)))$ and $\max(\mathrm{Dom}(G(i))) < \min(\mathrm{Dom}(G(j)))$.

Definition 2-8. *Assignment of the set of segment sequences for* $\mathfrak{H}$ *(SGS)*

$\mathrm{SGS} = \{(\mathfrak{H}, X) \mid \mathfrak{H} \in \mathrm{SEQ}$ and $X = \{G \mid G$ is a segment sequence for $\mathfrak{H}\}\}$

Theorem 2-14. *A sentence sequence* $\mathfrak{H}$ *is non-empty if and only if there is a non-empty segment sequence for* $\mathfrak{H}$

If $\mathfrak{H} \in$ SEQ, then: $\mathfrak{H} \neq \emptyset$ iff there is a $G \in \mathrm{SGS}(\mathfrak{H})$ with $G \neq \emptyset$.

Proof: Suppose $\mathfrak{H} \in$ SEQ. (*L-R*): Suppose $\mathfrak{H} \neq \emptyset$. Then we have $\emptyset \neq \{(i, \{(i, \mathfrak{H}_i)\}) \mid i \in \mathrm{Dom}(\mathfrak{H})\} \in \mathrm{SGS}(\mathfrak{H})$. (*R-L*): Now, suppose there is a $G \in \mathrm{SGS}(\mathfrak{H})$ such that $G \neq \emptyset$. Then there is an $i \in \mathrm{Dom}(G)$. Also, we have $\mathrm{Ran}(G) \subseteq \mathrm{SG}(\mathfrak{H})$ and thus $G(i) \in \mathrm{SG}(\mathfrak{H})$. With Theorem 2-1, we then have $\mathfrak{H} \neq \emptyset$. ■

Theorem 2-15. $\emptyset$ is a *segment sequence for all sequences*

If $\mathfrak{H} \in$ SEQ, then $\emptyset \in \mathrm{SGS}(\mathfrak{H})$.

Proof: Suppose $\mathfrak{H} \in$ SEQ. Then we have that $\emptyset$ is a sequence with $\mathrm{Ran}(\emptyset) = \emptyset \subseteq \mathrm{SG}(\mathfrak{H})$ and for all $i, j \in \mathrm{Dom}(\emptyset) = \emptyset$ we trivially have: If $i < j$, then $\min(\mathrm{Dom}(\emptyset(i))) < \min(\mathrm{Dom}(\emptyset(j)))$ and $\max(\mathrm{Dom}(\emptyset(i))) < \min(\mathrm{Dom}(\emptyset(j)))$. ■

Theorem 2-16. *Properties of segment sequences*

If $\mathfrak{H} \in$ SEQ and $G \in \mathrm{SGS}(\mathfrak{H})$, then:

(i) G is an injection of $\mathrm{Dom}(G)$ into $\mathrm{Ran}(G)$,
(ii) G is a bijection between $\mathrm{Dom}(G)$ and $\mathrm{Ran}(G)$,
(iii) $\mathrm{Dom}(G) = |\mathrm{Ran}(G)|$, and
(iv) G is a finite sequence.

Proof: Suppose $\mathfrak{H} \in$ SEQ and $G \in \mathrm{SGS}(\mathfrak{H})$. Then we have that G is a sequence with $\mathrm{Ran}(G) \subseteq \mathrm{SG}(\mathfrak{H})$ and for all $i, j \in \mathrm{Dom}(G)$: If $i < j$, then $\min(\mathrm{Dom}(G(i))) < \min(\mathrm{Dom}(G(j)))$ and $\max(\mathrm{Dom}(G(i))) < \min(\mathrm{Dom}(G(j)))$.

Ad (i): Now, suppose i, $j \in$ Dom(G) and suppose $G(i) = G(j)$. Then we have min(Dom($G(i)$)) = min(Dom($G(j)$)). Suppose for contradiction that $i \neq j$. Then we would have $i < j$ or $j < i$ and thus we would have min(Dom($G(i)$)) < min(Dom($G(j)$)) or min(Dom($G(j)$)) < min(Dom($G(i)$)), which both contradict min(Dom($G(i)$)) = min(Dom($G(j)$)). Therefore we have for i, $j \in$ Dom(G) with $G(i) = G(j)$ that $i = j$. Hence G is an injection of Dom(G) in Ran(G).

Ad (ii): G is a surjection of Dom(G) onto Ran(G) and with (i) G is then a bijection between Dom(G) and Ran(G).

Ad (iii): Since G is a sequence, it holds with (ii): Dom(G) = |Ran(G)|

Ad (iv): G is a sequence and with (iii) G is then a finite sequence, because we have Ran(G) $\subseteq$ SG($\mathfrak{H}$) $\subseteq$ POT($\mathfrak{H}$) and hence (because with $\mathfrak{H} \in$ SEQ it holds that $|\mathfrak{H}| \in \mathbb{N}$): Dom($G$) = |Ran($G$)| $\leq$ |SG($\mathfrak{H}$)| $\leq$ |POT($\mathfrak{H}$)| = $2^{|\mathfrak{H}|} \in \mathbb{N}$. ■

Theorem 2-17. *Existence of segment sequences that enumerate all elements of a set of disjunct segments*
If $\mathfrak{H} \in$ SEQ and $X \subseteq$ SG($\mathfrak{H}$) and for all $\mathfrak{A}$, $\mathfrak{A}' \in X$ it holds that if $\mathfrak{A} \neq \mathfrak{A}'$, then $\mathfrak{A} \cap \mathfrak{A}' = \emptyset$, then: There is a $G \in$ SGS($\mathfrak{H}$) such that Ran(G) = X.

Proof: Suppose $\mathfrak{H} \in$ SEQ and $X \subseteq$ SG($\mathfrak{H}$) and suppose for all $\mathfrak{A}$, $\mathfrak{A}' \in X$: If $\mathfrak{A} \neq \mathfrak{A}'$, then $\mathfrak{A} \cap \mathfrak{A}' = \emptyset$. We have $\mathfrak{B} = \{(l, \mathfrak{H}_l)$ | There is an $\mathfrak{A} \in X$ and l = min(Dom($\mathfrak{A}$))$\} \subseteq \mathfrak{H}$. According to Theorem 2-10, there is thus a suitable sequence of natural numbers g for $\mathfrak{B}$. With Theorem 2-11, we then have that g is a bijection between Dom(g) and Dom($\mathfrak{B}$). According to the definition of $\mathfrak{B}$, we then have for all $\mathfrak{A} \in X$: min(Dom($\mathfrak{A}$)) = $g(i)$ for an $i \in$ Dom(g). Because g is strictly monotone increasing we also have: If i, $j \in$ Dom(g) and $i < j$, then $g(i) < g(j)$.

We then have for all $i \in$ Dom(g): There is exactly one $\mathfrak{A} \in X$ such that $g(i)$ = min(Dom($\mathfrak{A}$)). To see this, suppose that $i \in$ Dom(g). Then we have $g(i)$ = min(Dom($\mathfrak{A}$)) for an $\mathfrak{A} \in X$. Now, suppose $\mathfrak{A}' \in X$ and $g(i)$ = min(Dom($\mathfrak{A}'$)). According to the hypothesis, we have $X \subseteq$ SG($\mathfrak{H}$) and hence, with Theorem 2-9, we have $\mathfrak{A} \cap \mathfrak{A}' \neq \emptyset$. By hypothesis, we have that $\mathfrak{A} = \mathfrak{A}'$.

Now, let $G = \{(i, \mathfrak{A}) \mid i \in \text{Dom}(g) \text{ and } \mathfrak{A} \in X \text{ and } g(i) = \min(\text{Dom}(\mathfrak{A}))\}$. First, we have that G is a sequence with $\text{Ran}(G) \subseteq X \subseteq \text{SG}(\mathfrak{H})$. Also, we have for all $i, j \in \text{Dom}(G)$: If $i < j$, then $\min(\text{Dom}(G(i))) < \min(\text{Dom}(G(j)))$ and $\max(\text{Dom}(G(i))) < \min(\text{Dom}(G(j)))$. To see this, suppose $i, j \in \text{Dom}(G)$ and suppose $i < j$. Then we have $\min(\text{Dom}(G(i))) = g(i) < g(j) = \min(\text{Dom}(G(j))$. Then we have $G(i) \neq G(j)$ and hence, by hypothesis, $G(i) \cap G(j) = \emptyset$. Furthermore, we have $G(i), G(j) \in \text{SG}(\mathfrak{H})$. Because of $\min(\text{Dom}(G(i))) < \min(\text{Dom}(G(j)))$, it then follows with Theorem 2-8 that $\max(\text{Dom}(G(i))) < \min(\text{Dom}(G(j)))$.

Last, we have $\text{Ran}(G) = X$. We already have $\text{Ran}(G) \subseteq X$. Now, suppose $\mathfrak{A} \in X$. Then we have $\min(\text{Dom}(\mathfrak{A})) = g(i)$ for an $i \in \text{Dom}(g)$. Then we have $(i, \mathfrak{A}) \in G$ and hence $\mathfrak{A} \in \text{Ran}(G)$. ■

Theorem 2-18. *Sufficient conditions for the identity of arguments of a segment sequence*
If $\mathfrak{H} \in \text{SEQ}$ and $G \in \text{SGS}(\mathfrak{H})$, then for all $i, j \in \text{Dom}(G)$:
(i) If $\min(\text{Dom}(G(i))) = \min(\text{Dom}(G(j)))$, then $i = j$, and
(ii) If $\max(\text{Dom}(G(i))) = \max(\text{Dom}(G(j)))$, then $i = j$.

Proof: Suppose $\mathfrak{H} \in \text{SEQ}$ and $G \in \text{SGS}(\mathfrak{H})$ and suppose $i, j \in \text{Dom}(G)$. Now, suppose $\min(\text{Dom}(G(i))) = \min(\text{Dom}(G(j))$. With Definition 2-7, it follows that if $i < j$, then $\min(\text{Dom}((G(i))) < \min(\text{Dom}(G(j)))$, and if $j < i$, then $\min(\text{Dom}((G(j))) < \min(\text{Dom}(G(i)))$. Both cases contradict the assumption. Therefore we have $i = j$.

Now, suppose $\max(\text{Dom}(G(i))) = \max(\text{Dom}(G(j)))$. If $i < j$ or $j < i$, then we would have $\max(\text{Dom}(G(i))) < \min(\text{Dom}(G(j)))$ or $\max(\text{Dom}(G(j))) < \min(\text{Dom}(G(i)))$. Therefore we would have $\max(\text{Dom}(G(i))) < \min(\text{Dom}(G(j))) \leq \max(\text{Dom}(G(j)))$ or $\max(\text{Dom}(G(j))) < \min(\text{Dom}(G(i))) \leq \max(\text{Dom}(G(i)))$. Both cases contradict the assumption. Therefore we have $i = j$. ■

Theorem 2-19. *Different members of a segment sequence are disjunct*
If $\mathfrak{H} \in$ SEQ and $G \in$ SGS($\mathfrak{H}$), then for all $i, j \in$ Dom(G): If $G(i) \neq G(j)$, then $G(i) \cap G(j) = \emptyset$.

Proof: Suppose $\mathfrak{H} \in$ SEQ and $G \in$ SGS($\mathfrak{H}$). Then G is a sequence with Ran(G) $\subseteq$ SG($\mathfrak{H}$) and for all $i, j \in$ Dom(G): If $i < j$, then min(Dom($G(i)$)) < min(Dom($G(j)$)) and max(Dom($G(i)$)) < min(Dom($G(j)$)). Let $i, j \in$ Dom(G). Then it holds that $G(i), G(j) \in$ SG($\mathfrak{H}$). Now, suppose $G(i) \neq G(j)$. With Theorem 2-16-(i) it then holds that $i \neq j$. Then we have $i < j$ or $j < i$. Thus we have

min(Dom($G(i)$)) < min(Dom($G(j)$)) and max(Dom($G(i)$)) < min(Dom($G(j)$))
or
min(Dom($G(j)$)) < min(Dom($G(i)$)) and max(Dom($G(j)$)) < min(Dom($G(i)$)).

With Theorem 2-8, we thus have $G(i) \cap G(j) = \emptyset$. ■

Definition 2-9. *AS-comprising segment sequence for a segment in* $\mathfrak{H}$
G <u>is an AS-comprising segment sequence for</u> $\mathfrak{A}$ <u>in</u> $\mathfrak{H}$
iff

(i) $\mathfrak{H} \in$ SEQ,
(ii) $\mathfrak{A} \in$ SG($\mathfrak{H}$),
(iii) $G \in$ SGS($\mathfrak{H}$)\{$\emptyset$}, and
  a) min(Dom($\mathfrak{A}$)) $\leq$ min(Dom($G(0)$)),
  b) max(Dom(G(max(Dom(G))))) $\leq$ max(Dom($\mathfrak{A}$)), and
  c) for all $l \in$ Dom(AS($\mathfrak{H}$)) $\cap$ Dom($\mathfrak{A}$) it holds that there is an $i \in$ Dom(G) such that $l \in$ Dom($G(i)$).

Definition 2-10. *Assignment of the set of AS-comprising segment sequences in* $\mathfrak{H}$ *(ASCS)*
ASCS = {($\mathfrak{H}$, X) | $\mathfrak{H} \in$ SEQ and X = {G | There is an $\mathfrak{A} \in$ SG($\mathfrak{H}$) and G is an AS-comprising segment sequence for $\mathfrak{A}$ in $\mathfrak{H}$}}

Theorem 2-20. *Existence of AS-comprising segment sequences for all segments*
If $\mathfrak{H} \in$ SEQ and $\mathfrak{A} \in$ SG($\mathfrak{H}$), then there is an AS-comprising segment sequence G for $\mathfrak{A}$ in $\mathfrak{H}$.

Proof: Suppose $\mathfrak{H} \in$ SEQ and $\mathfrak{A} \in$ SG($\mathfrak{H}$). Then we have that {(0, $\mathfrak{A}$)} is an AS-comprising segment sequence for $\mathfrak{A}$ in $\mathfrak{H}$. ■

Theorem 2-21. *A sentence sequence* $\mathfrak{H}$ *is non-empty if and only if* ASCS($\mathfrak{H}$) *is non-empty*
If $\mathfrak{H} \in$ SEQ, then: $\mathfrak{H} \neq \emptyset$ iff ASCS($\mathfrak{H}$) $\neq \emptyset$.

Proof: Suppose $\mathfrak{H} \in$ SEQ. Suppose $\mathfrak{H} \neq \emptyset$. Then there is with Theorem 2-1 an $\mathfrak{A}$ such that $\mathfrak{A} \in$ SG($\mathfrak{H}$). With Theorem 2-20, we then have ASCS($\mathfrak{H}$) $\neq \emptyset$. Now, suppose ASCS($\mathfrak{H}$) $\neq \emptyset$. According to Definition 2-10 there is then an $\mathfrak{A} \in$ SG($\mathfrak{H}$). From this it follows with Theorem 2-1 that $\mathfrak{H} \neq \emptyset$. ■

Theorem 2-22. *Properties of AS-comprising segment sequences*
If $\mathfrak{H} \in$ SEQ and $G \in$ ASCS($\mathfrak{H}$), then:
(i) G is an injection of Dom(G) into Ran(G),
(ii) G is a bijection between Dom(G) and Ran(G),
(iii) Dom(G) = |Ran(G)|, and
(iv) G is a finite sequence.

Proof: Suppose $\mathfrak{H} \in$ SEQ and $G \in$ ASCS($\mathfrak{H}$). With Definition 2-9, we have that $G \in$ SGS($\mathfrak{H}$)\{$\emptyset$}. From this, the statement follows with Theorem 2-16. ■

Theorem 2-23. *All members of an AS-comprising segment sequence lie within the respective segment*
If G is an AS-comprising segment sequence for $\mathfrak{A}$ in $\mathfrak{H}$, then for all $i \in$ Dom(G): min(Dom($\mathfrak{A}$)) $\leq$ min(Dom($G(i)$)) and max(Dom($G(i)$)) $\leq$ max(Dom($\mathfrak{A}$)).

Proof: Suppose G is an AS-comprising segment sequence for $\mathfrak{A}$ in $\mathfrak{H}$ and suppose $i \in$ Dom(G). Then we have $0 \leq i \leq$ max(Dom(G)). According to Definition 2-9, we have that $G \in$ SGS($\mathfrak{H}$)\{$\emptyset$}. With Definition 2-7 we then have that for all k, $j \in$ Dom(G): If $k < j$, then min(Dom($G(k)$)) < min(Dom($G(j)$)) and max(Dom($G(k)$)) < min(Dom($G(j)$)). Therefore we have that min(Dom($G(0)$)) $\leq$ min(Dom($G(i)$)) and max(Dom($G(i)$)) $\leq$ max(Dom(G(max(Dom(G))))). It also follows from the assumption and Definition 2-9 that min(Dom($\mathfrak{A}$)) $\leq$ min(Dom($G(0)$) and max(Dom(G(max(Dom(G)))) $\leq$ max(Dom($\mathfrak{A}$)). Thus it then follows that: min(Dom($\mathfrak{A}$)) $\leq$ min(Dom($G(i)$)) and max(Dom($G(i)$)) $\leq$ max(Dom($\mathfrak{A}$)). ■

Theorem 2-24. *All members of an AS-comprising segment sequence are subsets of the respective segment*
If G is an AS-comprising segment sequence for $\mathfrak{A}$ in $\mathfrak{H}$, then for all $i \in \mathrm{Dom}(G)$: $G(i) \subseteq \mathfrak{A}$.

Proof: Suppose G is an AS-comprising segment sequence for $\mathfrak{A}$ in $\mathfrak{H}$ and suppose $i \in \mathrm{Dom}(G)$. With Definition 2-9 and Definition 2-7 we then have $\mathrm{Ran}(G) \subseteq \mathrm{SG}(\mathfrak{H})$ and thus that $G(i)$ is a segment in $\mathfrak{H}$. With Theorem 2-23 we also have that $\min(\mathrm{Dom}(\mathfrak{A})) \leq \min(\mathrm{Dom}(G(i)))$ and $\max(\mathrm{Dom}(G(i))) \leq \max(\mathrm{Dom}(\mathfrak{A}))$. It then follows with Theorem 2-5 that $G(i) \subseteq \mathfrak{A}$. ■

Theorem 2-25. *Non-empty restrictions of AS-comprising segment sequences are AS-comprising segment sequences*
If G is an AS-comprising segment sequence for $\mathfrak{A}$ in $\mathfrak{H}$, then for all $j \in \mathrm{Dom}(G)$: $G{\restriction}(j+1)$ is an AS-comprising segment sequence for $\mathfrak{A}{\restriction}(\max(\mathrm{Dom}(G(j)))+1)$.

Proof: Suppose G is an AS-comprising segment sequence for $\mathfrak{A}$ in $\mathfrak{H}$ and suppose $j \in \mathrm{Dom}(G)$. According to Definition 2-9 we then have that $\mathfrak{H} \in \mathrm{SEQ}$ and $\mathfrak{A} \in \mathrm{SG}(\mathfrak{H})$ and $G \in \mathrm{SGS}(\mathfrak{H})\setminus\{\emptyset\}$ and $\min(\mathrm{Dom}(\mathfrak{A})) \leq \min(\mathrm{Dom}(G(0))$ and $\max(\mathrm{Dom}(G(\max(\mathrm{Dom}(G)))))$ $\leq \max(\mathrm{Dom}(\mathfrak{A}))$ and that it holds for all $l \in \mathrm{Dom}(\mathrm{AS}(\mathfrak{H})) \cap \mathrm{Dom}(\mathfrak{A})$ that there is an $i \in \mathrm{Dom}(G)$ such that $l \in \mathrm{Dom}(G(i))$. With Definition 2-7, we can easiliy show that $G{\restriction}(j+1) \in \mathrm{SGS}(\mathfrak{H})\setminus\{\emptyset\}$. With Theorem 2-23, we have that $\min(\mathrm{Dom}(\mathfrak{A})) \leq \min(\mathrm{Dom}(G(j))) \leq \max(\mathrm{Dom}(G(j))) \leq \max(\mathrm{Dom}(\mathfrak{A}))$ and thus that $\max(\mathrm{Dom}(G(j))) \in \mathrm{Dom}(\mathfrak{A})$. With Theorem 2-6, we thus have that $\mathfrak{A}{\restriction}(\max(\mathrm{Dom}(G(j)))+1) \in \mathrm{SG}(\mathfrak{H})$.

Now, the three sub-clauses of clause (iii) of Definition 2-9 have to be shown. *Ad a)*: First, we have $0 < j+1$. Thus we have $0 \in \mathrm{Dom}(G{\restriction}(j+1))$ and hence $(G{\restriction}(j+1))(0) = G(0)$ and thus $\min(\mathrm{Dom}(\mathfrak{A}{\restriction}(\max(\mathrm{Dom}(G(j)))+1))) = \min(\mathrm{Dom}(\mathfrak{A})) \leq \min(\mathrm{Dom}(G(0))) \leq \min(\mathrm{Dom}((G{\restriction}(j+1))(0)))$. *Ad b)*: $\max(\mathrm{Dom}((G{\restriction}(j+1))(\max(\mathrm{Dom}(G{\restriction}(j+1)))))) = \max(\mathrm{Dom}(G(j))) = \max(\mathrm{Dom}(\mathfrak{A}{\restriction}(\max(\mathrm{Dom}(G(j)))+1)))$. *Ad c)*: Now, suppose $l \in \mathrm{Dom}(\mathrm{AS}(\mathfrak{H})) \cap \mathrm{Dom}(\mathfrak{A}{\restriction}(\max(\mathrm{Dom}(G(j)))+1))$. Then there is an $i \in \mathrm{Dom}(G)$ such that $l \in \mathrm{Dom}(G(i))$. Suppose for contradiction that $j+1 \leq i$. With $G \in \mathrm{SGS}(\mathfrak{H})$ and Definition 2-7, we would then have that $\max(\mathrm{Dom}(G(j))) < \min(\mathrm{Dom}(G(i))) \leq l \leq \max(\mathrm{Dom}(G(i)))$

and, at the same time, we would have that $l \leq \max(\mathrm{Dom}(G(j)))$. Contradiction! Therefore we have $i < j+1$ and thus $G(i) = (G{\upharpoonright}(j+1))(i)$. Therefore we have that for all $l \in \mathrm{Dom}(\mathrm{AS}(\mathfrak{H})) \cap \mathrm{Dom}(\mathfrak{A}{\upharpoonright}(\max(\mathrm{Dom}(G(j)))+1))$ it holds that there is an $i \in \mathrm{Dom}(G{\upharpoonright}(j+1))$ such that $l \in \mathrm{Dom}((G{\upharpoonright}(j+1))(i))$. According to Definition 2-9, we thus have that $G{\upharpoonright}(j+1)$ is an AS-comprising segment sequence for $\mathfrak{A}{\upharpoonright}(\max(\mathrm{Dom}(G(j)))+1)$. ■

Theorem 2-26. *Sufficient conditions for the identity of arguments of an AS-comprising segment sequence*
If $\mathfrak{H} \in$ SEQ and $G \in$ ASCS($\mathfrak{H}$), then for all $i, j \in \mathrm{Dom}(G)$:
(i) If $\min(\mathrm{Dom}(G(i))) = \min(\mathrm{Dom}(G(j)))$, then $i = j$, and
(ii) If $\max(\mathrm{Dom}(G(i))) = \max(\mathrm{Dom}(G(j)))$, then $i = j$.

Proof: Suppose $\mathfrak{H} \in$ SEQ and $G \in$ ASCS($\mathfrak{H}$). It then follows with Definition 2-9 and Definition 2-10 that $G \in \mathrm{SGS}(\mathfrak{H})\setminus\{\emptyset\}$ and thus the theorem follows with Theorem 2-18. ■

Theorem 2-27. *Different members of an AS-comprising segment sequence are disjunct*
If $\mathfrak{H} \in$ SEQ and $G \in$ ASCS($\mathfrak{H}$), then for all $i, j \in \mathrm{Dom}(G)$: If $G(i) \neq G(j)$, then $G(i) \cap G(j) = \emptyset$.

Proof: Suppose $\mathfrak{H} \in$ SEQ and $G \in$ ASCS($\mathfrak{H}$). It then follows with Definition 2-9 and Definition 2-10 that $G \in \mathrm{SGS}(\mathfrak{H})\setminus\{\emptyset\}$ and thus the theorem follows with Theorem 2-19. ■

2.2 Closed Segments

In the following section, we introduce CdI-, NI- and RA-like segments. These kinds of segments show forms that are connected to inferences by conditional introduction (CdI-like), negation introduction (NI-like) and particular-quantifier elimination (RA-like), respectively. Among these segments, we will then distinguish so called minimal CdI-, NI, and PE-closed segments, which will form the minimal closed segments. Then, we will define the generation relation GEN, with which we can generate further non-redundant CdI-, NI- and RA-like segments from minimal closed segments. Then, we will define the set of GEN-inductive relations. The intersection of the set of GEN-inductive relations will then be singled out as that relation which assigns a sentence sequence all and only those segments that are closed in this sentence sequence. Thus, closed segments in a sentence sequence will be exactly those CdI-, NI- and RA-like segments in this sequence that are either minimal closed segments or that can be generated by the generation relation from minimal closed segments.

Then, we will prove some general theorems on closed segments. Subsequently, we will define CdI-, NI- and PE-closed segments. This will be done in such a way that CdI-, NI- and PE-closed segments will be closed segments that are CdI-, NI- and RA-like, respectively, and that all closed segements will be CdI- or NI- or PE-closed. At the end of the chapter, we will prove theorems (Theorem 2-66, Theorem 2-67, Theorem 2-68, Theorem 2-69), with which we can later show that CdI-, NI-, PE-closed segments (and thus any closed segments) can be generated by (and only by) CdI, NI and PE, respectively. In the next chapter (2.3), the availability conception will be established with direct recourse to this chapter: A proposition Γ will be available in a sequence $\mathfrak{H}$ at a position i if and only if Γ is the proposition of $\mathfrak{H}_i$ and $(i, \mathfrak{H}_i)$ lies in all closed segments in $\mathfrak{H}$ at most at the end. We will then have that assumptions can be discharged by and only by CdI, NI and PE.

The first three definitions introduce CdI-, NI- and RA-like segments. Then, following some theorems, we will define minimal (CdI- resp. NI- resp. PE-)closed segments.

Definition 2-11. *CdI-like segment*

$\mathfrak{A}$ is a CdI-like segment in $\mathfrak{H}$

iff

$\mathfrak{H} \in$ SEQ, $\mathfrak{A} \in$ SG($\mathfrak{H}$) and there are $\Delta, \Gamma \in$ CFORM such that

(i) $\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}))} = \ulcorner$Suppose $\Delta\urcorner$,

(ii) $\mathrm{P}(\mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{A}))-1}) = \Gamma$, and

(iii) $\mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{A}))} = \ulcorner$Therefore $\Delta \rightarrow \Gamma\urcorner$.

Definition 2-12. *NI-like segment*

$\mathfrak{A}$ is an NI-like segment in $\mathfrak{H}$

iff

$\mathfrak{H} \in$ SEQ, $\mathfrak{A} \in$ SG($\mathfrak{H}$) and there are $\Delta, \Gamma \in$ CFORM and $i \in$ Dom($\mathfrak{H}$) such that

(i) $\min(\mathrm{Dom}(\mathfrak{A})) \leq i < \max(\mathrm{Dom}(\mathfrak{A}))$,

(ii) $\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}))} = \ulcorner$Suppose $\Delta\urcorner$,

(iii) $\mathrm{P}(\mathfrak{H}_i) = \Gamma$ and $\mathrm{P}(\mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{A}))-1}) = \ulcorner\neg\Gamma\urcorner$
oder
$\mathrm{P}(\mathfrak{H}_i) = \ulcorner\neg\Gamma\urcorner$ and $\mathrm{P}(\mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{A}))-1}) = \Gamma$, and

(iv) $\mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{A}))} = \ulcorner$Therefore $\neg\Delta\urcorner$.

In clause (iii) of Definition 2-12, two contradictory propositions, such as one needs for negation introduction, are localised in the respective sentence sequence. Either the negative ($\ulcorner\neg\Gamma\urcorner$) or the positive ($\Gamma$) part of the contradiction is the proposition of the penultimate member of the respective segment $\mathfrak{A}$. The position of the other part of the contradiction is left open. It is only required that this other part occurs at some position (i) between the first and the penultimate member of the segment. This is unproblematic in the case of minimal NI-closed segments (Definition 2-15). However, if we want to generate not-minimal closed segments from closed segments, we have to take care that the part of the contradiction whose exact position is not specified does not lie in a proper subsegment of $\mathfrak{A}$ that is already closed. This we have to keep in mind when we construct the generation relation (cf. especially Definition 2-18).

Definition 2-13. *RA-like segment*

$\mathfrak{A}$ <u>is an RA-like segment in</u> $\mathfrak{H}$
iff
$\mathfrak{H} \in$ SEQ, $\mathfrak{A} \in$ SG($\mathfrak{H}$) and there is $\xi \in$ VAR, $\Delta \in$ FORM, where FV(Δ) $\subseteq \{\xi\}$, $\beta \in$ PAR, $\Gamma \in$ CFORM and $\mathfrak{B} \in$ SG($\mathfrak{H}$) such that

(i) $P(\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))}) = \ulcorner\bigvee\xi\Delta\urcorner$,

(ii) $\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))+1} = \ulcorner$Suppose $[\beta, \xi, \Delta]\urcorner$,

(iii) $P(\mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{B}))-1}) = \Gamma$,

(iv) $\mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{B}))} = \ulcorner$Therefore $\Gamma\urcorner$,

(v) $\beta \notin$ STSF($\{\Delta, \Gamma\}$),

(vi) There is no j such that $j \leq \min(\mathrm{Dom}(\mathfrak{B}))$ and $\beta \in$ ST($\mathfrak{H}_j$), and

(vii) $\mathfrak{A} = \mathfrak{B}\setminus\{(\min(\mathrm{Dom}(\mathfrak{B})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))})\}$.

Note: 'RA' stands for **r**epresentative instance **a**ssumption, that is, for the representative instance assumption one has to make before one can carry out a particular-quantifier elimination.

Theorem 2-28. *No segment is at the same time a CdI- and an NI- or a CdI- and an RA-like segment*

(i) There are no $\mathfrak{A}$, $\mathfrak{H}$ such that $\mathfrak{A}$ is a CdI- and an NI-like segment in $\mathfrak{H}$,

(ii) There are no $\mathfrak{A}$, $\mathfrak{H}$ such that $\mathfrak{A}$ is a CdI- and an RA-like segment in $\mathfrak{H}$.

Proof: Follows from the definitions and the theorems on unique readability (Theorem 1-10 to Theorem 1-12). ■

Note that it is possible that an $\mathfrak{A}$ is an NI- and RA-like segment in $\mathfrak{H}$. This is for example the case if the assumption for an indirect proof does not contain parameters and provides one part of the contradiction, while the (empty) particular-quantification of the indirect assumption has been gained immediately before this assumption.

Theorem 2-29. *The last member of a CdI- or NI- or RA-like segment is not an assumption-sentence*

If $\mathfrak{A}$ is a CdI- or NI- or RA-like segment in $\mathfrak{H}$, then $\max(\mathrm{Dom}(\mathfrak{A})) \notin \mathrm{Dom}(\mathrm{AS}(\mathfrak{H}))$.

Proof: Follows from Definition 2-11-(iii), Definition 2-12-(iv), Definition 2-13-(iv) and the theorem on the unique readability of sentences (Theorem 1-12). ■

Theorem 2-30. *All assumption-sentences in a CdI- or NI- or RA-like segment lie in a proper subsegment that does not include the last member of the respective segment*

If $\mathfrak{A}$ is a CdI- or NI- or RA-like segment in $\mathfrak{H}$, and $i \in \mathrm{Dom}(\mathfrak{A}) \cap \mathrm{Dom}(\mathrm{AS}(\mathfrak{H}))$, then $\min(\mathrm{Dom}(\mathfrak{A})) \leq i < \max(\mathrm{Dom}(\mathfrak{A}))$.

Proof: Follows from Theorem 2-29. ■

Theorem 2-31. *Cardinality of CdI-, NI-, and RA-like segments*

(i) If $\mathfrak{A}$ is a CdI- or RA-like segment in $\mathfrak{H}$, then $2 \leq |\mathfrak{A}|$, and

(ii) If $\mathfrak{A}$ is an NI-like segment in $\mathfrak{H}$, then $3 \leq |\mathfrak{A}|$.

Proof: The theorem follows with the theorems on unique readability (Theorem 1-10 to Theorem 1-12) directly from Definition 2-11, Definition 2-12 and Definition 2-13. ■

Definition 2-14. *Minimal CdI-closed segment*

$\mathfrak{A}$ is a minimal CdI-closed segment in $\mathfrak{H}$

iff

$\mathfrak{A}$ is a CdI-like segment in $\mathfrak{H}$ and

(i) $\mathrm{AS}(\mathfrak{H}) \cap \mathfrak{A} = \{(\min(\mathrm{Dom}(\mathfrak{A})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}))})\}$, and

(ii) For all $i \in \mathrm{Dom}(\mathfrak{A})$ it holds that $\mathfrak{A}\upharpoonright i$ is not a CdI- or NI- or RA-like segment in $\mathfrak{H}$.

Definition 2-15. *Minimal NI-closed segment*

$\mathfrak{A}$ is a minimal NI-closed segment in $\mathfrak{H}$

iff

$\mathfrak{A}$ is an NI-like segment in $\mathfrak{H}$ and

(i) $\mathrm{AS}(\mathfrak{H}) \cap \mathfrak{A} = \{(\min(\mathrm{Dom}(\mathfrak{A})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}))})\}$, and

(ii) For all $i \in \mathrm{Dom}(\mathfrak{A})$ it holds that $\mathfrak{A}\upharpoonright i$ is not a CdI- or NI- or RA-like segment in $\mathfrak{H}$.

Definition 2-16. *Minimal PE-closed segment*

$\mathfrak{A}$ is a minimal PE-closed segment in $\mathfrak{H}$

iff

$\mathfrak{A}$ is a RA-like segment in $\mathfrak{H}$ and

(i) $\mathrm{AS}(\mathfrak{H}) \cap \mathfrak{A} = \{(\min(\mathrm{Dom}(\mathfrak{A})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}))})\}$, and

(ii) For all $i \in \mathrm{Dom}(\mathfrak{A})$ holds that $\mathfrak{A}\upharpoonright i$ is not a CdI- or NI- or RA-like segment in $\mathfrak{H}$.

Definition 2-17. *Minimal closed segment*

$\mathfrak{A}$ <u>is a minimal closed segment in</u> $\mathfrak{H}$

iff

$\mathfrak{A}$ is a minimal CdI- or a minimal NI- or a minimal PE-closed segment in $\mathfrak{H}$.

Theorem 2-32. *CdI-, NI- and RA-like segments with just one assumption-sentence have a minimal closed segment as an initial segment*

If $\mathfrak{A}$ is a CdI- or NI- or RA-like segment in $\mathfrak{H}$ and $|\mathrm{AS}(\mathfrak{H}) \cap \mathfrak{A}| = 1$, then $\mathfrak{A}$ is a minimal closed segment in $\mathfrak{H}$ or there is an $i \in \mathrm{Dom}(\mathfrak{A})$ such that $\mathfrak{A}\restriction i$ is a minimal closed segment in $\mathfrak{H}$.

Proof: Suppose $\mathfrak{A}$ is a CdI- or NI- or RA-like segment in $\mathfrak{H}$ and $|\mathrm{AS}(\mathfrak{H}) \cap \mathfrak{A}| = 1$. With Definition 2-11, Definition 2-12 and Definition 2-13, we then have $\mathrm{AS}(\mathfrak{H}) \cap \mathfrak{A} = \{(\min(\mathrm{Dom}(\mathfrak{A})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}))})\}$. Suppose $\mathfrak{A}$ is not a minimal closed segment in $\mathfrak{H}$. By hypothesis, we then have, with Definition 2-17 and Definition 2-14, Definition 2-15 and Definition 2-16, that there is a $j \in \mathrm{Dom}(\mathfrak{A})$ such that $\mathfrak{A}\restriction j$ is a CdI- or NI- or RA-like segment in $\mathfrak{H}$. Now, let $i = \min(\{j \mid j \in \mathrm{Dom}(\mathfrak{A})$ and $\mathfrak{A}\restriction j$ is a CdI-, NI- or RA-like segment in $\mathfrak{H}\})$. Then we have $\mathrm{AS}(\mathfrak{H}) \cap \mathfrak{A}\restriction i \subseteq \mathrm{AS}(\mathfrak{H}) \cap \mathfrak{A}$ and, with Theorem 2-7, we have $\min(\mathrm{Dom}(\mathfrak{A}\restriction i)) = \min(\mathrm{Dom}(\mathfrak{A}))$ and thus $\mathrm{AS}(\mathfrak{H}) \cap \mathfrak{A}\restriction i = \{(\min(\mathrm{Dom}(\mathfrak{A}\restriction i)), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}\restriction i))})\}$. Because of the minimality of i, we also have that for all $l \in \mathrm{Dom}(\mathfrak{A}\restriction i)$ it holds that $(\mathfrak{A}\restriction i)\restriction l = \mathfrak{A}\restriction l$ is not a CdI-, NI- or RA-like segment in $\mathfrak{H}$. Thus we have that $\mathfrak{A}\restriction i$ is a minimal CdI- or NI- or PE-closed segment and thus a minimal closed segment in $\mathfrak{H}$. ■

Theorem 2-33. *Ratio of inference- and assumption-sentences in minimal closed segments*

If $\mathfrak{A}$ is a minimal closed segment in $\mathfrak{H}$, then $|\mathrm{AS}(\mathfrak{H}) \cap \mathfrak{A}| \leq |\mathrm{IS}(\mathfrak{H}) \cap \mathfrak{A}|$.

Proof: Suppose $\mathfrak{A}$ is a minimal closed segment and thus a minimal CdI- or NI- or PE-closed segment in $\mathfrak{H}$. Then it holds with the definitions and Theorem 2-29 that $|\mathrm{AS}(\mathfrak{H}) \cap \mathfrak{A}| = 1 \leq |\mathrm{IS}(\mathfrak{H}) \cap \mathfrak{A}|$. ■

Now, we will define a generation relation for segments with which we can generate further non-redundant CdI-, NI-, and RA-like segments from minimal closed segments, where all assumption-sentences of the generated segments are first members of a non-redundant CdI-, NI- or RA-like subsegment. To do this, we first define the following proto-generation relation:

Definition 2-18. *Proto-generation relation for non-redundant CdI-, NI- and RA-like segments in sequences (PGEN)*

PGEN = $\{(\langle\mathfrak{H}, G\rangle, X) \mid \mathfrak{H} \in$ SEQ and $G \in$ ASCS($\mathfrak{H}$) and $X = \{\mathfrak{A} \mid \mathfrak{A} \in$ SG($\mathfrak{H}$) and there is a $\mathfrak{B} \in$ SG($\mathfrak{H}$) such that

(i) G is an AS-comprising segment sequence for $\mathfrak{B}$ in $\mathfrak{H}$,

(ii) AS($\mathfrak{H}$) $\cap\ \mathfrak{B} \neq \emptyset$,

(iii) min(Dom($\mathfrak{A}$))+1 = min(Dom($\mathfrak{B}$)) and max(Dom($\mathfrak{A}$)) = max(Dom($\mathfrak{B}$))+1,

(iv) $\mathfrak{A}$ is a CdI- or NI- or RA-like segment in $\mathfrak{H}$ and if $\mathfrak{A}$ is an NI-like segment in $\mathfrak{H}$, then there are $\Delta, \Gamma \in$ CFORM and $i \in$ Dom($\mathfrak{H}$) such that

a) min(Dom($\mathfrak{A}$)) $\leq i <$ max(Dom($\mathfrak{A}$)),

b) $\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}))}$ = ⌜Suppose Δ⌝,

c) P($\mathfrak{H}_i$) = Γ and P($\mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{A}))-1}$) = ⌜$\neg\Gamma$⌝
or
P($\mathfrak{H}_i$) = ⌜$\neg\Gamma$⌝ and P($\mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{A}))-1}$) = Γ,

d) For all $r \in$ Dom(G): $i <$ min(Dom($G(r)$)) or max(Dom($G(r)$)) $\leq i$,

e) $\mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{A}))}$ = ⌜Therefore $\neg\Delta$⌝, and

(v) For all $i \in$ Dom($\mathfrak{A}$): $\mathfrak{A}\upharpoonright i$ is not a minimal closed segment in $\mathfrak{H}\}\}$.

In clause (iv) of Definition 2-18, a special requirement is made for NI-like segments. The reason is that the values of the AS-comprising segment sequence G are to be the ›material‹ when we construct further closed segments from closed segments. In the NI-case, we have to make sure that only such segments $\mathfrak{A}$ are generated as NI-closed in which both parts of the required contradiction actually lie in $\mathfrak{A}\upharpoonright$max(Dom($\mathfrak{A}$)) and are both not included in any closed subsegment of $\mathfrak{A}\upharpoonright$max(Dom($\mathfrak{A}$)). For the first part of the contradiction, this is ensured by (iv-d) (cf. the proof of Theorem 2-68).

Theorem 2-34. *Some properties of PGEN*

If $\mathfrak{H} \in$ SEQ and $G \in$ ASCS($\mathfrak{H}$) and $\mathfrak{A} \in$ PGEN($\langle\mathfrak{H}, G\rangle$), then:

(i) There is $\mathfrak{B} \in$ SG($\mathfrak{H}$) such that G is an AS-comprising segment sequence for $\mathfrak{B}$ in $\mathfrak{H}$ and AS($\mathfrak{H}$) $\cap\ \mathfrak{B} \neq \emptyset$, min(Dom($\mathfrak{A}$))+1 = min(Dom($\mathfrak{B}$)) and max(Dom($\mathfrak{A}$)) = max(Dom($\mathfrak{B}$))+1,

(ii) $\mathfrak{A} \in$ SG($\mathfrak{H}$) is a CdI- or NI- or RA-like segment in $\mathfrak{H}$,

(iii) For all $i \in$ Dom($\mathfrak{A}$): $\mathfrak{A}\upharpoonright i$ is not a minimal closed segment in $\mathfrak{H}$,

(iv) There is an $i \in$ Dom($\mathfrak{A}$) such that min(Dom($\mathfrak{A}$)) $< i$ and $i \in$ Dom(AS($\mathfrak{H}$)),

(v) $\mathfrak{A}$ is not a minimal closed segment in $\mathfrak{H}$,

(vi) $G \neq \emptyset$, and

(vii) For all $\mathfrak{C} \in \mathrm{PGEN}(\langle\mathfrak{H}, G\rangle)$ it holds that $\min(\mathrm{Dom}(\mathfrak{C})) = \min(\mathrm{Dom}(\mathfrak{A}))$.

Proof: Suppose $\mathfrak{H} \in \mathrm{SEQ}$ and $G \in \mathrm{ASCS}(\mathfrak{H})$ and $\mathfrak{A} \in \mathrm{PGEN}(\langle\mathfrak{H}, G\rangle)$. Then clauses (i)-(iii) follow directly from Definition 2-18. Now, suppose $\mathfrak{B}$ satisfies clause (i). Then we have $\mathrm{AS}(\mathfrak{H}) \cap \mathfrak{B} \neq \emptyset$ and hence there is an $i \in \mathrm{Dom}(\mathrm{AS}(\mathfrak{H})) \cap \mathrm{Dom}(\mathfrak{B}) \subseteq \mathrm{Dom}(\mathrm{AS}(\mathfrak{H})) \cap \mathrm{Dom}(\mathfrak{A})$ where, because of $\min(\mathrm{Dom}(\mathfrak{A}))+1 = \min(\mathrm{Dom}(\mathfrak{B}))$, we have that $\min(\mathrm{Dom}(\mathfrak{A})) < i$. It then follows that clause (iv) holds. From this follows with Definition 2-14, Definition 2-15, Definition 2-16 and Definition 2-17 that clause (v) also holds. With $\mathrm{AS}(\mathfrak{H}) \cap \mathfrak{B} \neq \emptyset$ and Definition 2-9, we also have that there is an $i \in \mathrm{Dom}(G)$, and hence that $G \neq \emptyset$. Therefore we have (vi).

According to Definition 2-9, we have that $\min(\mathrm{Dom}(\mathfrak{B})) \leq \min(\mathrm{Dom}(G(0))) \leq \max(\mathrm{Dom}(\mathfrak{B}))$ and thus that $\min(\mathrm{Dom}(\mathfrak{A})) < \min(\mathrm{Dom}(G(0)))$. Now, suppose $\mathfrak{C} \in \mathrm{PGEN}(\langle\mathfrak{H}, G\rangle)$. Then there is a $\mathfrak{B}' \in \mathrm{SG}(\mathfrak{H})$ such that G is an AS-comprising segment sequence for $\mathfrak{B}'$ in $\mathfrak{H}$ and $\min(\mathrm{Dom}(\mathfrak{C}))+1 = \min(\mathrm{Dom}(\mathfrak{B}'))$ and $\max(\mathrm{Dom}(\mathfrak{C})) = \max(\mathrm{Dom}(\mathfrak{B}'))+1$ and $\mathfrak{C}$ is a CdI- or NI- or RA-like segment in $\mathfrak{H}$. Then we have $\min(\mathrm{Dom}(\mathfrak{A})), \min(\mathrm{Dom}(\mathfrak{C})) \in \mathrm{Dom}(\mathrm{AS}(\mathfrak{H}))$. According to Definition 2-9, we have that $\min(\mathrm{Dom}(\mathfrak{B}')) \leq \min(\mathrm{Dom}(G(0))) \leq \max(\mathrm{Dom}(\mathfrak{B}'))$ and thus $\min(\mathrm{Dom}(\mathfrak{C})) < \min(\mathrm{Dom}(G(0)))$. It thus follows that $\min(\mathrm{Dom}(\mathfrak{A})), \min(\mathrm{Dom}(\mathfrak{C})) < \min(\mathrm{Dom}(G(0))) \leq \max(\mathrm{Dom}(\mathfrak{B})), \max(\mathrm{Dom}(\mathfrak{B}'))$.

Now, suppose for contradiction that $\min(\mathrm{Dom}(\mathfrak{C})) < \min(\mathrm{Dom}(\mathfrak{A}))$. Then we would have that $\min(\mathrm{Dom}(\mathfrak{B}')) \leq \min(\mathrm{Dom}(\mathfrak{A})) \leq \max(\mathrm{Dom}(\mathfrak{B}'))$. Then we would also have that $\min(\mathrm{Dom}(\mathfrak{A})) \in \mathrm{Dom}(\mathrm{AS}(\mathfrak{H})) \cap \mathrm{Dom}(\mathfrak{B}')$. Now, G is an AS-comprising segment sequence for $\mathfrak{B}'$ in $\mathfrak{H}$. With Definition 2-9, we would thus have that $\min(\mathrm{Dom}(\mathfrak{A})) \in \mathrm{Dom}(G(l))$ for an $l \in \mathrm{Dom}(G)$. Since G is an AS-comprising segment sequence for $\mathfrak{B}$ in $\mathfrak{H}$, we would have, with Theorem 2-24, that $\min(\mathrm{Dom}(\mathfrak{A}))+1 = \min(\mathrm{Dom}(\mathfrak{B})) \leq \min(\mathrm{Dom}(\mathfrak{A}))$. Contradiction! Now, suppose for contradiction that $\min(\mathrm{Dom}(\mathfrak{A})) < \min(\mathrm{Dom}(\mathfrak{C}))$. Then we would have that $\min(\mathrm{Dom}(\mathfrak{B})) \leq \min(\mathrm{Dom}(\mathfrak{C})) \leq \max(\mathrm{Dom}(\mathfrak{B}))$. Thus we would now have $\min(\mathrm{Dom}(\mathfrak{C})) \in \mathrm{Dom}(\mathrm{AS}(\mathfrak{H})) \cap \mathrm{Dom}(\mathfrak{B})$ and thus $\min(\mathrm{Dom}(\mathfrak{C})) \in \mathrm{Dom}(G(l'))$ for an $l' \in \mathrm{Dom}(G)$ and thus $\min(\mathrm{Dom}(\mathfrak{C}))+1 = \min(\mathrm{Dom}(\mathfrak{B}')) \leq \min(\mathrm{Dom}(\mathfrak{C}))$. Contradiction! Therefore we have $\min(\mathrm{Dom}(\mathfrak{C})) = \min(\mathrm{Dom}(\mathfrak{A}))$ and hence that clause (vii) holds. ■

For given $\mathfrak{H}$, G, the desired generation relation singles out the non-redundant segments from PGEN($\langle\mathfrak{H}, G\rangle$):

Definition 2-19. *Generation relation for non-redundant CdI-, NI- and RA-like segments in sequences (GEN)*

GEN = $\{(\langle\mathfrak{H}, G\rangle, X) \mid \mathfrak{H} \in$ SEQ, $G \in$ ASCS($\mathfrak{H}$) and $X = \{\mathfrak{A} \mid \mathfrak{A} \in$ PGEN($\langle\mathfrak{H}, G\rangle$) and there is no $i \in$ Dom($\mathfrak{A}$) and $j \in$ Dom(G) such that $\mathfrak{A}\upharpoonright i \in$ PGEN($\langle\mathfrak{H}, G\upharpoonright(j+1)\rangle$)$\}\}$.

GEN is a 2-ary function that assigns each sentence sequence $\mathfrak{H}$ and AS-comprising segment sequence G for a segment $\mathfrak{B}$ in $\mathfrak{H}$ a subset X of the set of CdI-, NI- or RA-like segments in $\mathfrak{H}$ that have the members of G as proper subsegments. This subset is then either empty or it is the singleton of the shortest segment that can be generated with PGEN for $\mathfrak{H}$ and restrictions of G on j+1 with $j \in$ Dom(G). This ensures later that not only minimal, but also GEN-generated and thus all closed segments are uniquely determined by their beginning (cf. Theorem 2-50). The following theorem sums up some properties of GEN for GEN($\langle\mathfrak{H}, G\rangle$) $\neq \emptyset$.

Theorem 2-35. *Some consequences of Definition 2-19*

If $\mathfrak{H} \in$ SEQ and $G \in$ ASCS($\mathfrak{H}$) and $\mathfrak{A} \in$ GEN($\langle\mathfrak{H}, G\rangle$), then:

(i) There is $\mathfrak{B} \in$ SG($\mathfrak{H}$) such that G is an AS-comprising segment sequence for $\mathfrak{B}$ in $\mathfrak{H}$ and AS($\mathfrak{H}$) $\cap$ $\mathfrak{B} \neq \emptyset$, min(Dom($\mathfrak{A}$))+1 = min(Dom($\mathfrak{B}$)) and max(Dom($\mathfrak{A}$)) = max(Dom($\mathfrak{B}$))+1,

(ii) $\mathfrak{A} \in$ SG($\mathfrak{H}$) is a CdI- or NI- or RA-like segment in $\mathfrak{H}$,

(iii) For all $i \in$ Dom($\mathfrak{A}$): $\mathfrak{A}\upharpoonright i$ is not a minimal closed segment in $\mathfrak{H}$,

(iv) There is an $i \in$ Dom($\mathfrak{A}$) such that min(Dom($\mathfrak{A}$)) $< i$ and $i \in$ Dom(AS($\mathfrak{H}$)),

(v) $\mathfrak{A}$ is not a minimal closed segment in $\mathfrak{H}$,

(vi) There is no $i \in$ Dom($\mathfrak{A}$) and $j \in$ Dom(G) such that $\mathfrak{A}\upharpoonright i \in$ PGEN($\langle\mathfrak{H}, G\upharpoonright(j+1)\rangle$), and

(vii) GEN($\langle\mathfrak{H}, G\rangle$) = $\{\mathfrak{A}\}$.

Proof: Suppose $\mathfrak{H} \in$ SEQ and $G \in$ ASCS($\mathfrak{H}$) and $\mathfrak{A} \in$ GEN($\langle\mathfrak{H}, G\rangle$). Then clauses (i)-(v) follow directly from Definition 2-19 and Theorem 2-34. Clause (vi) follows directly from Definition 2-19. Now, suppose $\mathfrak{C} \in$ GEN($\langle\mathfrak{H}, G\rangle$). With Definition 2-19, we then have with $\mathfrak{A}$, $\mathfrak{C} \in$ GEN($\langle\mathfrak{H}, G\rangle$), that also $\mathfrak{A}$, $\mathfrak{C} \in$ PGEN($\langle\mathfrak{H}, G\rangle$) and thus with Theorem 2-34-(vii) that min(Dom($\mathfrak{A}$)) = min(Dom($\mathfrak{C}$)). Now, suppose for contradiction that max(Dom($\mathfrak{A}$)) $<$ max(Dom($\mathfrak{C}$)). Then we would have that min(Dom($\mathfrak{C}$)) $\leq$

max(Dom($\mathfrak{A}$))+1 ≤ max(Dom($\mathfrak{C}$)) and thus max(Dom($\mathfrak{A}$))+1 ∈ Dom($\mathfrak{C}$). At the same time we would have that $\mathfrak{C}$↾max(Dom($\mathfrak{A}$))+1 = $\mathfrak{A}$ ∈ PGEN(⟨$\mathfrak{H}$, G⟩) = PGEN(⟨$\mathfrak{H}$, G↾(max(Dom(G))+1)⟩). With Definition 2-19, we would thus have $\mathfrak{C}$ ∉ GEN(⟨$\mathfrak{H}$, G⟩). Contradiction! For max(Dom($\mathfrak{C}$)) < max(Dom($\mathfrak{A}$)), a contradiction follows analogously. Therefore we have that also max(Dom($\mathfrak{C}$)) = max(Dom($\mathfrak{A}$)) and thus, with Theorem 2-4, that $\mathfrak{C}$ = $\mathfrak{A}$ ∈ {$\mathfrak{A}$}. Therefore we have GEN(⟨$\mathfrak{H}$, G⟩) ⊆ {$\mathfrak{A}$}. Also, we have by hypothesis {$\mathfrak{A}$} ⊆ GEN(⟨$\mathfrak{H}$, G⟩) and hence: GEN(⟨$\mathfrak{H}$, G⟩) = {$\mathfrak{A}$} and thus (vii). ■

Theorem 2-36. *GEN-generated segments are greater than the members of the respective AS-comprising segment sequence*
If $\mathfrak{H}$ ∈ SEQ and G ∈ ASCS($\mathfrak{H}$), then for all $\mathfrak{C}$ ∈ Ran(G) and $\mathfrak{A}$ ∈ GEN(⟨$\mathfrak{H}$, G⟩): |$\mathfrak{C}$| < |$\mathfrak{A}$|.

Proof: Suppose $\mathfrak{H}$ ∈ SEQ and G ∈ ASCS($\mathfrak{H}$). Now, suppose $\mathfrak{C}$ ∈ Ran(G) and $\mathfrak{A}$ ∈ GEN(⟨$\mathfrak{H}$, G⟩). Then there is a $\mathfrak{B}$ ∈ SG($\mathfrak{H}$) such that G is an AS-comprising segment sequence for $\mathfrak{B}$ in $\mathfrak{H}$ and min(Dom($\mathfrak{A}$))+1 = min(Dom($\mathfrak{B}$)) and max(Dom($\mathfrak{A}$)) = max(Dom($\mathfrak{B}$))+1 and $\mathfrak{A}$ is a CdI- or NI- or RA-like segment in $\mathfrak{H}$. Then we have |$\mathfrak{B}$| < |$\mathfrak{A}$|. Because of $\mathfrak{C}$ ∈ Ran(G), we also have, with Theorem 2-24, that |$\mathfrak{C}$| ≤ |$\mathfrak{B}$| and hence that |$\mathfrak{C}$| < |$\mathfrak{A}$|. ■

Theorem 2-37. *Preparatory theorem for Theorem 2-39 (a)*
{($\mathfrak{H}$, $\mathfrak{A}$) | $\mathfrak{A}$ is a minimal closed segment in $\mathfrak{H}$} ⊆ SEQ × {$\mathfrak{A}$ | $\mathfrak{A}$ is a segment}.

Proof: Suppose ($\mathfrak{H}$, $\mathfrak{A}$) ∈ {($\mathfrak{H}$, $\mathfrak{A}$) | $\mathfrak{A}$ is a minimal closed segment in $\mathfrak{H}$}. It then follows from Definition 2-14, Definition 2-15 and Definition 2-16 that $\mathfrak{A}$ is a segment in $\mathfrak{H}$ and thus that $\mathfrak{H}$ ∈ SEQ. Thus: ($\mathfrak{H}$, $\mathfrak{A}$) ∈ SEQ × {$\mathfrak{A}$ | $\mathfrak{A}$ is a segment}. ■

Theorem 2-38. *Preparatory for Theorem 2-39 (b)*
For all $\mathfrak{H}$ ∈ SEQ and G ∈ ASCS($\mathfrak{H}$) it holds that {$\mathfrak{H}$} × GEN(⟨$\mathfrak{H}$, G⟩) ⊆ SEQ × {$\mathfrak{A}$ | $\mathfrak{A}$ is a segment}.

Proof: Suppose $\mathfrak{H}$ ∈ SEQ and G ∈ ASCS($\mathfrak{H}$). Now, suppose ($\mathfrak{H}$, $\mathfrak{A}$) ∈ {$\mathfrak{H}$} × GEN(⟨$\mathfrak{H}$, G⟩). It then follows by hypothesis and Theorem 2-35-(ii) that $\mathfrak{A}$ ∈ SG($\mathfrak{H}$) and thus follows the whole statement. ■

Now, we can define the set of GEN-inductive relations:

Definition 2-20. *The set of GEN-inductive relations (CSR)*
CSR = $\{R \mid R \subseteq \text{SEQ} \times \{\mathfrak{A} \mid \mathfrak{A} \text{ is a segment}\}$ and

(i) $\{(\mathfrak{H}, \mathfrak{A}) \mid \mathfrak{A}$ is a minimal closed segment in $\mathfrak{H}\} \subseteq R$, and

(ii) For all $\mathfrak{H} \in \text{SEQ}$ and $G \in \text{ASCS}(\mathfrak{H})$ with $\{\mathfrak{H}\} \times \text{Ran}(G) \subseteq \text{R}$ it holds that $\{\mathfrak{H}\} \times \text{GEN}(\langle\mathfrak{H}, G\rangle) \subseteq R\}$.

Definition 2-20 is essentially a supporting definition for Definition 2-21, in which we define the relation that relates a sentence sequence to all and only the segments that are closed in this sequence. Informally, we can say that CSR consists of all relations R that relate a given sentence sequence $\mathfrak{H}$ to all minimal closed segments in $\mathfrak{H}$ (if such segments exist) and further to all segments $\mathfrak{A}$ in $\mathfrak{H}$ that can be generated by GEN from segments $\mathfrak{B}_0, \ldots, \mathfrak{B}_{n-1}$ with $\{(\mathfrak{H}, \mathfrak{B}_0), \ldots, (\mathfrak{H}, \mathfrak{B}_{n-1})\} \subseteq R$.

Theorem 2-39. *Preparatory theorem for Theorem 2-40*
$\text{SEQ} \times \{\mathfrak{A} \mid \mathfrak{A} \text{ is a segment}\} \in \text{CSR}$.

Proof: First, we have $\text{SEQ} \times \{\mathfrak{A} \mid \mathfrak{A} \text{ is a segment}\} \subseteq \text{SEQ} \times \{\mathfrak{A} \mid \mathfrak{A} \text{ is a segment}\}$. With Theorem 2-37, we also have that $\{(\mathfrak{H}, \mathfrak{A}) \mid \mathfrak{A}$ is a minimal closed segment in $\mathfrak{H}\} \subseteq \text{SEQ} \times \{\mathfrak{A} \mid \mathfrak{A} \text{ is a segment}\}$. With Theorem 2-38, we also have that for all $\mathfrak{H} \in \text{SEQ}$ and $G \in \text{ASCS}(\mathfrak{H})$ with $\{\mathfrak{H}\} \times \text{Ran}(G) \subseteq \text{SEQ} \times \{\mathfrak{A} \mid \mathfrak{A} \text{ is a segment}\}$ it holds that $\{\mathfrak{H}\} \times \text{GEN}(\langle\mathfrak{H}, G\rangle) \subseteq \text{SEQ} \times \{\mathfrak{A} \mid \mathfrak{A} \text{ is a segment}\}$. ■

Now, we define the relation that relates a given sentence sequence $\mathfrak{H}$ to all and only the segments that are minimal closed segments in $\mathfrak{H}$ or that can be generated from minimal closed segments in $\mathfrak{H}$ by successive applications of GEN:

Definition 2-21. *The smallest GEN-inductive relation (CS)*
CS = $\bigcap$CSR.

The following theorem assures us that CS is, first, indeed a relation, that relates a given sentence sequence $\mathfrak{H}$ to all and only the segments that are minimal closed segments in $\mathfrak{H}$ or that can be generated from minimal closed segments in $\mathfrak{H}$ by successive applications of GEN, and, second, that CS is a subset of all such relations and hence the smallest such relation. Thus, we have that CS relates a given sentence sequence only to segments of the kind indicated.

Theorem 2-40. *CS is the smallest GEN-inductive relation*

(i) CS $\in$ CSR and

(ii) If $R \in$ CSR, then CS $\subseteq R$.

Proof: (ii) follows from Definition 2-21. *Ad (i)*: We have to show that a) CS $\subseteq$ SEQ $\times$ {$\mathfrak{A}$ | $\mathfrak{A}$ is a segment}, b) {($\mathfrak{H}$, $\mathfrak{A}$) | $\mathfrak{A}$ is a minimal closed segment in $\mathfrak{H}$} $\subseteq$ CS and c) for all $\mathfrak{H} \in$ SEQ and $G \in$ ASCS($\mathfrak{H}$) with {$\mathfrak{H}$} $\times$ Ran(G) $\subseteq$ CS it holds that {$\mathfrak{H}$} $\times$ GEN($\langle\mathfrak{H}, G\rangle$) $\subseteq$ CS.

a), i.e. CS $\subseteq$ SEQ $\times$ {$\mathfrak{A}$ | $\mathfrak{A}$ is a segment}, follows with Theorem 2-39 and (ii). Since for all $R \in$ CSR we have that {($\mathfrak{H}$, $\mathfrak{A}$) | $\mathfrak{A}$ is a minimal closed segment in $\mathfrak{H}$} $\subseteq R$, we have, with Definition 2-21, also b), i.e. {($\mathfrak{H}$, $\mathfrak{A}$) | $\mathfrak{A}$ is a minimal closed segment in $\mathfrak{H}$} $\subseteq$ CS.

We still have to show that c), i.e. that for all $\mathfrak{H} \in$ SEQ and $G \in$ ASCS($\mathfrak{H}$) with {$\mathfrak{H}$} $\times$ Ran(G) $\subseteq$ CS it holds holds that {$\mathfrak{H}$} $\times$ GEN($\langle\mathfrak{H}, G\rangle$) $\subseteq$ CS. For this, suppose first that $\mathfrak{H} \in$ SEQ and $G \in$ ASCS($\mathfrak{H}$) and {$\mathfrak{H}$} $\times$ Ran(G) $\subseteq$ CS. According to Definition 2-21, what we have to show in order to prove that {$\mathfrak{H}$} $\times$ GEN($\langle\mathfrak{H}, G\rangle$) $\subseteq$ CS is that for all $R \in$ CSR it holds that {$\mathfrak{H}$} $\times$ GEN($\langle\mathfrak{H}, G\rangle$) $\subseteq R$. Now, suppose $R \in$ CSR. It then follows from {$\mathfrak{H}$} $\times$ Ran(G) $\subseteq$ CS (from our first hypothesis) and (ii) that {$\mathfrak{H}$} $\times$ Ran(G) $\subseteq R$. By hypothesis, we have $R \in$ CSR. With Definition 2-20, we thus have {$\mathfrak{H}$} $\times$ GEN($\langle\mathfrak{H}, G\rangle$) $\subseteq R$. Therefore we have for all $R \in$ CSR that {$\mathfrak{H}$} $\times$ GEN($\langle\mathfrak{H}, G\rangle$) $\subseteq R$ and thus we have that {$\mathfrak{H}$} $\times$ GEN($\langle\mathfrak{H}, G\rangle$) $\subseteq$ CS. Therefore we have for all $\mathfrak{H} \in$ SEQ and $G \in$ ASCS($\mathfrak{H}$) with {$\mathfrak{H}$} $\times$ Ran(G) $\subseteq$ CS: {$\mathfrak{H}$} $\times$ GEN($\langle\mathfrak{H}, G\rangle$) $\subseteq$ CS. ■

With the preceding theorem, we can informally say that the following definition characterises exactly those segments in a sentence sequence as segments that are closed in this sequence that are minimal closed segments in this sequence or that can be generated from these minimal segments by successive application of GEN.

Definition 2-22. *Closed segments*
$\mathfrak{A}$ <u>is a closed segment in</u> $\mathfrak{H}$ iff $(\mathfrak{H}, \mathfrak{A}) \in$ CS.

Theorem 2-41. *Closed segments are minimal or GEN-generated*
$(\mathfrak{H}, \mathfrak{A}) \in$ CS
iff

(i) $\mathfrak{A}$ is a minimal closed segment in $\mathfrak{H}$

or

(ii) $\mathfrak{H} \in$ SEQ and there is a $G \in$ ASCS($\mathfrak{H}$) with $\{\mathfrak{H}\} \times$ Ran(G) $\subseteq$ CS and $\mathfrak{A} \in$ GEN($\langle\mathfrak{H}, G\rangle$).

Proof: The right-left-direction follows with Theorem 2-40-(i) and Definition 2-20. Now, for the left-right-direction, suppose $X = \{(\mathfrak{H}, \mathfrak{A}) \mid \mathfrak{A}$ is a minimal closed segment in $\mathfrak{H}$ or $\mathfrak{H} \in$ SEQ and there is a $G \in$ ASCS($\mathfrak{H}$) with $\{\mathfrak{H}\} \times$ Ran(G) $\subseteq$ CS and $\mathfrak{A} \in$ GEN($\langle\mathfrak{H}, G\rangle$)$\} \cap$ CS. To prove the theorem, it suffices to show that $X \in$ CSR, then the statement follows with Theorem 2-40-(ii).

With Theorem 2-40-(i), we have CS $\in$ CSR. According to Definition 2-20 and the definition of X, we then have $X \subseteq$ CS $\subseteq$ SEQ $\times$ $\{\mathfrak{A} \mid \mathfrak{A}$ is a segment$\}$ and $\{(\mathfrak{H}, \mathfrak{A}) \mid \mathfrak{A}$ is a minimal closed segment in $\mathfrak{H}\} \subseteq X$.

We still have to show that for all $\mathfrak{H} \in$ SEQ and $G \in$ ASCS($\mathfrak{H}$) with $\{\mathfrak{H}\} \times$ Ran(G) $\subseteq X$ it holds that $\{\mathfrak{H}\} \times$ GEN($\langle\mathfrak{H}, G\rangle$) $\subseteq X$. First, suppose $\mathfrak{H} \in$ SEQ and $G \in$ ASCS($\mathfrak{H}$) and $\{\mathfrak{H}\} \times$ Ran(G) $\subseteq X$. Then we have that $\{\mathfrak{H}\} \times$ Ran(G) $\subseteq$ CS and thus, with Theorem 2-40-(i) and Definition 2-20, that also $\{\mathfrak{H}\} \times$ GEN($\langle\mathfrak{H}, G\rangle$) $\subseteq$ CS. Now, suppose $(\mathfrak{H}, \mathfrak{A}) \in \{\mathfrak{H}\} \times$ GEN($\langle\mathfrak{H}, G\rangle$). Then we have $\mathfrak{A} \in$ GEN($\langle\mathfrak{H}, G\rangle$). Thus there is a $G \in$ ASCS($\mathfrak{H}$) with $\{\mathfrak{H}\} \times$ Ran(G) $\subseteq$ CS and $\mathfrak{A} \in$ GEN($\langle\mathfrak{H}, G\rangle$) and we also have $(\mathfrak{H}, \mathfrak{A}) \in$ CS. Therefore we have $(\mathfrak{H}, \mathfrak{A}) \in X$. Hence we have $X \in$ CSR. ■

Theorem 2-42. *Closed segments are CdI- or NI- or RA-like segments*
If $(\mathfrak{H}, \mathfrak{A}) \in$ CS, then $\mathfrak{A}$ is a CdI-, NI- or RA-like segment in $\mathfrak{H}$.

Proof: Suppose $(\mathfrak{H}, \mathfrak{A}) \in$ CS. Then it holds with Theorem 2-41 and Theorem 2-37 that $\mathfrak{H} \in$ SEQ and that $\mathfrak{A}$ is a minimal closed segment in $\mathfrak{H}$ or that there is a $G \in$ ASCS($\mathfrak{H}$) with

$\{\mathfrak{H}\} \times \mathrm{Ran}(G) \subseteq \mathrm{CS}$ and $\mathfrak{A} \in \mathrm{GEN}(\langle\mathfrak{H}, G\rangle)$. The statement then follows immediately with Definition 2-14, Definition 2-15, Definition 2-16, Definition 2-17 and Theorem 2-35-(ii). ■

Theorem 2-43. ∅ is neither in Dom(CS) nor in Ran(CS)
If $(\mathfrak{H}, \mathfrak{A}) \in \mathrm{CS}$, then $\mathfrak{H} \neq \emptyset$ and $\mathfrak{A} \neq \emptyset$.

Proof: Suppose $(\mathfrak{H}, \mathfrak{A}) \in \mathrm{CS}$. It then holds with Theorem 2-42 that $\mathfrak{A}$ is a CdI- or an NI- or an RA-like segment in $\mathfrak{H}$. It then holds with Definition 2-11, Definition 2-12 and Definition 2-13 that $\mathfrak{H} \in \mathrm{SEQ}$ und $\mathfrak{A} \in \mathrm{SG}(\mathfrak{H})$. With Theorem 2-1 and Definition 2-1, we then have $\mathfrak{H} \neq \emptyset$ und $\mathfrak{A} \neq \emptyset$. ■

Theorem 2-42 shows that CS only contains pairs of sentence sequences and CdI- or NI- or RA-like segments in these sequences. So, the first and last members of the segments give them the form that is known from the corresponding patterns of inference (for NE with the contradictory statements included in a proper intial segment of the respective segment and for PE with the particular-quantification before the respective RA-like segment). However, not every pair of a sentence sequence and a segment in this sentence sequence that shows such a form is in CS. This can be shown using Theorem 2-41 and Theorem 2-42. Here an example for a sentence sequence and a CdI-like segment in this sequence for which the ordered pair of both is not an element of CS:

Example [2.1] Let $\mathfrak{H}^{[2.1]}$ be the following sequence:

0	Suppose	$P_{1.1}(c_1)$
1	Suppose	$P_{1.1}(c_1)$
2	Therefore	$P_{1.1}(c_1) \rightarrow P_{1.1}(c_1)$

Comment: Suppose $(\mathfrak{H}^{[2.1]}, \mathfrak{H}^{[2.1]}) \in \mathrm{CS}$. According to Theorem 2-41, we would then have that $\mathfrak{H}^{[2.1]}$ is a minimal closed segment in $\mathfrak{H}^{[2.1]}$ or that there would be a $G \in \mathrm{ASCS}(\mathfrak{H}^{[2.1]})$ with $\{\mathfrak{H}^{[2.1]}\} \times \mathrm{Ran}(G) \subseteq \mathrm{CS}$ and $\mathfrak{H}^{[2.1]} \in \mathrm{GEN}(\langle\mathfrak{H}^{[2.1]}, G\rangle)$. Since $|\mathrm{AS}(\mathfrak{H}^{[2.1]})| = 2$, $\mathfrak{H}^{[2.1]}$ is not a minimal closed segment in $\mathfrak{H}^{[2.1]}$. Therefore there has to be a $G \in \mathrm{ASCS}(\mathfrak{H}^{[2.1]})$ with $\{\mathfrak{H}^{[2.1]}\} \times \mathrm{Ran}(G) \subseteq \mathrm{CS}$ and $\mathfrak{H}^{[2.1]} \in \mathrm{GEN}(\langle\mathfrak{H}^{[2.1]}, G\rangle)$.

Then we have $\mathfrak{H}^{[2.1]} \in \mathrm{GEN}(\langle\mathfrak{H}^{[2.1]}, G\rangle)$. Then there is a $\mathfrak{B} \in \mathrm{SG}(\mathfrak{H}^{[2.1]})$ such that G is an AS-comprising segment sequence for $\mathfrak{B}$ in $\mathfrak{H}^{[2.1]}$ and $\min(\mathrm{Dom}(\mathfrak{H}^{[2.1]}))+1$ =

min(Dom(𝔅)) and max(Dom($\mathfrak{H}^{[2.1]}$)) = max(Dom(𝔅))+1. Then we have 𝔅 = {(1, ⌜Suppose $P_{1.1}(c_1)$⌝)}. Since G is an AS-comprising segment sequence for 𝔅 in $\mathfrak{H}^{[2.1]}$, we then have Ran(G) = {{(1, ⌜Suppose $P_{1.1}(c_1)$⌝)}}.

Yet, {(1, ⌜Suppose $P_{1.1}(c_1)$⌝)} is not a CdI- or NI- or RA-like segment in $\mathfrak{H}^{[2.1]}$. By hypothesis, however, we have {$\mathfrak{H}^{[2.1]}$} × Ran(G) ⊆ CS and thus ($\mathfrak{H}^{[2.1]}$, {(1, ⌜Suppose $P_{1.1}(c_1)$⌝)}) ∈ CS. According to Theorem 2-42, we would then have that {(1, ⌜Suppose $P_{1.1}(c_1)$⌝)} is a CdI- or NI- or RA-like segment in $\mathfrak{H}^{[2.1]}$. Thus, the assumption that ($\mathfrak{H}^{[2.1]}$, $\mathfrak{H}^{[2.1]}$) ∈ CS leads to a contradiction. Therefore ($\mathfrak{H}^{[2.1]}$, $\mathfrak{H}^{[2.1]}$) ∉ CS. ■

Theorem 2-44. *Closed segments have at least two elements*
If (𝔥, 𝔄) ∈ CS, then 2 ≤ |𝔄|.

Proof: With Theorem 2-31 it holds for all CdI- or NI- or RA-like segments 𝔄 in 𝔥 that 2 ≤ |𝔄|. From this the theorem follows with Theorem 2-42. ■

Theorem 2-45. *Every closed segment has a minimal closed segment as subsegment*
If (𝔥, 𝔄) ∈ CS, then there is a minimal closed segment 𝔅 in 𝔥 such that 𝔅 ⊆ 𝔄.

Proof: Let X = {(𝔥, 𝔄) | There is a minimal closed segment 𝔅 in 𝔥 such that 𝔅 ⊆ 𝔄} ∩ CS. To prove the theorem, it suffices to show that X ∈ CSR, then the statement follows with Theorem 2-40-(ii).

First, we have X ⊆ CS ⊆ SEQ × {𝔄 | 𝔄 is a segment} and {(𝔥, 𝔄) | 𝔄 is a minimal closed segment in 𝔥} ⊆ X.

We still have to show that it holds for all 𝔥 ∈ SEQ and G ∈ ASCS(𝔥) with {𝔥} × Ran(G) ⊆ X that {𝔥} × GEN(⟨𝔥, G⟩) ⊆ X. First, suppose 𝔥 ∈ SEQ and G ∈ ASCS(𝔥) and {𝔥} × Ran(G) ⊆ X. Then we have {𝔥} × Ran(G) ⊆ CS. Now, suppose (𝔥, 𝔄) ∈ {𝔥} × GEN(⟨𝔥, G⟩). Then we have (𝔥, 𝔄) ∈ CS. Because of 𝔄 ∈ GEN(⟨𝔥, G⟩) there is then, with Theorem 2-35, a 𝔅 ∈ SG(𝔥) such that G is an AS-comprising segment sequence for 𝔅 in 𝔥, AS(𝔥) ∩ 𝔅 ≠ ∅ and min(Dom(𝔄))+1 = min(Dom(𝔅)) and max(Dom(𝔄)) = max(Dom(𝔅))+1 and 𝔄 is a CdI- or NI- or RA-like segment in 𝔥.

Then there is an i ∈ Dom(AS(𝔥)) ∩ Dom(𝔅). We have that G is an AS-comprising segment sequence for 𝔅. With Definition 2-9, it thus holds for all r ∈ Dom(AS(𝔥)) ∩

Dom($\mathfrak{B}$) that there is an $s \in$ Dom(G) such that $r \in$ Dom($G(s)$). Therefore there is such an s for i. By hypothesis, we have $\{\mathfrak{H}\} \times$ Ran(G) $\subseteq X$ and hence ($\mathfrak{H}$, $G(s)$) $\in X$ and thus there is a minimal closed segment $\mathfrak{C}$ in $\mathfrak{H}$ such that $\mathfrak{C} \subseteq G(s)$. With Theorem 2-24, we have $G(s) \subseteq \mathfrak{B}$ and hence $\mathfrak{C} \subseteq \mathfrak{B}$ and thus, because of $\mathfrak{B} \subseteq \mathfrak{A}$, we have $\mathfrak{C} \subseteq \mathfrak{A}$. Hence we have ($\mathfrak{H}$, $\mathfrak{A}$) $\in X$. ■

Theorem 2-46. *Ratio of inference- and assumption-sentences in closed segments*
If ($\mathfrak{H}$, $\mathfrak{A}$) $\in$ CS, then |AS($\mathfrak{H}$) $\cap$ $\mathfrak{A}$| $\leq$ |IS($\mathfrak{H}$) $\cap$ $\mathfrak{A}$|.

Proof: Let X = {($\mathfrak{H}$, $\mathfrak{A}$) | If $\mathfrak{A}$ is a CdI- or NI- or RA-like segment in $\mathfrak{H}$, then |AS($\mathfrak{H}$) $\cap$ $\mathfrak{A}$| $\leq$ |IS($\mathfrak{H}$) $\cap$ $\mathfrak{A}$|} $\cap$ CS. To prove the theorem, it suffices to show that $X \in$ CSR, then the statement follows with Theorem 2-40-(ii) and Theorem 2-42.

First, we have $X \subseteq$ CS $\subseteq$ SEQ $\times$ {$\mathfrak{A}$ | $\mathfrak{A}$ is a segment}. With Theorem 2-33, we also have {($\mathfrak{H}$, $\mathfrak{A}$) | $\mathfrak{A}$ is a minimal closed segment in $\mathfrak{H}$} $\subseteq X$.

We have to show that for all $\mathfrak{H} \in$ SEQ and $G \in$ ASCS($\mathfrak{H}$) with $\{\mathfrak{H}\} \times$ Ran(G) $\subseteq X$ it holds that $\{\mathfrak{H}\} \times$ GEN($\langle\mathfrak{H}, G\rangle$) $\subseteq X$. First, suppose $\mathfrak{H} \in$ SEQ and $G \in$ ASCS($\mathfrak{H}$) and $\{\mathfrak{H}\} \times$ Ran(G) $\subseteq X$. Then we have $\{\mathfrak{H}\} \times$ Ran(G) $\subseteq$ CS. Now, suppose ($\mathfrak{H}$, $\mathfrak{A}$) $\in \{\mathfrak{H}\} \times$ GEN($\langle\mathfrak{H}, G\rangle$). Then we have ($\mathfrak{H}$, $\mathfrak{A}$) $\in$ CS. Because of $\mathfrak{A} \in$ GEN($\langle\mathfrak{H}, G\rangle$), there is then, with Theorem 2-35, a $\mathfrak{B} \in$ SG($\mathfrak{H}$) such that G is an AS-comprising segment sequence for $\mathfrak{B}$ in $\mathfrak{H}$ and min(Dom($\mathfrak{A}$))+1 = min(Dom($\mathfrak{B}$)) and max(Dom($\mathfrak{A}$)) = max(Dom($\mathfrak{B}$))+1 and $\mathfrak{A}$ is a CdI- or NI- or RA-like segment in $\mathfrak{H}$. With Theorem 2-29, we then have |AS($\mathfrak{H}$) $\cap$ $\mathfrak{A}$| $\leq$ 1+|AS($\mathfrak{H}$) $\cap$ $\mathfrak{B}$| and 1+|IS($\mathfrak{H}$) $\cap$ $\mathfrak{B}$| $\leq$ |IS($\mathfrak{H}$) $\cap$ $\mathfrak{A}$|. With Definition 2-9-(iii-c), we have for all $l \in$ Dom(AS($\mathfrak{H}$)) $\cap$ Dom($\mathfrak{B}$): There is an $i \in$ Dom(G) such that $l \in$ Dom($G(i)$) and with Theorem 2-24 it holds for all $i \in$ Dom(G) that $G(i) \subseteq \mathfrak{B}$. Thus we have $\bigcup$ {AS($\mathfrak{H}$) $\cap$ $G(i)$ | $i \in$ Dom(G)} = AS($\mathfrak{H}$) $\cap$ $\mathfrak{B}$. Also, we have $\bigcup$ {IS($\mathfrak{H}$) $\cap$ $G(i)$ | $i \in$ Dom(G)} $\subseteq$ IS($\mathfrak{H}$) $\cap$ $\mathfrak{B}$.

Because of $\{\mathfrak{H}\} \times$ Ran(G) $\subseteq X$, we have that for all $i \in$ Dom(G) it holds that ($\mathfrak{H}$, $G(i)$) $\in X$ and thus that |AS($\mathfrak{H}$) $\cap$ $G(i)$| $\leq$ |IS($\mathfrak{H}$) $\cap$ $G(i)$|. With Theorem 2-22-(i) and Theorem 2-27, it holds for all i, $j \in$ Dom(G) that if $i \neq j$, then $G(i) \cap G(j) = \emptyset$. Thus we have for all i, $j \in \mathrm{Dom}(G)$: If $i \neq j$, then $(\mathrm{AS}(\mathfrak{H}) \cap G(i)) \cap (\mathrm{AS}(\mathfrak{H}) \cap G(j)) = \emptyset$ and $(\mathrm{IS}(\mathfrak{H}) \cap G(i)) \cap (\mathrm{IS}(\mathfrak{H}) \cap G(j)) = \emptyset$.

Hence we have

$$|\bigcup\{\mathrm{AS}(\mathfrak{H}) \cap G(j) \mid j \in \mathrm{Dom}(G)\}| = \sum_{j=0}^{\mathrm{Dom}(G)-1} |\mathrm{AS}(\mathfrak{H}) \cap G(j)|$$

and

$$|\bigcup\{\mathrm{IS}(\mathfrak{H}) \cap G(j) \mid j \in \mathrm{Dom}(G)\}| = \sum_{j=0}^{\mathrm{Dom}(G)-1} |\mathrm{IS}(\mathfrak{H}) \cap G(j)|.$$

Because of $|\mathrm{AS}(\mathfrak{H}) \cap G(j)| \leq |\mathrm{IS}(\mathfrak{H}) \cap G(j)|$ for all $j \in \mathrm{Dom}(G)$, we also have:

$$\sum_{j=0}^{\mathrm{Dom}(G)-1} |\mathrm{AS}(\mathfrak{H}) \cap G(j)| \leq \sum_{j=0}^{\mathrm{Dom}(G)-1} |\mathrm{IS}(\mathfrak{H}) \cap G(j)|.$$

Thus we have

$$|\mathrm{AS}(\mathfrak{H}) \cap \mathfrak{A}| \leq 1+|\mathrm{AS}(\mathfrak{H}) \cap \mathfrak{B}| = 1+\sum_{j=0}^{\mathrm{Dom}(G)-1} |\mathrm{AS}(\mathfrak{H}) \cap G(j)| \leq$$

$$1+\sum_{j=0}^{\mathrm{Dom}(G)-1} |\mathrm{IS}(\mathfrak{H}) \cap G(j)| \leq 1+|\mathrm{IS}(\mathfrak{H}) \cap \mathfrak{B}| \leq |\mathrm{IS}(\mathfrak{H}) \cap \mathfrak{A}|.$$

Therefore we have $(\mathfrak{H}, \mathfrak{A}) \in X$. ■

Theorem 2-47. *Every assumption-sentence in a closed segment* $\mathfrak{A}$ *lies at the beginning of* $\mathfrak{A}$ *or at the beginning of a proper closed subsegment of* $\mathfrak{A}$

If $(\mathfrak{H}, \mathfrak{A}) \in \mathrm{CS}$, then for all $i \in \mathrm{Dom}(\mathrm{AS}(\mathfrak{H})) \cap \mathrm{Dom}(\mathfrak{A})$:

(i) $i = \min(\mathrm{Dom}(\mathfrak{A}))$

or

(ii) There is a $\mathfrak{B}$ with $(\mathfrak{H}, \mathfrak{B}) \in \mathrm{CS}$ such that

a) $i = \min(\mathrm{Dom}(\mathfrak{B}))$ and

b) $\min(\mathrm{Dom}(\mathfrak{A})) < \min(\mathrm{Dom}(\mathfrak{B})) < \max(\mathrm{Dom}(\mathfrak{B})) < \max(\mathrm{Dom}(\mathfrak{A}))$.

Proof: Let $X = \{(\mathfrak{H}, \mathfrak{A}) \mid$ For all $i \in \mathrm{Dom}(\mathrm{AS}(\mathfrak{H})) \cap \mathrm{Dom}(\mathfrak{A})$: $i = \min(\mathrm{Dom}(\mathfrak{A}))$ or there is a $\mathfrak{B}$ with $(\mathfrak{H}, \mathfrak{B}) \in \mathrm{CS}$ such that $i = \min(\mathrm{Dom}(\mathfrak{B}))$ and $\min(\mathrm{Dom}(\mathfrak{A})) < \min(\mathrm{Dom}(\mathfrak{B})) < \max(\mathrm{Dom}(\mathfrak{B})) < \max(\mathrm{Dom}(\mathfrak{A}))\} \cap \mathrm{CS}$. To prove the theorem, it suffices to show that $X \in \mathrm{CSR}$, then the statement follows with Theorem 2-40-(ii).

First, we have $X \subseteq \mathrm{CS} \subseteq \mathrm{SEQ} \times \{\mathfrak{A} \mid \mathfrak{A}$ is segment$\}$ and with Definition 2-17, Definition 2-14-(i), Definition 2-15-(i), Definition 2-16-(i) and Theorem 2-41 it holds that $\{(\mathfrak{H}, \mathfrak{A}) \mid \mathfrak{A}$ is a minimal closed segment in $\mathfrak{H}\} \subseteq X$.

We still have to show that for all $\mathfrak{H} \in$ SEQ and $G \in$ ASCS($\mathfrak{H}$) with $\{\mathfrak{H}\} \times$ Ran(G) $\subseteq X$ it holds that $\{\mathfrak{H}\} \times$ GEN($\langle\mathfrak{H}, G\rangle$) $\subseteq X$. First, suppose $\mathfrak{H} \in$ SEQ and $G \in$ ASCS($\mathfrak{H}$) and $\{\mathfrak{H}\} \times$ Ran(G) $\subseteq X$. Then we have $\{\mathfrak{H}\} \times$ Ran(G) $\subseteq$ CS. Now, suppose ($\mathfrak{H}$, $\mathfrak{A}$) $\in \{\mathfrak{H}\} \times$ GEN($\langle\mathfrak{H}, G\rangle$). Then we have ($\mathfrak{H}$, $\mathfrak{A}$) $\in$ CS. With $\mathfrak{A} \in$ GEN($\langle\mathfrak{H}, G\rangle$), there is then a $\mathfrak{B} \in$ SG($\mathfrak{H}$) such that G is an AS-comprising segment sequence for $\mathfrak{B}$ in $\mathfrak{H}$, AS($\mathfrak{H}$) $\cap\, \mathfrak{B} \neq \emptyset$ and min(Dom($\mathfrak{A}$))+1 = min(Dom($\mathfrak{B}$)) and max(Dom($\mathfrak{A}$)) = max(Dom($\mathfrak{B}$))+1 and $\mathfrak{A}$ is a CdI- or NI- or RA-like segment in $\mathfrak{H}$.

Now, suppose $i \in$ Dom(AS($\mathfrak{H}$)) $\cap$ Dom($\mathfrak{A}$) and $i \neq$ min(Dom($\mathfrak{A}$)). With Theorem 2-30, we then have min(Dom($\mathfrak{A}$)) $< i <$ max(Dom($\mathfrak{A}$)). Then we have min(Dom($\mathfrak{B}$)) $\leq i \leq$ max(Dom($\mathfrak{B}$)). Then we have $i \in$ Dom(AS($\mathfrak{H}$)) $\cap$ Dom($\mathfrak{B}$). We have that G is an AS-comprising segment sequence for $\mathfrak{B}$. With Definition 2-9, we therefore have that for all $r \in$ Dom(AS($\mathfrak{H}$)) $\cap$ Dom($\mathfrak{B}$) there is an $s \in$ Dom(G) such that $r \in$ Dom($G(s)$). Therefore there is such an s for i. Then we have $i \in$ Dom(AS($\mathfrak{H}$)) $\cap$ Dom($G(s)$) and according to Theorem 2-24 we have $G(s) \subseteq \mathfrak{B} \subseteq \mathfrak{A}$. By hypothesis, we have $\{\mathfrak{H}\} \times$ Ran(G) $\subseteq X$ and hence ($\mathfrak{H}$, $G(s)$) $\in X$. Therefore we have that for all $r \in$ Dom(AS($\mathfrak{H}$)) $\cap$ Dom($G(s)$) it holds that r = min(Dom($G(s)$)) or that there is a $\mathfrak{C}$ with ($\mathfrak{H}$, $\mathfrak{C}$) $\in$ CS such that r = min(Dom($\mathfrak{C}$)) and min(Dom($G(s)$)) < min(Dom($\mathfrak{C}$)) < max(Dom($\mathfrak{C}$)) < max(Dom($G(s)$)). Therefore we have i = min(Dom($G(s)$)) or there is a suitable $\mathfrak{C}$. In the first case, $G(s)$) itself is the desired segment, because with ($\mathfrak{H}$, $G(s)$) $\in X$ we also have ($\mathfrak{H}$, $G(s)$) $\in$ CS. Moreover, it then follows by hypothesis that min(Dom($\mathfrak{A}$)) $< i$ = min(Dom($G(s)$)) and max(Dom($G(s)$)) $\leq$ max(Dom($\mathfrak{B}$)) < max(Dom($\mathfrak{B}$))+1 = max(Dom($\mathfrak{A}$))). With Theorem 2-44, we also have min(Dom($G(s)$)) < max(Dom($G(s)$)). Suppose for the second case that $\mathfrak{C}$ is as required. Then we have min(Dom($\mathfrak{A}$)) $< i$ = min(Dom($\mathfrak{C}$)) < max(Dom($\mathfrak{C}$)) < max(Dom($G(s)$)) $\leq$ max(Dom($\mathfrak{B}$)) < max(Dom($\mathfrak{A}$)) and hence $\mathfrak{C}$ is the desired segment.

Therefore we have for all $i \in$ Dom(AS($\mathfrak{H}$)) $\cap$ Dom($\mathfrak{A}$): i = min(Dom($\mathfrak{A}$)) or there is a $\mathfrak{B}$ with ($\mathfrak{H}$, $\mathfrak{B}$) $\in$ CS such that i = min(Dom($\mathfrak{B}$)) and min(Dom($\mathfrak{A}$)) < min(Dom($\mathfrak{B}$)) < max(Dom($\mathfrak{B}$)) < max(Dom($\mathfrak{A}$)). Hence we have ($\mathfrak{H}$, $\mathfrak{A}$) $\in X$. ■

Theorem 2-48. *Every closed segment is a minimal closed segment or a CdI- or NI- or RA-like segment whose assumption-sentences lie at the beginning or in a proper closed subsegment*
If $(\mathfrak{H}, \mathfrak{A}) \in$ CS, then:

(i) $\mathfrak{A}$ is a minimal closed segment in $\mathfrak{H}$

or

(ii) $\mathfrak{A}$ is a CdI- or NI- or RA-like segment $\mathfrak{H}$, where for all $i \in$ Dom(AS($\mathfrak{H}$)) $\cap$ Dom($\mathfrak{A}$) with min(Dom($\mathfrak{A}$)) $< i$ it holds that there is a $\mathfrak{B}$ such that

a) $(i, \mathfrak{H}_i) \in \mathfrak{B}$,

b) $(\mathfrak{H}, \mathfrak{B}) \in$ CS,

c) i = min(Dom($\mathfrak{B}$)) and

d) min(Dom($\mathfrak{A}$)) < min(Dom($\mathfrak{B}$)) < max(Dom($\mathfrak{B}$)) < max(Dom($\mathfrak{A}$)).

Proof: Suppose $(\mathfrak{H}, \mathfrak{A}) \in$ CS. Now, suppose $\mathfrak{A}$ is not a minimal closed segment in $\mathfrak{H}$. Then it holds with Theorem 2-42 that $\mathfrak{A}$ is a CdI- or NI- or RA-like segment in $\mathfrak{H}$ and, with Theorem 2-47, that for all $i \in$ Dom(AS($\mathfrak{H}$)) $\cap$ Dom($\mathfrak{A}$) with min(Dom($\mathfrak{A}$)) $< i$ there is a suitable $\mathfrak{B}$. ■

Theorem 2-49. *Closed segments are non-redundant, i.e. proper initial segments of closed segments are not closed segments*
If $(\mathfrak{H}, \mathfrak{A}) \in$ CS, then for all $i \in$ Dom($\mathfrak{A}$): $(\mathfrak{H}, \mathfrak{A}\restriction i) \notin$ CS.

Proof: Suppose X = {$(\mathfrak{H}, \mathfrak{A})$ | $(\mathfrak{H}, \mathfrak{A}) \in$ CS and for all $i \in$ Dom($\mathfrak{A}$): $(\mathfrak{H}, \mathfrak{A}\restriction i) \notin$ CS }. To prove the theorem, it suffices to show that $X \in$ CSR, then the statement follows with Theorem 2-40-(ii).

First, we have $X \subseteq$ CS $\subseteq$ SEQ $\times$ {$\mathfrak{A}$ | $\mathfrak{A}$ is a segment} and with Definition 2-17, Definition 2-14-(ii), Definition 2-15-(ii), Definition 2-16-(ii), Theorem 2-41 and Theorem 2-42 it holds that {$(\mathfrak{H}, \mathfrak{A})$ | $\mathfrak{A}$ is a minimal closed segment in $\mathfrak{H}$} $\subseteq X$.

We have to show that for all $\mathfrak{H} \in$ SEQ and $G \in$ ASCS($\mathfrak{H}$) with $\{\mathfrak{H}\} \times$ Ran(G) $\subseteq X$ it holds that $\{\mathfrak{H}\} \times$ GEN($\langle\mathfrak{H}, G\rangle$) $\subseteq X$. First, suppose $\mathfrak{H} \in$ SEQ and $G \in$ ASCS($\mathfrak{H}$) and $\{\mathfrak{H}\} \times$ Ran(G) $\subseteq X$. Then we have $\{\mathfrak{H}\} \times$ Ran(G) $\subseteq$ CS. Now, suppose $(\mathfrak{H}, \mathfrak{A}) \in \{\mathfrak{H}\} \times$ GEN($\langle\mathfrak{H}, G\rangle$). Then we have $\mathfrak{A} \in$ GEN($\langle\mathfrak{H}, G\rangle$) and thus $(\mathfrak{H}, \mathfrak{A}) \in$ CS. Also, there is then a $\mathfrak{B} \in$ SG($\mathfrak{H}$) such that G is an AS-comprising segment sequence for $\mathfrak{B}$ in $\mathfrak{H}$ and AS($\mathfrak{H}$) $\cap$ $\mathfrak{B} \neq \emptyset$ and min(Dom($\mathfrak{A}$))+1 = min(Dom($\mathfrak{B}$)) and max(Dom($\mathfrak{A}$)) = max(Dom($\mathfrak{B}$))+1 and $\mathfrak{A}$ is a CdI- or NI- or RA-like segment in $\mathfrak{H}$. Now, suppose for contradiction that $(\mathfrak{H},$

$\mathfrak{A}\restriction i) \in$ CS for an $i \in$ Dom($\mathfrak{A}$). Then we have that $\mathfrak{A}\restriction i$ is a segment in $\mathfrak{H}$. With Theorem 2-7, we then have min(Dom($\mathfrak{A}\restriction i$)) = min(Dom($\mathfrak{A}$)) and thus with Theorem 2-23 that for all $j \in$ Dom(G) it holds that min(Dom($\mathfrak{A}\restriction i$)) < min(Dom($\mathfrak{B}$)) $\leq$ min(Dom($G(j)$)).

With Theorem 2-35-(iii), we then have that $\mathfrak{A}\restriction i$ is not a minimal closed segment in $\mathfrak{H}$. Then it holds with Theorem 2-41 that there is a $G^* \in$ ASCS($\mathfrak{H}$) with $\{\mathfrak{H}\} \times$ Ran(G^*) $\subseteq$ CS and $\mathfrak{A}\restriction i \in$ GEN($\langle\mathfrak{H}, G^*\rangle$). With Theorem 2-35, we then have that there is a $\mathfrak{B}' \in$ SG($\mathfrak{H}$) such that min(Dom($\mathfrak{A}$))+1 = min(Dom($\mathfrak{A}\restriction i$))+1 = min(Dom($\mathfrak{B}'$)) and max(Dom($\mathfrak{A}\restriction i$)) = i-1 = max(Dom($\mathfrak{B}'$))+1. We will now show that there is an $s \in$ Dom(G) such that $\mathfrak{A}\restriction i \in$ PGEN($\langle\mathfrak{H}, G(\restriction s+1)\rangle$), which, according to Theorem 2-35-(vi), contradicts $\mathfrak{A} \in$ GEN($\langle\mathfrak{H}, G\rangle$).

It holds with Theorem 2-35-(iv) that there is an $l \in$ Dom(AS($\mathfrak{H}$)) $\cap$ Dom($\mathfrak{A}\restriction i$) such that min(Dom($\mathfrak{A}\restriction i$)) = min(Dom($\mathfrak{A}$)) < l. Now, suppose l_0 = max($\{l \mid l \in$ Dom(AS($\mathfrak{H}$)) $\cap$ Dom($\mathfrak{A}\restriction i$) and min(Dom($\mathfrak{A}\restriction i$)) < $l\}$. It then follows with $i \leq$ max(Dom($\mathfrak{A}$)) and Dom($\mathfrak{A}\restriction i$) $\subseteq$ Dom($\mathfrak{A}$) that min(Dom($\mathfrak{A}$)) = min(Dom($\mathfrak{A}\restriction i$)) < l_0 < max(Dom($\mathfrak{A}$)). Then we have min(Dom($\mathfrak{B}$)) $\leq l_0 \leq$ max(Dom($\mathfrak{B}$)). Then we have $l_0 \in$ Dom(AS($\mathfrak{H}$)) $\cap$ Dom($\mathfrak{B}$). We have that G is an AS-comprising segment sequence for $\mathfrak{B}$. With Definition 2-9, it therefore holds that there is an $s \in$ Dom(G) such that $l_0 \in$ Dom($G(s)$). Then we have that $l_0 \in$ Dom(AS($\mathfrak{H}$)) $\cap$ Dom($G(s)$) and hence, because of $\{\mathfrak{H}\} \times$ Ran(G) $\subseteq X \subseteq$ CS and with Theorem 2-47, that min(Dom($G(s)$)) $\leq l_0$ < max(Dom($G(s)$)). We also have that ($\mathfrak{H}$, $\mathfrak{A}\restriction i$) $\in$ CS and thus, with Theorem 2-47, that l_0 < i-1. Hence, we have that min(Dom($\mathfrak{A}\restriction i$)) < min(Dom($G(s)$)) < i-1.

Now, suppose $k \leq s$. Since G is an AS-comprising segment sequence for $\mathfrak{B}$ in $\mathfrak{H}$, it then follows with Definition 2-9 and Definition 2-7 that min(Dom($\mathfrak{A}\restriction i$)) < min(Dom($G(k)$)) $\leq$ min(Dom($G(s)$)) < i-1 and thus min(Dom($G(k)$)) $\in$ Dom($\mathfrak{B}'$). Since $\{\mathfrak{H}\} \times$ Ran(G) $\subseteq X$ $\subseteq$ CS, it then holds with Theorem 2-42 that min(Dom($G(k)$)) $\in$ Dom(AS($\mathfrak{H}$)) $\cap$ Dom($\mathfrak{B}'$). Since G^* is an AS-comprising segment sequence for $\mathfrak{B}'$ in $\mathfrak{H}$, there is then an r $\in$ Dom(G^*) such that min(Dom($G(k)$)) $\in$ Dom($G^*(r)$). Then we have min(Dom($G(k)$)) $\in$ Dom(AS($\mathfrak{H}$)) $\cap$ Dom($G^*(r)$). Suppose min(Dom($G^*(r)$)) = min(Dom($G(k)$)). Then it holds with $\{\mathfrak{H}\} \times$ Ran(G) $\subseteq X$ and $\{\mathfrak{H}\} \times$ Ran(G^*) $\subseteq$ CS that max(Dom($G(k)$)) $\leq$ max(Dom($G^*(r)$)). Suppose min(Dom($G^*(r)$)) $\neq$ min(Dom($G(k)$)). Then it holds with

$\{\mathfrak{H}\} \times \mathrm{Ran}(G^*) \subseteq \mathrm{CS}$ and Theorem 2-47 that there is a $\mathfrak{C}$ such that $(\mathfrak{H}, \mathfrak{C}) \in \mathrm{CS}$ and $\min(\mathrm{Dom}(G(k))) = \min(\mathrm{Dom}(\mathfrak{C}))$ and $\min(\mathrm{Dom}(G^*(r))) < \min(\mathrm{Dom}(\mathfrak{C})) < \max(\mathrm{Dom}(\mathfrak{C})) < \max(\mathrm{Dom}(G^*(r)))$. Then it holds with $\{\mathfrak{H}\} \times \mathrm{Ran}(G) \subseteq X$ that $\max(\mathrm{Dom}(G(k))) \leq \max(\mathrm{Dom}(\mathfrak{C}))$. Thus holds with Theorem 2-5-(i) in both cases $G(k) \subseteq G^*(r)$. Therefore we have for all $k \leq s$ that there is an $r \in \mathrm{Dom}(G^*)$ such that $G(k) \subseteq G^*(r)$.

Since G^* is an AS-comprising segment sequence for $\mathfrak{B}'$ and $\max(\mathrm{Dom}(\mathfrak{B}')) = i$-2 we thus have in particular that $\max(\mathrm{Dom}(G(s))) \leq i$-2. We also have that if $\mathfrak{A}\restriction i$ is an NI-like segment in $\mathfrak{H}$, then there is $j \in \mathrm{Dom}(\mathfrak{A}\restriction i)$ such that $\mathrm{P}(\mathfrak{H}_j) = \Gamma$ and $\mathrm{P}(\mathfrak{H}_{i\text{-}2}) = \ulcorner\neg\Gamma\urcorner$ or $\mathrm{P}(\mathfrak{H}_j) = \ulcorner\neg\Gamma\urcorner$ and $\mathrm{P}(\mathfrak{H}_{i\text{-}2}) = \Gamma$ and for all $r \in \mathrm{Dom}(G^*)$ it holds that $j < \min(\mathrm{Dom}(G^*(r)))$ or $\max(\mathrm{Dom}(G^*(r))) \leq j$. If there was a $k \leq s$ such that $\min(\mathrm{Dom}(G(k))) \leq j < \max(\mathrm{Dom}(G(k)))$, then there would be, as we have just shown, an $r \in \mathrm{Dom}(G^*)$ such that $G(k) \subseteq G^*(r)$ and thus $\min(\mathrm{Dom}(G^*(r))) \leq j < \max(\mathrm{Dom}(G^*(r)))$. Therefore, if $\mathfrak{A}\restriction i$ is an NI-like segment in $\mathfrak{H}$, then there is $j \in \mathrm{Dom}(\mathfrak{A}\restriction i)$ such that $\mathrm{P}(\mathfrak{H}_j) = \Gamma$ and $\mathrm{P}(\mathfrak{H}_{i\text{-}2}) = \ulcorner\neg\Gamma\urcorner$ or $\mathrm{P}(\mathfrak{H}_j) = \ulcorner\neg\Gamma\urcorner$ and $\mathrm{P}(\mathfrak{H}_{i\text{-}2}) = \Gamma$ and for all $k \leq s$ it holds that $j < \min(\mathrm{Dom}(G(k)))$ or $\max(\mathrm{Dom}(G(k))) \leq j$. Also, we have for all $l \in \mathrm{Dom}(\mathrm{AS}(\mathfrak{H})) \cap \mathrm{Dom}(\mathfrak{B}')$ that there is a $k \leq s$ such that $l \in \mathrm{Dom}(G(k))$. First, we have $\mathfrak{B}' \subseteq \mathfrak{B}$ and thus there is for every such l a $k \in \mathrm{Dom}(G)$ such that $l \in \mathrm{Dom}(G(k))$. Also, if $s < k$, we would have, with Definition 2-9 and Definition 2-7, that $l_0 < \max(\mathrm{Dom}(G(s))) < \min(\mathrm{Dom}(G(k))) \leq l$, while, on the other hand, we have $l \leq l_0$.

With Definition 2-9 and Definition 2-7, we can easily show that $G\restriction(s+1) \in \mathrm{SGS}(\mathfrak{H})$. Hence, we have that $G\restriction(s+1)$ is an AS-comprising segment sequence for $\mathfrak{B}'$ and thus also that $G\restriction(s+1) \in \mathrm{ASCS}(\mathfrak{H})$ and hence that $\mathfrak{A}\restriction i \in \mathrm{PGEN}(\langle\mathfrak{H}, G\restriction(s+1)\rangle)$. This, however contradicts Theorem 2-35-(vi). Therefore there is no $i \in \mathrm{Dom}(\mathfrak{A})$ such that $(\mathfrak{H}, \mathfrak{A}\restriction i) \in \mathrm{CS}$ and, because $(\mathfrak{H}, \mathfrak{A}) \in \mathrm{CS}$, we have $(\mathfrak{H}, \mathfrak{A}) \in X$. ■

Theorem 2-50. *Closed segments are uniquely determined by their beginnings*
If $\mathfrak{A}$, $\mathfrak{A}'$ are closed segments in $\mathfrak{H}$ and $\min(\mathrm{Dom}(\mathfrak{A})) = \min(\mathrm{Dom}(\mathfrak{A}'))$, then $\mathfrak{A} = \mathfrak{A}'$.

Proof: Let $\mathfrak{A}$, $\mathfrak{A}'$ be closed segments in $\mathfrak{H}$ and $\min(\mathrm{Dom}(\mathfrak{A})) = \min(\mathrm{Dom}(\mathfrak{A}'))$. Suppose for contradiction that $\max(\mathrm{Dom}(\mathfrak{A})) < \max(\mathrm{Dom}(\mathfrak{A}'))$. Then we would have have min(Dom($\mathfrak{A}$')) = min(Dom($\mathfrak{A}$)) < max(Dom($\mathfrak{A}$))+1 ≤ max(Dom($\mathfrak{A}$')). Since $\mathfrak{A}$' is a segment, we would thus have max(Dom($\mathfrak{A}$))+1 ∈ Dom($\mathfrak{A}$') and thus that $\mathfrak{A}$'↾(max(Dom($\mathfrak{A}$))+1) = $\mathfrak{A}$ is a closed segment in $\mathfrak{H}$. Together with Theorem 2-49 this contradicts our assumption that $\mathfrak{A}$' is a closed segment in $\mathfrak{H}$. In the same way, it follows for max(Dom($\mathfrak{A}$')) < max(Dom($\mathfrak{A}$)) that $\mathfrak{A}$ would not be a closed segment in $\mathfrak{H}$. Therefore we have max(Dom($\mathfrak{A}$)) = max(Dom($\mathfrak{A}$')) and thus $\mathfrak{A}$ = $\mathfrak{A}$'. ■

Theorem 2-51. *AS-comprising segment sequences for one and the same segment for which all values are closed segments are identical.*
If $\mathfrak{A}$ is a segment in $\mathfrak{H}$ and G, G* are AS-comprising segment sequences for $\mathfrak{A}$ in $\mathfrak{H}$ and {$\mathfrak{H}$} × Ran(G) ⊆ CS and {$\mathfrak{H}$} × Ran(G*) ⊆ CS, then G = G*.

Proof: Suppose $\mathfrak{A}$ is a segment in $\mathfrak{H}$ and suppose G, G* are AS-comprising segment sequences for $\mathfrak{A}$ in $\mathfrak{H}$ and {$\mathfrak{H}$} × Ran(G) ⊆ CS and {$\mathfrak{H}$} × Ran(G*) ⊆ CS. With Definition 2-9, we then have G, G* ∈ SGS($\mathfrak{H}$)\{∅} and with Theorem 2-24 it holds for all i ∈ Dom(G) that G(i) ⊆ $\mathfrak{A}$, and for all j ∈ Dom(G*) that G*(j) ⊆ $\mathfrak{A}$. Also, we have Ran(G) ⊆ Ran(G*). To see this, suppose i ∈ Dom(G). Then we have ($\mathfrak{H}$, G(i)) ∈ CS and thus we have that min(Dom(G(i))) ∈ Dom(AS($\mathfrak{H}$)) ∩ Dom($\mathfrak{A}$). Thus there is a j ∈ Dom(G*) such that min(Dom(G(i))) ∈ Dom(G*(j)). With ($\mathfrak{H}$, G*(j)) ∈ CS and Theorem 2-47 and Theorem 2-49, we then have G(i) ⊆ G*(j). Analogously, it follows that there is an i* ∈ Dom(G) such that G*(j) ⊆ G(i*). Then we have G(i) ⊆ G(i*). Since we have, with Theorem 2-43, that G(i) ≠ ∅ and thus G(i) ∩ G(i*) ≠ ∅, it then follows with Theorem 2-27 that G(i) = G(i*) and thus that G*(j) ⊆ G(i). Hence we have G*(j) = G(i). Therefore we have G(i) ∈ Ran(G*). Hence, we have Ran(G) ⊆ Ran(G*). Analogously, it follows that Ran(G*) ⊆ Ran(G). Hence, we have Ran(G) = Ran(G*). With Theorem 2-22-(iii), it then follows that Dom(G) = Dom(G*).

Now, we show by induction on i that it holds for all i ∈ Dom(G) = Dom(G*) that G(i) = G*(i) and thus that G = G*. For this, suppose that for all l < i it holds that if l ∈ Dom(G), then G(l) = G*(l). Now, suppose i ∈ Dom(G). Suppose for contradiction that G(i) ≠ G*(i). With ($\mathfrak{H}$, G(i)) ∈ CS and ($\mathfrak{H}$, G*(i)) ∈ CS and with Theorem 2-50, we then have min(Dom(G(i))) ≠ min(Dom(G*(i))). Suppose min(Dom(G(i))) <

min(Dom($G^*(i)$)). It holds with ($\mathfrak{H}$, $G(i)$) ∈ CS that min(Dom($G(i)$)) ∈ Dom(AS($\mathfrak{H}$)) ∩ Dom($\mathfrak{A}$). Thus there is a j ∈ Dom(G^*) such that min(Dom($G(i)$)) ∈ Dom($G^*(j)$). In the same way as above, it then follows that $G^*(j) = G(i)$. Since, by hypothesis, $G(i) \neq G^*(i)$, we then have $G^*(j) \neq G^*(i)$ and thus $j \neq i$. Since G, G^* ∈ SGS($\mathfrak{H}$), it then follows with Definition 2-7 and min(Dom($G^*(j)$)) = min(Dom($G(i)$)) < min(Dom($G^*(i)$)) that $j < i$. According to the I.H., it then follows that $G(j) = G^*(j) = G(i)$, whereas it holds with Theorem 2-22-(i) and $j < i$ that $G(j) \neq G(i)$. Contradiction! Using the I.H., we can show a contradiction for min(Dom($G^*(i)$)) < min(Dom($G(i)$)) in the same way. Hence we have min(Dom($G(i)$)) = min(Dom($G^*(i)$)) and thus we have $G(i) = G^*(i)$. ■

Theorem 2-52. *If the beginning of a closed segments* $\mathfrak{A}'$ *lies in a closed segment* $\mathfrak{A}$*, then* $\mathfrak{A}'$ *is a subsegment of* $\mathfrak{A}$

If $\mathfrak{A}$, $\mathfrak{A}'$ are closed segments in $\mathfrak{H}$ and min(Dom($\mathfrak{A}'$)) ∈ Dom($\mathfrak{A}$), then $\mathfrak{A}' \subseteq \mathfrak{A}$.

Proof: Let $\mathfrak{A}$, $\mathfrak{A}'$ be closed segments in $\mathfrak{H}$ and suppose min(Dom($\mathfrak{A}'$)) ∈ Dom($\mathfrak{A}$). Then we have min(Dom($\mathfrak{A}'$)) ∈ Dom(AS($\mathfrak{H}$)) ∩ Dom($\mathfrak{A}$). With Theorem 2-47, there is then a $\mathfrak{B} \subseteq \mathfrak{A}$ such that $\mathfrak{B}$ is a closed segment in $\mathfrak{H}$ and min(Dom($\mathfrak{A}'$)) = min(Dom($\mathfrak{B}$)). It then follows with Theorem 2-50 that $\mathfrak{A}' = \mathfrak{B}$ and therefore that $\mathfrak{A}' \subseteq \mathfrak{A}$. ■

Theorem 2-53. *Closed segments are uniquely determined by their end*

If $\mathfrak{A}$, $\mathfrak{A}'$ are closed segments in $\mathfrak{H}$ and max(Dom($\mathfrak{A}$)) = max(Dom($\mathfrak{A}'$)), then $\mathfrak{A} = \mathfrak{A}'$.

Proof: Let $\mathfrak{A}$, $\mathfrak{A}'$ be closed segments in $\mathfrak{H}$ and max(Dom($\mathfrak{A}$)) = max(Dom($\mathfrak{A}'$)). Suppose min(Dom($\mathfrak{A}$)) < min(Dom($\mathfrak{A}'$)). Then we have min(Dom($\mathfrak{A}$)) < min(Dom($\mathfrak{A}'$)) < max(Dom($\mathfrak{A}'$)) = max(Dom($\mathfrak{A}$)). Then we have min(Dom($\mathfrak{A}'$)) ∈ Dom(AS($\mathfrak{H}$)) ∩ Dom($\mathfrak{A}$)) and min(Dom($\mathfrak{A}$)) < min(Dom($\mathfrak{A}'$)). With Theorem 2-48 there is thus a closed segment $\mathfrak{B}$ in $\mathfrak{H}$ such that min(Dom($\mathfrak{A}'$)) = min(Dom($\mathfrak{B}$)) and min(Dom($\mathfrak{A}$)) < min(Dom($\mathfrak{B}$)) < max(Dom($\mathfrak{B}$)) < max(Dom($\mathfrak{A}$)). It then holds with Theorem 2-50 that $\mathfrak{A}' = \mathfrak{B}$. But then we have max(Dom($\mathfrak{A}'$)) = max(Dom($\mathfrak{B}$)) < max(Dom($\mathfrak{A}$)), which contradicts the hypothesis. Therefore we have min(Dom($\mathfrak{A}'$)) ≤ min(Dom($\mathfrak{A}$)). In the same way, we can show that for min(Dom($\mathfrak{A}'$)) < min(Dom($\mathfrak{A}$)) we would have max(Dom($\mathfrak{A}$)) < max(Dom($\mathfrak{A}'$)), which also contradicts the assumption. Hence we have min(Dom($\mathfrak{A}'$)) ≤

min(Dom($\mathfrak{A}$)) and min(Dom($\mathfrak{A}$)) $\leq$ min(Dom($\mathfrak{A}$')) and thus min(Dom($\mathfrak{A}$)) = min(Dom($\mathfrak{A}$')). From this, it follows with Theorem 2-50 that $\mathfrak{A}$ = $\mathfrak{A}$'. ■

Theorem 2-54. *Proper subsegment relation between closed segments*
If $\mathfrak{A}$, $\mathfrak{A}$' are closed segments in $\mathfrak{H}$, then:
min(Dom($\mathfrak{A}$')) $\in$ Dom($\mathfrak{A}$)\{min(Dom($\mathfrak{A}$))}
iff
$\mathfrak{A}$' $\subset$ $\mathfrak{A}$.

Proof: Let $\mathfrak{A}$, $\mathfrak{A}$' be closed segments in $\mathfrak{H}$. (*L-R*): Suppose min(Dom($\mathfrak{A}$')) $\in$ Dom($\mathfrak{A}$))\{min(Dom($\mathfrak{A}$))}. Hence min(Dom($\mathfrak{A}$')) $\neq$ min(Dom($\mathfrak{A}$)) and therefore $\mathfrak{A}$' $\neq$ $\mathfrak{A}$. Furthermore min(Dom($\mathfrak{A}$')) $\in$ Dom($\mathfrak{A}$) and hence by Theorem 2-52 $\mathfrak{A}$' $\subseteq$ $\mathfrak{A}$. Thus $\mathfrak{A}$' $\subset$ $\mathfrak{A}$.

(*R-L*): Now, suppose $\mathfrak{A}$' $\subset$ $\mathfrak{A}$. Then we have min(Dom($\mathfrak{A}$')) $\in$ Dom($\mathfrak{A}$). We also have min(Dom($\mathfrak{A}$')) $\neq$ min(Dom($\mathfrak{A}$)), because otherwise it would hold with Theorem 2-50 that $\mathfrak{A}$' = $\mathfrak{A}$. Hence we have min(Dom($\mathfrak{A}$')) $\in$ Dom($\mathfrak{A}$)\{min(Dom($\mathfrak{A}$))}. ■

Theorem 2-55. *Proper and improper subsegment relations between closed segments*
If $\mathfrak{A}$, $\mathfrak{A}$' are closed segments in $\mathfrak{H}$ and min(Dom($\mathfrak{A}$')) $\in$ Dom($\mathfrak{A}$), then $\mathfrak{A}$' $\subset$ $\mathfrak{A}$ or $\mathfrak{A}$' = $\mathfrak{A}$.

Proof: Let $\mathfrak{A}$, $\mathfrak{A}$' be closed segments in $\mathfrak{H}$ and suppose min(Dom($\mathfrak{A}$')) $\in$ Dom($\mathfrak{A}$). Suppose min(Dom($\mathfrak{A}$')) $\in$ Dom($\mathfrak{A}$))\{min(Dom($\mathfrak{A}$))}. With Theorem 2-54, we then have $\mathfrak{A}$' $\subset$ $\mathfrak{A}$. Suppose min(Dom($\mathfrak{A}$')) = min(Dom($\mathfrak{A}$)). With Theorem 2-50, we then have $\mathfrak{A}$' = $\mathfrak{A}$. ■

Theorem 2-56. *Inclusion relations between non-disjunct closed segments*
If $\mathfrak{A}$, $\mathfrak{A}$' are closed segments in $\mathfrak{H}$ and $\mathfrak{A}$ $\cap$ $\mathfrak{A}$' $\neq \emptyset$, then:

(i) min(Dom($\mathfrak{A}$)) < min(Dom($\mathfrak{A}$')) iff $\mathfrak{A}$' $\subset$ $\mathfrak{A}$,
(ii) min(Dom($\mathfrak{A}$)) = min(Dom($\mathfrak{A}$')) iff $\mathfrak{A}$' = $\mathfrak{A}$,
(iii) min(Dom($\mathfrak{A}$)) < min(Dom($\mathfrak{A}$')) iff max(Dom($\mathfrak{A}$')) < max(Dom($\mathfrak{A}$)),
(iv) min(Dom($\mathfrak{A}$)) = min(Dom($\mathfrak{A}$')) iff max(Dom($\mathfrak{A}$)) = max(Dom($\mathfrak{A}$')).

Proof: Let $\mathfrak{A}$ and $\mathfrak{A}$' be closed segments in $\mathfrak{H}$ and let $\mathfrak{A}$ $\cap$ $\mathfrak{A}$' $\neq \emptyset$.

Ad (i): (*L-R*): Suppose min(Dom($\mathfrak{A}$)) < min(Dom($\mathfrak{A}$')). Since $\mathfrak{A}$ and $\mathfrak{A}$' are segments and $\mathfrak{A}$ $\cap$ $\mathfrak{A}$' $\neq \emptyset$, it holds with Theorem 2-9 that min(Dom($\mathfrak{A}$)) $\in$ Dom($\mathfrak{A}$') or min(Dom($\mathfrak{A}$')) $\in$ Dom($\mathfrak{A}$). With the hypothesis, it then holds that min(Dom($\mathfrak{A}$')) $\in$

$\text{Dom}(\mathfrak{A})\backslash\{\min(\text{Dom}(\mathfrak{A}))\}$. With Theorem 2-54, we thus have $\mathfrak{A}' \subset \mathfrak{A}$. (*R-L*): Suppose $\mathfrak{A}' \subset \mathfrak{A}$. Again with Theorem 2-54, we then have $\min(\text{Dom}(\mathfrak{A}')) \in \text{Dom}(\mathfrak{A}))\backslash\{\min(\text{Dom}(\mathfrak{A}))\}$ and therefore: $\min(\text{Dom}(\mathfrak{A})) < \min(\text{Dom}(\mathfrak{A}'))$.

Ad (ii): Follows with Theorem 2-50

Ad (iii): (*L-R*): Suppose $\min(\text{Dom}(\mathfrak{A})) < \min(\text{Dom}(\mathfrak{A}'))$. Then we have with (i) that $\mathfrak{A}' \subset \mathfrak{A}$. With Theorem 2-5-(i) we then have $\max(\text{Dom}(\mathfrak{A}')) \leq \max(\text{Dom}(\mathfrak{A}))$. With $\mathfrak{A}' \subset \mathfrak{A}$ and Theorem 2-53, we then have $\max(\text{Dom}(\mathfrak{A}')) \neq \max(\text{Dom}(\mathfrak{A}))$. Hence we have $\max(\text{Dom}(\mathfrak{A}')) < \max(\text{Dom}(\mathfrak{A}))$. (*R-L*): Suppose $\max(\text{Dom}(\mathfrak{A}')) < \max(\text{Dom}(\mathfrak{A}))$. It then holds with Theorem 2-5-(i) that $\mathfrak{A} \not\subseteq \mathfrak{A}'$. With (i) and (ii) we then have that neither $\min(\text{Dom}(\mathfrak{A}')) < \min(\text{Dom}(\mathfrak{A}))$ nor $\min(\text{Dom}(\mathfrak{A}')) = \min(\text{Dom}(\mathfrak{A}))$. Therefore we have $\min(\text{Dom}(\mathfrak{A})) < \min(\text{Dom}(\mathfrak{A}'))$.

Ad (iv): Follows with (ii) and Theorem 2-53. ■

Theorem 2-57. *Closed segments are either disjunct or one is a subsegment of the other.*
If $\mathfrak{A}$ and $\mathfrak{A}'$ are closed segments in $\mathfrak{H}$, then: $\mathfrak{A} \cap \mathfrak{A}' = \emptyset$ or $\mathfrak{A} \subseteq \mathfrak{A}'$ or $\mathfrak{A}' \subseteq \mathfrak{A}$.

Proof: Let $\mathfrak{A}$ and $\mathfrak{A}'$ be closed segments in $\mathfrak{H}$. Suppose $\mathfrak{A} \cap \mathfrak{A}' \neq \emptyset$. Then we have $\min(\text{Dom}(\mathfrak{A}')) \leq \min(\text{Dom}(\mathfrak{A}))$ or $\min(\text{Dom}(\mathfrak{A})) \leq \min(\text{Dom}(\mathfrak{A}'))$. With Theorem 2-56-(i) and -(ii), it then follows that $\mathfrak{A} \subseteq \mathfrak{A}'$ or $\mathfrak{A}' \subseteq \mathfrak{A}$. ■

Theorem 2-58. *A minimal closed segment $\mathfrak{A}'$ is either disjunct from a closed segment $\mathfrak{A}$ or it is a subsegment of $\mathfrak{A}$*
If $\mathfrak{A}$ is a closed segment in $\mathfrak{H}$ and $\mathfrak{A}'$ is a minimal closed segment in $\mathfrak{H}$, then: $\mathfrak{A} \cap \mathfrak{A}' = \emptyset$ or $\mathfrak{A}' \subseteq \mathfrak{A}$.

Proof: Let $\mathfrak{A}$ be a closed segment in $\mathfrak{H}$ and suppose $\mathfrak{A}'$ is a minimal closed segment in $\mathfrak{H}$. Then $\mathfrak{A}'$ is also a closed segment in $\mathfrak{H}$. Suppose $\mathfrak{A} \cap \mathfrak{A}' \neq \emptyset$. Then we have $\min(\text{Dom}(\mathfrak{A})) \leq \min(\text{Dom}(\mathfrak{A}'))$. For if $\min(\text{Dom}(\mathfrak{A}')) < \min(\text{Dom}(\mathfrak{A}))$, we would have with Theorem 2-56-(i) that $\mathfrak{A} \subset \mathfrak{A}'$. Then we would have with Theorem 2-54 $\min(\text{Dom}(\mathfrak{A})) \in \text{Dom}(\mathfrak{A}'))\backslash\{\min(\text{Dom}(\mathfrak{A}'))\}$. Thus we would have $\min(\text{Dom}(\mathfrak{A})) \neq \min(\text{Dom}(\mathfrak{A}'))$. Since $\mathfrak{A}$ is a closed segment, we would also have that $\min(\text{Dom}(\mathfrak{A})) \in \text{Dom}(\mathfrak{A}') \cap \text{Dom}(AS(\mathfrak{H}))$ and thus, according to Definition 2-17, Definition 2-14, Definition 2-15 and Definition 2-16, that $\min(\text{Dom}(\mathfrak{A})) = \min(\text{Dom}(\mathfrak{A}'))$. Contradiction! Therefore $\min(\text{Dom}(\mathfrak{A})) \leq \min(\text{Dom}(\mathfrak{A}'))$. With $\mathfrak{A} \cap \mathfrak{A}' \neq \emptyset$ and Theorem 2-56-(i) and -(ii), it then follows that $\mathfrak{A}' \subseteq \mathfrak{A}$. ■

The next theorem tells us that for every segment $\mathfrak{A}$ that contains at least one assumption-sentence and in which for every assumption-sentence there is a closed subsegment of $\mathfrak{A}$ that contains this assumption-sentence there is an AS-comprising segment sequence G for $\mathfrak{A}$ that enumerates the greatest closed disjunct subsegments of $\mathfrak{A}$ in such a way that all closed subsegments of $\mathfrak{A}$ are covered

Theorem 2-59 will play an important role in the proofs of Theorem 2-67, Theorem 2-68, Theorem 2-69, which are crucial for arriving at a proof of the correctness and completeness of the Speech Act Calculus: With these theorems we can later show that assumptions can be discharged by CdI, NI and PE and only by CdI, NI and PE. Theorem 2-59 itself is essential for showing that CdI, NI and PE can discharge assumptions and thus for the proof of completeness.

Theorem 2-59. *GEN-material-provision theorem*
If $\mathfrak{A}$ is a segment in $\mathfrak{H}$, $\mathrm{AS}(\mathfrak{H}) \cap \mathfrak{A} \neq \emptyset$, and for every $i \in \mathrm{Dom}(\mathfrak{A}) \cap \mathrm{Dom}(\mathrm{AS}(\mathfrak{H}))$ there is a closed segment $\mathfrak{B}$ in $\mathfrak{H}$ such that $(i, \mathfrak{H}_i) \in \mathfrak{B}$ and $\mathfrak{B} \subseteq \mathfrak{A}$, then:
There is a $G \in \mathrm{ASCS}(\mathfrak{H})$ such that

(i) G is an AS-comprising segment sequence for $\mathfrak{A}$ in $\mathfrak{H}$,

(ii) $\bigcup\mathrm{Ran}(G) = \bigcup\{\mathfrak{B} \mid \mathfrak{B} \subseteq \mathfrak{A}$ is a closed segment in $\mathfrak{H}\}$, and

(iii) $\{\mathfrak{H}\} \times \mathrm{Ran}(G) \subseteq \{\mathfrak{H}\} \times \{\mathfrak{B} \mid \mathfrak{B} \subseteq \mathfrak{A}$ is a closed segment in $\mathfrak{H}\} \subseteq \mathrm{CS}$.

Proof: Suppose $\mathfrak{A}$ is a segment in $\mathfrak{H}$, $\mathrm{AS}(\mathfrak{H}) \cap \mathfrak{A} \neq \emptyset$, and for every $i \in \mathrm{Dom}(\mathfrak{A}) \cap \mathrm{Dom}(\mathrm{AS}(\mathfrak{H}))$ there is a closed segment $\mathfrak{B}$ in $\mathfrak{H}$ such that $(i, \mathfrak{H}_i) \in \mathfrak{B}$ and $\mathfrak{B} \subseteq \mathfrak{A}$. It follows with Definition 2-1 that $\mathfrak{H} \in \mathrm{SEQ}$.

Suppose $X = \{\mathfrak{B} \mid \mathfrak{B} \subseteq \mathfrak{A}$ and $(\mathfrak{H}, \mathfrak{B}) \in \mathrm{CS}$ and for all $\mathfrak{C} \subseteq \mathfrak{A}$: If $(\mathfrak{H}, \mathfrak{C}) \in \mathrm{CS}$ and $\mathfrak{B} \subseteq \mathfrak{C}$, then $\mathfrak{B} = \mathfrak{C}\}$. Then it holds that $X \subseteq \mathrm{SG}(\mathfrak{H})$. To apply Theorem 2-17 we show that for all $\mathfrak{A}^*, \mathfrak{A}' \in X$ with $\mathfrak{A}^* \neq \mathfrak{A}'$ it holds, that $\mathfrak{A}^* \cap \mathfrak{A}' = \emptyset$. To that end suppose $\mathfrak{A}^*, \mathfrak{A}' \in X$ and $\mathfrak{A}^* \neq \mathfrak{A}'$. From $\mathfrak{A}^*, \mathfrak{A}' \in X$ it follows that $(\mathfrak{H}, \mathfrak{A}^*), (\mathfrak{H}, \mathfrak{A}') \in \mathrm{CS}$. Theorem 2-57 yields $\mathfrak{A}^* \cap \mathfrak{A}' = \emptyset$ or $\mathfrak{A}^* \subseteq \mathfrak{A}'$ or $\mathfrak{A}' \subseteq \mathfrak{A}^*$. The second and the third alternative lead to a contradiction: Assume $\mathfrak{A}^* \subseteq \mathfrak{A}'$. Since $\mathfrak{A}^* \in X$ we have that for all $\mathfrak{C} \subseteq \mathfrak{A}$: If $(\mathfrak{H}, \mathfrak{C}) \in \mathrm{CS}$ and $\mathfrak{A}^* \subseteq \mathfrak{C}$, then $\mathfrak{A}^* = \mathfrak{C}$. Since $\mathfrak{A}' \in X$ we have $\mathfrak{A}' \subseteq \mathfrak{A}$ and $(\mathfrak{H}, \mathfrak{A}') \in \mathrm{CS}$. From the last assumption we can derive $\mathfrak{A}^* = \mathfrak{A}'$, which contradicts an earlier assumption. From the assumption of $\mathfrak{A}' \subseteq \mathfrak{A}^*$ we can analogously derive a contradiction. Hence $\mathfrak{A}^* \cap \mathfrak{A}' = \emptyset$

must be the case. So we have for all $\mathfrak{A}^*$, $\mathfrak{A}' \in X$ with $\mathfrak{A}^* \neq \mathfrak{A}'$, that $\mathfrak{A}^* \cap \mathfrak{A}' = \emptyset$. With Theorem 2-17 it holds that there is a $G \in \mathrm{SGS}(\mathfrak{H})$ such that $\mathrm{Ran}(G) = X$.

Now we can show that G satisfies conditions (i) to (iii). From (i) it follows that $G \in \mathrm{ASCS}(\mathfrak{H})$. *Ad (i)*: We have to show that

a) $G \neq \emptyset$,

b) $\min(\mathrm{Dom}(\mathfrak{A})) \leq \min(\mathrm{Dom}(G(0)))$,

c) $\max(\mathrm{Dom}(G(\max(\mathrm{Dom}(G))))) \leq \max(\mathrm{Dom}(\mathfrak{A}))$, and

d) for all $l \in \mathrm{Dom}(\mathrm{AS}(\mathfrak{H})) \cap \mathrm{Dom}(\mathfrak{A})$ it holds that there is an $i \in \mathrm{Dom}(G)$ such that $l \in \mathrm{Dom}(G(i))$.

By Definition 2-9 it then follows that G is an AS-comprising segment sequence for $\mathfrak{A}$ in $\mathfrak{H}$. Since $\mathrm{AS}(\mathfrak{H}) \cap \mathfrak{A} \neq \emptyset$ and thus $\mathrm{Dom}(\mathrm{AS}(\mathfrak{H})) \cap \mathrm{Dom}(\mathfrak{A}) \neq \emptyset$, we get a) from d). Furthermore since for every $i \in \mathrm{Dom}(\mathfrak{A}) \cap \mathrm{Dom}(\mathrm{AS}(\mathfrak{H}))$ there is a closed segment $\mathfrak{B}$ in $\mathfrak{H}$ such that $(i, \mathfrak{H}_i) \in \mathfrak{B}$ and $\mathfrak{B} \subseteq \mathfrak{A}$, both d) and a) follow from

e) for all $\mathfrak{B} \subseteq \mathfrak{A}$ with $(\mathfrak{H}, \mathfrak{B}) \in \mathrm{CS}$: There is an $i \in \mathrm{Dom}(G)$, such that $\mathfrak{B} \subseteq G(i)$.

Ad e): Suppose $\mathfrak{B} \subseteq \mathfrak{A}$ with $(\mathfrak{H}, \mathfrak{B}) \in \mathrm{CS}$, such that there is no $i \in \mathrm{Dom}(G)$ with $\mathfrak{B} \subseteq G(i)$. Suppose $k = \min(\{j \mid$ There is a $\mathfrak{C} \subseteq \mathfrak{A}$ with $(\mathfrak{H}, \mathfrak{C}) \in \mathrm{CS}$, such that there is no $i \in \mathrm{Dom}(G)$ with $\mathfrak{C} \subseteq G(i)$, and $j = \min(\mathrm{Dom}(\mathfrak{C}))\})$. Then there is a $\mathfrak{C} \subseteq \mathfrak{A}$ with $(\mathfrak{H}, \mathfrak{C}) \in \mathrm{CS}$, such that there is no $i \in \mathrm{Dom}(G)$ with $\mathfrak{C} \subseteq G(i)$, and $k = \min(\mathrm{Dom}(\mathfrak{C}))$. Now suppose $\mathfrak{C}' \subseteq \mathfrak{A}$ and $(\mathfrak{H}, \mathfrak{C}') \in \mathrm{CS}$ and $\mathfrak{C} \subseteq \mathfrak{C}'$. Then we have $\min(\mathrm{Dom}(\mathfrak{C}')) \leq k$. From that it follows that there is no $i \in \mathrm{Dom}(G)$, such that $\mathfrak{C}' \subseteq G(i)$, else it would also hold that $\mathfrak{C} \subseteq G(i)$ for the same i. Since k is minimal, we get $\min(\mathrm{Dom}(\mathfrak{C}')) = k$. With Theorem 2-50 we can derive that $\mathfrak{C} = \mathfrak{C}'$. Hence for all $\mathfrak{C}' \subseteq \mathfrak{A}$ with $(\mathfrak{H}, \mathfrak{C}') \in \mathrm{CS}$ and $\mathfrak{C} \subseteq \mathfrak{C}'$ we get $\mathfrak{C} = \mathfrak{C}'$. Therefore $\mathfrak{C} \in X$ and by that there is an $i \in \mathrm{Dom}(G)$, such that $\mathfrak{C} = G(i)$. Contradiction! Thus for all $\mathfrak{B} \subseteq \mathfrak{A}$ with $(\mathfrak{H}, \mathfrak{B}) \in \mathrm{CS}$ there is an $i \in \mathrm{Dom}(G)$, such that $\mathfrak{B} \subseteq G(i)$. *Ad b)*: For all $\mathfrak{B} \in \mathrm{Ran}(G) = X$ it holds that $\mathfrak{B} \subseteq \mathfrak{A}$. Because of $G \neq \emptyset$ we get $G(0) \in \mathrm{Ran}(G) = X$ and thereby $G(0) \subseteq \mathfrak{A}$. Hence $\min(\mathrm{Dom}(\mathfrak{A})) \leq \min(\mathrm{Dom}(G(0)))$. *Ad c)*: With $G \neq \emptyset$ we get $\max(\mathrm{Dom}(G)) \in \mathrm{Dom}(G)$ and thereby $G(\max(\mathrm{Dom}(G))) \in \mathrm{Ran}(G) = X$. Hence $\max(\mathrm{Dom}(G(\max(\mathrm{Dom}(G))))) \leq \max(\mathrm{Dom}(\mathfrak{A}))$.

Ad (ii): Suppose $(i, \mathfrak{H}_i) \in \bigcup\mathrm{Ran}(G)$. Therefore $(i, \mathfrak{H}_i) \in \bigcup X$. Hence we have a $\mathfrak{B} \in X$ with $(i, \mathfrak{H}_i) \in \mathfrak{B}$. From that we can infer $\mathfrak{B} \subseteq \mathfrak{A}$ and $(\mathfrak{H}, \mathfrak{B}) \in \mathrm{CS}$. Thus $\mathfrak{B} \in \{\mathfrak{B} \mid \mathfrak{B} \subseteq \mathfrak{A}$ is a closed segment in $\mathfrak{H}\}$ and $(i, \mathfrak{H}_i) \in \bigcup\{\mathfrak{B} \mid \mathfrak{B} \subseteq \mathfrak{A}$ is a closed segment in $\mathfrak{H}\}$. From e) we get vice versa $\bigcup\{\mathfrak{B} \mid \mathfrak{B} \subseteq \mathfrak{A}$ is a closed segment in $\mathfrak{H}\} \subseteq \bigcup\mathrm{Ran}(G)$.

Ad (iii): (iii) follows from the definition of X and $\mathrm{Ran}(G) = X$. ■

Theorem 2-60. *If all members of an AS-comprising segment sequence for* $\mathfrak{A}$ *are closed segments, then every closed subsegment of* $\mathfrak{A}$ *is a subsegment of a sequence member*

If $\mathfrak{H} \in \mathrm{SEQ}$, $\mathfrak{A} \in \mathrm{SG}(\mathfrak{H})$ and $G \in \mathrm{ASCS}(\mathfrak{H})$ is an AS-comprising segment sequence for $\mathfrak{A}$ in $\mathfrak{H}$ and $\{\mathfrak{H}\} \times \mathrm{Ran}(G) \subseteq \mathrm{CS}$, then for all $\mathfrak{C}$: If $\mathfrak{C} \subseteq \mathfrak{A}$ is a closed segment in $\mathfrak{H}$, then there is an $i \in \mathrm{Dom}(G)$ such that $\mathfrak{C} \subseteq G(i)$.

Proof: Suppose $\mathfrak{H} \in \mathrm{SEQ}$, $\mathfrak{A} \in \mathrm{SG}(\mathfrak{H})$ and $G \in \mathrm{ASCS}(\mathfrak{H})$ is an AS-comprising segment sequence for $\mathfrak{A}$ in $\mathfrak{H}$ and $\{\mathfrak{H}\} \times \mathrm{Ran}(G) \subseteq \mathrm{CS}$. Now, suppose $\mathfrak{C} \subseteq \mathfrak{A}$ is a closed segment in $\mathfrak{H}$. With Definition 2-11 to Definition 2-13 and Theorem 2-42, we then have $\min(\mathrm{Dom}(\mathfrak{C})) \in \mathrm{Dom}(\mathrm{AS}(\mathfrak{H}) \cap \mathfrak{A})$. According to Definition 2-9-(iii-c), there is thus an $i \in \mathrm{Dom}(G)$ such that $\min(\mathrm{Dom}(\mathfrak{C})) \in \mathrm{Dom}(G(i))$. By hypothesis, we have $(\mathfrak{H}, G(i)) \in \mathrm{CS}$. It then follows with Theorem 2-52 that $\mathfrak{C} \subseteq G(i)$. ■

Up to now, we have primarily proved theorems that hold for all closed segments. Later, we will also and mostly be interested in those properties of closed segments that depend on whether they are the result of the application of conditional introduction (CdI-closed) or negation introduction (NI-closed) or particular-quantifier elimination (PE-closed). Accordingly, we will now define different predicates for these kinds of closed segments. We will then have that every closed segment belongs to one of these kinds (Theorem 2-61).

Definition 2-23. *CdI-closed segment*

$\mathfrak{A}$ is a CdI-closed segment in $\mathfrak{H}$

iff

$\mathfrak{A}$ is a closed segment and a CdI-like segment in $\mathfrak{H}$.

Definition 2-24. *NI-closed segment*

𝔄 <u>is an NI-closed segment in</u> 𝔋
iff
𝔄 is a closed segment and an NI-like segment in 𝔋.

Definition 2-25. *PE-closed segment*

𝔄 <u>is a PE-closed segment in</u> 𝔋
iff
𝔄 is a closed segment and an RA-like segment in 𝔋.

Theorem 2-61. *CdI-, NI- and PE-closed segments and only these are closed segments*

𝔄 is a closed segment in 𝔋
iff
𝔄 is a CdI- or NI- or PE-closed segment in 𝔋.

Proof: Follows from Definition 2-22, Definition 2-23, Definition 2-24, Definition 2-25 and Theorem 2-42. ■

Theorem 2-62. *Monotony of '(F-)closed segment'-predicates*

If 𝔋, 𝔋' ∈ SEQ and 𝔋 ⊆ 𝔋', then:

(i) If 𝔄 is a CdI-closed segment in 𝔋, then 𝔄 is a CdI-closed segment in 𝔋',
(ii) If 𝔄 is an NI-closed segment in 𝔋, then 𝔄 is an NI-closed segment in 𝔋',
(iii) If 𝔄 is a PE-closed segment in 𝔋, then 𝔄 is a PE-closed segment in 𝔋',
(iv) If 𝔄 is a minimal CdI-closed segment in 𝔋, then 𝔄 is a minimal CdI-closed segment in 𝔋',
(v) If 𝔄 is a minimal NI-closed segment in 𝔋, then 𝔄 a minimal NI-closed segment in 𝔋',
(vi) If 𝔄 is a minimal PE-closed segment in 𝔋, then 𝔄 is a minimal PE-closed segment in 𝔋',
(vii) If 𝔄 is a minimal closed segment in 𝔋, then 𝔄 is a minimal closed segment in 𝔋', and
(viii) If 𝔄 is a closed segment in 𝔋, then 𝔄 is a closed segment in 𝔋'.

Proof: See Remark 2-1. ■

Theorem 2-63. *Closed segments in the first sequence of a concatenation remain closed*
If $\mathfrak{H}'$, $\mathfrak{H} \in$ SEQ, then:

(i) If $\mathfrak{A}$ is a CdI-closed segment in $\mathfrak{H}$, then $\mathfrak{A}$ is a CdI-closed segment in $\mathfrak{H}^\frown\mathfrak{H}'$,
(ii) If $\mathfrak{A}$ is an NI-closed segment in $\mathfrak{H}$, then $\mathfrak{A}$ is an NI-closed segment in $\mathfrak{H}^\frown\mathfrak{H}'$,
(iii) If $\mathfrak{A}$ is a PE-closed segment in $\mathfrak{H}$, then $\mathfrak{A}$ is a PE-closed segment in $\mathfrak{H}^\frown\mathfrak{H}'$, and
(iv) If $\mathfrak{A}$ is a closed segment in $\mathfrak{H}$, then $\mathfrak{A}$ is a closed segment in $\mathfrak{H}^\frown\mathfrak{H}'$.

Proof: Follows with $\mathfrak{H} \subseteq \mathfrak{H}^\frown\mathfrak{H}'$ and Theorem 2-62-(i), -(ii), -(iii) and -(viii). ■

Theorem 2-64. *(F-)closed segments in restrictions*
If $\mathfrak{H}$ is a sequence, then:

(i) $\mathfrak{A}$ is a CdI-closed segment in $\mathfrak{H}$ iff $\mathfrak{A}$ is a CdI-closed segment in $\mathfrak{H}\upharpoonright$max(Dom($\mathfrak{A}$))+1,
(ii) $\mathfrak{A}$ is an NI-closed segment in $\mathfrak{H}$ iff $\mathfrak{A}$ is an NI-closed segment in $\mathfrak{H}\upharpoonright$max(Dom($\mathfrak{A}$))+1,
(iii) $\mathfrak{A}$ is a PE-closed segment in $\mathfrak{H}$ iff $\mathfrak{A}$ is a PE-closed segment in $\mathfrak{H}\upharpoonright$max(Dom($\mathfrak{A}$))+1,
(iv) $\mathfrak{A}$ is a minimal CdI-closed segment in $\mathfrak{H}$ iff $\mathfrak{A}$ is a minimal CdI-closed segment in $\mathfrak{H}\upharpoonright$max(Dom($\mathfrak{A}$))+1,
(v) $\mathfrak{A}$ is a minimal NI-closed segment in $\mathfrak{H}$ iff $\mathfrak{A}$ is a minimal NI-closed segment in $\mathfrak{H}\upharpoonright$max(Dom($\mathfrak{A}$))+1,
(vi) $\mathfrak{A}$ is a minimal PE-closed segment in $\mathfrak{H}$ iff $\mathfrak{A}$ is a minimal PE-closed segment in $\mathfrak{H}\upharpoonright$max(Dom($\mathfrak{A}$))+1,
(vii) $\mathfrak{A}$ is a minimal closed segment in $\mathfrak{H}$ iff $\mathfrak{A}$ is a minimal closed segment in $\mathfrak{H}\upharpoonright$max(Dom($\mathfrak{A}$))+1, and
(viii) $\mathfrak{A}$ is a closed segment in $\mathfrak{H}$ iff $\mathfrak{A}$ is a closed segment in $\mathfrak{H}\upharpoonright$max(Dom($\mathfrak{A}$))+1.

Proof: See Remark 2-2. ■

Theorem 2-65. *Preparatory theorem for Theorem 2-67, Theorem 2-68 and Theorem 2-69*
If $\mathfrak{A}$ is a segment in $\mathfrak{H}$ and if it holds for all closed segments $\mathfrak{B}$ in $\mathfrak{H}\upharpoonright$max(Dom($\mathfrak{A}$)) that min(Dom($\mathfrak{A}$)) < min(Dom($\mathfrak{B}$)) or max(Dom($\mathfrak{B}$)) ≤ min(Dom($\mathfrak{A}$)), then for all $i \in$ Dom($\mathfrak{A}$):

(i) $\mathfrak{A}\upharpoonright i$ is not a closed segment in $\mathfrak{H}$, and
(ii) There is no $G \in$ ASCS($\mathfrak{H}$) such that $\{\mathfrak{H}\} \times$ Ran(G) $\subseteq$ CS and $\mathfrak{A}\upharpoonright i \in$ PGEN($\langle\mathfrak{H}, G\rangle$).

Proof: Suppose $\mathfrak{A}$ is a segment in $\mathfrak{H}$ and suppose it holds for all closed segments $\mathfrak{B}$ in $\mathfrak{H}\upharpoonright$max(Dom($\mathfrak{A}$)) that min(Dom($\mathfrak{A}$)) < min(Dom($\mathfrak{B}$)) or max(Dom($\mathfrak{B}$)) ≤ min(Dom($\mathfrak{A}$)). Next, suppose $i \in$ Dom($\mathfrak{A}$). First, we have $\mathfrak{H} \in$ SEQ. *Ad (i)*: Suppose for contradiction that $\mathfrak{A}\upharpoonright i$ is a closed segment in $\mathfrak{H}$. With Theorem 2-64-(viii), we would then have that $\mathfrak{A}\upharpoonright i$ is a closed segment in $\mathfrak{H}\upharpoonright i$. Furthermore, we have $i \leq$ max(Dom($\mathfrak{A}$)) and hence $\mathfrak{H}\upharpoonright i \subseteq \mathfrak{H}\upharpoonright$max(Dom($\mathfrak{A}$)) and thus it holds with Theorem 2-62-(viii) that $\mathfrak{A}\upharpoonright i$ is a closed segment in $\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A}))$. With Theorem 2-7, we have that $\min(\mathrm{Dom}(\mathfrak{A}\upharpoonright i)) = \min(\mathrm{Dom}(\mathfrak{A}))$ and hence, with Theorem 2-31, that neither $\min(\mathrm{Dom}(\mathfrak{A})) < \min(\mathrm{Dom}(\mathfrak{A}\upharpoonright i))$ nor $\max(\mathrm{Dom}(\mathfrak{A}\upharpoonright i)) \leq \min(\mathrm{Dom}(\mathfrak{A}))$, which contradicts the hypothesis.

Ad (ii): Suppose for contradiction that there is a $G \in \mathrm{ASCS}(\mathfrak{H})$ such that $\{\mathfrak{H}\} \times \mathrm{Ran}(G) \subseteq \mathrm{CS}$ and $\mathfrak{A}\upharpoonright i \in \mathrm{PGEN}(\langle\mathfrak{H}, G\rangle)$. Now, suppose $j = \min(\{i \mid i \in \mathrm{Dom}(\mathfrak{A})$ and there is $G \in \mathrm{ASCS}(\mathfrak{H})$ such that $\{\mathfrak{H}\} \times \mathrm{Ran}(G) \subseteq \mathrm{CS}$ and $\mathfrak{A}\upharpoonright i \in \mathrm{PGEN}(\langle\mathfrak{H}, G\rangle)\})$. Then there is a G^* $\in \mathrm{ASCS}(\mathfrak{H})$ such that $\{\mathfrak{H}\} \times \mathrm{Ran}(G^*) \subseteq \mathrm{CS}$ and $\mathfrak{A}\upharpoonright j \in \mathrm{PGEN}(\langle\mathfrak{H}, G^*\rangle)$. Now, suppose for contradiction that there are a $k \in \mathrm{Dom}(\mathfrak{A}\upharpoonright j)$ and an $l \in \mathrm{Dom}(G^*)$ such that $\mathfrak{A}\upharpoonright k \in \mathrm{PGEN}(\langle\mathfrak{H}, G^*\upharpoonright(l+1)\rangle)$. According to Theorem 2-25, $G^*\upharpoonright(l+1)$ is then an AS-comprising segment sequence for $\mathfrak{A}\upharpoonright\max(\mathrm{Dom}(G^*(l)))+1$. According to Definition 2-10, we then have that $G^*\upharpoonright(l+1) \in \mathrm{ASCS}(\mathfrak{H})$ and, by hypothesis, that $\mathfrak{A}\upharpoonright k \in \mathrm{PGEN}(\langle\mathfrak{H}, G^*\upharpoonright(l+1)\rangle)$. On the other hand, we also have $k < j$. Thus, we have a contradiction to the minimality of j. Therefore there are no $k \in \mathrm{Dom}(\mathfrak{A}\upharpoonright j)$ and $l \in \mathrm{Dom}(G^*)$ such that $\mathfrak{A}\upharpoonright k \in \mathrm{PGEN}(\langle\mathfrak{H}, G^*\upharpoonright(l+1)\rangle)$. According to Definition 2-19, we then have that $\mathfrak{A}\upharpoonright j \in \mathrm{GEN}(\langle\mathfrak{H}, G^*\rangle)$ and thus, with $\{\mathfrak{H}\} \times \mathrm{Ran}(G^*) \subseteq \mathrm{CS}$ and Theorem 2-41, that $(\mathfrak{H}, \mathfrak{A}\upharpoonright j) \in \mathrm{CS}$ and therefore that $\mathfrak{A}\upharpoonright j$ is a closed segment in $\mathfrak{H}$, which contradicts (i). ■

We close ch. 2.2 with four theorems that provide the basis for the proof of the correctness and the completeness of the Speech Act Calculus. With these theorems we can later show that CdI, NI and PE and only CdI, NI and PE can generate CdI-, NI- and PE-closed segments and thus any closed segments.

Theorem 2-66. *Every closed segment is a minimal closed segment or a CdI- or NI- or PE-closed segment whose assumption-sentences lie at the beginning or in a proper closed sub-segment*

If $\mathfrak{A}$ is a closed segment in $\mathfrak{H}$, then:

(i) $\mathfrak{A}$ is a minimal closed segment in $\mathfrak{H}$

or

(ii) $\mathfrak{A}$ is a CdI- or NI- or PE-closed segment in $\mathfrak{H}$, where for all $i \in \text{Dom}(\text{AS}(\mathfrak{H})) \cap \text{Dom}(\mathfrak{A})$ with $\min(\text{Dom}(\mathfrak{A})) < i$ it holds that there is a $\mathfrak{B}$ such that

a) $(i, \mathfrak{H}_i) \in \mathfrak{B}$,

b) $\mathfrak{B}$ is a closed segment in $\mathfrak{H}$,

c) $i = \min(\text{Dom}(\mathfrak{B}))$, and

d) $\min(\text{Dom}(\mathfrak{A})) < \min(\text{Dom}(\mathfrak{B})) < \max(\text{Dom}(\mathfrak{B})) < \max(\text{Dom}(\mathfrak{A}))$.

Proof: Follows from Definition 2-22, Definition 2-23, Definition 2-24, Definition 2-25 and Theorem 2-48. ■

Theorem 2-67. *Lemma for Theorem 2-91*

$\mathfrak{A}$ is a segment in $\mathfrak{H}$ and there are $\Delta, \Gamma \in$ CFORM such that

(i) $\mathfrak{H}_{\min(\text{Dom}(\mathfrak{A}))} = \ulcorner\text{Suppose } \Delta\urcorner$,

(ii) For all closed segments $\mathfrak{B}$ in $\mathfrak{H}\restriction\max(\text{Dom}(\mathfrak{A}))$: $\min(\text{Dom}(\mathfrak{A})) < \min(\text{Dom}(\mathfrak{B}))$ or $\max(\text{Dom}(\mathfrak{B})) \leq \min(\text{Dom}(\mathfrak{A}))$,

(iii) $\text{P}(\mathfrak{H}_{\max(\text{Dom}(\mathfrak{A}))-1}) = \Gamma$,

(iv) For every $r \in \text{Dom}(\text{AS}(\mathfrak{H})) \cap \text{Dom}(\mathfrak{A})$ with $\min(\text{Dom}(\mathfrak{A})) < r \leq \max(\text{Dom}(\mathfrak{A}))-1$ there is a closed segment $\mathfrak{B}$ in $\mathfrak{H}\restriction\max(\text{Dom}(\mathfrak{A}))$ such that $(r, \mathfrak{H}_r) \in \mathfrak{B}$, and

(v) $\mathfrak{H}_{\max(\text{Dom}(\mathfrak{A}))} = \ulcorner\text{Therefore } \Delta \rightarrow \Gamma\urcorner$,

iff

$\mathfrak{A}$ is a CdI-closed segment in $\mathfrak{H}$.

Proof: (*L-R*): Let $\mathfrak{H}$ and $\mathfrak{A}$ satsify the requirements and let Δ and Γ be as demanded. First, we have $\mathfrak{H} \in$ SEQ. With Definition 2-11, we have that $\mathfrak{A}$ is a CdI-like segment in $\mathfrak{H}$. Also, from clause (ii) of our hypothesis and Theorem 2-65-(i), it follows for all $k \in \text{Dom}(\mathfrak{A})$ that $\mathfrak{A}\restriction k$ is not a closed segment in $\mathfrak{H}$.

We have that $\text{AS}(\mathfrak{H}) \cap \mathfrak{A} = \{(\min(\text{Dom}(\mathfrak{A})), \mathfrak{H}_{\min(\text{Dom}(\mathfrak{A}))})\}$ or that there is an $i \in \text{Dom}(\text{AS}(\mathfrak{H})) \cap \text{Dom}(\mathfrak{A})$ with $\min(\text{Dom}(\mathfrak{A})) < i \leq \max(\text{Dom}(\mathfrak{A}))-1$.

Now, suppose $\text{AS}(\mathfrak{H}) \cap \mathfrak{A} = \{(\min(\text{Dom}(\mathfrak{A})), \mathfrak{H}_{\min(\text{Dom}(\mathfrak{A}))})\}$. Because we have for all $k \in \text{Dom}(\mathfrak{A})$ that $\mathfrak{A}\restriction k$ is not a closed segment in $\mathfrak{H}$, we have, with Theorem 2-32, that $\mathfrak{A}$ is a minimal closed and thus a closed segment in $\mathfrak{H}$. Since $\mathfrak{A}$ is a CdI-like segment in $\mathfrak{H}$, $\mathfrak{A}$ is thus a CdI-closed segment in $\mathfrak{H}$.

Now, suppose there is an $i \in$ Dom(AS($\mathfrak{H}$)) $\cap$ Dom($\mathfrak{A}$) with min(Dom($\mathfrak{A}$)) $< i \leq$ max(Dom($\mathfrak{A}$))-1. Now, let $\mathfrak{C} = \{(l, \mathfrak{H}_l) \mid$ min(Dom($\mathfrak{A}$))+1 $\leq l \leq$ max(Dom($\mathfrak{A}$))-1$\}$. Then $\mathfrak{C}$ is a segment in $\mathfrak{H}$ and $i \in$ Dom(AS($\mathfrak{H}$)) $\cap$ Dom($\mathfrak{C}$). Also, for every $r \in$ Dom(AS($\mathfrak{H}$)) $\cap$ Dom($\mathfrak{C}$) there is a closed segment $\mathfrak{B}$ in $\mathfrak{H}$ such that $(r, \mathfrak{H}_r) \in \mathfrak{B}$ and $\mathfrak{B} \subseteq \mathfrak{C}$. To see this, suppose $r \in$ Dom(AS($\mathfrak{H}$)) $\cap$ Dom($\mathfrak{C}$). Then we have min(Dom($\mathfrak{A}$)) $< r \leq$ max(Dom($\mathfrak{A}$))-1. According to clause (iv) of our hypothesis, there is thus a closed segment $\mathfrak{B}$ in $\mathfrak{H}\restriction$max(Dom($\mathfrak{A}$)) such that $(r, \mathfrak{H}_r) \in \mathfrak{B}$. Then we have min(Dom($\mathfrak{C}$)) $\leq$ min(Dom($\mathfrak{B}$)), because otherwise we would have min(Dom($\mathfrak{B}$)) $\leq$ min(Dom($\mathfrak{A}$)) $< r \leq$ max(Dom($\mathfrak{B}$)), which contradicts clause (ii). From $\mathfrak{B}$ being a segment in $\mathfrak{H}\restriction$max(Dom($\mathfrak{A}$)), we then have max(Dom($\mathfrak{B}$)) $\leq$ max(Dom($\mathfrak{A}$))-1 = max(Dom($\mathfrak{C}$)). With Theorem 2-5, we hence have $\mathfrak{B} \subseteq \mathfrak{C}$.

Thus $\mathfrak{C}$ satisfies the requirements of Theorem 2-59. Therefore there is a $G \in$ ASCS($\mathfrak{H}$) such that G is an AS-comprising segment sequence for $\mathfrak{C}$ in $\mathfrak{H}$ and $\{\mathfrak{H}\} \times$ Ran(G) $\subseteq$ CS. According to the definition of $\mathfrak{C}$, we have $\mathfrak{C} \in$ SG($\mathfrak{H}$) and min(Dom($\mathfrak{A}$))+1 = min(Dom($\mathfrak{C}$)) and max(Dom($\mathfrak{A}$)) = max(Dom($\mathfrak{C}$))+1 and AS($\mathfrak{H}$) $\cap\ \mathfrak{C} \neq \emptyset$. We also have that $\mathfrak{A}$ is a CdI-like segment in $\mathfrak{H}$. It thus holds with Theorem 2-28 that $\mathfrak{A}$ is not an NI-like segment in $\mathfrak{H}$. Furthermore, we have that it holds for all $i \in$ Dom($\mathfrak{A}$) that $\mathfrak{A}\restriction i$ is not a closed segment in $\mathfrak{H}$. Thus we also have for all $i \in$ Dom($\mathfrak{A}$) that $\mathfrak{A}\restriction i$ is not a minimal closed segment in $\mathfrak{H}$.

According to Definition 2-18, we thus have $\mathfrak{A} \in$ PGEN($\langle\mathfrak{H}, G\rangle$). Now, suppose for contradiction that there are $k \in$ Dom($\mathfrak{A}$) and $l \in$ Dom(G) such that $\mathfrak{A}\restriction k \in$ PGEN($\langle\mathfrak{H}, G\restriction(l+1)\rangle$). According to Theorem 2-25, $G\restriction(l+1)$ is an AS-comprising segment sequence for $\mathfrak{A}\restriction$max(Dom($G(l)$))+1, and thus, with Definition 2-10, we have $G\restriction(l+1) \in$ ASCS($\mathfrak{H}$). By hypothesis, we have $\mathfrak{A}\restriction k \in$ PGEN($\langle\mathfrak{H}, G\restriction(l+1)\rangle$) and we have $\mathfrak{H} \in$ SEQ and $\{\mathfrak{H}\} \times$ Ran($G\restriction(l+1)$) $\subseteq \{\mathfrak{H}\} \times$ Ran(G) $\subseteq$ CS. Altogether, we would thus have a contradiction to Theorem 2-65-(ii). Therefore there are no $k \in$ Dom($\mathfrak{A}$) and $l \in$ Dom(G) such that $\mathfrak{A}\restriction k \in$ PGEN($\langle\mathfrak{H}, G\restriction(l+1)\rangle$). According to Definition 2-19, we thus have $\mathfrak{A} \in$ GEN($\langle\mathfrak{H}, G\rangle$). Since $\{\mathfrak{H}\} \times$ Ran(G) $\subseteq$ CS, it thus follows with Theorem 2-41 that $(\mathfrak{H}, \mathfrak{A}) \in$ CS. Hence

$\mathfrak{A}$ is a closed segment in $\mathfrak{H}$ and a CdI-like segment in $\mathfrak{H}$ and thus a CdI-closed segment in $\mathfrak{H}$.

(*R-L*): Now, suppose $\mathfrak{A}$ is a CdI-closed segment in $\mathfrak{H}$. Then $\mathfrak{A}$ is a closed segment and a CdI-like segment in $\mathfrak{H}$. From $\mathfrak{A}$ being a CdI-like segment in $\mathfrak{H}$ it then follows that there are Δ, $\Gamma \in$ CFORM such that (i), (iii) and (v) are satisfied. With Theorem 2-48, we also have that (iv) holds. (If $\mathfrak{A}$ is a minimal closed segment, (iv) holds trivially.)

Now, suppose $\mathfrak{B}$ is a closed segment in $\mathfrak{H}\restriction\max(\mathrm{Dom}(\mathfrak{A}))$. Suppose $\min(\mathrm{Dom}(\mathfrak{B})) \leq \min(\mathrm{Dom}(\mathfrak{A}))$ and $\min(\mathrm{Dom}(\mathfrak{A})) < \max(\mathrm{Dom}(\mathfrak{B}))$. Then we would have $\min(\mathrm{Dom}(\mathfrak{A})) \in \mathrm{Dom}(\mathfrak{B})$ and hence $\mathfrak{A} \cap \mathfrak{B} \neq \emptyset$ and $\min(\mathrm{Dom}(\mathfrak{B})) \leq \min(\mathrm{Dom}(\mathfrak{A}))$. With Theorem 2-56-(i) and -(ii), we would thus have $\mathfrak{A} \subseteq \mathfrak{B}$. But then we would have $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{H}\restriction\max(\mathrm{Dom}(\mathfrak{A}))$ and hence $\max(\mathrm{Dom}(\mathfrak{A})) \notin \mathrm{Dom}(\mathfrak{A}) \neq \emptyset$. Contradiction! Therefore we have $\min(\mathrm{Dom}(\mathfrak{A})) < \min(\mathrm{Dom}(\mathfrak{B}))$ or $\max(\mathrm{Dom}(\mathfrak{B})) \leq \min(\mathrm{Dom}(\mathfrak{A}))$. Therefore we also have (iii). ■

Theorem 2-68. *Lemma for Theorem 2-92*

$\mathfrak{A}$ is a segment in $\mathfrak{H}$ and there are Δ, $\Gamma \in$ CFORM and $i \in \mathrm{Dom}(\mathfrak{H})$ such that

(i) $\min(\mathrm{Dom}(\mathfrak{A})) \leq i < \max(\mathrm{Dom}(\mathfrak{A}))$,

(ii) $\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}))}$ = ⌜Suppose Δ⌝,

(iii) For all closed segments $\mathfrak{B}$ in $\mathfrak{H}\restriction\max(\mathrm{Dom}(\mathfrak{A}))$: $\min(\mathrm{Dom}(\mathfrak{A})) < \min(\mathrm{Dom}(\mathfrak{B}))$ or $\max(\mathrm{Dom}(\mathfrak{B})) \leq \min(\mathrm{Dom}(\mathfrak{A}))$,

(iv) $\mathrm{P}(\mathfrak{H}_i) = \Gamma$ and $\mathrm{P}(\mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{A}))-1})$ = ⌜$\neg\Gamma$⌝
or
$\mathrm{P}(\mathfrak{H}_i)$ = ⌜$\neg\Gamma$⌝ and $\mathrm{P}(\mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{A}))-1}) = \Gamma$,

(v) For all closed segments $\mathfrak{B}$ in $\mathfrak{H}\restriction\max(\mathrm{Dom}(\mathfrak{A}))$: $i < \min(\mathrm{Dom}(\mathfrak{B}))$ or $\max(\mathrm{Dom}(\mathfrak{B})) \leq i$,

(vi) For every $r \in \mathrm{Dom}(\mathrm{AS}(\mathfrak{H})) \cap \mathrm{Dom}(\mathfrak{A})$ with $\min(\mathrm{Dom}(\mathfrak{A})) < r \leq \max(\mathrm{Dom}(\mathfrak{A}))-1$ there is a closed segment $\mathfrak{B}$ in $\mathfrak{H}\restriction\max(\mathrm{Dom}(\mathfrak{A}))$ such that $(r, \mathfrak{H}_r) \in \mathfrak{B}$, and

(vii) $\mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{A}))}$ = ⌜Therefore $\neg\Delta$⌝

iff

$\mathfrak{A}$ is an NI-closed segment in $\mathfrak{H}$.

Proof: (*L-R*): Let $\mathfrak{H}$ and $\mathfrak{A}$ satsify the requirements and let Δ, Γ and i be as demanded. First, we have $\mathfrak{H} \in$ SEQ. With Definition 2-12, we have that $\mathfrak{A}$ is an NI-like segment in $\mathfrak{H}$. Also, from clause (iii) of our hypothesis and Theorem 2-65-(i), it follows for all $k \in \mathrm{Dom}(\mathfrak{A})$ that $\mathfrak{A}\restriction k$ is not a closed segment in $\mathfrak{H}$.

We have that $AS(\mathfrak{H}) \cap \mathfrak{A} = \{(\min(Dom(\mathfrak{A})), \mathfrak{H}_{\min(Dom(\mathfrak{A}))})\}$ or that there is an $i \in Dom(AS(\mathfrak{H})) \cap Dom(\mathfrak{A})$ with $\min(Dom(\mathfrak{A})) < i \leq \max(Dom(\mathfrak{A}))$-1.

Now, suppose $AS(\mathfrak{H}) \cap \mathfrak{A} = \{(\min(Dom(\mathfrak{A})), \mathfrak{H}_{\min(Dom(\mathfrak{A}))})\}$.Because we have for all $k \in Dom(\mathfrak{A})$ that $\mathfrak{A}\restriction k$ is not a closed segment in $\mathfrak{H}$, we have, with Theorem 2-32, that $\mathfrak{A}$ is a minimal closed and thus a closed segment in $\mathfrak{H}$. Since $\mathfrak{A}$ is an NI-like segment in $\mathfrak{H}$, $\mathfrak{A}$ is thus an NI-closed segment in $\mathfrak{H}$.

Now, suppose ther is an $s \in Dom(AS(\mathfrak{H})) \cap Dom(\mathfrak{A})$ with $\min(Dom(\mathfrak{A})) < s \leq \max(Dom(\mathfrak{A}))$-1. Now, let $\mathfrak{C} = \{(l, \mathfrak{H}_l) \mid \min(Dom(\mathfrak{A}))+1 \leq l \leq \max(Dom(\mathfrak{A}))-1\}$. Then we have that $\mathfrak{C}$ is a segment in $\mathfrak{H}$ and $s \in Dom(AS(\mathfrak{H})) \cap Dom(\mathfrak{C})$. Also, there is for every $r \in Dom(AS(\mathfrak{H})) \cap Dom(\mathfrak{C})$ a closed segment $\mathfrak{B}$ in $\mathfrak{H}$ such that $(r, \mathfrak{H}_r) \in \mathfrak{B}$ and $\mathfrak{B} \subseteq \mathfrak{C}$. To see this, suppose $r \in Dom(AS(\mathfrak{H})) \cap Dom(\mathfrak{C})$. Then we have $\min(Dom(\mathfrak{A})) < r \leq \max(Dom(\mathfrak{A}))$-1 and hence there is, according to clause (vi), a closed segment $\mathfrak{B}$ in $\mathfrak{H}\restriction\max(Dom(\mathfrak{A}))$ such that $(r, \mathfrak{H}_r) \in \mathfrak{B}$. Then we have $\min(Dom(\mathfrak{C})) \leq \min(Dom(\mathfrak{B}))$, because otherwise we would have $\min(Dom(\mathfrak{B})) \leq \min(Dom(\mathfrak{A})) < r \leq \max(Dom(\mathfrak{B}))$, which contradicts clause (iii). It also follows from $\mathfrak{B}$ being a segment in $\mathfrak{H}\restriction\max(Dom(\mathfrak{A}))$ that $\max(Dom(\mathfrak{B})) \leq \max(Dom(\mathfrak{A}))-1 = \max(Dom(\mathfrak{C}))$. With Theorem 2-5, we therefore have $\mathfrak{B} \subseteq \mathfrak{C}$.

Thus $\mathfrak{C}$ satisfies the conditions of Theorem 2-59. Therefore there is a $G \in ASCS(\mathfrak{H})$ such that G is an AS-comprising segment sequence for $\mathfrak{C}$ in $\mathfrak{H}$ and $\{\mathfrak{H}\} \times Ran(G) \subseteq \{\mathfrak{H}\} \times \{\mathfrak{C}^* \mid \mathfrak{C}^* \subseteq \mathfrak{C}$ is a closed segment in $\mathfrak{H}\} \subseteq \{\mathfrak{H}\} \times \{\mathfrak{C}^* \mid \mathfrak{C}^* \subseteq \mathfrak{A}$ is a closed segment in $\mathfrak{H}\} \subseteq CS$. According to the definition of $\mathfrak{C}$, we have that $\mathfrak{C} \in SG(\mathfrak{H})$ and that $\min(Dom(\mathfrak{A}))+1 = \min(Dom(\mathfrak{C}))$ and $\max(Dom(\mathfrak{A})) = \max(Dom(\mathfrak{C}))+1$ and we have that $\mathfrak{A}$ is an NI-like segment in $\mathfrak{H}$. Also, we have for all $r \in Dom(G)$: $i < \min(Dom(G(r)))$ or $\max(Dom(G(r))) \leq i$. To see this, suppose $r \in Dom(G)$. Then we have $G(r) \subseteq \mathfrak{C}$ is a closed segment in $\mathfrak{H}\restriction\max(Dom(\mathfrak{A}))$. By clause (v), we then have $i < \min(Dom(G(r)))$ or $\max(Dom(G(r))) \leq i$. Furthermore, because for all $i \in Dom(\mathfrak{A})$ it holds that $\mathfrak{A}\restriction i$ is not a closed segment in $\mathfrak{H}$, we also have that for all $i \in Dom(\mathfrak{A})$ it holds that $\mathfrak{A}\restriction i$ is not a minimal closed segment in $\mathfrak{H}$.

Thus, according to Definition 2-18, we have $\mathfrak{A} \in PGEN(\langle\mathfrak{H}, G\rangle)$. Now, suppose for contradiction that there are a $k \in Dom(\mathfrak{A})$ and an $l \in Dom(G)$ such that $\mathfrak{A}\restriction k \in PGEN(\langle\mathfrak{H}, G\restriction(l+1)\rangle)$. According to Theorem 2-25, $G\restriction(l+1)$ is an AS-comprising segment sequence for $\mathfrak{A}\upharpoonright$max(Dom($G(l)$))+1 and thus we have, according to Definition 2-10, that $G\upharpoonright(l+1) \in$ ASCS($\mathfrak{H}$). By hypothesis, we have $\mathfrak{A}\upharpoonright k \in$ PGEN($\langle\mathfrak{H}, G\upharpoonright(l+1)\rangle$). On the other hand, we have $\mathfrak{H} \in$ SEQ and $\{\mathfrak{H}\} \times$ Ran($G\upharpoonright(l+1)$) $\subseteq \{\mathfrak{H}\} \times$ Ran(G) $\subseteq$ CS. Altogether, we would thus have a contradiction to Theorem 2-65-(ii). Therefore there are no $k \in$ Dom($\mathfrak{A}$) and $l \in$ Dom(G) such that $\mathfrak{A}\upharpoonright k \in$ PGEN($\langle\mathfrak{H}, G\upharpoonright(l+1)\rangle$). According to Definition 2-19, we thus have $\mathfrak{A} \in$ GEN($\langle\mathfrak{H}, G\rangle$) and thus with $\{\mathfrak{H}\} \times$ Ran(G) $\subseteq$ CS and Theorem 2-41 ($\mathfrak{H}, \mathfrak{A}$) $\in$ CS. Hence we have that $\mathfrak{A}$ is a closed segment in $\mathfrak{H}$ and an NI-like segment in $\mathfrak{H}$ and thus an NI-closed segment in $\mathfrak{H}$.

(*R-L*): Now, suppose $\mathfrak{A}$ is an NI-closed segment in $\mathfrak{H}$. Then $\mathfrak{A}$ is a closed segment and an NI-like segment in $\mathfrak{H}$. We have AS($\mathfrak{H}$) $\cap\ \mathfrak{A} = \{$(min(Dom($\mathfrak{A}$)), $\mathfrak{H}_{\text{min(Dom}(\mathfrak{A}))}$)$\}$ or there is a $j \in$ Dom(AS($\mathfrak{H}$)) $\cap$ Dom($\mathfrak{A}$) with min(Dom($\mathfrak{A}$)) $< j \leq$ max(Dom($\mathfrak{A}$))-1.

First case: Suppose AS($\mathfrak{H}$) $\cap\ \mathfrak{A} = \{$(min(Dom($\mathfrak{A}$)), $\mathfrak{H}_{\text{min(Dom}(\mathfrak{A}))}$)$\}$. Then it holds, with Theorem 2-35-(iv) and Theorem 2-41, that $\mathfrak{A}$ is a minimal closed segment in $\mathfrak{H}$. Since $\mathfrak{A}$ is an NI-like segment in $\mathfrak{H}$, we then have that $\mathfrak{A}$ is a minimal NI-closed segment in $\mathfrak{H}$. From this it follows that there are Δ, $\Gamma \in$ CFORM and $i \in$ Dom($\mathfrak{H}$) such that (i), (ii), (iv) and (vii) hold. Also, we have trivially that (vi) holds. Let now Δ, Γ and i be as demanded in clauses (i), (ii), (iv) and (vii).

Then we also have (iii) and (v). To see this, suppose $\mathfrak{B}$ is a closed segment in $\mathfrak{H}\upharpoonright$max(Dom($\mathfrak{A}$)). Then we have for l = min(Dom($\mathfrak{A}$)) or $l = i$ that $l <$ min(Dom($\mathfrak{B}$)) or max(Dom($\mathfrak{B}$)) $\leq l$. Since $\mathfrak{A}$ is a minimal NI-closed segment and thus a minimal closed segment in $\mathfrak{H}$, it holds with Theorem 2-58 that $\mathfrak{B} \cap \mathfrak{A} = \emptyset$ or $\mathfrak{A} \subseteq \mathfrak{B}$. Since, by hypothesis, we have $\mathfrak{B} \subseteq \mathfrak{H}\upharpoonright$max(Dom($\mathfrak{A}$)), it follows that $\{$(max(Dom($\mathfrak{A}$)), $\mathfrak{H}_{\text{max(Dom}(\mathfrak{A}))}$)$\} \in \mathfrak{A}\setminus\mathfrak{B}$ and hence that $\mathfrak{A} \not\subseteq \mathfrak{B}$ and thus that $\mathfrak{B} \cap \mathfrak{A} = \emptyset$. On the other hand, for l = min(Dom($\mathfrak{A}$)) or $l = i$ and min(Dom($\mathfrak{B}$)) $\leq l <$ max(Dom($\mathfrak{B}$)) we would have $\mathfrak{B} \cap \mathfrak{A} \neq \emptyset$ and thus a contradiction.

Second case: Now, suppose there is a $j \in$ Dom(AS($\mathfrak{H}$)) $\cap$ Dom($\mathfrak{A}$) with min(Dom($\mathfrak{A}$)) $< j \leq$ max(Dom($\mathfrak{A}$))-1. Then $\mathfrak{A}$ is not a minimal closed segment in $\mathfrak{H}$. With Theorem 2-41, there is then a $G \in$ ASCS($\mathfrak{H}$) with $\{\mathfrak{H}\} \times$ Ran(G) $\subseteq$ CS and $\mathfrak{A} \in$ GEN($\langle\mathfrak{H}, G\rangle$). Then G is an AS-comprising segment sequence for $\mathfrak{C} = \{(l, \mathfrak{H}_l) \mid$ min(Dom($\mathfrak{A}$))+1 $\leq l \leq$ max(Dom($\mathfrak{A}$))-1$\}$ in $\mathfrak{H}$. We have that $\mathfrak{A}$ is an NI-like segment in $\mathfrak{H}$ and thus, according to Definition 2-18 and Definition 2-19:

There is Δ, $\Gamma \in$ CFORM and $i \in \mathrm{Dom}(\mathfrak{H})$ such that

a) $\min(\mathrm{Dom}(\mathfrak{A})) \leq i < \max(\mathrm{Dom}(\mathfrak{A}))$,

b) $\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}))} = \ulcorner$Suppose $\Delta\urcorner$,

c) $\mathrm{P}(\mathfrak{H}_i) = \Gamma$ and $\mathrm{P}(\mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{A}))-1}) = \ulcorner\neg\Gamma\urcorner$
or
$\mathrm{P}(\mathfrak{H}_i) = \ulcorner\neg\Gamma\urcorner$ and $\mathrm{P}(\mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{A}))-1}) = \Gamma$,

d) For all $r \in \mathrm{Dom}(G)$: $i < \min(\mathrm{Dom}(G(r)))$ or $\max(\mathrm{Dom}(G(r))) \leq i$,

e) $\mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{A}))} = \ulcorner$Therefore $\neg\Delta\urcorner$.

Then clauses (i), (ii), (iv) and (vii) are satisfied. With Theorem 2-48, we also have (vi).

Also, we have (iii) and (v). To see this, suppose $\mathfrak{B}$ is a closed segment in $\mathfrak{H}\restriction\max(\mathrm{Dom}(\mathfrak{A}))$. Then it holds that $\mathfrak{B} \subseteq \mathfrak{H}\restriction\max(\mathrm{Dom}(\mathfrak{A}))$ and hence that $\{(\max(\mathrm{Dom}(\mathfrak{A})), \mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{A}))})\} \in \mathfrak{A}\backslash\mathfrak{B}$ and hence that $\mathfrak{A} \not\subseteq \mathfrak{B}$. It also follows that $\max(\mathrm{Dom}(\mathfrak{B})) < \max(\mathrm{Dom}(\mathfrak{A}))$. Thus we have that $\mathfrak{B} \cap \mathfrak{A} = \emptyset$ or $\mathfrak{B} \subseteq \mathfrak{C}$. To see this, suppose $\mathfrak{B} \cap \mathfrak{A} \neq \emptyset$. Because of $\mathfrak{A} \not\subseteq \mathfrak{B}$, we then have, with Theorem 2-57, that $\mathfrak{B} \subset \mathfrak{A}$ and hence, with Theorem 2-56, that $\min(\mathrm{Dom}(\mathfrak{A})) < \min(\mathrm{Dom}(\mathfrak{B}))$. Altogether, we thus have $\min(\mathrm{Dom}(\mathfrak{C})) = \min(\mathrm{Dom}(\mathfrak{A}))+1 \leq \min(\mathrm{Dom}(\mathfrak{B})) < \max(\mathrm{Dom}(\mathfrak{B})) \leq \max(\mathrm{Dom}(\mathfrak{A}))-1 = \max(\mathrm{Dom}(\mathfrak{C}))$ and hence, with Theorem 2-5, $\mathfrak{B} \subseteq \mathfrak{C}$.

With Theorem 2-52 it then follows immediately that (iii) holds, i.e. that $\min(\mathrm{Dom}(\mathfrak{A})) < \min(\mathrm{Dom}(\mathfrak{B}))$ or $\max(\mathrm{Dom}(\mathfrak{B})) \leq \min(\mathrm{Dom}(\mathfrak{A}))$. Furthermore, we also have (v), i.e. that $i < \min(\mathrm{Dom}(\mathfrak{B}))$ or $\max(\mathrm{Dom}(\mathfrak{B})) \leq i$. To see this, suppose for contradiction that $\min(\mathrm{Dom}(\mathfrak{B})) \leq i < \max(\mathrm{Dom}(\mathfrak{B}))$. Then we would have $(i, \mathfrak{H}_i) \in \mathfrak{B}$. We have that $\mathfrak{B} \subseteq \mathfrak{A}$ is a closed segment in $\mathfrak{H}$ and thus, with Theorem 2-60, that there is an $r \in \mathrm{Dom}(G)$ such that $\mathfrak{B} \subseteq G(r)$. Then we would have $\min(\mathrm{Dom}(G(r))) \leq \min(\mathrm{Dom}(\mathfrak{B})) \leq i < \max(\mathrm{Dom}(\mathfrak{B})) \leq \max(\mathrm{Dom}(G(r)))$. But, because of d) we would also have that $i < \min(\mathrm{Dom}(G(r)))$ or $\max(\mathrm{Dom}(G(r))) \leq i$. Contradiction! Therefore we have $i < \min(\mathrm{Dom}(\mathfrak{B}))$ or $\max(\mathrm{Dom}(\mathfrak{B})) \leq i$. ■

Theorem 2-69. *Lemma for Theorem 2-93*

$\mathfrak{A}$ is a segment in $\mathfrak{H}$ and there are $\xi \in$ VAR, $\beta \in$ PAR, $\Delta \in$ FORM, where FV(Δ) $\subseteq \{\xi\}$, $\Gamma \in$ CFORM and $\mathfrak{B} \in$ SG($\mathfrak{H}$) such that

(i) $P(\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))}) = \ulcorner\bigvee\xi\Delta\urcorner$,

(ii) For all closed segments $\mathfrak{C}$ in $\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A}))$: $\min(\mathrm{Dom}(\mathfrak{B})) < \min(\mathrm{Dom}(\mathfrak{C}))$ or $\max(\mathrm{Dom}(\mathfrak{C})) \leq \min(\mathrm{Dom}(\mathfrak{B}))$,

(iii) $\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))+1} = \ulcorner$Suppose $[\beta, \xi, \Delta]\urcorner$,

(iv) For all closed segments $\mathfrak{C}$ in $\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A}))$: $\min(\mathrm{Dom}(\mathfrak{B}))+1 < \min(\mathrm{Dom}(\mathfrak{C}))$ or $\max(\mathrm{Dom}(\mathfrak{C})) \leq \min(\mathrm{Dom}(\mathfrak{B}))+1$,

(v) $P(\mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{B}))-1}) = \Gamma$,

(vi) $\mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{B}))} = \ulcorner$Therefore $\Gamma\urcorner$,

(vii) $\beta \notin$ STSF($\{\Delta, \Gamma\}$),

(viii) There is no $j \leq \min(\mathrm{Dom}(\mathfrak{B}))$ such that $\beta \in$ ST($\mathfrak{H}_j$),

(ix) $\mathfrak{A} = \mathfrak{B}\setminus\{(\min(\mathrm{Dom}(\mathfrak{B})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))})\}$, and

(x) For every $r \in \mathrm{Dom}(\mathrm{AS}(\mathfrak{H})) \cap \mathrm{Dom}(\mathfrak{A})$ with $\min(\mathrm{Dom}(\mathfrak{A})) < r \leq \max(\mathrm{Dom}(\mathfrak{A}))$-1 there is a closed segment $\mathfrak{C}$ in $\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A}))$ such that $(r, \mathfrak{H}_r) \in \mathfrak{C}$

iff

$\mathfrak{A}$ is a PE-closed segment in $\mathfrak{H}$.

Proof: (*L-R*): Let $\mathfrak{A}$ be a segment in $\mathfrak{H}$ and let ξ, β, Δ, Γ and $\mathfrak{B}$ be as demanded. Then we have $\mathfrak{H} \in$ SEQ. With Definition 2-13, we have that $\mathfrak{A}$ is an RA-like segment in $\mathfrak{H}$ and we have $\min(\mathrm{Dom}(\mathfrak{A})) = \min(\mathrm{Dom}(\mathfrak{B}))+1$. With clause (iv) of our hypothesis and Theorem 2-65-(i), we have that for all $k \in \mathrm{Dom}(\mathfrak{A})$ it holds that $\mathfrak{A}\upharpoonright k$ is not a closed segment in $\mathfrak{H}$.

We have that $\mathrm{AS}(\mathfrak{H}) \cap \mathfrak{A} = \{(\min(\mathrm{Dom}(\mathfrak{A})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}))})\}$ or that there is an $i \in \mathrm{Dom}(\mathrm{AS}(\mathfrak{H})) \cap \mathrm{Dom}(\mathfrak{A})$ with $\min(\mathrm{Dom}(\mathfrak{A})) < i \leq \max(\mathrm{Dom}(\mathfrak{A}))$-1.

Suppose $\mathrm{AS}(\mathfrak{H}) \cap \mathfrak{A} = \{(\min(\mathrm{Dom}(\mathfrak{A})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}))})\}$. Since it holds for all $k \in \mathrm{Dom}(\mathfrak{A})$ that $\mathfrak{A}\upharpoonright k$ is not a closed segment in $\mathfrak{H}$, we have, with Theorem 2-32, that $\mathfrak{A}$ is a minimal closed and thus a closed segment in $\mathfrak{H}$. Since $\mathfrak{A}$ is an RA-like segment in $\mathfrak{H}$, $\mathfrak{A}$ is thus a PE-closed segment in $\mathfrak{H}$.

Now, suppose there is an $i \in \mathrm{Dom}(\mathrm{AS}(\mathfrak{H})) \cap \mathrm{Dom}(\mathfrak{A})$ with $\min(\mathrm{Dom}(\mathfrak{A})) < i \leq \max(\mathrm{Dom}(\mathfrak{A}))$-1. Now, let $\mathfrak{C}^* = \{(l, \mathfrak{H}_l) \mid \min(\mathrm{Dom}(\mathfrak{A}))+1 \leq l \leq \max(\mathrm{Dom}(\mathfrak{A}))\text{-}1\}$. Then we have that $\mathfrak{C}^*$ is a segment in $\mathfrak{H}$ and $i \in \mathrm{Dom}(\mathrm{AS}(\mathfrak{H})) \cap \mathrm{Dom}(\mathfrak{C}^*)$. We also have that for every $r \in \mathrm{Dom}(\mathrm{AS}(\mathfrak{H})) \cap \mathrm{Dom}(\mathfrak{C}^*)$ there is a closed segment $\mathfrak{C}$ in $\mathfrak{H}$ such that $(r, \mathfrak{H}_r) \in \mathfrak{C}$ and $\mathfrak{C} \subseteq \mathfrak{C}^*$. To see this, suppose $r \in \mathrm{Dom}(\mathrm{AS}(\mathfrak{H})) \cap \mathrm{Dom}(\mathfrak{C}^*)$. Then we have $\min(\mathrm{Dom}(\mathfrak{A})) < r \leq \max(\mathrm{Dom}(\mathfrak{A}))$-1 and hence there, is according to clause (x), a closed segment $\mathfrak{C}$ in $\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A}))$ such that $(r, \mathfrak{H}_r) \in \mathfrak{C}$. Then we have $\min(\mathrm{Dom}(\mathfrak{C}^*)) \leq$

min(Dom($\mathfrak{C}$)), because otherwise we would have min(Dom($\mathfrak{C}$)) ≤ min(Dom($\mathfrak{A}$)) < r ≤ max(Dom($\mathfrak{C}$)), which contradicts clause (iv). On the other hand, it follows from $\mathfrak{C}$ being a segment in $\mathfrak{H}\restriction$max(Dom($\mathfrak{A}$)) that max(Dom($\mathfrak{C}$)) ≤ max(Dom($\mathfrak{A}$))-1 = max(Dom($\mathfrak{C}$*)). With Theorem 2-5, we therefore have $\mathfrak{C} \subseteq \mathfrak{C}$*.

Thus $\mathfrak{C}$* satisfies the requirements of Theorem 2-59. Therefore there is a $G \in$ ASCS($\mathfrak{H}$) such that G is an AS-comprising segment sequence for $\mathfrak{C}$* in $\mathfrak{H}$ and {$\mathfrak{H}$} × Ran(G) ⊆ CS. According to the definition of $\mathfrak{C}$*, we have that $\mathfrak{C}$* ∈ SG($\mathfrak{H}$) and min(Dom($\mathfrak{A}$))+1 = min(Dom($\mathfrak{C}$*)) and max(Dom($\mathfrak{A}$)) = max(Dom($\mathfrak{C}$*))+1 and that $\mathfrak{A}$ is an RA-like segment in $\mathfrak{H}$. Suppose, $\mathfrak{A}$ is an NI-like segment in $\mathfrak{H}$. Then we have Γ = ⌜¬[β, ξ Δ]⌝ and P($\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}))}$) = [β, ξ, Δ] and P($\mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{A}))-1}$) = ⌜¬[β, ξ, Δ]⌝. Also, we have that for all r ∈ Dom(G) it holds that min(Dom($\mathfrak{A}$)) < min(Dom($\mathfrak{C}$*)) ≤ min(Dom(G(r)). Furthermore, since it holds for all i ∈ Dom($\mathfrak{A}$) that $\mathfrak{A}\restriction i$ is not a closed segment in $\mathfrak{H}$, we also have that for all i ∈ Dom($\mathfrak{A}$) it holds that $\mathfrak{A}\restriction i$ is not a minimal closed segment in $\mathfrak{H}$.

According to Definition 2-18, we thus have $\mathfrak{A}$ ∈ PGEN(⟨$\mathfrak{H}$, G⟩). Now, suppose for contradiction that there are a k ∈ Dom($\mathfrak{A}$) and an l ∈ Dom(G) such that $\mathfrak{A}\restriction k$ ∈ PGEN(⟨$\mathfrak{H}$, $G\restriction(l+1)$⟩). According to Theorem 2-25, $G\restriction(l+1)$ is an AS-comprising segment sequence for $\mathfrak{A}\restriction$max(Dom(G(l)))+1 and thus, according to Definition 2-10, we have $G\restriction(l+1)$ ∈ ASCS($\mathfrak{H}$). By hypothesis, we have $\mathfrak{A}\restriction k$ ∈ PGEN(⟨$\mathfrak{H}$, $G\restriction(l+1)$⟩). On the other hand, we have $\mathfrak{H}$ ∈ SEQ and {$\mathfrak{H}$} × Ran($G\restriction(l+1)$) ⊆ {$\mathfrak{H}$} × Ran(G) ⊆ CS. Altogether, we thus have a contradiction to Theorem 2-65-(ii). Therefore there are no k ∈ Dom($\mathfrak{A}$) and l ∈ Dom(G) such that $\mathfrak{A}\restriction k$ ∈ PGEN(⟨$\mathfrak{H}$, $G\restriction(l+1)$⟩). According to Definition 2-19, we hence have that $\mathfrak{A}$ ∈ GEN(⟨$\mathfrak{H}$, G⟩) and thus, with {$\mathfrak{H}$} × Ran(G) ⊆ CS and Theorem 2-41, that ($\mathfrak{H}$, $\mathfrak{A}$) ∈ CS. Hence $\mathfrak{A}$ is a closed segment in $\mathfrak{H}$ and an RA-like segment in $\mathfrak{H}$ and thus a PE-closed segment in $\mathfrak{H}$.

(*R-L*): Now, suppose $\mathfrak{A}$ is a PE-closed segment in $\mathfrak{H}$. Then we have that $\mathfrak{A}$ is a closed segment and an RA-like segment in $\mathfrak{H}$. From $\mathfrak{A}$ being an RA-like segment in $\mathfrak{H}$ it follows that there are ξ ∈ VAR, β ∈ PAR, Δ ∈ FORM, where FV(Δ) ⊆ {ξ}, Γ ∈ CFORM and a $\mathfrak{B}$ ∈ SG($\mathfrak{H}$) for which clauses (i), (iii), and (v)-(ix) are satisfied. We also have with Theorem 2-48 that (x) holds (if $\mathfrak{A}$ is a minimal closed segment, (x) holds trivially). Also, we have that min(Dom($\mathfrak{A}$)) = min(Dom($\mathfrak{B}$))+1.

Now, we still have to show that clauses (ii) and (iv) hold. For this, we first show (iv). Suppose $\mathfrak{C}$ is a closed segment in $\mathfrak{H}\restriction\max(\mathrm{Dom}(\mathfrak{A}))$. Suppose for contradiction that $\min(\mathrm{Dom}(\mathfrak{C})) \leq \min(\mathrm{Dom}(\mathfrak{A})) < \max(\mathrm{Dom}(\mathfrak{C}))$. Then we would have $\min(\mathrm{Dom}(\mathfrak{A})) \in \mathrm{Dom}(\mathfrak{C})$ and hence $\mathfrak{A} \cap \mathfrak{C} \neq \emptyset$. With Theorem 2-56, we would then have $\mathfrak{A} \subseteq \mathfrak{C}$. Thus we would have $\mathfrak{A} \subseteq \mathfrak{C} \subseteq \mathfrak{H}\restriction\max(\mathrm{Dom}(\mathfrak{A}))$ and hence $\max(\mathrm{Dom}(\mathfrak{A})) \notin \mathrm{Dom}(\mathfrak{A}) \neq \emptyset$. Contradiction! Therefore we have $\min(\mathrm{Dom}(\mathfrak{A})) < \min(\mathrm{Dom}(\mathfrak{C}))$ or $\max(\mathrm{Dom}(\mathfrak{C})) \leq \min(\mathrm{Dom}(\mathfrak{A}))$.

We still have to show (ii). Suppose again that $\mathfrak{C}$ is a closed segment in $\mathfrak{H}\restriction\max(\mathrm{Dom}(\mathfrak{A}))$. Suppose $\min(\mathrm{Dom}(\mathfrak{C})) \leq \min(\mathrm{Dom}(\mathfrak{B})) < \max(\mathrm{Dom}(\mathfrak{C}))$. Then we would have $\min(\mathrm{Dom}(\mathfrak{C})) < \min(\mathrm{Dom}(\mathfrak{A})) \leq \max(\mathrm{Dom}(\mathfrak{C}))$. As we have just shown, it holds with (iv) that $\min(\mathrm{Dom}(\mathfrak{A})) < \min(\mathrm{Dom}(\mathfrak{C}))$ or $\max(\mathrm{Dom}(\mathfrak{C})) \leq \min(\mathrm{Dom}(\mathfrak{A}))$. Since the first case is exluded, it follows that $\max(\mathrm{Dom}(\mathfrak{C})) \leq \min(\mathrm{Dom}(\mathfrak{A}))$ and thus that $\max(\mathrm{Dom}(\mathfrak{C})) = \min(\mathrm{Dom}(\mathfrak{A}))$. Then we would have $\max(\mathrm{Dom}(\mathfrak{C})) \in \mathrm{Dom}(\mathrm{AS}(\mathfrak{H}))$. But with Theorem 2-42, $\mathfrak{C}$ is a CdI- or NI- or RA-like segment in $\mathfrak{H}$ and thus we have, with Theorem 2-29, that $\max(\mathrm{Dom}(\mathfrak{C})) \notin \mathrm{Dom}(\mathrm{AS}(\mathfrak{H}))$. Contradiction! Thus we have $\min(\mathrm{Dom}(\mathfrak{B})) < \min(\mathrm{Dom}(\mathfrak{C}))$ or $\max(\mathrm{Dom}(\mathfrak{C})) \leq \min(\mathrm{Dom}(\mathfrak{B}))$. Therefore we also have (ii). ■

2.3 AVS, AVAS, AVP and AVAP

Now, the availability conception is established with recourse to ch. 2.2. This is done in such a way that a proposition is available in a sentence sequence $\mathfrak{H}$ at an $i \in \mathrm{Dom}(\mathfrak{H})$ if and only if $(i, \mathfrak{H}_i)$ does not lie within a proper initial segment of any closed segment in $\mathfrak{H}$ (Definition 2-26). Of all the propositions of the members of a closed segment $\mathfrak{A}$ in $\mathfrak{H}$ it is thus at most the proposition of the last member of $\mathfrak{A}$ that is available in $\mathfrak{H}$ at any $i \in \mathrm{Dom}(\mathfrak{A})$, namely at $\max(\mathrm{Dom}(\mathfrak{A}))$. The function AVS then assigns exactly that subset of $\mathfrak{H}$ to a sentence sequence $\mathfrak{H}$ for whose elements $(i, \mathfrak{H}_i)$ it holds that the proposition of $\mathfrak{H}_i$ is available in $\mathfrak{H}$ at i (Definition 2-28). The propositions of the sentences from AVS($\mathfrak{H}$) are then collected by the function AVP to form AVP($\mathfrak{H}$), the set of the propositions that are available in $\mathfrak{H}$ at some position (Definition 2-30). The function AVAS assigns a sentence sequence $\mathfrak{H}$ that subset of $\mathfrak{H}$ for whose elements $(i, \mathfrak{H}_i)$ it holds that $\mathfrak{H}_i$ is an assumption-sentence and that the proposition of $\mathfrak{H}_i$ is available in $\mathfrak{H}$ at i (Definition 2-29). The propositions of the assumption-sentences from AVAS($\mathfrak{H}$) are then collected by the function AVAP to form AVAP($\mathfrak{H}$), the set of propositions that have been assumed in $\mathfrak{H}$ at some position and are still available at that position, i.e. the set of available assumptions of $\mathfrak{H}$ (Definition 2-31).

Then, we will prove some theorems which will, on the one hand, establish connections between AVS, AVAS, AVP and AVAP and, on the other hand, show connections between the extension of a sentence sequence and changes of availability. The most important theorems for the understanding of the calculus and for the further development are Theorem 2-82, Theorem 2-83, Theorem 2-91, Theorem 2-92 and Theorem 2-93. With this chapter, we will finish our preparations so that we can then develop and analyse the Speech Act Calculus in the next chapters.

Definition 2-26. *Availability of a proposition in a sentence sequence at a position*

Γ <u>is available in</u> $\mathfrak{H}$ <u>at</u> i
iff
$\Gamma \in$ CFORM and $\mathfrak{H} \in$ SEQ and

(i) $i \in \mathrm{Dom}(\mathfrak{H})$,
(ii) $\Gamma = \mathrm{P}(\mathfrak{H}_i)$, and
(iii) There is no closed segment $\mathfrak{A}$ in $\mathfrak{H}$ such that $\min(\mathrm{Dom}(\mathfrak{A})) \leq i < \max(\mathrm{Dom}(\mathfrak{A}))$.

Definition 2-27. *Availability of a proposition in a sentence sequence*
Γ <u>is available in</u> $\mathfrak{H}$
iff
There is an $i \in \mathrm{Dom}(\mathfrak{H})$ such that Γ is available in $\mathfrak{H}$ at i.

Note: If it is obvious to which sentence sequence we are referring, we will also use the shorter formulations 'Γ is available at i' or 'Γ is available'.

Definition 2-28. *Assignment of the set of available sentences (AVS)*
$\mathrm{AVS} = \{(\mathfrak{H}, X) \mid \mathfrak{H} \in \mathrm{SEQ}$ and $X = \{(i, \mathfrak{H}_i) \mid i \in \mathrm{Dom}(\mathfrak{H})$ and $\mathrm{P}(\mathfrak{H}_i)$ is available in $\mathfrak{H}$ at $i\}\}$.

Definition 2-29. *Assignment of the set of available assumption-sentences (AVAS)*
$\mathrm{AVAS} = \{(\mathfrak{H}, X) \mid \mathfrak{H} \in \mathrm{SEQ}$ and $X = \mathrm{AVS}(\mathfrak{H}) \cap \mathrm{AS}(\mathfrak{H})\}$.

Note: The titles 'assignment of the set of … sentences' are misleading insofar AVS and AVAS do not assign sets of sentences to sentence sequences but subsets of these sequences, thus sets of ordered pairs, whose second projections are then the respective sentences.

Theorem 2-70. *Relation of AVAS, AVS and respective sentence sequence*
If $\mathfrak{H} \in \mathrm{SEQ}$, then:
(i) $\mathrm{AVAS}(\mathfrak{H}) = \mathrm{AVS}(\mathfrak{H}) \cap \mathrm{AS}(\mathfrak{H})$ and
(ii) $\mathrm{AVAS}(\mathfrak{H}) \subseteq \mathrm{AVS}(\mathfrak{H}) \subseteq \mathfrak{H}$.

Proof: Follows directly from the definitions. ■

Definition 2-30. *Assignment of the set of available propositions (AVP)*
$\mathrm{AVP} = \{(\mathfrak{H}, X) \mid \mathfrak{H} \in \mathrm{SEQ}$ and $X = \{\Gamma \mid$ There is an $i \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}))$ and $\Gamma = \mathrm{P}(\mathfrak{H}_i)\}\}$.

Definition 2-31. *Assignment of the set of available assumptions (AVAP)*
$\mathrm{AVAP} = \{(\mathfrak{H}, X) \mid \mathfrak{H} \in \mathrm{SEQ}$ and $X = \{\Gamma \mid$ There is an $i \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}))$ and $\Gamma = \mathrm{P}(\mathfrak{H}_i)\}\}$.

Theorem 2-71. *Relation of AVAP and AVP*
If $\mathfrak{H} \in \mathrm{SEQ}$, then $\mathrm{AVAP}(\mathfrak{H}) \subseteq \mathrm{AVP}(\mathfrak{H})$.

Proof: Follows with Theorem 2-70 directly from the definitions. ■

Theorem 2-72. *AVS-inclusion implies AVAS-inclusion*
If $\mathfrak{H}$, $\mathfrak{H}' \in$ SEQ and AVS($\mathfrak{H}$) $\subseteq$ AVS($\mathfrak{H}'$), then AVAS($\mathfrak{H}$) $\subseteq$ AVAS($\mathfrak{H}'$).

Proof: Suppose $\mathfrak{H}$, $\mathfrak{H}' \in$ SEQ and suppose AVS($\mathfrak{H}$) $\subseteq$ AVS($\mathfrak{H}'$). Now, suppose $(i, \mathfrak{H}_i) \in$ AVAS($\mathfrak{H}$). Then we have $(i, \mathfrak{H}_i) \in$ AVS($\mathfrak{H}$) $\cap$ AS($\mathfrak{H}$). Then we have $(i, \mathfrak{H}_i) \in$ AVS($\mathfrak{H}$) and $\mathfrak{H}_i \in$ ASENT. By hypothesis, we then have $(i, \mathfrak{H}_i) \in$ AVS($\mathfrak{H}'$) and hence also $(i, \mathfrak{H}_i) \in \mathfrak{H}'$. Since $\mathfrak{H}_i \in$ ASENT, we then also have $(i, \mathfrak{H}_i) \in$ AS($\mathfrak{H}'$) and thus $(i, \mathfrak{H}_i) \in$ AVS($\mathfrak{H}'$) $\cap$ AS($\mathfrak{H}'$) = AVAS($\mathfrak{H}'$). ■

Theorem 2-73. *AVAS-reduction implies AVS-reduction*
If $\mathfrak{H}$, $\mathfrak{H}' \in$ SEQ and AVAS($\mathfrak{H}$)\AVAS($\mathfrak{H}'$) $\neq \emptyset$, then AVS($\mathfrak{H}$)\AVS($\mathfrak{H}'$) $\neq \emptyset$.

Proof: Suppose $\mathfrak{H}$, $\mathfrak{H}' \in$ SEQ and suppose AVAS($\mathfrak{H}$)\AVAS($\mathfrak{H}'$) $\neq \emptyset$. Hence AVAS($\mathfrak{H}$) $\not\subseteq$ AVAS($\mathfrak{H}'$) and with Theorem 2-72 we get AVS($\mathfrak{H}$) $\not\subseteq$ AVS($\mathfrak{H}'$). It follows immediately that AVS($\mathfrak{H}$)\AVS($\mathfrak{H}'$) $\neq \emptyset$. ■

Theorem 2-74. *AVS-inclusion implies AVP-inclusion*
If $\mathfrak{H}$, $\mathfrak{H}' \in$ SEQ and AVS($\mathfrak{H}$) $\subseteq$ AVS($\mathfrak{H}'$), then AVP($\mathfrak{H}$) $\subseteq$ AVP($\mathfrak{H}'$).

Proof: Suppose $\mathfrak{H}$, $\mathfrak{H}' \in$ SEQ and suppose AVS($\mathfrak{H}$) $\subseteq$ AVS($\mathfrak{H}'$). Now, suppose $\Gamma \in$ AVP($\mathfrak{H}$). Then there is an $i \in$ Dom(AVS($\mathfrak{H}$)) such that Γ = P($\mathfrak{H}_i$). Then we have $(i, \mathfrak{H}_i) \in$ AVS($\mathfrak{H}$). By hypothesis, we then have $(i, \mathfrak{H}_i) \in$ AVS($\mathfrak{H}'$). We have AVS($\mathfrak{H}'$) $\subseteq \mathfrak{H}'$ and hence $(i, \mathfrak{H}_i) \in \mathfrak{H}'$ and therefore $\mathfrak{H}_i = \mathfrak{H}'_i$. Hence we have Γ = P($\mathfrak{H}_i$) = P($\mathfrak{H}'_i$). Therefore we have $i \in$ Dom(AVS($\mathfrak{H}'$)) and Γ = P($\mathfrak{H}'_i$). Therefore we have $\Gamma \in$ AVP($\mathfrak{H}'$). ■

Theorem 2-75. *AVAS-inclusion implies AVAP-inclusion*
If $\mathfrak{H}$, $\mathfrak{H}' \in$ SEQ and AVAS($\mathfrak{H}$) $\subseteq$ AVAS($\mathfrak{H}'$), then AVAP($\mathfrak{H}$) $\subseteq$ AVAP($\mathfrak{H}'$).

Proof: Suppose $\mathfrak{H}$, $\mathfrak{H}' \in$ SEQ and suppose AVAS($\mathfrak{H}$) $\subseteq$ AVAS($\mathfrak{H}'$). Now, suppose $\Gamma \in$ AVAP($\mathfrak{H}$). Then there is an $i \in$ Dom(AVAS($\mathfrak{H}$)) such that Γ = P($\mathfrak{H}_i$). Then we have $(i, \mathfrak{H}_i) \in$ AVAS($\mathfrak{H}$). By hypothesis, we then have $(i, \mathfrak{H}_i) \in$ AVAS($\mathfrak{H}'$). We have AVAS($\mathfrak{H}'$) $\subseteq \mathfrak{H}'$ and hence $(i, \mathfrak{H}_i) \in \mathfrak{H}'$ and therefore $\mathfrak{H}_i = \mathfrak{H}'_i$. Hence we then have Γ = P($\mathfrak{H}_i$) = P($\mathfrak{H}'_i$). Therefore we have $i \in$ Dom(AVAS($\mathfrak{H}'$)) and Γ = P($\mathfrak{H}'_i$). Therefore we have $\Gamma \in$ AVAP($\mathfrak{H}'$). ■

Theorem 2-76. *AVAP is at most as great as AVAS*
For all $\mathfrak{H} \in$ SEQ: $|\text{AVAP}(\mathfrak{H})| \leq |\text{AVAS}(\mathfrak{H})|$.

Proof: Suppose $\mathfrak{H} \in$ SEQ. According to Definition 2-31, we then have that f : AVAP($\mathfrak{H}$) $\rightarrow$ AVAS($\mathfrak{H}$), $f(\Gamma) = (\min(\{i \mid i \in \text{Dom}(\text{AVAS}(\mathfrak{H}))$ and $\text{P}(\mathfrak{H}_i) = \Gamma\}), \mathfrak{H}_{\min(\{i \mid i \in \text{Dom}(\text{AVAS}(\mathfrak{H})) \text{ and } \text{P}(\mathfrak{H}_i) = \Gamma\})})$ is an injection of AVAP($\mathfrak{H}$) into AVAS($\mathfrak{H}$). ■

Theorem 2-77. *AVAP is empty if and only if AVAS is empty*
For all $\mathfrak{H} \in$ SEQ: $|\text{AVAP}(\mathfrak{H})| = 0$ iff $|\text{AVAS}(\mathfrak{H})| = 0$.

Proof: Suppose $\mathfrak{H} \in$ SEQ. Suppose $|\text{AVAP}(\mathfrak{H})| \neq 0$. With Theorem 2-76, we then have $|\text{AVAS}(\mathfrak{H})| \neq 0$. Now, suppose $|\text{AVAS}(\mathfrak{H})| \neq 0$. Then there is $(i, \mathfrak{H}_i) \in$ AVAS($\mathfrak{H}$). With Definition 2-31, we then have $\text{P}(\mathfrak{H}_i) \in$ AVAP($\mathfrak{H}$) and thus $|\text{AVAP}(\mathfrak{H})| \neq 0$. Thus we have $|\text{AVAP}(\mathfrak{H})| \neq 0$ iff $|\text{AVAS}(\mathfrak{H})| \neq 0$, from which the statement follows immediately. ■

Theorem 2-78. *If AVAS is non-redundant, every assumption is available as an assumption at exactly one position*
If $\mathfrak{H} \in$ SEQ and $|\text{AVAP}(\mathfrak{H})| = |\text{AVAS}(\mathfrak{H})|$, then it holds for all $\Gamma \in$ AVAP($\mathfrak{H}$) that there is exactly one $j \in \text{Dom}(\text{AVAS}(\mathfrak{H}))$ such that $\Gamma = \text{P}(\mathfrak{H}_j)$.

Proof: Suppose $\mathfrak{H} \in$ SEQ and $|\text{AVAP}(\mathfrak{H})| = |\text{AVAS}(\mathfrak{H})|$. With Theorem 2-70-(ii), we have AVAS($\mathfrak{H}$) $\subseteq \mathfrak{H}$ and thus, with $\mathfrak{H} \in$ SEQ and Definition 1-24 and Definition 1-23, that $|\text{AVAP}(\mathfrak{H})| = |\text{AVAS}(\mathfrak{H})| = k$ for a $k \in \mathbb{N}$. Now, suppose $\Gamma \in$ AVAP($\mathfrak{H}$). Then we have $k > 0$. According to Definition 2-31, there is then a $j \in \text{Dom}(\text{AVAS}(\mathfrak{H}))$ such that $\Gamma = \text{P}(\mathfrak{H}_j)$. Now, suppose $i \in \text{Dom}(\text{AVAS}(\mathfrak{H}))$ and $\Gamma = \text{P}(\mathfrak{H}_i)$. Suppose for contradiction that $i \neq j$. Then we would have $|\text{AVAS}(\mathfrak{H})\setminus\{(j, \mathfrak{H}_j)\}| = k\text{-}1$, while, on the other hand, f : AVAP($\mathfrak{H}$) $\rightarrow$ AVAS($\mathfrak{H}$)$\setminus\{(j, \mathfrak{H}_j)\}$, $f(\text{B}) = (\min(\{l \mid l \in \text{Dom}(\text{AVAS}(\mathfrak{H})\setminus\{(j, \mathfrak{H}_j)\})$ and $\text{P}(\mathfrak{H}_l) = \text{B}\}), \mathfrak{H}_{\min(\{l \mid l \in \text{Dom}(\text{AVAS}(\mathfrak{H})\setminus\{(j, \mathfrak{H}_j)\}) \text{ and } \text{P}(\mathfrak{H}_l) = \text{B}\})})$ would be an injection of AVAP($\mathfrak{H}$) into AVAS($\mathfrak{H}$)$\setminus\{(j, \mathfrak{H}_j)\}$) and hence $k = |\text{AVAP}(\mathfrak{H})| \leq k\text{-}1$. Contradiction! ■

Theorem 2-79. *AVS, AVAS, AVP and AVAP in concatenations with one-member sentence sequences*

If $\mathfrak{H}, \mathfrak{H}' \in$ SEQ and Dom($\mathfrak{H}'$) = 1, then:

(i) $\text{AVS}(\mathfrak{H}^\frown\mathfrak{H}') \subseteq \text{AVS}(\mathfrak{H}) \cup \{(\text{Dom}(\mathfrak{H}), \mathfrak{H}'_0)\}$,

(ii) $\text{AVAS}(\mathfrak{H}^\frown\mathfrak{H}') \subseteq \text{AVAS}(\mathfrak{H}) \cup \{(\text{Dom}(\mathfrak{H}), \mathfrak{H}'_0)\}$,

(iii) $\text{AVP}(\mathfrak{H}^\frown\mathfrak{H}') \subseteq \text{AVP}(\mathfrak{H}) \cup \{\text{C}(\mathfrak{H}')\}$,

(iv) $\text{AVAP}(\mathfrak{H}^\frown\mathfrak{H}') \subseteq \text{AVAP}(\mathfrak{H}) \cup \{\text{C}(\mathfrak{H}')\}$.

Proof: Suppose $\mathfrak{H}, \mathfrak{H}' \in$ SEQ and suppose Dom($\mathfrak{H}'$) = 1.

Ad (i): Suppose $(i, (\mathfrak{H}^\frown\mathfrak{H}')_i) \in \text{AVS}(\mathfrak{H}^\frown\mathfrak{H}')$. Then we have that $i \in \text{Dom}(\mathfrak{H}^\frown\mathfrak{H}')$ and $\text{P}((\mathfrak{H}^\frown\mathfrak{H}')_i)$ is available in $\mathfrak{H}^\frown\mathfrak{H}'$ at i. We have $i \in \text{Dom}(\mathfrak{H})$ or $i = \text{Dom}(\mathfrak{H})$.

Suppose $i \in \text{Dom}(\mathfrak{H})$. Then we have $(\mathfrak{H}^\frown\mathfrak{H}')_i = \mathfrak{H}_i$. Suppose for contradiction that $\text{P}(\mathfrak{H}_i) = \text{P}((\mathfrak{H}^\frown\mathfrak{H}')_i)$ is not available in $\mathfrak{H}$ at i. According to Definition 2-26, there would then be an $\mathfrak{A}$ such that $\mathfrak{A}$ is a closed segment in $\mathfrak{H}$ and $\min(\text{Dom}(\mathfrak{A})) \leq i < \max(\text{Dom}(\mathfrak{A}))$. Because of $\mathfrak{H} \subseteq \mathfrak{H}^\frown\mathfrak{H}'$, we would then, with Theorem 2-62-(viii), have that $\mathfrak{A}$ is also a closed segment in $\mathfrak{H}^\frown\mathfrak{H}'$ and $\min(\text{Dom}(\mathfrak{A})) \leq i < \max(\text{Dom}(\mathfrak{A}))$. But then $\text{P}((\mathfrak{H}^\frown\mathfrak{H}')_i)$ would not be in $\mathfrak{H}^\frown\mathfrak{H}'$ at i. Therefore we have $i \in \text{Dom}(\mathfrak{H})$ and $\text{P}((\mathfrak{H}^\frown\mathfrak{H}')_i)$ is available in $\mathfrak{H}$ at i and hence $(i, (\mathfrak{H}^\frown\mathfrak{H}')_i) \in \text{AVS}(\mathfrak{H})$.

Now, suppose $i = \text{Dom}(\mathfrak{H})$. Then we have $(\mathfrak{H}^\frown\mathfrak{H}')_i = (\mathfrak{H}^\frown\mathfrak{H}')_{\text{Dom}(\mathfrak{H})} = \mathfrak{H}'_0$ and thus $(i, (\mathfrak{H}^\frown\mathfrak{H}')_i) = (\text{Dom}(\mathfrak{H}), \mathfrak{H}'_0) \in \{(\text{Dom}(\mathfrak{H}), \mathfrak{H}'_0)\}$.

Ad (ii): Suppose $(i, (\mathfrak{H}^\frown\mathfrak{H}')_i) \in \text{AVAS}(\mathfrak{H}^\frown\mathfrak{H}')$. With Theorem 2-70, we then have $(i, (\mathfrak{H}^\frown\mathfrak{H}')_i) \in \text{AVS}(\mathfrak{H}^\frown\mathfrak{H}')$ and $(\mathfrak{H}^\frown\mathfrak{H}')_i \in$ ASENT. With (i), we then have $(i, (\mathfrak{H}^\frown\mathfrak{H}')_i) \in \text{AVS}(\mathfrak{H}) \cup \{(\text{Dom}(\mathfrak{H}), \mathfrak{H}'_0)\}$. Suppose $(i, (\mathfrak{H}^\frown\mathfrak{H}')_i) \notin \{(\text{Dom}(\mathfrak{H}), \mathfrak{H}'_0)\}$ and thus $(i, (\mathfrak{H}^\frown\mathfrak{H}')_i) \in \text{AVS}(\mathfrak{H})$. Then we have $(i, (\mathfrak{H}^\frown\mathfrak{H}')_i) \in \text{AVS}(\mathfrak{H})$ and $(\mathfrak{H}^\frown\mathfrak{H}')_i \in$ ASENT and thus we have that $(i, (\mathfrak{H}^\frown\mathfrak{H}')_i) \in \text{AVAS}(\mathfrak{H})$.

Ad (iii): Suppose $\Gamma \in \text{AVP}(\mathfrak{H}^\frown\mathfrak{H}')$. Then there is an $i \in \text{Dom}(\mathfrak{H}^\frown\mathfrak{H}')$ such that Γ is available in $\mathfrak{H}^\frown\mathfrak{H}'$ at i. Then we have $\Gamma = \text{P}((\mathfrak{H}^\frown\mathfrak{H}')_i)$ and $(i, (\mathfrak{H}^\frown\mathfrak{H}')_i) \in \text{AVS}(\mathfrak{H}^\frown\mathfrak{H}')$. With (i), we then have $(i, (\mathfrak{H}^\frown\mathfrak{H}')_i) \in \text{AVS}(\mathfrak{H}) \cup \{(\text{Dom}(\mathfrak{H}), \mathfrak{H}'_0)\}$. Now, suppose $(i, (\mathfrak{H}^\frown\mathfrak{H}')_i) \in \text{AVS}(\mathfrak{H})$. Then we have $i \in \text{Dom}(\text{AVS}(\mathfrak{H}))$ and $\mathfrak{H}_i = (\mathfrak{H}^\frown\mathfrak{H}')_i$ and hence $\Gamma = \text{P}(\mathfrak{H}_i) \in \text{AVP}(\mathfrak{H})$. Now, suppose $(i, (\mathfrak{H}^\frown\mathfrak{H}')_i) \in \{(\text{Dom}(\mathfrak{H}), \mathfrak{H}'_0)\}$. Then we have $i = \text{Dom}(\mathfrak{H})$ and $(\mathfrak{H}^\frown\mathfrak{H}')_i = \mathfrak{H}'_0$ and hence $\Gamma = \text{P}(\mathfrak{H}'_0) = \text{C}(\mathfrak{H}') \in \{\text{C}(\mathfrak{H}')\}$.

Ad (iv): Suppose $\Gamma \in \text{AVAP}(\mathfrak{H}^\frown\mathfrak{H}')$. Then there is an $i \in \text{Dom}(\text{AVAS}(\mathfrak{H}^\frown\mathfrak{H}'))$ and $\Gamma = \text{P}((\mathfrak{H}^\frown\mathfrak{H}')_i)$. Then we have $(i, (\mathfrak{H}^\frown\mathfrak{H}')_i) \in \text{AVAS}(\mathfrak{H}^\frown\mathfrak{H}')$. With (ii), we then have $(i, (\mathfrak{H}^\frown\mathfrak{H}')_i) \in \text{AVAS}(\mathfrak{H}) \cup \{(\text{Dom}(\mathfrak{H}), \mathfrak{H}'_0)\}$. Now, suppose $(i, (\mathfrak{H}^\frown\mathfrak{H}')_i) \in \text{AVAS}(\mathfrak{H})$. Then we have $i \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{S}))$ and $\mathfrak{S}_i = (\mathfrak{S}^\frown\mathfrak{S}')_i$ and hence $\Gamma = \mathrm{P}(\mathfrak{S}_i) \in \mathrm{AVAP}(\mathfrak{S})$. Now, suppose $(i, (\mathfrak{S}^\frown\mathfrak{S}')_i) \in \{(\mathrm{Dom}(\mathfrak{S}), \mathfrak{S}'_0)\}$. Then we have $i = \mathrm{Dom}(\mathfrak{S})$ and $(\mathfrak{S}^\frown\mathfrak{S}')_i = \mathfrak{S}'_0$ and hence $\Gamma = \mathrm{P}(\mathfrak{S}'_0) = \mathrm{C}(\mathfrak{S}') \in \{\mathrm{C}(\mathfrak{S}')\}$. ■

Theorem 2-80. *AVS, AVAS, AVP and AVAP in concatenations with sentence sequences*
If $\mathfrak{S}, \mathfrak{S}' \in \mathrm{SEQ}$, then:

(i) $\mathrm{AVS}(\mathfrak{S}^\frown\mathfrak{S}') \subseteq \mathrm{AVS}(\mathfrak{S}) \cup \{(\mathrm{Dom}(\mathfrak{S})+i, \mathfrak{S}'_i) \mid i \in \mathrm{Dom}(\mathfrak{S}')\}$,

(ii) $\mathrm{AVAS}(\mathfrak{S}^\frown\mathfrak{S}') \subseteq \mathrm{AVAS}(\mathfrak{S}) \cup \{(\mathrm{Dom}(\mathfrak{S})+i, \mathfrak{S}'_i) \mid i \in \mathrm{Dom}(\mathfrak{S}')\}$.

Proof: By induction on $\mathrm{Dom}(\mathfrak{S}')$. For $\mathrm{Dom}(\mathfrak{S}') = 0$, the induction basis follows with $\mathfrak{S}^\frown\mathfrak{S}' = \mathfrak{S}$. Now, suppose, the statement holds for all $\mathfrak{S}^* \in \mathrm{SEQ}$ with $\mathrm{Dom}(\mathfrak{S}^*) = j$. For (i), we thus have $\mathrm{AVS}(\mathfrak{S}^\frown\mathfrak{S}^*) \subseteq \mathrm{AVS}(\mathfrak{S}) \cup \{(\mathrm{Dom}(\mathfrak{S})+i, \mathfrak{S}^*_i) \mid i \in \mathrm{Dom}(\mathfrak{S}^*)\}$ for all $\mathfrak{S}^* \in \mathrm{SEQ}$ with $\mathrm{Dom}(\mathfrak{S}^*) = j$. Now, suppose $\mathrm{Dom}(\mathfrak{S}') = j+1$. Then we have $\mathrm{Dom}(\mathfrak{S}'\upharpoonright\mathrm{Dom}(\mathfrak{S}')\text{-}1) = j$. According to the I.H., we thus have $\mathrm{AVS}(\mathfrak{S}^\frown(\mathfrak{S}'\upharpoonright\mathrm{Dom}(\mathfrak{S}')\text{-}1)) \subseteq \mathrm{AVS}(\mathfrak{S}) \cup \{(\mathrm{Dom}(\mathfrak{S})+i, (\mathfrak{S}'\upharpoonright\mathrm{Dom}(\mathfrak{S}')\text{-}1)_i) \mid i \in \mathrm{Dom}(\mathfrak{S}'\upharpoonright\mathrm{Dom}(\mathfrak{S}')\text{-}1)\} = \mathrm{AVS}(\mathfrak{S}) \cup \{(\mathrm{Dom}(\mathfrak{S})+i, \mathfrak{S}'_i) \mid i \in \mathrm{Dom}(\mathfrak{S}')\text{-}1\}$. We have $\mathrm{AVS}(\mathfrak{S}^\frown\mathfrak{S}') = \mathrm{AVS}(\mathfrak{S}^\frown(\mathfrak{S}'\upharpoonright\mathrm{Dom}(\mathfrak{S}')\text{-}1)^\frown\{(0, \mathfrak{S}'_{\mathrm{Dom}(\mathfrak{S}')\text{-}1})\})$. According to Theorem 2-79, we have $\mathrm{AVS}(\mathfrak{S}^\frown(\mathfrak{S}'\upharpoonright\mathrm{Dom}(\mathfrak{S}')\text{-}1)^\frown\{(0, \mathfrak{S}'_{\mathrm{Dom}(\mathfrak{S}')\text{-}1})\}) \subseteq \mathrm{AVS}(\mathfrak{S}^\frown(\mathfrak{S}'\upharpoonright\mathrm{Dom}(\mathfrak{S}')\text{-}1)) \cup \{(\mathrm{Dom}(\mathfrak{S}^\frown(\mathfrak{S}'\upharpoonright\mathrm{Dom}(\mathfrak{S}')\text{-}1)), \mathfrak{S}'_{\mathrm{Dom}(\mathfrak{S}')\text{-}1})\} = \mathrm{AVS}(\mathfrak{S}^\frown(\mathfrak{S}'\upharpoonright\mathrm{Dom}(\mathfrak{S}')\text{-}1)) \cup \{(\mathrm{Dom}(\mathfrak{S})+(\mathrm{Dom}(\mathfrak{S}')\text{-}1), \mathfrak{S}'_{\mathrm{Dom}(\mathfrak{S}')\text{-}1})\}$. Altogether, we thus have $\mathrm{AVS}(\mathfrak{S}^\frown\mathfrak{S}') \subseteq \mathrm{AVS}(\mathfrak{S}) \cup \{(\mathrm{Dom}(\mathfrak{S})+i, \mathfrak{S}'_i) \mid i \in \mathrm{Dom}(\mathfrak{S}')\text{-}1\} \cup \{(\mathrm{Dom}(\mathfrak{S})+(\mathrm{Dom}(\mathfrak{S}')\text{-}1), \mathfrak{S}'_{\mathrm{Dom}(\mathfrak{S}')\text{-}1})\}$ and thus $\mathrm{AVS}(\mathfrak{S}^\frown\mathfrak{S}') \subseteq \mathrm{AVS}(\mathfrak{S}) \cup \{(\mathrm{Dom}(\mathfrak{S})+i, \mathfrak{S}'_i) \mid i \in \mathrm{Dom}(\mathfrak{S}')\}$. The proof of (ii) is carried out analogously. ■

Theorem 2-81. *AVS, AVAS, AVP and AVAP in restrictions on* $\mathrm{Dom}(\mathfrak{S})\text{-}1$
If $\mathfrak{S} \in \mathrm{SEQ}$, then:

(i) $\mathrm{AVS}(\mathfrak{S}) \subseteq \mathrm{AVS}(\mathfrak{S}\upharpoonright\mathrm{Dom}(\mathfrak{S})\text{-}1) \cup \{(\mathrm{Dom}(\mathfrak{S})\text{-}1, \mathfrak{S}_{\mathrm{Dom}(\mathfrak{S})\text{-}1})\}$,

(ii) $\mathrm{AVAS}(\mathfrak{S}) \subseteq \mathrm{AVAS}(\mathfrak{S}\upharpoonright\mathrm{Dom}(\mathfrak{S})\text{-}1) \cup \{(\mathrm{Dom}(\mathfrak{S})\text{-}1, \mathfrak{S}_{\mathrm{Dom}(\mathfrak{S})\text{-}1})\}$,

(iii) $\mathrm{AVP}(\mathfrak{S}) \subseteq \mathrm{AVP}(\mathfrak{S}\upharpoonright\mathrm{Dom}(\mathfrak{S})\text{-}1) \cup \{\mathrm{P}(\mathfrak{S}_{\mathrm{Dom}(\mathfrak{S})\text{-}1})\}$,

(iv) $\mathrm{AVAP}(\mathfrak{S}) \subseteq \mathrm{AVAP}(\mathfrak{S}\upharpoonright\mathrm{Dom}(\mathfrak{S})\text{-}1) \cup \{\mathrm{P}(\mathfrak{S}_{\mathrm{Dom}(\mathfrak{S})\text{-}1})\}$.

Proof: Suppose $\mathfrak{S} \in \mathrm{SEQ}$. For $\mathfrak{S} = \emptyset$, we have that $\mathrm{AVS}(\mathfrak{S}) \cup \mathrm{AVAS}(\mathfrak{S}) \cup \mathrm{AVP}(\mathfrak{S}) \cup \mathrm{AVAP}(\mathfrak{S}) = \emptyset$ and thus the theorem holds. Now, suppose $\mathfrak{S} \neq \emptyset$. Then we have $\mathfrak{S} = (\mathfrak{S}\upharpoonright\mathrm{Dom}(\mathfrak{S})\text{-}1)^\frown\{(0, \mathfrak{S}_{\mathrm{Dom}(\mathfrak{S})\text{-}1})\}$ and the theorem follows with Theorem 2-79. ■

Theorem 2-82. *The conclusion is always available*
If $\mathfrak{H} \in \text{SEQ}\setminus\{\emptyset\}$, then $\text{C}(\mathfrak{H})$ is available in $\mathfrak{H}$ at $\text{Dom}(\mathfrak{H})$-1.

Proof: Suppose $\mathfrak{H} \in \text{SEQ}\setminus\{\emptyset\}$. Then it holds for all closed segments $\mathfrak{A}$ in $\mathfrak{H}$ that $\max(\text{Dom}(\mathfrak{A})) \leq \text{Dom}(\mathfrak{H})\text{-1}$ and therefore there is no closed segment $\mathfrak{A}$ in $\mathfrak{H}$ such that $\min(\text{Dom}(\mathfrak{A})) \leq \text{Dom}(\mathfrak{H})\text{-1} < \max(\text{Dom}(\mathfrak{A}))$. Therefore $\text{P}(\mathfrak{H}_{\text{Dom}(\mathfrak{H})\text{-1}}) = \text{C}(\mathfrak{H})$ is available in $\mathfrak{H}$ at $\text{Dom}(\mathfrak{H})$-1. ■

Theorem 2-83. *Connections between non-availability and the emergence of a closed segment in the transition from* $\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-1}$ *to* $\mathfrak{H}$
If $\mathfrak{H} \in \text{SEQ}$ and $\text{AVS}(\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-1})\setminus\text{AVS}(\mathfrak{H}) \neq \emptyset$, then:
There is a $\mathfrak{B}$ such that $\mathfrak{B}$ is a closed segment in $\mathfrak{H}$ and

(i) $\min(\text{Dom}(\mathfrak{B})) \leq \text{Dom}(\mathfrak{H})\text{-2}$ and $\max(\text{Dom}(\mathfrak{B})) = \text{Dom}(\mathfrak{H})\text{-1}$,
(ii) For all closed segments $\mathfrak{C}$ in $\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-1}$ it holds that $\mathfrak{B}\upharpoonright\text{Dom}(\mathfrak{H})\text{-1} \cap \mathfrak{C} = \emptyset$ or $\min(\text{Dom}(\mathfrak{B})) < \min(\text{Dom}(\mathfrak{C}))$ and $\max(\text{Dom}(\mathfrak{C})) < \text{Dom}(\mathfrak{H})\text{-1}$,
(iii) For all closed segments $\mathfrak{C}^*$ in $\mathfrak{H}$: If $\mathfrak{C}^*$ is not a closed segment in $\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-1}$, then $\mathfrak{C}^* = \mathfrak{B}$,
(iv) $\text{AVS}(\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-1})\setminus\text{AVS}(\mathfrak{H}) \subseteq \{(j, \mathfrak{H}_j) \mid \min(\text{Dom}(\mathfrak{B})) \leq j < \text{Dom}(\mathfrak{H})\text{-1}\}$,
(v) $\text{AVS}(\mathfrak{H}) = (\text{AVS}(\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-1})\setminus\{(j, \mathfrak{H}_j) \mid \min(\text{Dom}(\mathfrak{B})) \leq j < \text{Dom}(\mathfrak{H})\text{-1}\}) \cup \{(\text{Dom}(\mathfrak{H})\text{-1}, \mathfrak{H}_{\text{Dom}(\mathfrak{H})\text{-1}})\}$,
(vi) $\text{AVAS}(\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-1})\setminus\text{AVAS}(\mathfrak{H}) = \{(\min(\text{Dom}(\mathfrak{B})), \mathfrak{H}_{\min(\text{Dom}(\mathfrak{B}))})\}$,
(vii) $\text{AVAS}(\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-1}) = \text{AVAS}(\mathfrak{H}) \cup \{(\min(\text{Dom}(\mathfrak{B})), \mathfrak{H}_{\min(\text{Dom}(\mathfrak{B}))})\}$,
(viii) $\text{AVP}(\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-1})\setminus\text{AVP}(\mathfrak{H}) \subseteq \{\text{P}(\mathfrak{H}_j) \mid \min(\text{Dom}(\mathfrak{B})) \leq j < \text{Dom}(\mathfrak{H})\text{-1}\}$,
(ix) $\text{AVP}(\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-1}) \subseteq \{\text{P}(\mathfrak{H}_j) \mid j \in \text{Dom}(\text{AVS}(\mathfrak{H})\upharpoonright\text{Dom}(\mathfrak{H})\text{-1})\} \cup \{\text{P}(\mathfrak{H}_j) \mid \min(\text{Dom}(\mathfrak{B})) \leq j < \text{Dom}(\mathfrak{H})\text{-1}\}$,
(x) $\text{AVAP}(\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-1})\setminus\text{AVAP}(\mathfrak{H}) \subseteq \{\text{P}(\mathfrak{H}_{\min(\text{Dom}(\mathfrak{B}))})\}$, and
(xi) $\text{AVAP}(\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-1}) = \text{AVAP}(\mathfrak{H}) \cup \{\text{P}(\mathfrak{H}_{\min(\text{Dom}(\mathfrak{B}))})\}$.

Proof: Suppose $\mathfrak{H} \in \text{SEQ}$ and suppose $\text{AVS}(\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-1})\setminus\text{AVS}(\mathfrak{H}) \neq \emptyset$. According to Definition 2-28, there is then an $i \in \text{Dom}(\mathfrak{H})\text{-1}$ such that $(i, \mathfrak{H}_i) \in \text{AVS}(\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-1})\setminus\text{AVS}(\mathfrak{H})$. Then we have $\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-1} \neq \emptyset$ and thus $\mathfrak{H} \neq \emptyset$.

According to Definition 2-28 and Definition 2-26, there is then no $\mathfrak{B}'$ such that $\mathfrak{B}'$ is a closed segment in $\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-1}$ and $\min(\text{Dom}(\mathfrak{B}')) \leq i < \max(\text{Dom}(\mathfrak{B}'))$, and that there is a $\mathfrak{B}$ such that $\mathfrak{B}$ is a closed segment in $\mathfrak{H}$ and $\min(\text{Dom}(\mathfrak{B})) \leq i < \max(\text{Dom}(\mathfrak{B}))$.

Ad (i): We have $\max(\text{Dom}(\mathfrak{B})) \leq \text{Dom}(\mathfrak{H})\text{-1}$. Suppose for contradiction that $\text{Dom}(\mathfrak{H})\text{-2} < \min(\text{Dom}(\mathfrak{B}))$. With Theorem 2-44, we would then have $\text{Dom}(\mathfrak{H})\text{-1} \leq \min(\text{Dom}(\mathfrak{B})) < \max(\text{Dom}(\mathfrak{B})) \leq \text{Dom}(\mathfrak{H})\text{-1}$. Contradiction! Therefore we have $\min(\text{Dom}(\mathfrak{B})) \leq$

$\mathrm{Dom}(\mathfrak{H})$-2. Now, suppose for contradiction that $\max(\mathrm{Dom}(\mathfrak{B})) < \mathrm{Dom}(\mathfrak{H})-1$. Then we would have $\min(\mathrm{Dom}(\mathfrak{B})) < \max(\mathrm{Dom}(\mathfrak{B})) < \mathrm{Dom}(\mathfrak{H})-1$. With Theorem 2-64-(viii) and Theorem 2-62-(viii), we would then have that $\mathfrak{B}$ is a closed segment in $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})-1$ and that $\min(\mathrm{Dom}(\mathfrak{B})) \leq i < \max(\mathrm{Dom}(\mathfrak{B}))$. But then we would have $(i, \mathfrak{H}_i) \notin \mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})-1)$. Therefore we have that $\max(\mathrm{Dom}(\mathfrak{B})) = \mathrm{Dom}(\mathfrak{H})-1$ and hence that $\min(\mathrm{Dom}(\mathfrak{B})) \leq \mathrm{Dom}(\mathfrak{H})-2$ and $\max(\mathrm{Dom}(\mathfrak{B})) = \mathrm{Dom}(\mathfrak{H})-1$.

Ad (ii): Suppose $\mathfrak{C}$ is a closed segment in $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})-1$. Now, suppose $\mathfrak{B}\upharpoonright\mathrm{Dom}(\mathfrak{H})-1 \cap \mathfrak{C} \neq \emptyset$. Then we have $\mathfrak{B} \cap \mathfrak{C} \neq \emptyset$. With Theorem 2-57, it then holds that $\mathfrak{B} \subseteq \mathfrak{C}$ or $\mathfrak{C} \subseteq \mathfrak{B}$. Since $\mathfrak{C} \subseteq \mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})-1$ and $(\mathrm{Dom}(\mathfrak{H})-1, \mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-1}) \in \mathfrak{B}$, we have $\mathfrak{B} \not\subseteq \mathfrak{C}$. Thus we have $\mathfrak{C} \subset \mathfrak{B}$. With Theorem 2-56-(i) and -(iii), we thus have $\min(\mathrm{Dom}(\mathfrak{B})) < \min(\mathrm{Dom}(\mathfrak{C}))$ and $\max(\mathrm{Dom}(\mathfrak{C})) < \max(\mathrm{Dom}(\mathfrak{B})) = \mathrm{Dom}(\mathfrak{H})-1$.

Ad (iii): Suppose $\mathfrak{C}^*$ is a closed segment in $\mathfrak{H}$, but not a closed segment in $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})-1$. Then we have $\max(\mathrm{Dom}(\mathfrak{C}^*)) = \mathrm{Dom}(\mathfrak{H})-1$. First, we have $\max(\mathrm{Dom}(\mathfrak{C}^*)) \leq \mathrm{Dom}(\mathfrak{H})-1$. If $\max(\mathrm{Dom}(\mathfrak{C}^*)) < \mathrm{Dom}(\mathfrak{H})-1$, then we would have, with Theorem 2-64-(viii) and Theorem 2-62-(viii), that $\mathfrak{C}^*$ is a closed segment in $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})-1$, which contradicts the hypothesis. Therefore we have $\mathrm{Dom}(\mathfrak{H})-1 \leq \max(\mathrm{Dom}(\mathfrak{C}^*))$ and hence $\max(\mathrm{Dom}(\mathfrak{C}^*)) = \mathrm{Dom}(\mathfrak{H})-1 = \max(\mathrm{Dom}(\mathfrak{B}))$. With Theorem 2-53, it then follows that $\mathfrak{C}^* = \mathfrak{B}$.

Ad (iv): Suppose $(i, \mathfrak{H}_i) \in \mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})-1)\setminus\mathrm{AVS}(\mathfrak{H})$. Then there is a closed segment $\mathfrak{C}$ in $\mathfrak{H}$ such that $\min(\mathrm{Dom}(\mathfrak{C})) \leq i < \max(\mathrm{Dom}(\mathfrak{C}))$ and $\mathfrak{C}$ is not a closed segment in $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})-1$. Then it holds with (iii) that $\mathfrak{C} = \mathfrak{B}$ and hence that $\min(\mathrm{Dom}(\mathfrak{B})) \leq i < \max(\mathrm{Dom}(\mathfrak{B})) = \mathrm{Dom}(\mathfrak{H})-1$. It then follows that $(i, \mathfrak{H}_i) \in \{(j, \mathfrak{H}_j) \mid \min(\mathrm{Dom}(\mathfrak{B})) \leq j < \mathrm{Dom}(\mathfrak{H})-1\}$.

Ad (v): First, suppose $(i, \mathfrak{H}_i) \in \mathrm{AVS}(\mathfrak{H})$. With Theorem 2-81-(i), we then have $(i, \mathfrak{H}_i) \in \mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})-1) \cup \{(\mathrm{Dom}(\mathfrak{H})-1, \mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-1})\}$. Also, we have that there is no closed segment $\mathfrak{C}$ in $\mathfrak{H}$ such that $\min(\mathrm{Dom}(\mathfrak{C})) \leq i < \max(\mathrm{Dom}(\mathfrak{C}))$. Since $\mathfrak{B}$ is a closed segment in $\mathfrak{H}$, it then follows with (i) that $(i, \mathfrak{H}_i) \notin \{(j, \mathfrak{H}_j) \mid \min(\mathrm{Dom}(\mathfrak{B})) \leq j < \mathrm{Dom}(\mathfrak{H})-1\}$. Hence we have $(i, \mathfrak{H}_i) \in (\mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})-1)\setminus\{(j, \mathfrak{H}_j) \mid \min(\mathrm{Dom}(\mathfrak{B})) \leq j < \mathrm{Dom}(\mathfrak{H})-1\}) \cup \{(\mathrm{Dom}(\mathfrak{H})-1, \mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-1})\}$.

Now, suppose $(i, \mathfrak{H}_i) \in (\mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})-1)\setminus\{(j, \mathfrak{H}_j) \mid \min(\mathrm{Dom}(\mathfrak{B})) \leq j < \mathrm{Dom}(\mathfrak{H})-1\}) \cup \{(\mathrm{Dom}(\mathfrak{H})-1, \mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-1})\}$. First, suppose $(i, \mathfrak{H}_i) \in \mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})-1)\setminus\{(j, \mathfrak{H}_j) \mid \min(\mathrm{Dom}(\mathfrak{B})) \leq j < \mathrm{Dom}(\mathfrak{H})-1\}$. If $(i, \mathfrak{H}_i) \notin \mathrm{AVS}(\mathfrak{H})$, we would have $(i, \mathfrak{H}_i) \in \mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})-1)\setminus\mathrm{AVS}(\mathfrak{H})$ and $(i, \mathfrak{H}_i) \notin \{(j, \mathfrak{H}_j) \mid \min(\mathrm{Dom}(\mathfrak{B})) \leq j < \mathrm{Dom}(\mathfrak{H})-1\}$, which contradicts (iv). In the first case, we thus have $(i, \mathfrak{H}_i) \in \mathrm{AVS}(\mathfrak{H})$. Now, suppose $(i, \mathfrak{H}_i) \in \{(\mathrm{Dom}(\mathfrak{H})\text{-}1, \mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})\text{-}1})\}$. Then we have $i = \mathrm{Dom}(\mathfrak{H})\text{-}1$ and $\mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})\text{-}1}) = \mathrm{C}(\mathfrak{H})$ and thus, with Theorem 2-82, that in the second case it holds as well that $(i, \mathfrak{H}_i) \in \mathrm{AVS}(\mathfrak{H})$.

Ad (vi): First, suppose $(i, \mathfrak{H}_i) \in \mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1))\backslash\mathrm{AVAS}(\mathfrak{H})$. Then we have $(i, \mathfrak{H}_i) \in (\mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) \cap \mathrm{AS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1))\backslash(\mathrm{AVS}(\mathfrak{H}) \cap \mathrm{AS}(\mathfrak{H}))$. Since $\mathrm{AS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) \subseteq \mathrm{AS}(\mathfrak{H})$, we have $(i, \mathfrak{H}_i) \in \mathrm{AS}(\mathfrak{H})$ and thus $(i, \mathfrak{H}_i) \notin \mathrm{AVS}(\mathfrak{H})$ and hence $(i, \mathfrak{H}_i) \in \mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\backslash\mathrm{AVS}(\mathfrak{H})$. With (iv) and (i), it thus holds that $(i, \mathfrak{H}_i) \in \mathfrak{B}$. Then we have $(i, \mathfrak{H}_i) \in \mathrm{AS}(\mathfrak{H}) \cap \mathfrak{B}$ and hence there is, with Theorem 2-47, a $\mathfrak{C} \subseteq \mathfrak{B}$ such that $\mathfrak{C}$ is a closed segment in $\mathfrak{H}$ and $i = \min(\mathrm{Dom}(\mathfrak{C}))$. Because of $(i, \mathfrak{H}_i) \in \mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$, $\mathfrak{C}$ is then not a closed segment in $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1$. With (iii), we then have $\mathfrak{C} = \mathfrak{B}$ and thus $i = \min(\mathrm{Dom}(\mathfrak{C})) = \min(\mathrm{Dom}(\mathfrak{B}))$. Then we have $(i, \mathfrak{H}_i) = (\min(\mathrm{Dom}(\mathfrak{B})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))})$.

Now, we have to show that $\{(\min(\mathrm{Dom}(\mathfrak{B})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))})\} \subseteq \mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\backslash\mathrm{AVAS}(\mathfrak{H})$. First, we have $(\min(\mathrm{Dom}(\mathfrak{B})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))}) \in \mathrm{AS}(\mathfrak{H})$. Suppose for contradiction that there is a closed segment $\mathfrak{C}$ in $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1$ such that $\min(\mathrm{Dom}(\mathfrak{C})) \leq \min(\mathrm{Dom}(\mathfrak{B})) < \max(\mathrm{Dom}(\mathfrak{C}))$. Then we would have $\mathfrak{C} \cap \mathfrak{B}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1 \neq \emptyset$. But with (ii), we would then have $\min(\mathrm{Dom}(\mathfrak{B})) < \min(\mathrm{Dom}(\mathfrak{C}))$. Contradiction! Therefore there is no such closed segment $\mathfrak{C}$ in $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1$ and hence we have $(\min(\mathrm{Dom}(\mathfrak{B})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))}) \in \mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. On the other hand, we have with $\mathfrak{B}$ itself a closed segment $\mathfrak{B}'$ in $\mathfrak{H}$ such that $\min(\mathrm{Dom}(\mathfrak{B}')) \leq \min(\mathrm{Dom}(\mathfrak{B})) < \max(\mathrm{Dom}(\mathfrak{B}'))$ and thus we have $(\min(\mathrm{Dom}(\mathfrak{B})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))}) \notin \mathrm{AVAS}(\mathfrak{H})$ and hence $(\min(\mathrm{Dom}(\mathfrak{B})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))}) \in \mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\backslash\mathrm{AVAS}(\mathfrak{H})$.

Ad (vii): First, suppose $(i, \mathfrak{H}_i) \in \mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. Then we have $(i, \mathfrak{H}_i) \in \mathrm{AVAS}(\mathfrak{H})$ or $(i, \mathfrak{H}_i) \notin \mathrm{AVAS}(\mathfrak{H})$. Now, suppose $(i, \mathfrak{H}_i) \notin \mathrm{AVAS}(\mathfrak{H})$. Then we have $(i, \mathfrak{H}_i) \in \mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\backslash\mathrm{AVAS}(\mathfrak{H})$ and thus, with (vi), $(i, \mathfrak{H}_i) \in \{(\min(\mathrm{Dom}(\mathfrak{B})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))})\}$. Therefore we have in both cases $(i, \mathfrak{H}_i) \in \mathrm{AVAS}(\mathfrak{H}) \cup \{(\min(\mathrm{Dom}(\mathfrak{B})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))})\}$.

Now, suppose $(i, \mathfrak{H}_i) \in \mathrm{AVAS}(\mathfrak{H}) \cup \{(\min(\mathrm{Dom}(\mathfrak{B})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))})\}$. First, suppose $(i, \mathfrak{H}_i) \in \mathrm{AVAS}(\mathfrak{H})$. Then we have $(i, \mathfrak{H}_i) \in \mathrm{AS}(\mathfrak{H})$. With Theorem 2-81-(ii), we also have $(i, \mathfrak{H}_i) \in \mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) \cup \{(\mathrm{Dom}(\mathfrak{H})\text{-}1, \mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})\text{-}1})\}$. With (i), it holds that $\max(\mathrm{Dom}(\mathfrak{B})) = \mathrm{Dom}(\mathfrak{H})\text{-}1$. Since $\mathfrak{B}$ is a closed segment in $\mathfrak{H}$ and thus a CdI- or NI- or RA-like segment in $\mathfrak{H}$, we have, with Theorem 2-29, that $(\mathrm{Dom}(\mathfrak{H})\text{-}1, \mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})\text{-}1}) \notin \mathrm{AS}(\mathfrak{H})$ and thus that $(i, \mathfrak{H}_i) \notin \{(\mathrm{Dom}(\mathfrak{H})\text{-}1, \mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})\text{-}1})\}$. Thus we have $(i, \mathfrak{H}_i) \in$

$\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. Now, suppose $(i, \mathfrak{H}_i) \in \{(\min(\mathrm{Dom}(\mathfrak{B})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))})\}$. With (vi), we then have again that $(i, \mathfrak{H}_i) \in \mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$.

Ad (viii): Suppose $\Gamma \in \mathrm{AVP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\backslash\mathrm{AVP}(\mathfrak{H})$. Then there is an $i \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1))$ and $\Gamma = \mathrm{P}(\mathfrak{H}_i)$. Then we have $(i, \mathfrak{H}_i) \in \mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ and $(i, \mathfrak{H}_i) \notin \mathrm{AVS}(\mathfrak{H})$, because otherwise we would have $\Gamma \in \mathrm{AVP}(\mathfrak{H})$. With (iv), it then holds that $(i, \mathfrak{H}_i) \in \{(j, \mathfrak{H}_j) \mid \min(\mathrm{Dom}(\mathfrak{B})) \leq j < \mathrm{Dom}(\mathfrak{H})\text{-}1\}$. Then we have $\Gamma \in \{\mathrm{P}(\mathfrak{H}_j) \mid \min(\mathrm{Dom}(\mathfrak{B})) \leq j < \mathrm{Dom}(\mathfrak{H})\text{-}1\}$.

Ad (ix): Suppose $\Gamma \in \mathrm{AVP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. Then there is an $i \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1))$ such that $\Gamma = \mathrm{P}(\mathfrak{H}_i)$. Then we have $(i, \mathfrak{H}_i) \in \mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ and thus also $i < \mathrm{Dom}(\mathfrak{H})\text{-}1$. We have that $\Gamma \in \{\mathrm{P}(\mathfrak{H}_j) \mid \min(\mathrm{Dom}(\mathfrak{B})) \leq j < \mathrm{Dom}(\mathfrak{H})\text{-}1\}$ or $\Gamma \notin \{\mathrm{P}(\mathfrak{H}_j) \mid \min(\mathrm{Dom}(\mathfrak{B})) \leq j < \mathrm{Dom}(\mathfrak{H})\text{-}1\}$. Now, suppose $\Gamma \notin \{\mathrm{P}(\mathfrak{H}_j) \mid \min(\mathrm{Dom}(\mathfrak{B})) \leq j < \mathrm{Dom}(\mathfrak{H})\text{-}1\}$. Then we have $(i, \mathfrak{H}_i) \notin \{(j, \mathfrak{H}_j) \mid \min(\mathrm{Dom}(\mathfrak{B})) \leq j < \mathrm{Dom}(\mathfrak{H})\text{-}1\}$ and thus $(i, \mathfrak{H}_i) \in \mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\backslash\{(j, \mathfrak{H}_j) \mid \min(\mathrm{Dom}(\mathfrak{B})) \leq j < \mathrm{Dom}(\mathfrak{H})\text{-}1\}$. With (v), we then have $(i, \mathfrak{H}_i) \in \mathrm{AVS}(\mathfrak{H})$ and, with $i < \mathrm{Dom}(\mathfrak{H})\text{-}1$, it then holds that $(i, \mathfrak{H}_i) \in \mathrm{AVS}(\mathfrak{H})\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1$. Therefore we have $i \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H})\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ and thus $\Gamma \in \{\mathrm{P}(\mathfrak{H}_j) \mid j \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H})\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\}$. Therefore we have in both cases $\Gamma \in \{\mathrm{P}(\mathfrak{H}_j) \mid j \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H})\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\} \cup \{\mathrm{P}(\mathfrak{H}_j) \mid \min(\mathrm{Dom}(\mathfrak{B})) \leq j < \mathrm{Dom}(\mathfrak{H})\text{-}1\}$.

Ad (x): Suppose $\Gamma \in \mathrm{AVAP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\backslash\mathrm{AVAP}(\mathfrak{H})$. Then there is an $i \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1))$ and $\Gamma = \mathrm{P}(\mathfrak{H}_i)$. Then we have $(i, \mathfrak{H}_i) \in \mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ and $(i, \mathfrak{H}_i) \notin \mathrm{AVAS}(\mathfrak{H})$, because otherwise we would have $\Gamma \in \mathrm{AVAP}(\mathfrak{H})$. With (vi), it then follows that $(i, \mathfrak{H}_i) = (\min(\mathrm{Dom}(\mathfrak{B})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))})$. Then we have $\Gamma = \mathrm{P}(\mathfrak{H}_i) = \mathrm{P}(\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))}) \in \{\mathrm{P}(\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))})\}$.

And last, ad (xi): With (vii) it holds that $\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) = \mathrm{AVAS}(\mathfrak{H}) \cup \{(\min(\mathrm{Dom}(\mathfrak{B})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))})\}$. We thus have: $\Gamma \in \mathrm{AVAP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ iff there is an $i \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1))$ and $\Gamma = \mathrm{P}(\mathfrak{H}_i)$ iff there is an $i \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})) \cup \{\min(\mathrm{Dom}(\mathfrak{B}))\}$ and $\Gamma = \mathrm{P}(\mathfrak{H}_i)$ iff $\Gamma \in \mathrm{AVAP}(\mathfrak{H}) \cup \{\mathrm{P}(\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))})\}$. Hence we have $\mathrm{AVAP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) = \mathrm{AVAP}(\mathfrak{H}) \cup \{\mathrm{P}(\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))})\}$. ■

Theorem 2-84. *AVS-reduction in the transition from* $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1$ *to* $\mathfrak{H}$ *if and only if a new closed segment emerges*

If $\mathfrak{H} \in \mathrm{SEQ}$, then:

$\mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\backslash\mathrm{AVS}(\mathfrak{H}) \neq \emptyset$

iff

There is a $\mathfrak{B}$ such that

(i) $\mathfrak{B}$ is a closed segment in $\mathfrak{H}$, and

(ii) $\min(\mathrm{Dom}(\mathfrak{B})) \leq \mathrm{Dom}(\mathfrak{H})\text{-}2$ and $\max(\mathrm{Dom}(\mathfrak{B})) = \mathrm{Dom}(\mathfrak{H})\text{-}1$.

Proof: Suppose $\mathfrak{H} \in \mathrm{SEQ}$. The left-right-direction follows immediately with Theorem 2-83. Now, for the right-left-direction, suppose there is a $\mathfrak{B}$ such that $\mathfrak{B}$ is a closed segment in $\mathfrak{H}$ and $\min(\mathrm{Dom}(\mathfrak{B})) \leq \mathrm{Dom}(\mathfrak{H})\text{-}2$ and $\max(\mathrm{Dom}(\mathfrak{B})) = \mathrm{Dom}(\mathfrak{H})\text{-}1$. Then it holds that $(\min(\mathrm{Dom}(\mathfrak{B})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))}) \in \mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\backslash\mathrm{AVS}(\mathfrak{H})$. First, we have $(\min(\mathrm{Dom}(\mathfrak{B})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))}) \notin \mathrm{AVS}(\mathfrak{H})$, because with $\mathfrak{B}$ itself there is a closed segment $\mathfrak{B}'$ in $\mathfrak{H}$ such that $\min(\mathrm{Dom}(\mathfrak{B}')) \leq \min(\mathrm{Dom}(\mathfrak{B})) < \max(\mathrm{Dom}(\mathfrak{B}'))$.

Now, suppose $\mathfrak{C}$ is a closed segment in $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1$. Because of $\mathfrak{C} \subseteq \mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1$ and $(\mathrm{Dom}(\mathfrak{H})\text{-}1, \mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})\text{-}1}) \in \mathfrak{B}$, we then have $\mathfrak{B} \not\subseteq \mathfrak{C}$. With Theorem 2-52, we then have $\min(\mathrm{Dom}(\mathfrak{B})) \notin \mathrm{Dom}(\mathfrak{C})$. Thus there is no closed segment $\mathfrak{C}$ in $\mathfrak{H}$ such that $\min(\mathrm{Dom}(\mathfrak{C})) \leq \min(\mathrm{Dom}(\mathfrak{B})) < \max(\mathrm{Dom}(\mathfrak{C}))$ and thus it holds that $(\min(\mathrm{Dom}(\mathfrak{B})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))}) \in \mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. Hence we have $(\min(\mathrm{Dom}(\mathfrak{B})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))}) \in \mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\backslash\mathrm{AVS}(\mathfrak{H})$. ■

Theorem 2-85. *AVAS-reduction in the transition from* $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1$ *to* $\mathfrak{H}$ *if and only if this involves the emergence of a new closed segment whose first member is exactly the now unavailable assumption-sentence and the maximal member in* $\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$

If $\mathfrak{H} \in \mathrm{SEQ}$, then:

$\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\backslash\mathrm{AVAS}(\mathfrak{H}) \neq \emptyset$

iff

There is a $\mathfrak{B}$ such that

(i) $\mathfrak{B}$ is a closed segment in $\mathfrak{H}$,

(ii) $\min(\mathrm{Dom}(\mathfrak{B})) \leq \mathrm{Dom}(\mathfrak{H})\text{-}2$ and $\max(\mathrm{Dom}(\mathfrak{B})) = \mathrm{Dom}(\mathfrak{H})\text{-}1$, and

(iii) $\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\backslash\mathrm{AVAS}(\mathfrak{H}) = \{(\min(\mathrm{Dom}(\mathfrak{B})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))})\} = \{(\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1))), \mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)))})\}$.

Proof: Suppose $\mathfrak{H} \in \mathrm{SEQ}$. (*L-R*): Suppose $\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\backslash\mathrm{AVAS}(\mathfrak{H}) \neq \emptyset$. With Theorem 2-73, we then have that also $\mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\backslash\mathrm{AVS}(\mathfrak{H}) \neq \emptyset$. With Theorem 2-83, there is then a $\mathfrak{B}$ such that $\mathfrak{B}$ is a closed segment in $\mathfrak{H}$ and $\min(\mathrm{Dom}(\mathfrak{B})) \leq$

Dom($\mathfrak{H}$)-2 and max(Dom($\mathfrak{B}$)) = Dom($\mathfrak{H}$)-1 and AVAS($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1)\AVAS($\mathfrak{H}$) = {(min(Dom($\mathfrak{B}$)), $\mathfrak{H}_{\text{min(Dom}(\mathfrak{B}))}$)}.

Then we have min(Dom($\mathfrak{B}$)) = max(Dom(AVAS($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1))). First, we have (min(Dom($\mathfrak{B}$)), $\mathfrak{H}_{\text{min(Dom}(\mathfrak{B}))}$) ∈ AVAS($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1) and thus min(Dom($\mathfrak{B}$)) ∈ Dom(AVAS($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1)). Now, suppose k ∈ Dom(AVAS($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1)) and suppose min(Dom($\mathfrak{B}$)) ≤ k. Then we have (k, $\mathfrak{H}_k$) ∈ AVAS($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1) and thus ($k$, $\mathfrak{H}_k$) ∈ AS($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1) and thus also ($k$, $\mathfrak{H}_k$) ∈ AS($\mathfrak{H}$). Also, we have min(Dom($\mathfrak{B}$)) ≤ k < Dom($\mathfrak{H}$)-1 = max(Dom($\mathfrak{B}$)). Thus we have k ∈ AS($\mathfrak{H}$) ∩ Dom($\mathfrak{B}$). With Theorem 2-66, we then have k = min(Dom($\mathfrak{B}$)) or there is a $\mathfrak{C}$ such that k = min(Dom($\mathfrak{C}$)) and min(Dom($\mathfrak{B}$)) < min(Dom($\mathfrak{C}$)) < max(Dom($\mathfrak{C}$)) < max(Dom($\mathfrak{B}$)) = Dom($\mathfrak{H}$)-1. The second case is, however, exluded, because otherwise there would be, with Theorem 2-64-(viii) and Theorem 2-62-(viii), a closed segment $\mathfrak{C}$ in $\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1 with min(Dom($\mathfrak{C}$)) ≤ k < max(Dom($\mathfrak{C}$)), and we would thus have (k, $\mathfrak{H}_k$) ∉ AVAS($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1). Therefore we have k = min(Dom($\mathfrak{B}$)). Hence we have min(Dom($\mathfrak{B}$)) = max(Dom(AVAS($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1))) and thus {(min(Dom($\mathfrak{B}$)), $\mathfrak{H}_{\text{min(Dom}(\mathfrak{B}))}$)} = {(max(Dom(AVAS($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1))), $\mathfrak{H}_{\text{max(Dom(AVAS}(\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-1})))}$)}.

(*R-L*): Now, suppose there is a closed segment $\mathfrak{B}$ in $\mathfrak{H}$ such that AVAS($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1))\AVAS($\mathfrak{H}$) = {(min(Dom($\mathfrak{B}$)), $\mathfrak{H}_{\text{min(Dom}(\mathfrak{B}))}$)}. Then we have AVAS($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1))\AVAS($\mathfrak{H}$) ≠ ∅. ■

Theorem 2-86. *If the last member of a closed segment $\mathfrak{B}$ in $\mathfrak{H}$ is identical to the last member of $\mathfrak{H}$, then the first member of $\mathfrak{B}$ is the maximal member of* AVAS($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1) *and is not any more available in $\mathfrak{H}$*

If $\mathfrak{B}$ is a closed segment in $\mathfrak{H}$ and max(Dom($\mathfrak{B}$)) = Dom($\mathfrak{H}$)-1, then it holds: AVAS($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1)\AVAS($\mathfrak{H}$) = {(min(Dom($\mathfrak{B}$)), $\mathfrak{H}_{\text{min(Dom}(\mathfrak{B}))}$)} = {(max(Dom(AVAS($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1))), $\mathfrak{H}_{\text{max(Dom(AVAS}(\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-1})))}$)}.

Proof: Suppose $\mathfrak{B}$ is a closed segment in $\mathfrak{H}$ and max(Dom($\mathfrak{B}$)) = Dom($\mathfrak{H}$)-1. Then $\mathfrak{B}$ is a CdI- or NI- or RA-like segment in $\mathfrak{H}$ and $\mathfrak{H}$ ∈ SEQ. With Theorem 2-31, we thus have min(Dom($\mathfrak{B}$)) < max(Dom($\mathfrak{B}$)) = Dom($\mathfrak{H}$)-1 and hence min(Dom($\mathfrak{B}$)) ≤ Dom($\mathfrak{H}$)-2. With Theorem 2-84, we then have AVS($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$))\AVS($\mathfrak{H}$) ≠ ∅. From this, we get with Theorem 2-83-(vi) that there is a $\mathfrak{C}$ such that $\mathfrak{C}$ is a closed segment in $\mathfrak{H}$ and AVAS($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1)\AVAS($\mathfrak{H}$) = {(min(Dom($\mathfrak{C}$)), $\mathfrak{H}_{\text{min(Dom}(\mathfrak{C}))}$)}. We have that $\mathfrak{B}$ is a closed segment in $\mathfrak{H}$ and, because of max(Dom($\mathfrak{B}$)) = Dom($\mathfrak{H}$)-1, $\mathfrak{B}$ is not a segment and thus not a closed segment in $\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1. With Theorem 2-83-(iii), we then have $\mathfrak{B} = \mathfrak{C}$ and thus AVAS($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1)\AVAS($\mathfrak{H}$) = {(min(Dom($\mathfrak{B}$)), $\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))}$)}. With Theorem 2-85, it follows that AVAS($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1)\AVAS($\mathfrak{H}$) = {(min(Dom($\mathfrak{B}$)), $\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))}$)} = {(max(Dom(AVAS($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1))), $\mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1)))}$)}. ■

Theorem 2-87. *In the transition from* $\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1 *to* $\mathfrak{H}$, *the number of available assumption-sentences is reduced at most by one.*
If $\mathfrak{H} \in$ SEQ, then |AVAS($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1)\AVAS($\mathfrak{H}$)| ≤ 1.

Proof: Suppose $\mathfrak{H} \in$ SEQ. Then we have AVAS($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1))\AVAS($\mathfrak{H}$) = ∅ or AVAS($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1)\AVAS($\mathfrak{H}$) ≠ ∅. In the first case, we have |(AVAS($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1)\AVAS($\mathfrak{H}$)| = 0. Now, suppose AVAS($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1))\AVAS($\mathfrak{H}$) ≠ ∅. With Theorem 2-85, there is then a closed segment $\mathfrak{B}$ in $\mathfrak{H}$ such that AVAS($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1))\AVAS($\mathfrak{H}$) = {(min(Dom($\mathfrak{B}$)), $\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))}$)}. Then we have |AVAS($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1)\AVAS($\mathfrak{H}$)| = 1. ■

Theorem 2-88. *In the transition from* $\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1 *to* $\mathfrak{H}$ *proper AVAP-inclusion implies proper AVAS-inclusion*
If $\mathfrak{H} \in$ SEQ and AVAP($\mathfrak{H}$) ⊂ AVAP($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1), then AVAS($\mathfrak{H}$) ⊂ AVAS($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1).

Proof: Suppose $\mathfrak{H} \in$ SEQ and suppose AVAP($\mathfrak{H}$) ⊂ AVAP($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1). Then there is a Γ ∈ CFORM such that Γ ∈ AVAP($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1)\AVAP($\mathfrak{H}$). Then there is an $i \in$ Dom(AVAS($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1)) such that Γ = P($\mathfrak{H}_i$). Then we have $i \notin$ Dom(AVAS($\mathfrak{H}$)), because otherwise we would have Γ ∈ AVAP($\mathfrak{H}$). Thus we have AVAS($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1)\AVAS($\mathfrak{H}$) ≠ ∅. With Theorem 2-85, there is then a closed segment $\mathfrak{B}$ in $\mathfrak{H}$ such that max(Dom($\mathfrak{B}$)) = Dom($\mathfrak{H}$)-1. Then $\mathfrak{B}$ is a CdI- or NI- or RA-like segment in $\mathfrak{H}$. It then follows, with Theorem 2-29, that (Dom($\mathfrak{H}$)-1, $\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})\text{-}1}$) ∉ AS($\mathfrak{H}$) and thus (Dom($\mathfrak{H}$)-1, $\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})\text{-}1}$) ∉ AVAS($\mathfrak{H}$). With Theorem 2-81, we have AVAS($\mathfrak{H}$) ⊆ AVAS($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1) ∪ {(Dom($\mathfrak{H}$)-1, $\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})\text{-}1}$)}. Then we have AVAS($\mathfrak{H}$) ⊆ AVAS($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1), and, with ($i$, $\mathfrak{H}_i$) ∈ AVAS($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1)\AVAS($\mathfrak{H}$), it follows that AVAS($\mathfrak{H}$) ⊂ AVAS($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1). ■

Theorem 2-89. *Preparatory theorem (a) for Theorem 2-91, Theorem 2-92 and Theorem 2-93*

If $\mathfrak{A}$ is a segment in $\mathfrak{H}$ and $l \in \mathrm{Dom}(\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A})))$, then:

$(l, \mathfrak{H}_l) \in \mathrm{AVS}(\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A})))$

iff

For all closed segments $\mathfrak{C}$ in $\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A}))$: $l < \min(\mathrm{Dom}(\mathfrak{C}))$ or $\max(\mathrm{Dom}(\mathfrak{C})) \leq l$.

Proof: Suppose $\mathfrak{A}$ is a segment in $\mathfrak{H}$ and $l \in \mathrm{Dom}(\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A})))$. (*L-R*): First, suppose $(l, \mathfrak{H}_l) \in \mathrm{AVS}(\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A})))$. Now, suppose $\mathfrak{C}$ is a closed segment in $\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A}))$. If $\min(\mathrm{Dom}(\mathfrak{C})) \leq l < \max(\mathrm{Dom}(\mathfrak{C}))$, then we would have $(l, \mathfrak{H}_l) \notin \mathrm{AVS}(\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A})))$, which contradicts the hypothesis. Therefore we have $l < \min(\mathrm{Dom}(\mathfrak{C}))$ or $\max(\mathrm{Dom}(\mathfrak{C})) \leq l$. (*R-L*): Now, suppose for all closed segments $\mathfrak{C}$ in $\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A}))$: $l < \min(\mathrm{Dom}(\mathfrak{C}))$ or $\max(\mathrm{Dom}(\mathfrak{C})) \leq l$. Then it holds for all closed segments $\mathfrak{C}$ in $\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A}))$ that it is not the case that $\min(\mathrm{Dom}(\mathfrak{C})) \leq l < \max(\mathrm{Dom}(\mathfrak{C}))$. By hypothesis, we have $l \in \mathrm{Dom}(\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A})))$ and thus $\mathrm{P}(\mathfrak{H}_l)$ is available in $\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A}))$ at l. Hence we have $(l, \mathfrak{H}_l) \in \mathrm{AVS}(\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A})))$. ■

Theorem 2-90. *Preparatory theorem (b) for Theorem 2-91, Theorem 2-92 and Theorem 2-93*

If $\mathfrak{A}$ is a segment in $\mathfrak{H}$ and $l \in \mathrm{Dom}(\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A})))$, then:

$(l, \mathfrak{H}_l) \in \mathrm{AVAS}(\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A})))$

iff

$(l, \mathfrak{H}_l) \in \mathrm{AS}(\mathfrak{H})$ and for all closed segments $\mathfrak{C}$ in $\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A}))$: $l < \min(\mathrm{Dom}(\mathfrak{C}))$ or $\max(\mathrm{Dom}(\mathfrak{C})) \leq l$.

Proof: Suppose $\mathfrak{A}$ is a segment in $\mathfrak{H}$ and $l \in \mathrm{Dom}(\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A})))$. (*L-R*): First, suppose $(l, \mathfrak{H}_l) \in \mathrm{AVAS}(\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A})))$. Then we have $(l, \mathfrak{H}_l) \in \mathrm{AVS}(\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A}))) \cap \mathrm{AS}(\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A})))$. Because of $\mathrm{AS}(\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A}))) \subseteq \mathrm{AS}(\mathfrak{H})$, we thus have $(l, \mathfrak{H}_l) \in \mathrm{AS}(\mathfrak{H})$. With $(l, \mathfrak{H}_l) \in \mathrm{AVS}(\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A})))$ and Theorem 2-89, it follows that for all closed segments $\mathfrak{C}$ in $\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A}))$: $l < \min(\mathrm{Dom}(\mathfrak{C}))$ or $\max(\mathrm{Dom}(\mathfrak{C})) \leq l$. (*R-L*): Now, suppose $(l, \mathfrak{H}_l) \in \mathrm{AS}(\mathfrak{H})$ and suppose for all closed segments $\mathfrak{C}$ in $\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A}))$: $l < \min(\mathrm{Dom}(\mathfrak{C}))$ or $\max(\mathrm{Dom}(\mathfrak{C})) \leq l$. By hypothesis, we have $l \in \mathrm{Dom}(\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A})))$ and thus we have $(l, \mathfrak{H}_l) \in \mathrm{AS}(\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A})))$. With Theorem 2-89, it follows that $(l, \mathfrak{H}_l) \in \mathrm{AVS}(\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A})))$ and hence we have $(l, \mathfrak{H}_l) \in \mathrm{AVAS}(\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A})))$. ■

Theorem 2-91. *CdI-closes!-Theorem*

$\mathfrak{A}$ is a segment in $\mathfrak{H}$ and there are $\Delta, \Gamma \in$ CFORM such that

(i) $P(\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}))}) = \Delta$ and $(\min(\mathrm{Dom}(\mathfrak{A})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}))}) \in \mathrm{AVAS}(\mathfrak{H}\restriction\max(\mathrm{Dom}(\mathfrak{A})))$,

(ii) $P(\mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{A}))-1}) = \Gamma$,

(iii) There is no r such that $\min(\mathrm{Dom}(\mathfrak{A})) < r \leq \max(\mathrm{Dom}(\mathfrak{A}))-1$ and $(r, \mathfrak{H}_r) \in \mathrm{AVAS}(\mathfrak{H}\restriction\max(\mathrm{Dom}(\mathfrak{A})))$, and

(iv) $\mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{A}))} = \ulcorner$Therefore $\Delta \rightarrow \Gamma\urcorner$

iff

$\mathfrak{A}$ is a CdI-closed segment in $\mathfrak{H}$.

Proof: Follows directly from Theorem 2-67, Theorem 2-89 and Theorem 2-90. ■

Theorem 2-92. *NI-closes!-Theorem*

$\mathfrak{A}$ is a segment in $\mathfrak{H}$ and there are $\Delta, \Gamma \in$ CFORM and $i \in \mathrm{Dom}(\mathfrak{H})$ such that

(i) $\min(\mathrm{Dom}(\mathfrak{A})) \leq i < \max(\mathrm{Dom}(\mathfrak{A}))$,

(ii) $P(\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}))}) = \Delta$ and $(\min(\mathrm{Dom}(\mathfrak{A})), \mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}))}) \in \mathrm{AVAS}(\mathfrak{H}\restriction\max(\mathrm{Dom}(\mathfrak{A})))$,

(iii) $P(\mathfrak{H}_i) = \Gamma$ and $P(\mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{A}))-1}) = \ulcorner\neg\Gamma\urcorner$
or
$P(\mathfrak{H}_i) = \ulcorner\neg\Gamma\urcorner$ and $P(\mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{A}))-1}) = \Gamma$,

(iv) $(i, \mathfrak{H}_i) \in \mathrm{AVS}(\mathfrak{H}\restriction\max(\mathrm{Dom}(\mathfrak{A})))$,

(v) There is no r such that $\min(\mathrm{Dom}(\mathfrak{A})) < r \leq \max(\mathrm{Dom}(\mathfrak{A}))-1$ and $(r, \mathfrak{H}_r) \in \mathrm{AVAS}(\mathfrak{H}\restriction\max(\mathrm{Dom}(\mathfrak{A})))$, and

(vi) $\mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{A}))} = \ulcorner$Therefore $\neg\Delta\urcorner$

iff

$\mathfrak{A}$ is an NI-closed segment in $\mathfrak{H}$.

Proof: Follows directly from Theorem 2-68, Theorem 2-89 and Theorem 2-90. ■

Theorem 2-93. *PE-closes!-Theorem*

$\mathfrak{A}$ is a segment in $\mathfrak{H}$ and there are $\xi \in$ VAR, $\beta \in$ PAR, $\Delta \in$ FORM, where FV(Δ) $\subseteq \{\xi\}$, $\Gamma \in$ CFORM and $\mathfrak{B} \in$ SG($\mathfrak{H}$) such that

(i) P($\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))}$) = ⌜⋁$\xi\Delta$⌝ and (min(Dom($\mathfrak{B}$)), $\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))}$) $\in$ AVS($\mathfrak{H}$↾max(Dom($\mathfrak{A}$))),

(ii) P($\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))+1}$) = [$\beta$, ξ, Δ] and (min(Dom($\mathfrak{B}$))+1, $\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))+1}$) $\in$ AVAS($\mathfrak{H}$↾max(Dom($\mathfrak{A}$))),

(iii) P($\mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{B}))-1}$) = Γ,

(iv) $\mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{B}))}$ = ⌜Therefore Γ⌝,

(v) $\beta \notin$ STSF({Δ, Γ}),

(vi) There is no $j \leq$ min(Dom($\mathfrak{B}$)) such that $\beta \in$ ST($\mathfrak{H}_j$),

(vii) $\mathfrak{A} = \mathfrak{B}\setminus\{$(min(Dom($\mathfrak{B}$)), $\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))}$)$\}$ and

(viii) There is no r such that min(Dom($\mathfrak{A}$)) $< r \leq$ max(Dom($\mathfrak{A}$))-1 and (r, $\mathfrak{H}_r$) $\in$ AVAS($\mathfrak{H}$↾max(Dom($\mathfrak{A}$)))

iff

$\mathfrak{A}$ is a PE-closed segment in $\mathfrak{H}$.

Proof: Follows directly from Theorem 2-69, Theorem 2-89 and Theorem 2-90. ■

3 The Speech Act Calculus

The meta-theory of the calculus is now sufficiently developed, so that the calculus can be established (3.1). Then, we will provide a derivation and a consequence concept for the calculus (3.2). The chapter closes with the proof of theorems that describe the working of the calculus and are useful for the further development (3.3).

3.1 The Calculus

With the Speech Act Calculus, the rules for assuming and inferring are established, which ultimately serve to govern the derivation of propositions from sets of propositions. In preparation, we note: An author assumes a proposition Γ by uttering the sentence ⌜Suppose Γ⌝, and an author infers a proposition Γ by uttering the sentence ⌜Therefore Γ⌝. An author utters the empty sentence sequence by not uttering anything. An author utters a non-empty sentence sequence $\mathfrak{H}$ by successively uttering $\mathfrak{H}_i$ for every $i \in$ Dom($\mathfrak{H}$). An author extends a sentence sequence $\mathfrak{H}$ to a sentence sequence $\mathfrak{H}$* if he has uttered $\mathfrak{H}$ and now utters a sentence sequence $\mathfrak{H}$' such that $\mathfrak{H}^* = \mathfrak{H}^\frown\mathfrak{H}'$. An author thus extends an uttered sentence sequence $\mathfrak{H}$ to the sentence sequence $\mathfrak{H} \cup$ {(Dom($\mathfrak{H}$), ⌜Suppose Γ⌝)}, by assuming Γ, i.e. by uttering ⌜Suppose Γ⌝, and an author extends an uttered sentence sequence $\mathfrak{H}$ to the sentence sequence $\mathfrak{H} \cup$ {(Dom($\mathfrak{H}$), ⌜Therefore Γ⌝)} by inferring Γ, i.e. by uttering ⌜Therefore Γ⌝.[12]

The rules of the calculus – and only these – are to allow one to extend an already uttered sentence sequence $\mathfrak{H}$ to a sentence sequence $\mathfrak{H}$' with Dom($\mathfrak{H}$') = Dom($\mathfrak{H}$)+1. After the establishment of the rules, a derivation and a consequence concept can be established, according to which derivations will be exactly those non-empty sentence sequences that can in principle be uttered in accordance with the rules of the calculus (↑ 3.2).

As is usual for pragmatised natural deduction calculi, there is a rule of assumption (Speech-act rule 3-1) and 16 inference rules (Speech-act rule 3-2 to Speech-act rule 3-17). Additionally, the calculus contains an interdiction clause (IDC, Speech-act rule 3-18),

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[12] For the relation between the performance of speech acts and sequences of speech acts and the uttering of sentences and sequences of sentences, see HINST, P.: *Logischer Grundkurs*, p. 58–71, SIEGWART, G.: *Vorfragen*, p. 25–32, *Denkwerkzeuge,* p. 39–52, and, most recent and in English, *Alethic Acts*. Here, we obviously assume that the expressions and concatenations thereof stipulated by Postulate 1-1 to Postulate 1-3 are utterable entities.

which forbids all extensions that are not permitted by one of the rules from Speech-act rule 3-1 to Speech-act rule 3-17. Among the rules of inference, there are two for each of the connectives, quantificators (resp. quantifiers) and for the identity predicate. One of the rules regulates the introduction of the respective operator and the other rule regulates its elimination.

A shorthand version of the availability conception may facilitate an easier understanding of the presentation of the calculus: If $\mathfrak{H}$ is a sentence sequence, then $(i, \mathfrak{H}_i)$ is in AVS($\mathfrak{H}$) if and only if the proposition of $\mathfrak{H}_i$ is available in $\mathfrak{H}$ at i. Furthermore, $(i, \mathfrak{H}_i)$ is in AVAS($\mathfrak{H}$) if and only if the proposition of $\mathfrak{H}_i$ is available in $\mathfrak{H}$ at i and $\mathfrak{H}_i$ is an assumption-sentence. Γ is an element of AVP($\mathfrak{H}$) if and only if there is $(i, \mathfrak{H}_i) \in$ AVS($\mathfrak{H}$) such that Γ is the proposition of $\mathfrak{H}_i$, and Γ is an element of AVAP($\mathfrak{H}$) if and only if there is $(i, \mathfrak{H}_i) \in$ AVAS($\mathfrak{H}$) such that Γ is the proposition of $\mathfrak{H}_i$.

In order to give an intuitively accessible short version of the rules, we stipulate: If one has uttered a sentence sequence $\mathfrak{H}$ and Γ is available in $\mathfrak{H}$ at i, then one has gained Γ in $\mathfrak{H}$ at i. If Δ is the last assumption made in uttering $\mathfrak{H}$ that is still available, and if one has gained Γ in $\mathfrak{H}$ after or with the assumption of Δ, then one has gained Γ in $\mathfrak{H}$ departing from the assumption of Δ. If one extends $\mathfrak{H}$ to $\mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \Sigma)\}$ and $\Delta = \mathrm{P}(\mathfrak{H}_i)$ is an assumption that is available in $\mathfrak{H}$ at i but that is not any more available in $\mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \Sigma)\}$ at i, then one has discharged the assumption of Δ at i.

Now the *short version of the rules*, in which all reference to sentence sequences, positions and all grammatical specifications are neglected: One may assume any proposition Γ (AR); if one has last gained Γ departing from the assumption of Δ, then one may infer $\ulcorner\Delta \rightarrow \Gamma\urcorner$ and thus discharge the assumption of Δ (CdI); if one has gained Δ and $\ulcorner\Delta \rightarrow \Gamma\urcorner$, then one may infer Γ (CdE); if one has gained Δ and Γ, then one may infer $\ulcorner\Delta \wedge \Gamma\urcorner$ (CI); if one has gained $\ulcorner\Delta \wedge \Gamma\urcorner$ or gained $\ulcorner\Gamma \wedge \Delta\urcorner$, then one may infer Γ (CE); if one has gained $\ulcorner\Delta \rightarrow \Gamma\urcorner$ and $\ulcorner\Gamma \rightarrow \Delta\urcorner$, then one may infer $\ulcorner\Delta \leftrightarrow \Gamma\urcorner$ (BI); if one has gained Δ and $\ulcorner\Delta \leftrightarrow \Gamma\urcorner$ or gained Δ and $\ulcorner\Gamma \leftrightarrow \Delta\urcorner$, then one may infer Γ (BE); if one has gained Γ or gained Δ, then one may infer $\ulcorner\Delta \vee \Gamma\urcorner$ (DI); if one has gained $\ulcorner\mathrm{B} \vee \Delta\urcorner$, $\ulcorner\mathrm{B} \rightarrow \Gamma\urcorner$ and $\ulcorner\Delta \rightarrow \Gamma\urcorner$, then one may infer Γ (DE); if one has gained either Γ and last $\ulcorner\neg\Gamma\urcorner$ or $\ulcorner\neg\Gamma\urcorner$ and last Γ departing from the assumption of Δ, then one may infer $\ulcorner\neg\Delta\urcorner$ and thus discharge the assumption of Δ (NI); if one has gained $\ulcorner\neg\neg\Gamma\urcorner$, then one may infer Γ (NE); if one has gained [β, ξ, Δ], where β is not a subterm of Δ or of any available assumption, then one may infer $\ulcorner \bigwedge \xi \Delta \urcorner$ (UI), if one has gained $\ulcorner \bigwedge \xi \Delta \urcorner$, then one may infer [θ, ξ, Δ] (UE); if one has gained [θ, ξ, Δ], then one may infer $\ulcorner \bigvee \xi \Delta \urcorner$ (PI); if one has gained $\ulcorner \bigvee \xi \Delta \urcorner$, next assumed [β, ξ, Δ], where β is a new parameter and not a subterm of Δ, and then, departing from the assumption of [β, ξ, Δ], last gained Γ, where β is not a subterm of Γ, then one may infer Γ and thus discharge the assumption of [β, ξ, Δ] (PE); one may infer $\ulcorner \theta = \theta \urcorner$ (II); if one has gained $\ulcorner \theta_0 = \theta_1 \urcorner$ and $[\theta_0, \xi, \Delta]$, then one may infer $[\theta_1, \xi, \Delta]$ (IE); that is all one is allowed to do (IDC).

Now follow the rules of the Speech Act Calculus in their *authoritative formulation*:

Speech-act rule 3-1. *Rule of Assumption (AR)*
If one has uttered $\mathfrak{H} \in$ SEQ and if $\Gamma \in$ CFORM, then one may extend $\mathfrak{H}$ to $\mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner \text{Suppose } \Gamma \urcorner)\}$.

Speech-act rule 3-2. *Rule of Conditional Introduction (CdI)*
If one has uttered $\mathfrak{H} \in$ SEQ and if $\Delta, \Gamma \in$ CFORM and $i \in \mathrm{Dom}(\mathfrak{H})$, and

(i) $P(\mathfrak{H}_i) = \Delta$ and $(i, \mathfrak{H}_i) \in \mathrm{AVAS}(\mathfrak{H})$,
(ii) $P(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-1}) = \Gamma$, and
(iii) There is no l such that $i < l \leq \mathrm{Dom}(\mathfrak{H})-1$ and $(l, \mathfrak{H}_l) \in \mathrm{AVAS}(\mathfrak{H})$,

then one may extend $\mathfrak{H}$ to $\mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner \text{Therefore } \Delta \rightarrow \Gamma \urcorner)\}$.

Note that applying the rule of conditional introduction generates CdI-closed segments according to Definition 2-23 (cf. Theorem 2-91). If one extends $\mathfrak{H}$ to $\mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner \text{Therefore } \Delta \rightarrow \Gamma \urcorner)\}$ by CdI, then none of the propositions that one inferred or assumed by uttering $\mathfrak{H}$ after (and *including*) the i^{th} member is available in $\mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner \text{Therefore } \Delta \rightarrow \Gamma \urcorner)\}$, except for propositions that were available in $\mathfrak{H}$ before the i^{th} member (cf. Definition 2-26). Of course, this does not apply to the newly available conditional $\ulcorner \Delta \rightarrow \Gamma \urcorner$, as it is the proposition of the new last member and thus available in the resulting sentence sequence in any case (cf. Theorem 2-82). Since the proposition of the last member of a sentence sequence $\mathfrak{H}$ is always available in $\mathfrak{H}$ at $\mathrm{Dom}(\mathfrak{H})-1$, it also suffices in clause (ii) of the rule to demand solely that the consequent of the conditional one wants to infer is the proposition of the last member of $\mathfrak{H}$, without additionally demanding that that proposition is also available there. Similar remarks apply to Speech-act rule 3-10 (NI) and Speech-act rule 3-15 (PE).

Speech-act rule 3-3. *Rule of Conditional Elimination (CdE)*
If one has uttered $\mathfrak{H} \in$ SEQ and if $\Delta, \Gamma \in$ CFORM and $\{\Delta, \ulcorner\Delta \rightarrow \Gamma\urcorner\} \subseteq$ AVP($\mathfrak{H}$), then one may extend $\mathfrak{H}$ to $\mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Gamma\urcorner)\}$.

Speech-act rule 3-4. *Rule of Conjunction Introduction (CI)*
If one has uttered $\mathfrak{H} \in$ SEQ and if $\Delta, \Gamma \in$ AVP($\mathfrak{H}$), then one may extend $\mathfrak{H}$ to $\mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Delta \wedge \Gamma\urcorner)\}$.

Speech-act rule 3-5. *Rule of Conjunction Elimination (CE)*
If one has uttered $\mathfrak{H} \in$ SEQ and if $\Delta, \Gamma \in$ CFORM and $\{\ulcorner\Delta \wedge \Gamma\urcorner, \ulcorner\Gamma \wedge \Delta\urcorner\} \cap$ AVP($\mathfrak{H}$) $\neq \emptyset$, then one may extend $\mathfrak{H}$ to $\mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Gamma\urcorner)\}$.

Speech-act rule 3-6. *Rule of Biconditional Introduction (BI)*
If one has uttered $\mathfrak{H} \in$ SEQ and if $\Delta, \Gamma \in$ CFORM and $\{\ulcorner\Delta \rightarrow \Gamma\urcorner, \ulcorner\Gamma \rightarrow \Delta\urcorner\} \subseteq$ AVP($\mathfrak{H}$), then one may extend $\mathfrak{H}$ to $\mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Delta \leftrightarrow \Gamma\urcorner)\}$.

Here, the meta-logical *requirement of separability*, according to which each rule is to regulate only one operator, is violated, because the rule-antecedent demands that certain conditionals are available. The rule of biconditional introduction is thus at the same time a rule for the elimination of conditionals in certain contexts.

Speech-act rule 3-7. *Rule of Biconditional Elimination (BE)*
If one has uttered $\mathfrak{H} \in$ SEQ and if $\Delta \in$ AVP($\mathfrak{H}$), $\Gamma \in$ CFORM, und $\{\ulcorner\Delta \leftrightarrow \Gamma\urcorner, \ulcorner\Gamma \leftrightarrow \Delta\urcorner\} \cap$ AVP($\mathfrak{H}$) $\neq \emptyset$, then one may extend $\mathfrak{H}$ to $\mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Gamma\urcorner)\}$.

Speech-act rule 3-8. *Rule of Disjunction Introduction (DI)*
If one has uttered $\mathfrak{H} \in$ SEQ and if $\Delta, \Gamma \in$ CFORM and $\{\Delta, \Gamma\} \cap$ AVP($\mathfrak{H}$) $\neq \emptyset$, then one may extend $\mathfrak{H}$ to $\mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Delta \vee \Gamma\urcorner)\}$.

Speech-act rule 3-9. *Rule of Disjunction Elimination (DE)*
If one has uttered $\mathfrak{H} \in$ SEQ and if $B, \Delta, \Gamma \in$ CFORM and $\{\ulcorner B \vee \Delta\urcorner, \ulcorner B \rightarrow \Gamma\urcorner, \ulcorner\Delta \rightarrow \Gamma\urcorner\} \subseteq$ AVP($\mathfrak{H}$), then one may extend $\mathfrak{H}$ to $\mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Gamma\urcorner)\}$.

Here, the meta-logical requirement of separability is violated a second time, as the rule-antecedent demands that certain conditionals are available. The rule of disjunction elimination is thus at the same time a rule for the elimination of conditionals in certain contexts.

Speech-act rule 3-10. *Rule of Negation Introduction (NI)*
If one has uttered $\mathfrak{H} \in$ SEQ and if $\Delta, \Gamma \in$ CFORM and $i, j \in$ Dom($\mathfrak{H}$) and

(i) $i \leq j$,

(ii) $P(\mathfrak{H}_i) = \Delta$ and $(i, \mathfrak{H}_i) \in$ AVAS($\mathfrak{H}$),

(iii) $P(\mathfrak{H}_j) = \Gamma$ and $P(\mathfrak{H}_{Dom(\mathfrak{H})-1}) = \ulcorner\neg\Gamma\urcorner$
or
$P(\mathfrak{H}_j) = \ulcorner\neg\Gamma\urcorner$ and $P(\mathfrak{H}_{Dom(\mathfrak{H})-1}) = \Gamma$,

(iv) $(j, \mathfrak{H}_j) \in$ AVS($\mathfrak{H}$), and

(v) There is no l, such that $i < l \leq$ Dom($\mathfrak{H}$)-1 and $(l, \mathfrak{H}_l) \in$ AVAS($\mathfrak{H}$),

then one may extend $\mathfrak{H}$ to $\mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \neg\Delta\urcorner)\}$.

Applying the rule of negation introduction generates NI-closed segments according to Definition 2-24 (cf. Theorem 2-92). Thus, if one extends $\mathfrak{H}$ to $\mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \neg\Delta\urcorner)\}$ by NI, then none of the propositions that one inferred or assumed by uttering $\mathfrak{H}$ after (and *including*) the i^{th} member is available in $\mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \neg\Delta\urcorner)\}$, except for propositions that were available in $\mathfrak{H}$ before the i^{th} member (cf. Definition 2-26). Of course, this does not apply to the newly available negation $\ulcorner\neg\Delta\urcorner$. Since the proposition of the last member of a sentence sequence $\mathfrak{H}$ is always available in $\mathfrak{H}$ at Dom($\mathfrak{H}$)-1 (cf. Theorem 2-82), it also suffices in clause (iii) of the rule to demand that one of he two contradictory statements is available at j and that the second part of the contradiction is the proposition of the last sentence of $\mathfrak{H}$.

Speech-act rule 3-11. *Rule of Negation Elimination (NE)*
If one has uttered $\mathfrak{H} \in$ SEQ and if $\Gamma \in$ CFORM and $\ulcorner\neg\neg\Gamma\urcorner \in$ AVP($\mathfrak{H}$), then one may extend $\mathfrak{H}$ to $\mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Gamma\urcorner)\}$.

Speech-act rule 3-12. *Rule of Universal-quantifier Introduction (UI)*
If one has uttered $\mathfrak{H} \in$ SEQ and if $\beta \in$ PAR, $\xi \in$ VAR, $\Delta \in$ FORM, where FV(Δ) $\subseteq \{\xi\}$, $[\beta, \xi, \Delta] \in$ AVP($\mathfrak{H}$) and $\beta \notin$ STSF($\{\Delta\} \cup$ AVAP($\mathfrak{H}$)), then one may extend $\mathfrak{H}$ to $\mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \bigwedge\xi\Delta\urcorner)\}$.

Speech-act rule 3-13. *Rule of Universal-quantifier Elimination (UE)*

If one has uttered $\mathfrak{H} \in$ SEQ and if $\theta \in$ CTERM, $\xi \in$ VAR, $\Delta \in$ FORM, where FV(Δ) $\subseteq \{\xi\}$, and $\ulcorner\bigwedge\xi\Delta\urcorner \in$ AVP($\mathfrak{H}$), then one may extend $\mathfrak{H}$ to $\mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner$Therefore $[\theta, \xi, \Delta]\urcorner)\}$.

Speech-act rule 3-14. *Rule of Particular-quantifier Introduction (PI)*

If one has uttered $\mathfrak{H} \in$ SEQ and if $\theta \in$ CTERM, $\xi \in$ VAR, $\Delta \in$ FORM, where FV(Δ) $\subseteq \{\xi\}$, and $[\theta, \xi, \Delta] \in$ AVP($\mathfrak{H}$), then one may extend $\mathfrak{H}$ to $\mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner$Therefore $\bigvee\xi\Delta\urcorner)\}$.

Speech-act rule 3-15. *Rule of Particular-quantifier Elimination (PE)*

If one has uttered $\mathfrak{H} \in$ SEQ and if $\beta \in$ PAR, $\xi \in$ VAR, $\Delta \in$ FORM, where FV(Δ) $\subseteq \{\xi\}$, $\Gamma \in$ CFORM and $i \in$ Dom($\mathfrak{H}$), and

(i) P($\mathfrak{H}_i$) = $\ulcorner\bigvee\xi\Delta\urcorner$ and $(i, \mathfrak{H}_i) \in$ AVS($\mathfrak{H}$),

(ii) P($\mathfrak{H}_{i+1}$) = $[\beta, \xi, \Delta]$ and $(i+1, \mathfrak{H}_{i+1}) \in$ AVAS($\mathfrak{H}$),

(iii) P($\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-1}$) = Γ,

(iv) $\beta \notin$ STSF($\{\Delta, \Gamma\}$),

(v) There is no $j \leq i$ such that $\beta \in$ ST($\mathfrak{H}_j$),

(vi) There is no m such that $i+1 < m \leq$ Dom($\mathfrak{H}$)-1 and $(m, \mathfrak{H}_m) \in$ AVAS($\mathfrak{H}$),

then one may extend $\mathfrak{H}$ to $\mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner$Therefore $\Gamma\urcorner)\}$.

Applying the rule of particular-quantifier elimination generates PE-closed segments according to Definition 2-25 (cf. Theorem 2-93). Thus, if one extends $\mathfrak{H}$ to $\mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}),$ $\ulcorner$Therefore $\Gamma\urcorner)\}$ by PE, then none of the propositions that one inferred or assumed by uttering $\mathfrak{H}$ *after* the i^{th} member is available in $\mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner$Therefore $\Gamma\urcorner)\}$, except for propositions that were available in $\mathfrak{H}$ before the $i+1^{\mathrm{th}}$ member (cf. Definition 2-26). Of course, this does not apply to the last inferred proposition, i.e. Γ, which is in any case available in the resulting sentence sequence. Since the proposition of the last member of a sentence sequence $\mathfrak{H}$ is always available in $\mathfrak{H}$ at Dom($\mathfrak{H}$)-1 (cf. Theorem 2-82), it also sufficises in clause (iii) of the rule, to demand solely that Γ is the proposition of the last member of $\mathfrak{H}$.

Speech-act rule 3-16. *Rule of Identity Introduction (II)*

If one has uttered $\mathfrak{H} \in$ SEQ and if $\theta \in$ CTERM, then one may extend $\mathfrak{H}$ to $\mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}),$ $\ulcorner$Therefore $\theta = \theta\urcorner)\}$.

Speech-act rule 3-17. *Rule of Identity Elimination (IE)*

If one has uttered $\mathfrak{H} \in$ SEQ and if $\xi \in$ VAR, $\Delta \in$ FORM, where FV(Δ) $\subseteq \{\xi\}$, θ_0, $\theta_1 \in$ CTERM and $\{\ulcorner\theta_0 = \theta_1\urcorner, [\theta_0, \xi, \Delta]\} \subseteq$ AVP($\mathfrak{H}$), then one may extend $\mathfrak{H}$ to $\mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } [\theta_1, \xi, \Delta]\urcorner)\}$.

Last, we formulate a prohibition that makes the interdictory status of the rules explicit. For this, all 17 rule-antecedents for the extension of $\mathfrak{H}$ to $\mathfrak{H}'$ are required to be unsatisfied. This condition is then sufficient for one not being allowed to extend $\mathfrak{H}$ to $\mathfrak{H}'$.

Speech-act rule 3-18. *Interdiction Clause (IDC)*

If $\mathfrak{H} \notin$ SEQ or if one has not uttered $\mathfrak{H}$ or if there are no B, Γ, $\Delta \in$ CFORM and θ_0, $\theta_1 \in$ CTERM and $\beta \in$ PAR and $\xi \in$ VAR and $\Delta' \in$ FORM, where FV(Δ') $\subseteq \{\xi\}$, and i, $j \in$ Dom($\mathfrak{H}$) such that

(i) $\mathfrak{H}' = \mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Suppose } \Gamma\urcorner)\}$ or

(ii) $P(\mathfrak{H}_i) = \Delta$, $(i, \mathfrak{H}_i) \in$ AVAS($\mathfrak{H}$), $P(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})\text{-}1}) = \Gamma$, there is no l such that $i < l \leq$ Dom($\mathfrak{H}$)-1 and $(l, \mathfrak{H}_l) \in$ AVAS($\mathfrak{H}$), and $\mathfrak{H}' = \mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Delta \rightarrow \Gamma\urcorner)\}$ or

(iii) $\{\Delta, \ulcorner\Delta \rightarrow \Gamma\urcorner\} \subseteq$ AVP($\mathfrak{H}$) and $\mathfrak{H}' = \mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Gamma\urcorner)\}$ or

(iv) $\{\Delta, \Gamma\} \subseteq$ AVP($\mathfrak{H}$) and $\mathfrak{H}' = \mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Delta \wedge \Gamma\urcorner)\}$ or

(v) $\{\ulcorner\Delta \wedge \Gamma\urcorner, \ulcorner\Gamma \wedge \Delta\urcorner\} \cap$ AVP($\mathfrak{H}$) $\neq \emptyset$ and $\mathfrak{H}' = \mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Gamma\urcorner)\}$ or

(vi) $\{\ulcorner\Delta \rightarrow \Gamma\urcorner, \ulcorner\Gamma \rightarrow \Delta\urcorner\} \subseteq$ AVP($\mathfrak{H}$) and $\mathfrak{H}' = \mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Delta \leftrightarrow \Gamma\urcorner)\}$ or

(vii) $\Delta \in$ AVP($\mathfrak{H}$), $\{\ulcorner\Delta \leftrightarrow \Gamma\urcorner, \ulcorner\Gamma \leftrightarrow \Delta\urcorner\} \cap$ AVP($\mathfrak{H}$) $\neq \emptyset$, and $\mathfrak{H}' = \mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Gamma\urcorner)\}$ or

(viii) $\{\Delta, \Gamma\} \cap$ AVP($\mathfrak{H}$) $\neq \emptyset$ and $\mathfrak{H}' = \mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Delta \vee \Gamma\urcorner)\}$ or

(ix) $\{\ulcorner B \vee \Delta\urcorner, \ulcorner B \rightarrow \Gamma\urcorner, \ulcorner\Delta \rightarrow \Gamma\urcorner\} \subseteq$ AVP($\mathfrak{H}$) and $\mathfrak{H}' = \mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Gamma\urcorner)\}$ or

(x) $i \leq j$, $P(\mathfrak{H}_i) = \Delta$, $(i, \mathfrak{H}_i) \in$ AVAS($\mathfrak{H}$), $P(\mathfrak{H}_j) = \Gamma$ and $P(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})\text{-}1}) = \ulcorner\neg\Gamma\urcorner$ or $P(\mathfrak{H}_j) = \ulcorner\neg\Gamma\urcorner$ and $P(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})\text{-}1}) = \Gamma$, $(j, \mathfrak{H}_j) \in$ AVS($\mathfrak{H}$), there is no l such that $i < l \leq$ Dom($\mathfrak{H}$)-1 and $(l, \mathfrak{H}_l) \in$ AVAS($\mathfrak{H}$), and $\mathfrak{H}' = \mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \neg\Delta\urcorner)\}$ or

(xi) $\ulcorner\neg\neg\Gamma\urcorner \in$ AVP($\mathfrak{H}$) and $\mathfrak{H}' = \mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Gamma\urcorner)\}$ or

(xii) $[\beta, \xi, \Delta'] \in$ AVP($\mathfrak{H}$), $\beta \notin$ STSF($\{\Delta'\} \cup$ AVAP($\mathfrak{H}$)) and $\mathfrak{H}' = \mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \bigwedge\xi\Delta'\urcorner)\}$ or

(xiii) $\ulcorner\bigwedge\xi\Delta'\urcorner \in$ AVP($\mathfrak{H}$) and $\mathfrak{H}' = \mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } [\theta_0, \xi, \Delta']\urcorner)\}$ or

(xiv) $[\theta_0, \xi, \Delta'] \in$ AVP($\mathfrak{H}$) and $\mathfrak{H}' = \mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \bigvee\xi\Delta'\urcorner)\}$ or

(xv) $P(\mathfrak{H}_i) = \ulcorner\bigvee\xi\Delta'\urcorner$, $(i, \mathfrak{H}_i) \in$ AVS($\mathfrak{H}$), $P(\mathfrak{H}_{i+1}) = [\beta, \xi, \Delta']$, $(i+1, \mathfrak{H}_{i+1}) \in$ AVAS($\mathfrak{H}$), $P(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})\text{-}1}) = \Gamma$, $\beta \notin$ STSF($\{\Delta', \Gamma\}$), there is no $l \leq i$ such that $\beta \in$ ST($\mathfrak{H}_l$), there is no m such that $i+1 < m \leq$ Dom($\mathfrak{H}$)-1 and $(m, \mathfrak{H}_m) \in$ AVAS($\mathfrak{H}$), and $\mathfrak{H}' = \mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Gamma\urcorner)\}$ or

(xvi) $\mathfrak{H}' = \mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \theta_0 = \theta_0\urcorner)\}$ or

(xvii) $\{\ulcorner\theta_0 = \theta_1\urcorner, [\theta_0, \xi, \Delta]\} \subseteq AVP(\mathfrak{H})$ and $\mathfrak{H}' = \mathfrak{H} \cup \{(Dom(\mathfrak{H}), \ulcorner\text{Therefore } [\theta_1, \xi, \Delta]\urcorner)\}$,
then one may not extend $\mathfrak{H}$ to $\mathfrak{H}'$.

Informally, Speech-act rule 3-18 says: If none of the rules from Speech-act rule 3-1 to Speech-act rule 3-17 allows the extension of $\mathfrak{H}$ to $\mathfrak{H}'$, then one may not extend $\mathfrak{H}$ to $\mathfrak{H}'$.

By setting the 18 rules, the calculus has now been established and can already be used. If one wants to add further rules later, e.g. rules for adducing-as-reason, stating, the positing-as-axiom or defining, one has to adapt Speech-act rule 3-18 accordingly. In the next section, we will now establish a derivation concept and a consequence concept for the calculus (3.2). Then, we will prove some theorems that shed some light on the way in which the calculus works (3.3).

3.2 Derivations and Deductive Consequence Relation

Having established the calculus, we now have to provide a derivation and a consequence concept and to prove the adequacy of the latter. Since the derivation and consequence relations are not to be tied to the actual utterance of sentence sequences, but only to their utterability in accordance with the rules, the derivation concept is not to be established with recourse to the full rules of the calculus – which always demand the utterance of a certain sentence sequence – but only with recourse to those parts of the rules that are specific to sentence sequences and indepedent of actual utterances.

To do this, we will first define a function for every rule of the calculus that assigns a sentence sequence $\mathfrak{H}$ the set of sentence sequences to which an author that has uttered $\mathfrak{H}$ may extend $\mathfrak{H}$ in compliance with the respective rule (Definition 3-1 to Definition 3-17). Based on these functions, we will then define the function RCE, which assigns a sentence sequence $\mathfrak{H}$ the set of rule-compliant extensions of $\mathfrak{H}$, i.e. the set of sentence sequences to which an author who has uttered $\mathfrak{H}$ might extend $\mathfrak{H}$ in accordance with one of the rules of the calculus (Definition 3-18). Then, we will define the set of rule-compliant sentence sequences, RCS, as the set of sentence sequences for which all non-empty restrictions are rule-compliant extensions of the immediately preceding restriction (Definition 3-19). A derivation of a proposition Γ from a set of propositions X will then be a non-empty RCS-element for which it holds that $C(\mathfrak{H}) = \Gamma$ and $AVAP(\mathfrak{H}) = X$ (Definition 3-20). Then, we will introduce the concept of deductive consequence and related concepts, where a proposition Γ will be a deductive consequence of a set of propositions X if and only if there is a derivation of Γ from a $Y \subseteq X$ (Definition 3-21).

As announced, we will first define functions analogous to the rules in 3.1:

Definition 3-1. *Assumption Function (AF)*

$AF = \{(\mathfrak{H}, X) \mid \mathfrak{H} \in SEQ$ and $X = \{\mathfrak{H}' \mid$ There is $\Gamma \in CFORM$ such that
$\mathfrak{H}' = \mathfrak{H} \cup \{(Dom(\mathfrak{H}), \ulcorner Suppose\ \Gamma\urcorner)\}\}\}$.

Cf. Speech-act rule 3-1. Since the set of closed formulas is not empty, we have as a corollary that $AF(\mathfrak{H})$ is not empty for any sentence sequence $\mathfrak{H}$.

Definition 3-2. *Conditional Introduction Function (CdIF)*

CdIF = $\{(\mathfrak{H}, X) \mid \mathfrak{H} \in \text{SEQ}$ and $X = \{\mathfrak{H}' \mid$ There are $\Delta, \Gamma \in$ CFORM and $i \in \text{Dom}(\mathfrak{H})$ such that

(i) $\text{P}(\mathfrak{H}_i) = \Delta$ and $(i, \mathfrak{H}_i) \in \text{AVAS}(\mathfrak{H})$,

(ii) $\text{P}(\mathfrak{H}_{\text{Dom}(\mathfrak{H})\text{-}1}) = \Gamma$,

(iii) There is no l such that $i < l \leq \text{Dom}(\mathfrak{H})\text{-}1$ and $(l, \mathfrak{H}_l) \in \text{AVAS}(\mathfrak{H})$, and

(iv) $\mathfrak{H}' = \mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Delta \rightarrow \Gamma\urcorner)\}\}\}$.

Cf. Speech-act rule 3-2.

Definition 3-3. *Conditional Elimination Function (CdEF)*

CdEF = $\{(\mathfrak{H}, X) \mid \mathfrak{H} \in \text{SEQ}$ and $X = \{\mathfrak{H}' \mid$ There are $\Delta, \Gamma \in$ CFORM such that $\{\Delta, \ulcorner\Delta \rightarrow \Gamma\urcorner\}$ $\subseteq \text{AVP}(\mathfrak{H})$ and $\mathfrak{H}' = \mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Gamma\urcorner)\}\}\}$.

Cf. Speech-act rule 3-3.

Definition 3-4. *Conjunction Introduction Function (CIF)*

CIF = $\{(\mathfrak{H}, X) \mid \mathfrak{H} \in \text{SEQ}$ and $X = \{\mathfrak{H}' \mid$ There are $\Delta, \Gamma \in \text{AVP}(\mathfrak{H})$ such that $\mathfrak{H}' = \mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Delta \wedge \Gamma\urcorner)\}\}\}$.

Cf. Speech-act rule 3-4.

Definition 3-5. *Conjunction Elimination Function (CEF)*

CEF = $\{(\mathfrak{H}, X) \mid \mathfrak{H} \in \text{SEQ}$ and $X = \{\mathfrak{H}' \mid$ There are $\Delta, \Gamma \in$ CFORM such that $\{\ulcorner\Delta \wedge \Gamma\urcorner, \ulcorner\Gamma \wedge \Delta\urcorner\} \cap \text{AVP}(\mathfrak{H}) \neq \emptyset$ and $\mathfrak{H}' = \mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Gamma\urcorner)\}\}\}$.

Cf. Speech-act rule 3-5.

Definition 3-6. *Biconditional Introduction Function (BIF)*

BIF = $\{(\mathfrak{H}, X) \mid \mathfrak{H} \in \text{SEQ}$ and $X = \{\mathfrak{H}' \mid$ There are $\Delta, \Gamma \in$ CFORM such that $\{\ulcorner\Delta \rightarrow \Gamma\urcorner, \ulcorner\Gamma \rightarrow \Delta\urcorner\} \subseteq \text{AVP}(\mathfrak{H})$ and $\mathfrak{H}' = \mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Delta \leftrightarrow \Gamma\urcorner)\}\}\}$.

Cf. Speech-act rule 3-6.

Definition 3-7. *Biconditional Elimination Function (BEF)*

BEF = $\{(\mathfrak{H}, X) \mid \mathfrak{H} \in \text{SEQ}$ and $X = \{\mathfrak{H}' \mid$ There are $\Delta \in \text{AVP}(\mathfrak{H})$ and $\Gamma \in$ CFORM such that $\{\ulcorner\Delta \leftrightarrow \Gamma\urcorner, \ulcorner\Gamma \leftrightarrow \Delta\urcorner\} \cap \text{AVP}(\mathfrak{H}) \neq \emptyset$ and $\mathfrak{H}' = \mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Gamma\urcorner)\}\}\}$.

Cf. Speech-act rule 3-7.

Definition 3-8. *Disjunction Introduction Function (DIF)*

DIF = $\{(\mathfrak{H}, X) \mid \mathfrak{H} \in$ SEQ and $X = \{\mathfrak{H}' \mid$ There are $\Delta, \Gamma \in$ CFORM such that $\{\Delta, \Gamma\} \cap$ AVP$(\mathfrak{H}) \neq \emptyset$ and $\mathfrak{H}' = \mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Delta \vee \Gamma\urcorner)\}\}\}$.

Cf. Speech-act rule 3-8.

Definition 3-9. *Disjunction Elimination Function (DEF)*

DEF = $\{(\mathfrak{H}, X) \mid \mathfrak{H} \in$ SEQ and $X = \{\mathfrak{H}' \mid$ There are B, $\Delta, \Gamma \in$ CFORM such that $\{\ulcorner B \vee \Delta\urcorner, \ulcorner B \rightarrow \Gamma\urcorner, \ulcorner\Delta \rightarrow \Gamma\urcorner\} \subseteq$ AVP$(\mathfrak{H})$ and $\mathfrak{H}' = \mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Gamma\urcorner)\}\}\}$.

Cf. Speech-act rule 3-9.

Definition 3-10. *Negation Introduction Function (NIF)*

NIF = $\{(\mathfrak{H}, X) \mid \mathfrak{H} \in$ SEQ and $X = \{\mathfrak{H}' \mid$ There are $\Delta, \Gamma \in$ CFORM and $i, j \in$ Dom$(\mathfrak{H})$ such that

- (i) $i \leq j$,
- (ii) $P(\mathfrak{H}_i) = \Delta$ and $(i, \mathfrak{H}_i) \in$ AVAS$(\mathfrak{H})$,
- (iii) $P(\mathfrak{H}_j) = \Gamma$ and $P(\mathfrak{H}_{\text{Dom}(\mathfrak{H})\text{-}1}) = \ulcorner\neg\Gamma\urcorner$
  or
  $P(\mathfrak{H}_j) = \ulcorner\neg\Gamma\urcorner$ and $P(\mathfrak{H}_{\text{Dom}(\mathfrak{H})\text{-}1}) = \Gamma$,
- (iv) $(j, \mathfrak{H}_j) \in$ AVS$(\mathfrak{H})$,
- (v) There is no l such that $i < l \leq$ Dom$(\mathfrak{H})$-1 and $(l, \mathfrak{H}_l) \in$ AVAS$(\mathfrak{H})$, and
- (vi) $\mathfrak{H}' = \mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \neg\Delta\urcorner)\}\}\}$.

Cf. Speech-act rule 3-10.

Definition 3-11. *Negation Elimination Function (NEF)*

NEF= $\{(\mathfrak{H}, X) \mid \mathfrak{H} \in$ SEQ and $X = \{\mathfrak{H}' \mid$ There is $\Gamma \in$ CFORM such that $\ulcorner\neg\neg\Gamma\urcorner \in$ AVP$(\mathfrak{H})$, and $\mathfrak{H}' = \mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Gamma\urcorner)\}\}\}$.

Cf. Speech-act rule 3-11.

Definition 3-12. *Universal-quantifier Introduction Function (UIF)*

UIF = $\{(\mathfrak{H}, X) \mid \mathfrak{H} \in$ SEQ and $X = \{\mathfrak{H}' \mid$ There are $\beta \in$ PAR, $\xi \in$ VAR and $\Delta \in$ FORM, where FV$(\Delta) \subseteq \{\xi\}$, such that

- (i) $[\beta, \xi, \Delta] \in$ AVP$(\mathfrak{H})$,
- (ii) $\beta \notin$ STSF$(\{\Delta\} \cup$ AVAP$(\mathfrak{H}))$, and
- (iii) $\mathfrak{H}' = \mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \bigwedge\xi\Delta\urcorner)\}\}\}$.

Cf. Speech-act rule 3-12.

Definition 3-13. *Universal-quantifier Elimination Function (UEF)*

UEF = {($\mathfrak{H}$, X) | $\mathfrak{H}$ ∈ SEQ and X = {$\mathfrak{H}$' | There are θ ∈ CTERM, ξ ∈ VAR, Δ ∈ FORM, where FV(Δ) ⊆ {ξ}, such that ⌜⋀ξΔ⌝ ∈ AVP($\mathfrak{H}$) and $\mathfrak{H}$' = $\mathfrak{H}$ ∪ {(Dom($\mathfrak{H}$), ⌜Therefore [θ, ξ, Δ]⌝)}}}.

Cf. Speech-act rule 3-13.

Definition 3-14. *Particular-quantifier Introduction Function (PIF)*

PIF = {($\mathfrak{H}$, X) | $\mathfrak{H}$ ∈ SEQ and X = {$\mathfrak{H}$' | There are ξ ∈ VAR, Δ ∈ FORM, where FV(Δ) ⊆ {ξ}, and θ ∈ CTERM such that [θ, ξ, Δ] ∈ AVP($\mathfrak{H}$) and $\mathfrak{H}$' = $\mathfrak{H}$ ∪ {(Dom($\mathfrak{H}$), ⌜Therefore ⋁ξΔ⌝)}}}.

Cf. Speech-act rule 3-14.

Definition 3-15. *Particular-quantifier Elimination Function (PEF)*

PEF = {($\mathfrak{H}$, X) | $\mathfrak{H}$ ∈ SEQ and X = {$\mathfrak{H}$' | There are β ∈ PAR, ξ ∈ VAR, Δ ∈ FORM, where FV(Δ) ⊆ {ξ}, Γ ∈ CFORM and i ∈ Dom($\mathfrak{H}$) such that

(i) P($\mathfrak{H}_i$) = ⌜⋁ξΔ⌝ and (i, $\mathfrak{H}_i$) ∈ AVS($\mathfrak{H}$),

(ii) P($\mathfrak{H}_{i+1}$) = [β, ξ, Δ] and (i+1, $\mathfrak{H}_{i+1}$) ∈ AVAS($\mathfrak{H}$),

(iii) P($\mathfrak{H}_{\text{Dom}(\mathfrak{H})-1}$) = Γ,

(iv) β ∉ STSF({Δ, Γ}),

(v) There is no $j \leq i$ such that β ∈ ST($\mathfrak{H}_j$),

(vi) There is no m such that $i+1 < m \leq$ Dom($\mathfrak{H}$)-1 and (m, $\mathfrak{H}_m$) ∈ AVAS($\mathfrak{H}$), and

(vii) $\mathfrak{H}$' = $\mathfrak{H}$ ∪ {(Dom($\mathfrak{H}$), ⌜Therefore Γ⌝)}}}.

Cf. Speech-act rule 3-15.

Definition 3-16. *Identity Introduction Function (IIF)*

IIF = {($\mathfrak{H}$, X) | $\mathfrak{H}$ ∈ SEQ and X = {$\mathfrak{H}$' | There is θ ∈ CTERM such that $\mathfrak{H}$' = $\mathfrak{H}$ ∪ {(Dom($\mathfrak{H}$), ⌜Therefore θ = θ⌝)}}}.

Cf. Speech-act rule 3-16. Since the set of closed terms is not empty, it follows as a corollary that, like AF($\mathfrak{H}$), IIF($\mathfrak{H}$) is not empty for any sentence sequence $\mathfrak{H}$. This state of affairs is reflected in Theorem 3-2.

Definition 3-17. *Identity Elimination Function (IEF)*

IEF = $\{(\mathfrak{H}, X) \mid \mathfrak{H} \in$ SEQ and $X = \{\mathfrak{H}' \mid$ There are $\theta_0, \theta_1 \in$ CTERM, $\xi \in$ VAR and $\Delta \in$ FORM, where FV(Δ) $\subseteq \{\xi\}$, such that $\{\ulcorner\theta_0 = \theta_1\urcorner, [\theta_0, \xi, \Delta]\} \subseteq$ AVP($\mathfrak{H}$) and $\mathfrak{H}' = \mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } [\theta_1, \xi, \Delta]\urcorner)\}\}\}$.

Cf. Speech-act rule 3-17.

In the following, we will define the set of rule-compliant sentence sequences, RCS (Definition 3-19), and then the derivation predicate: '.. is a derivation of .. from ..' (Definition 3-20). We will do this in such a way that RCS will contain the empty sentence sequence and all and only those sentence sequences to which one can in principle extend the empty sentence sequence in compliance with the rules of the calculus. Based on the assumption function and the introduction and elimination functions we have just defined, RCS will thus be definined in such a way that RCS is the set of sentence sequences for which all non-empty restrictions are rule-compliant extensions of the immediately preceding restriction. To do this, we first definie the function RCE:

Definition 3-18. *Assignment of the set of rule-compliant assumption- and inference-extensions of a sentence sequence (RCE)*

RCE = $\{(\mathfrak{H}, X) \mid \mathfrak{H} \in$ SEQ and $X = \bigcup\{$AF($\mathfrak{H}$), CdIF($\mathfrak{H}$), CdEF($\mathfrak{H}$), CIF($\mathfrak{H}$), CEF($\mathfrak{H}$), BIF($\mathfrak{H}$), BEF($\mathfrak{H}$), DIF($\mathfrak{H}$), DEF($\mathfrak{H}$), NIF($\mathfrak{H}$), NEF($\mathfrak{H}$), UIF($\mathfrak{H}$), UEF($\mathfrak{H}$), PIF($\mathfrak{H}$), PEF($\mathfrak{H}$), IIF($\mathfrak{H}$), IEF($\mathfrak{H}$)$\}\}$.

RCE is defined in such a way that an author who has uttered $\mathfrak{H} \in$ SEQ may extend $\mathfrak{H}$ to $\mathfrak{H}'$ if and only if $\mathfrak{H}' \in$ RCE($\mathfrak{H}$). Before we defined the set of rule-compliant sentence sequences, RCS, we will prove some theorems about RCE.

Theorem 3-1. *RCE-extensions of sentence sequences are non-empty sentence sequences*

If $\mathfrak{H} \in$ SEQ, then RCE($\mathfrak{H}$) $\subseteq$ SEQ$\setminus\{\emptyset\}$.

Proof: Suppose $\mathfrak{H} \in$ SEQ. Suppose $\mathfrak{H}' \in$ RCE($\mathfrak{H}$). Then we have $\mathfrak{H}' \in$ AF($\mathfrak{H}$) or $\mathfrak{H}' \in$ CdIF($\mathfrak{H}$) or $\mathfrak{H}' \in$ CdEF($\mathfrak{H}$) or $\mathfrak{H}' \in$ CIF($\mathfrak{H}$) or $\mathfrak{H}' \in$ CEF($\mathfrak{H}$) or $\mathfrak{H}' \in$ BIF($\mathfrak{H}$) or $\mathfrak{H}' \in$ BEF($\mathfrak{H}$) or $\mathfrak{H}' \in$ DIF($\mathfrak{H}$) or $\mathfrak{H}' \in$ DEF($\mathfrak{H}$) or $\mathfrak{H}' \in$ NIF($\mathfrak{H}$) or $\mathfrak{H}' \in$ NEF($\mathfrak{H}$) or $\mathfrak{H}' \in$ UIF($\mathfrak{H}$) or $\mathfrak{H}' \in$ UEF($\mathfrak{H}$) or $\mathfrak{H}' \in$ PIF($\mathfrak{H}$) or $\mathfrak{H}' \in$ PEF($\mathfrak{H}$) or $\mathfrak{H}' \in$ IIF($\mathfrak{H}$) or $\mathfrak{H}' \in$ IEF($\mathfrak{H}$). It then follows from Definition 3-1 to Definition 3-17 that $\mathfrak{H}' = \mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \Sigma)\}$ for a $\Sigma \in$ SENT. In all cases, it then holds with Definition 1-23 and Definition 1-24 that $\mathfrak{H}' \in$ SEQ$\setminus\{\emptyset\}$. ■

Next, we want to show that RCE($\mathfrak{H}$) is not empty for any sentence sequence $\mathfrak{H}$ and that therefore every sentence sequence can be extended in some way.

Theorem 3-2. *RCE is not empty for any sentence sequence*
If $\mathfrak{H} \in$ SEQ, then RCE($\mathfrak{H}$) $\neq \emptyset$.

Proof: Suppose $\mathfrak{H} \in$ SEQ. We have that $\ulcorner x_0 \urcorner \in$ CTERM. According to Definition 3-16, we thus have $\mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } x_0 = x_0\urcorner)\} \in$ IIF($\mathfrak{H}$). Hence we have $\mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } x_0 = x_0\urcorner)\} \in$ RCE($\mathfrak{H}$) $\neq \emptyset$. ■

Theorem 3-3. *The elements of RCE($\mathfrak{H}$) are extensions of $\mathfrak{H}$ by exactly one sentence*
If $\mathfrak{H} \in$ SEQ and $\mathfrak{H}' \in$ RCE($\mathfrak{H}$), then there are $\Xi \in$ PERF and $\Gamma \in$ CFORM such that $\mathfrak{H}' = \mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\Xi\ \Gamma\urcorner)\}$.

Proof: Suppose $\mathfrak{H} \in$ SEQ and $\mathfrak{H}' \in$ RCE($\mathfrak{H}$). Then we have $\mathfrak{H}' \in$ AF($\mathfrak{H}$) or $\mathfrak{H}' \in$ CdIF($\mathfrak{H}$) or $\mathfrak{H}' \in$ CdEF($\mathfrak{H}$) or $\mathfrak{H}' \in$ CIF($\mathfrak{H}$) or $\mathfrak{H}' \in$ CEF($\mathfrak{H}$) or $\mathfrak{H}' \in$ BIF($\mathfrak{H}$) or $\mathfrak{H}' \in$ BEF($\mathfrak{H}$) or $\mathfrak{H}' \in$ DIF($\mathfrak{H}$) or $\mathfrak{H}' \in$ DEF($\mathfrak{H}$) or $\mathfrak{H}' \in$ NIF($\mathfrak{H}$) or $\mathfrak{H}' \in$ NEF($\mathfrak{H}$) or $\mathfrak{H}' \in$ UIF($\mathfrak{H}$) or $\mathfrak{H}' \in$ UEF($\mathfrak{H}$) or $\mathfrak{H}' \in$ PIF($\mathfrak{H}$) or $\mathfrak{H}' \in$ PEF($\mathfrak{H}$) or $\mathfrak{H}' \in$ IIF($\mathfrak{H}$) or $\mathfrak{H}' \in$ IEF($\mathfrak{H}$).

Suppose $\mathfrak{H}' \in$ AF($\mathfrak{H}$). According to Definition 3-1, there is then $\Gamma \in$ CFORM such that $\mathfrak{H}' = \mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Suppose } \Gamma\urcorner)\}$. Then we have $\mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H})} = \ulcorner\text{Suppose } \Gamma\urcorner$ and thus there are $\Xi \in$ PERF and $\Gamma \in$ CFORM such that $\mathfrak{H}' = \mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\Xi\ \Gamma\urcorner)\}$.

Suppose $\mathfrak{H}' \in$ CdIF($\mathfrak{H}$) or $\mathfrak{H}' \in$ CdEF($\mathfrak{H}$) or $\mathfrak{H}' \in$ CIF($\mathfrak{H}$) or $\mathfrak{H}' \in$ CEF($\mathfrak{H}$) or $\mathfrak{H}' \in$ BIF($\mathfrak{H}$) or $\mathfrak{H}' \in$ BEF($\mathfrak{H}$) or $\mathfrak{H}' \in$ DIF($\mathfrak{H}$) or $\mathfrak{H}' \in$ DEF($\mathfrak{H}$) or $\mathfrak{H}' \in$ NIF($\mathfrak{H}$) or $\mathfrak{H}' \in$ NEF($\mathfrak{H}$) or $\mathfrak{H}' \in$ UIF($\mathfrak{H}$) or $\mathfrak{H}' \in$ UEF($\mathfrak{H}$) or $\mathfrak{H}' \in$ PIF($\mathfrak{H}$) or $\mathfrak{H}' \in$ PEF($\mathfrak{H}$) or $\mathfrak{H}' \in$ IIF($\mathfrak{H}$) or $\mathfrak{H}' \in$ IEF($\mathfrak{H}$). According to Definition 3-2 to Definition 3-17, there is in each case a $\Gamma \in$ CFORM such that $\mathfrak{H}' = \mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Gamma\urcorner)\}$. Then we have $\mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H})} = \ulcorner\text{Therefore } \Gamma\urcorner$ and thus there are again $\Xi \in$ PERF and $\Gamma \in$ CFORM such that $\mathfrak{H}' = \mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\Xi\ \Gamma\urcorner)\}$. ■

Theorem 3-4. *RCE-extensions of sentence sequences are greater by exactly one than the initial sentence sequences*
If $\mathfrak{H} \in$ SEQ and $\mathfrak{H}' \in$ RCE($\mathfrak{H}$), then $\mathrm{Dom}(\mathfrak{H}') = \mathrm{Dom}(\mathfrak{H})+1$.

Proof: Suppose $\mathfrak{H} \in$ SEQ and $\mathfrak{H}' \in$ RCE($\mathfrak{H}$). With Theorem 3-3, there are $\Xi \in$ PERF and $\Gamma \in$ CFORM such that $\mathfrak{H}' = \mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\Xi\ \Gamma\urcorner)\}$ and thus we have $\mathrm{Dom}(\mathfrak{H}') = \mathrm{Dom}(\mathfrak{H})+1$. ■

Theorem 3-5. *Unique RCE-predecessors*
If $\mathfrak{H} \in$ SEQ and $\mathfrak{H}' \in$ RCE($\mathfrak{H}$), then $\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1 = \mathfrak{H}$.

Proof: Follows immediately from Theorem 3-3 and Theorem 3-4. ■

Definition 3-19. *The set of rule-compliant sentence sequences (RCS)*
RCS = $\{\mathfrak{H} \mid \mathfrak{H} \in$ SEQ and for all $j < \mathrm{Dom}(\mathfrak{H})$ it holds that $\mathfrak{H}\upharpoonright j\text{+}1 \in \mathrm{RCE}(\mathfrak{H}\upharpoonright j)\}$.

Theorem 3-6. *A sentence sequence $\mathfrak{H}$ is in RCS if and only if $\mathfrak{H}$ is empty or if $\mathfrak{H}$ is a rule-compliant extension of* $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1$ and $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1$ *is an RCS-element*
$\mathfrak{H} \in$ RCS
iff
$\mathfrak{H} = \emptyset$ or $\mathfrak{H} \in \mathrm{RCE}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ and $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1 \in$ RCS.

Proof: (*L-R*): Suppose $\mathfrak{H} \in$ RCS and $\mathfrak{H} \neq \emptyset$. Then we have $\mathfrak{H} \in \mathrm{SEQ}\setminus\{\emptyset\}$. We also have $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1 \in$ SEQ. It also holds that $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1 \subseteq \mathfrak{H}$ and that for all $j < \mathrm{Dom}(\mathfrak{H})$: $(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\upharpoonright j = \mathfrak{H}\upharpoonright j$. Because of $\mathfrak{H} \in$ RCS, we have with Definition 3-19 that for all $j < \mathrm{Dom}(\mathfrak{H})$ it holds that $\mathfrak{H}\upharpoonright j\text{+}1 \in \mathrm{RCE}(\mathfrak{H}\upharpoonright j)$. Thus we have, first, that $\mathfrak{H} = \mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1\text{+}1 \in \mathrm{RCE}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. Second, it then follows that for all $j < \mathrm{Dom}(\mathfrak{H})\text{-}1 = \mathrm{Dom}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ it holds that $(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\upharpoonright j\text{+}1 = \mathfrak{H}\upharpoonright j\text{+}1 \in \mathrm{RCE}(\mathfrak{H}\upharpoonright j) = \mathrm{RCE}((\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\upharpoonright j)$. According to Definition 3-19, we hence have $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1 \in$ RCS.

(*R-L*): Suppose $\mathfrak{H} = \emptyset$ or $\mathfrak{H} \in \mathrm{RCE}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ and $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1 \in$ RCS. If $\mathfrak{H} = \emptyset$, then $\mathfrak{H} \in$ SEQ and it holds trivially that $\mathfrak{H}\upharpoonright j\text{+}1 \in \mathrm{RCE}(\mathfrak{H}\upharpoonright j)$ for all $j < \mathrm{Dom}(\mathfrak{H})$ and thus we have $\mathfrak{H} \in$ RCS. Now, suppose $\mathfrak{H} \neq \emptyset$ and $\mathfrak{H} \in \mathrm{RCE}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ and $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1 \in$ RCS. According to Definition 3-19, we then have $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1 \in$ SEQ and $(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\upharpoonright j\text{+}1 \in \mathrm{RCE}((\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\upharpoonright j)$ for all $j < \mathrm{Dom}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$, and, moreover, $\mathfrak{H} \in \mathrm{RCE}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. According to Theorem 3-1, we then have $\mathfrak{H} \in$ SEQ and thus, with $\mathfrak{H} \neq \emptyset$, $\mathrm{Dom}(\mathfrak{H}) = \mathrm{Dom}(\mathfrak{H})\text{-}1\text{+}1 = \mathrm{Dom}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\text{+}1$. Then we have for all $j < \mathrm{Dom}(\mathfrak{H})$: $\mathfrak{H}\upharpoonright j = (\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\upharpoonright j$. Thus we have $\mathfrak{H}\upharpoonright j\text{+}1 = (\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\upharpoonright j\text{+}1 \in \mathrm{RCE}((\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\upharpoonright j) = \mathrm{RCE}(\mathfrak{H}\upharpoonright j)$ for all $j < \mathrm{Dom}(\mathfrak{H})\text{-}1$. If $j = \mathrm{Dom}(\mathfrak{H})\text{-}1$, then we have $\mathfrak{H}\upharpoonright j\text{+}1 = \mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1\text{+}1 = \mathfrak{H} \in \mathrm{RCE}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) = \mathrm{RCE}(\mathfrak{H}\upharpoonright j)$. Altogether we then have fo all $j < \mathrm{Dom}(\mathfrak{H})$ that $\mathfrak{H}\upharpoonright j\text{+}1 \in \mathrm{RCE}(\mathfrak{H}\upharpoonright j)$ and hence we have $\mathfrak{H} \in$ RCS. ■

The following theorem will often be used in the following chapters, without always being explicitly adduced as a reason:

Theorem 3-7. *The rule-compliant extension of a RCS-element results in a non-empty RCS-element*
If $\mathfrak{H} \in$ RCS and $\mathfrak{H}' \in$ AF($\mathfrak{H}$) $\cup$ CdIF($\mathfrak{H}$) $\cup$ CdEF($\mathfrak{H}$) $\cup$ CIF($\mathfrak{H}$) $\cup$ CEF($\mathfrak{H}$) $\cup$ BIF($\mathfrak{H}$) $\cup$ BEF($\mathfrak{H}$) $\cup$ DIF($\mathfrak{H}$) $\cup$ DEF($\mathfrak{H}$) $\cup$ NIF($\mathfrak{H}$) $\cup$ NEF($\mathfrak{H}$) $\cup$ UIF($\mathfrak{H}$) $\cup$ UEF($\mathfrak{H}$) $\cup$ PIF($\mathfrak{H}$) $\cup$ PEF($\mathfrak{H}$) $\cup$ IIF($\mathfrak{H}$) $\cup$ IEF($\mathfrak{H}$), then $\mathfrak{H}' \in$ RCS$\setminus\{\emptyset\}$.

Proof: Suppose $\mathfrak{H} \in$ RCS and $\mathfrak{H}' \in$ AF($\mathfrak{H}$) $\cup$ CdIF($\mathfrak{H}$) $\cup$ CdEF($\mathfrak{H}$) $\cup$ CIF($\mathfrak{H}$) $\cup$ CEF($\mathfrak{H}$) $\cup$ BIF($\mathfrak{H}$) $\cup$ BEF($\mathfrak{H}$) $\cup$ DIF($\mathfrak{H}$) $\cup$ DEF($\mathfrak{H}$) $\cup$ NIF($\mathfrak{H}$) $\cup$ NEF($\mathfrak{H}$) $\cup$ UIF($\mathfrak{H}$) $\cup$ UEF($\mathfrak{H}$) $\cup$ PIF($\mathfrak{H}$) $\cup$ PEF($\mathfrak{H}$) $\cup$ IIF($\mathfrak{H}$) $\cup$ IEF($\mathfrak{H}$). According to Definition 3-18, we then have $\mathfrak{H}' \in$ RCE($\mathfrak{H}$). With Theorem 3-5, we have $\mathfrak{H} = \mathfrak{H}'\upharpoonright$Dom($\mathfrak{H}'$)-1. Because of $\mathfrak{H} \in$ RCS and with Theorem 3-6, we then have $\mathfrak{H}' \in$ RCS. With Theorem 3-1, we then have $\mathfrak{H}' \neq \emptyset$ and thus $\mathfrak{H}' \in$ RCS$\setminus\{\emptyset\}$. ■

Theorem 3-8. *$\mathfrak{H}$ is a non-empty RCS-element if and only if $\mathfrak{H}$ is a non-empty sentence sequence and all non-empty initial segments of $\mathfrak{H}$ are non-empty RCS-elements*
$\mathfrak{H} \in$ RCS$\setminus\{\emptyset\}$ iff $\mathfrak{H} \in$ SEQ$\setminus\{\emptyset\}$ and for all $i \in$ Dom($\mathfrak{H}$): $\mathfrak{H}\upharpoonright i$+1 $\in$ RCS$\setminus\{\emptyset\}$.

Proof: *(L-R)*: Suppose $\mathfrak{H} \in$ RCS$\setminus\{\emptyset\}$. According to Definition 3-19, we then have $\mathfrak{H} \in$ SEQ and for all $i \in$ Dom($\mathfrak{H}$) that $\mathfrak{H}\upharpoonright(i$+1) $\in$ RCE($\mathfrak{H}\upharpoonright i$). With our hypothesis, we then have $\mathfrak{H} \in$ SEQ$\setminus\{\emptyset\}$. Suppose 0 $\in$ Dom($\mathfrak{H}$). Then we have $\mathfrak{H}\upharpoonright$1 $\in$ RCE($\mathfrak{H}\upharpoonright$0) = RCE($\emptyset$). With Theorem 3-6, we have $\emptyset \in$ RCS and thus we have, with $\mathfrak{H}\upharpoonright$1 $\in$ RCE($\emptyset$) and with Theorem 3-6, that $\mathfrak{H}\upharpoonright$1 $\in$ RCS. With 0 $\in$ Dom($\mathfrak{H}\upharpoonright$1) we then have $\mathfrak{H}\upharpoonright$1 $\in$ RCS$\setminus\{\emptyset\}$. Now, suppose for i it holds that if $i \in$ Dom($\mathfrak{H}$), then $\mathfrak{H}\upharpoonright i$+1 $\in$ RCS$\setminus\{\emptyset\}$. Now, suppose i+1 $\in$ Dom($\mathfrak{H}$). Then we have $i \in$ Dom($\mathfrak{H}$) and thus, according to the I.H., also $\mathfrak{H}\upharpoonright i$+1 $\in$ RCS$\setminus\{\emptyset\}$. Also, we have $\mathfrak{H}\upharpoonright i$+2 $\in$ RCE($\mathfrak{H}\upharpoonright i$+1). Because of $\mathfrak{H} \in$ SEQ and i+1 $\in$ Dom($\mathfrak{H}$), we have $\mathfrak{H}\upharpoonright i$+1 = ($\mathfrak{H}\upharpoonright(i$+2))$\upharpoonright$Dom($\mathfrak{H}\upharpoonright(i$+2))-1. With Theorem 3-6 and Theorem 3-1, we then have $\mathfrak{H}\upharpoonright i$+2 $\in$ RCS$\setminus\{\emptyset\}$.

(R-L): Now, suppose $\mathfrak{H} \in$ SEQ$\setminus\{\emptyset\}$ for all $i \in$ Dom($\mathfrak{H}$): $\mathfrak{H}\upharpoonright i$+1 $\in$ RCS$\setminus\{\emptyset\}$. With $\mathfrak{H} \in$ SEQ$\setminus\{\emptyset\}$, we then have Dom($\mathfrak{H}$)-1 $\in$ Dom($\mathfrak{H}$) and hence $\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1+1 = $\mathfrak{H} \in$ RCS$\setminus\{\emptyset\}$. ■

Based on Definition 3-19, we will now introduce a derivation concept. Subsequently, after having proved some theorems and considered an example concerning the derivation concept, we will establish a corresponding consequence concept.

Definition 3-20. *Derivation*
$\mathfrak{H}$ is a derivation of Γ from X
iff

(i) $\mathfrak{H} \in \text{RCS}\setminus\{\emptyset\}$,
(ii) $\Gamma = \text{C}(\mathfrak{H})$ and
(iii) $X = \text{AVAP}(\mathfrak{H})$.

If we take into account Definition 3-19, we now have characterised exatly those non-empty sentence sequences as derivations of a proposition from a set of propositions that can in principle be uttered by successively applying the rules of the Speech Act Calculus.

Theorem 3-9. *Properties of derivations*
If $\mathfrak{H}$ is a derivation of Γ from X, then:

(i) $\mathfrak{H} \in \text{SEQ}\setminus\{\emptyset\}$,
(ii) $\Gamma \in \text{CFORM}$ and
(iii) $X \subseteq \text{CFORM}$ and $|X| \in \mathbb{N}$.

Proof: Suppose $\mathfrak{H}$ is a derivation of Γ from X. Then we have $\mathfrak{H} \in \text{RCS}\setminus\{\emptyset\}$ and $\text{C}(\mathfrak{H}) = \Gamma$ and $X = \text{AVAP}(\mathfrak{H})$. With Definition 3-19, we have $\mathfrak{H} \in \text{SEQ}\setminus\{\emptyset\}$. According to Definition 1-25, Definition 1-24, Definition 1-23, Definition 1-18 and Definition 1-16, we have that $\text{C}(\mathfrak{H}) = \Gamma \in \text{CFORM}$. According to Definition 1-23 and Definition 1-24, we have $\text{Dom}(\mathfrak{H}) \in \mathbb{N}$. With Definition 2-31, Definition 2-29, Definition 2-28 and Definition 2-26, we thus also have $X = \text{AVAP}(\mathfrak{H}) \subseteq \text{CFORM}$ and $|X| = |\text{AVAP}(\mathfrak{H})| \in \mathbb{N}$. ■

Theorem 3-10. *In non-empty RCS-elements all non-empty initial segments are derivations of their respective conclusions*
If $\mathfrak{H} \in \text{RCS}\setminus\{\emptyset\}$, then it holds for all $i \in \text{Dom}(\mathfrak{H})$ that $\mathfrak{H}\restriction i{+}1$ is a derivation of $\text{P}(\mathfrak{H}_i)$ from $\text{AVAP}(\mathfrak{H}\restriction i{+}1)$.

Proof: Suppose $\mathfrak{H} \in \text{RCS}\setminus\{\emptyset\}$. With Theorem 3-8, it then holds for all $i \in \text{Dom}(\mathfrak{H})$ that $\mathfrak{H}\restriction i{+}1 \in \text{RCS}\setminus\{\emptyset\}$. Also, we have for all $i \in \text{Dom}(\mathfrak{H})$: $\text{P}(\mathfrak{H}_i) = \text{C}(\mathfrak{H}\restriction i{+}1)$ and $\text{AVAP}(\mathfrak{H}\restriction i{+}1) = \text{AVAP}(\mathfrak{H}\restriction i{+}1)$. ■

Theorem 3-11. *Uniqueness-theorem for the Speech Act Calculus*[13]
If $\mathfrak{H} \in$ SEQ, then:
(i) There is no Γ and no X such that $\mathfrak{H}$ is a derivation of Γ from X
or
(ii) There is exactly one Γ and exactly one X such that $\mathfrak{H}$ is a derivation of Γ from X.

Proof: Suppose $\mathfrak{H} \in$ SEQ. Then there is no Γ and no X such that $\mathfrak{H}$ is a derivation of Γ from X or there are a Γ and an X such that $\mathfrak{H}$ is a derivation of Γ from X. In the first case, the statement holds. Now, for the second case, suppose there are a Γ and an X such that $\mathfrak{H}$ is a derivation of Γ from X. According to Definition 3-20, we then have $\mathfrak{H} \in$ RCS\{∅}, Γ = C($\mathfrak{H}$) and AVAP($\mathfrak{H}$) = X. We still have to show uniqueness. For this, supoose $\mathfrak{H}$ is a derivation of Γ' from X'. Then we have Γ' = C($\mathfrak{H}$) = Γ and X' = AVAP($\mathfrak{H}$) = X. ■

Now, let us illutsrate this result with an example. Suppose $\xi \in$ VAR, $\Delta \in$ FORM, where FV(Δ) ⊆ {ξ}, and suppose $\beta \in$ PAR\ST(Δ). Now, let $\mathfrak{H}^{[3.1]}$ be the following sentence sequence:

Example [3.1]

0	Suppose	$\bigwedge\xi\neg\Delta$
1	Suppose	$\bigvee\xi\Delta$
2	Suppose	$[\beta, \xi, \Delta]$
3	Suppose	$\bigvee\xi\Delta$
4	Therefore	$\bigvee\xi\Delta \wedge [\beta, \xi, \Delta]$
5	Therefore	$[\beta, \xi, \Delta]$
6	Therefore	$\neg[\beta, \xi, \Delta]$
7	Therefore	$\neg\bigvee\xi\Delta$
8	Therefore	$\neg\bigvee\xi\Delta$
9	Therefore	$\neg\bigvee\xi\Delta$

Commentary: According to Theorem 3-11, there should either be no Γ and no X such that $\mathfrak{H}^{[3.1]}$ is a derivation of Γ from X or we should be able to find unique Γ and X such that

13 For the formulation of a corresponding theorem for a regulation of the predicate '.. is a derivation of .. from ..' according to which the set of propositions named at the third place has to be a superset of the set of assumptions that actually occur in the respective sentence sequence and are not eliminated there, see footnote 4.

$\mathfrak{H}^{[3.1]}$ is a derivation of Γ from X. This is actually the case as $\mathfrak{H}^{[3.1]}$ is a derivation of ⌜¬⋁ξΔ⌝ from {⌜⋀ξ¬Δ⌝}, where both are uniquely determined. This can be made clearer by an informal inspection of the sentence sequence. To do this, we first furnish the sentence sequence with comments that will then be explained.

Example [3.2]

				available
0	Suppose	⋀ξ¬Δ	(AR)	0
1	Suppose	⋁ξΔ	(AR)	0, 1
2	Suppose	[β, ξ, Δ]	(AR)	0, 1, 2
3	Suppose	⋁ξΔ	(AR)	0, 1, 2, 3
4	Therefore	⋁ξΔ ∧ [β, ξ, Δ]	(CI); 2, 3	0, 1, 2, 3, 4
5	Therefore	[β, ξ, Δ]	(CE); 4	0, 1, 2, 3, 4, 5
6	Therefore	¬[β, ξ, Δ]	(UE); 1	0, 1, 2, 3, 4, 5, 6
7	Therefore	¬⋁ξΔ	(NI); 5, 6	0, 1, 2, 7
8	Therefore	¬⋁ξΔ	(PE); 1, 7	0, 1, 8
9	Therefore	¬⋁ξΔ	(NI); 1, 8	0, 9

Explanation: In the second column from the right, the rules by which one may extend an already uttered sequence and the respective premise lines are given (cf. ch. 3.1). The uttermost right column displays the line numbers of those lines whose propositions are available in the restriction of $\mathfrak{H}^{[3.1]}$ on the successor of the current line number. Note that the propositions and assumptions that are available in $\mathfrak{H}^{[3.1]}\restriction i$ $(1 \leq i \leq 10)$ are always uniquely determined.

Also, we have that, for example, the inference in line 8 may only be carried out by PE and the inference in line 9 may only be carried out by NI, in both cases with uniquely determined premise lines. In line 8, NI is not an option, because, on the one hand, the proposition assumed in line 2 is still available in $\mathfrak{H}^{[3.1]}\restriction 8$ so that 1 cannot serve as an initial assumption for NI, while, on the other hand, 3 cannot serve as an initial assumption for NI, because the proposition assumed there is not any more available in $\mathfrak{H}^{[3.1]}\restriction 8$ at this position. Obversly, PE may not be carried out in line 9 (and NI may be carried out), because the representative instance assumption in line 2 is not any more available in $\mathfrak{H}^{[3.1]}\restriction 9$ at this position (and at all).

If one checks all other lines, one can easily convince oneself that $\mathfrak{H}^{[3.1]} \in \text{RCS}\backslash\{\emptyset\}$. The set of the assumptions that are available in $\mathfrak{H}^{[3.1]}$ is uniquely determined and determinable, because, with Definition 2-26, Definition 2-28, Definition 2-29 and Definition 2-31, one can check for every proposition A that has been assumed in $\mathfrak{H}^{[3.1]}$ whether A $\in$ AVAP($\mathfrak{H}^{[3.1]}$). As desired, one can easily convince oneself that AVAP($\mathfrak{H}^{[3.1]}$) = {⌜⋀ξ¬Δ⌝}. Obviously, we have $\mathfrak{H}^{[3.1]}{}_{\mathrm{Dom}(\mathfrak{H}^{[3.1]})-1}$ = ⌜Therefore ¬⋁ξΔ⌝ so that Theorem 3-11 is confirmed.

Note that the comments in the right columns do not serve to disambiguate from which set of propositions the proposition in the last line has been derived, but only serve to facilitate an easier traceability and understanding. Note that the rule-commentary to $\mathfrak{H}^{[3.1]}$ is uniquely determined by coincidence and that there are other sentence sequences for which different rule-commentaries may be produced: There are circumstances under which a transition may be carried out in accordance with different rules, e.g. UE and PE. However, it is not the case that the possibility of alternative rule-commentaries has any effects on the uniqueness of the availability-commentary. Available propositions (or lines) are not determined with recourse to the rule-commentary, but according to the definition of availability and thus, eventually, according to the definition of closed segments. The separate definition of availability excludes that we arrive at different availabilities for one and the same transition, even if that transisition can be carried out in accordance with more than one rule. Thus, it is always uniquely determined and determinable if a given sentence sequence is a derivation of a certain proposition from a certain set of propositions.

Closed segments emerge if and only if one may apply CdI, NI or PE (cf. Theorem 3-23 and Theorem 3-24). Thus, if a transition is covered by more than one rule, e.g. UE and PE, availabilities change as they do in a transition by PE. Thus, a user of the Speech Act Calculus is restriced in the preformance of certain inferences: For example, one is not free to carry out an assumption-discharging inference by PE as a not assumption-discharging inference by UE.

One may deem that this makes the Speech Act Calculus a bit unhandy, however, this shortcoming, if it is one, comes with the advantage that for every utterance of a sentence sequence by an author, we can uniquely determine if that author has uttered a derivation of a certain proposition from a certain set of propositions: The possibility to describe the utterance of one and the same sentence sequence in different ways so that, for example the utterance of a sentence sequence $\mathfrak{H}$ can be described as an utterance of a derivation of Γ from X and can also described as the utterance of a sentence sequence that is not a derivation of Γ from X, which exists for some calculi, does not exist for the Speech Act Calculus. If one utters derivations in accordance with the rules of the Speech Act Calculus, one does not have to use graphical means for the marking of subderivations nor meta-theoretical rule- or dependence-commentaries: In the framework of the Speech Act Calculus utterances of sentence sequences are not up for interpretation.

Now, we will introduce the deductive consequence concept and some other usual meta-logical concepts. In ch. 4, we will then prove some properties of the deductive consequence relation, such as reflexivity, transitivity and closure under introduction and elimination. Subsequently, in ch. 6, we will then provide an adequacy proof for the calculus relative to the classical model-theoretic consequence relation. This relation itself will be established in ch. 5. Now, for the definition of the consequence relation:

Definition 3-21. *Deductive consequence relation*
$X \vdash \Gamma$
iff
$X \subseteq$ CFORM and there is an $\mathfrak{H}$ such that
(i) $\mathfrak{H}$ is a derivation of Γ from AVAP($\mathfrak{H}$), and
(ii) AVAP($\mathfrak{H}$) $\subseteq X$.

With Theorem 3-9-(iii), it then follows, as usual, that for $X \subseteq$ CFORM it holds that $X \vdash \Gamma$ if and only if there is a finite $Y \subseteq X$ such that $Y \vdash \Gamma$. From this and Definition 3-23, it then follows that X is consistent if and only if all finite $Y \subseteq X$ are consistent, and, with Definition 3-24, that $X \subseteq$ CFORM is inconsistent if and only if there is a finite $Y \subseteq X$ such that Y is inconsistent. Under Definition 3-20, the following theorem is equivalent to Definition 3-21:

Theorem 3-12. Γ *is a deductive consequence of a set of propositions* X *if and only if there is a non-empty RCS-element* $\mathfrak{H}$ *such that* Γ *is the conclusion of* $\mathfrak{H}$ *and* AVAP($\mathfrak{H}$) $\subseteq X$
$X \vdash \Gamma$ iff $X \subseteq$ CFORM and there is $\mathfrak{H} \in$ RCS\{∅} such that Γ = C($\mathfrak{H}$) and AVAP($\mathfrak{H}$) $\subseteq X$.

Proof: Follows directly from Definition 3-20 and Definition 3-21. ■

Definition 3-22. *Logical provability*
$\vdash \Gamma$ iff $\emptyset \vdash \Gamma$.

Definition 3-23. *Consistency*
X is consistent
iff
$X \subseteq$ CFORM and there is no $\Gamma \in$ CFORM such that $X \vdash \Gamma$ and $X \vdash \ulcorner\neg\Gamma\urcorner$.

Definition 3-24. *Inconsistency*
X is inconsistent
iff
$X \subseteq$ CFORM and there is a $\Gamma \in$ CFORM such that $X \vdash \Gamma$ and $X \vdash \ulcorner\neg\Gamma\urcorner$.

Theorem 3-13. *Sets of propositions are inconsistent if and only if they are not consistent*
If $X \subseteq$ CFORM, then: X is inconsistent iff X is not consistent.

Proof: Follows directly from Definition 3-23 and Definition 3-24. ■

Definition 3-25. *Deductive consequence for sets*
$X \,_{M}\!\vdash Y$ iff $X \cup Y \subseteq$ CFORM and for all $\Delta \in Y$ it holds that $X \vdash \Delta$.

Definition 3-26. *Logical provability for sets*
$_{M}\!\vdash X$ iff $\emptyset \,_{M}\!\vdash X$.

Definition 3-27. *The closure of a set of propositions under deductive consequence*
$X^{\vdash} = \{\Delta \mid \Delta \in$ CFORM and $X \vdash \Delta\}$.

Before proving the usual properties for the deductive consequence relation in ch. 4 and ch. 6, we will prove some theorems that illustrate the working of the calculus in the following ch. 3.3.

3.3 AVS, AVAS, AVP and AVAP in Derivations and in Individual Transitions

Now, we will establish some theorems for the rules (cf. ch. 3.1) and operations (cf. ch. 3.2) respectively that describe the working of the Speech Act Calculus. More exactly, we will prove theorems that provide an account of the connections between changes in availabilities (AVS, AVAS, AVP, AVAP) in rule-compliant transitions from a sentence sequence $\mathfrak{H}$ to a sentence sequence $\mathfrak{H}'$ and the respective rule or operation. At the same time, these theorems provide the basis for the theorems about the deductive consequence relation that are proved in ch. 4 and for the proof of the correctness and the completeness of the Speech Act Calculus in ch. 6. At the end of the chapter, Theorem 3-30 offers an overview of the form of derivations and the availability conditions in derivations in the Speech Act Calculus.

Theorem 3-14. *AVS, AVAS, AVP, AVAP and RCE*

If $\mathfrak{H} \in$ SEQ and $\mathfrak{H}' \in$ RCE($\mathfrak{H}$), then:

(i) $\text{AVS}(\mathfrak{H}') \subseteq \text{AVS}(\mathfrak{H}) \cup \{(\text{Dom}(\mathfrak{H}), \mathfrak{H}'_{\text{Dom}(\mathfrak{H})})\}$,

(ii) $\text{AVAS}(\mathfrak{H}') \subseteq \text{AVAS}(\mathfrak{H}) \cup \{(\text{Dom}(\mathfrak{H}), \mathfrak{H}'_{\text{Dom}(\mathfrak{H})})\}$,

(iii) $\text{AVP}(\mathfrak{H}') \subseteq \text{AVP}(\mathfrak{H}) \cup \{\text{C}(\mathfrak{H}')\}$, and

(iv) $\text{AVAP}(\mathfrak{H}') \subseteq \text{AVAP}(\mathfrak{H}) \cup \{\text{C}(\mathfrak{H}')\}$.

Proof: Suppose $\mathfrak{H} \in$ SEQ and $\mathfrak{H}' \in$ RCE($\mathfrak{H}$). With Theorem 3-3, there are then $\Xi \in$ PERF and $\Gamma \in$ CFORM such that $\mathfrak{H}' = \mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\Xi\ \Gamma\urcorner)\} = \mathfrak{H}^\frown\{(0, \ulcorner\Xi\ \Gamma\urcorner)\}$ and the statement follows with Theorem 2-79. ■

Theorem 3-15. *AVS, AVAS, AVP, AVAP and AR*

If $\mathfrak{H} \in$ SEQ and $\mathfrak{H}' \in$ AF($\mathfrak{H}$), then:

(i) $\text{AVS}(\mathfrak{H}')\setminus\text{AVS}(\mathfrak{H}) = \{(\text{Dom}(\mathfrak{H}), \mathfrak{H}'_{\text{Dom}(\mathfrak{H})})\}$,

(ii) $\text{AVS}(\mathfrak{H}') = \text{AVS}(\mathfrak{H}) \cup \{(\text{Dom}(\mathfrak{H}), \mathfrak{H}'_{\text{Dom}(\mathfrak{H})})\}$,

(iii) $\text{AVAS}(\mathfrak{H}')\setminus\text{AVAS}(\mathfrak{H}) = \{(\text{Dom}(\mathfrak{H}), \mathfrak{H}'_{\text{Dom}(\mathfrak{H})})\}$,

(iv) $\text{AVAS}(\mathfrak{H}') = \text{AVAS}(\mathfrak{H}) \cup \{(\text{Dom}(\mathfrak{H}), \mathfrak{H}'_{\text{Dom}(\mathfrak{H})})\}$,

(v) $\text{AVP}(\mathfrak{H}')\setminus\text{AVP}(\mathfrak{H}) \subseteq \{\text{C}(\mathfrak{H}')\}$,

(vi) $\text{AVP}(\mathfrak{H}') = \text{AVP}(\mathfrak{H}) \cup \{\text{C}(\mathfrak{H}')\}$,

(vii) $AVAP(\mathfrak{H}')\backslash AVAP(\mathfrak{H}) \subseteq \{C(\mathfrak{H}')\}$, and

(viii) $AVAP(\mathfrak{H}') = AVAP(\mathfrak{H}) \cup \{C(\mathfrak{H}')\}$.

Proof: Suppose $\mathfrak{H} \in SEQ$ and $\mathfrak{H}' \in AF(\mathfrak{H})$. With Definition 3-18, it then holds that $\mathfrak{H}' \in RCE(\mathfrak{H})$. With Definition 3-1, we have that there is $\Gamma \in CFORM$ such that $\mathfrak{H}' = \mathfrak{H} \cup \{(Dom(\mathfrak{H}), \ulcorner Suppose\ \Gamma \urcorner)\}$. Thus we have $\mathfrak{H}' \upharpoonright Dom(\mathfrak{H}')\text{-}1 = \mathfrak{H}' \upharpoonright Dom(\mathfrak{H}) = \mathfrak{H}$.

Ad (i): Suppose $(i, \mathfrak{H}'_i) \in AVS(\mathfrak{H}')\backslash AVS(\mathfrak{H})$. With Theorem 3-14-(i), we then have $(i, \mathfrak{H}'_i) \in \{(Dom(\mathfrak{H}), \mathfrak{H}'_{Dom(\mathfrak{H})})\}$. With Theorem 2-82, we have $(Dom(\mathfrak{H}), \mathfrak{H}'_{Dom(\mathfrak{H})}) \in AVS(\mathfrak{H}')$ and we have $(Dom(\mathfrak{H}), \mathfrak{H}'_{Dom(\mathfrak{H})}) \notin AVS(\mathfrak{H}) \subseteq \mathfrak{H}$. Hence we have $(Dom(\mathfrak{H}), \mathfrak{H}'_{Dom(\mathfrak{H})}) \in AVS(\mathfrak{H}')\backslash AVS(\mathfrak{H})$.

Ad (ii): With Theorem 3-14-(i), it holds that $AVS(\mathfrak{H}') \subseteq AVS(\mathfrak{H}) \cup \{(Dom(\mathfrak{H}), \mathfrak{H}'_{Dom(\mathfrak{H})})\}$. Also, we have that $(Dom(\mathfrak{H}), \mathfrak{H}'_{Dom(\mathfrak{H})}) = (Dom(\mathfrak{H}), \ulcorner Suppose\ \Gamma \urcorner) \in AS(\mathfrak{H}')$. It then holds, with Theorem 2-30, that there is no CdI- or NI- or RA-like and thus no closed segment $\mathfrak{B}$ in $\mathfrak{H}'$ such that $min(Dom(\mathfrak{B})) \leq Dom(\mathfrak{H})\text{-}1 = Dom(\mathfrak{H}')\text{-}2$ and $max(Dom(\mathfrak{B})) = Dom(\mathfrak{H}) = Dom(\mathfrak{H}')\text{-}1$. With Theorem 2-84, we then have $AVS(\mathfrak{H})\backslash AVS(\mathfrak{H}') = \emptyset$ and thus $AVS(\mathfrak{H}) \subseteq AVS(\mathfrak{H}')$. With (i), we have $(Dom(\mathfrak{H}), \mathfrak{H}'_{Dom(\mathfrak{H})}) \in AVS(\mathfrak{H}')$ and hence we have $AVS(\mathfrak{H}) \cup \{(Dom(\mathfrak{H}), \mathfrak{H}'_{Dom(\mathfrak{H})})\} \subseteq AVS(\mathfrak{H}')$.

Ad (iii): Suppose $(i, \mathfrak{H}'_i) \in AVAS(\mathfrak{H}')\backslash AVAS(\mathfrak{H})$. With Theorem 3-14-(ii), it then follows that $(i, \mathfrak{H}'_i) \in \{(Dom(\mathfrak{H}), \mathfrak{H}'_{Dom(\mathfrak{H})})\}$. With (i), we also have $(Dom(\mathfrak{H}), \mathfrak{H}'_{Dom(\mathfrak{H})}) \in AVS(\mathfrak{H}')$. Also, we have $(Dom(\mathfrak{H}), \mathfrak{H}'_{Dom(\mathfrak{H})}) = (Dom(\mathfrak{H}), \ulcorner Suppose\ \Gamma \urcorner) \in AS(\mathfrak{H}')$ and thus we have $(Dom(\mathfrak{H}), \mathfrak{H}'_{Dom(\mathfrak{H})}) \in AVAS(\mathfrak{H}')$ and $(Dom(\mathfrak{H}), \mathfrak{H}'_{Dom(\mathfrak{H})}) \notin AVAS(\mathfrak{H}) \subseteq \mathfrak{H}$.

Ad (iv): With (iii), we have $(Dom(\mathfrak{H}), \mathfrak{H}'_{Dom(\mathfrak{H})}) \in AVAS(\mathfrak{H}') = AVS(\mathfrak{H}') \cap AS(\mathfrak{H}')$. With (ii), we thus have $AVAS(\mathfrak{H}) \cup \{(Dom(\mathfrak{H}), \mathfrak{H}'_{Dom(\mathfrak{H})})\} = (AVS(\mathfrak{H}) \cap AS(\mathfrak{H})) \cup (\{(Dom(\mathfrak{H}), \mathfrak{H}'_{Dom(\mathfrak{H})})\} \cap AS(\mathfrak{H}')) = (AVS(\mathfrak{H}) \cup \{(Dom(\mathfrak{H}), \mathfrak{H}'_{Dom(\mathfrak{H})})\}) \cap AS(\mathfrak{H}') = AVS(\mathfrak{H}') \cap AS(\mathfrak{H}') = AVAS(\mathfrak{H}')$.

Ad (v), (vi), (vii), (viii): (v) follows with Theorem 3-14-(iii), and (vii) follows with Theorem 3-14-(iv). (vi) follows with Definition 2-30 and (ii). (viii) follows with Definition 2-31 and (iv). ■

Theorem 3-16. *AVAS-increase only for AR*

If $\mathfrak{H} \in$ SEQ and $\mathfrak{H}' \in$ RCE($\mathfrak{H}$), then:

(i) If AVAS($\mathfrak{H}$) $\subset$ AVAS($\mathfrak{H}'$), then $\mathfrak{H}' \in$ AF($\mathfrak{H}$), and

(ii) If AVAP($\mathfrak{H}$) $\subset$ AVAP($\mathfrak{H}'$), then $\mathfrak{H}' \in$ AF($\mathfrak{H}$).

Proof: Suppose $\mathfrak{H} \in$ SEQ and $\mathfrak{H}' \in$ RCE($\mathfrak{H}$). *Ad (i)*: Suppose AVAS($\mathfrak{H}$) $\subset$ AVAS($\mathfrak{H}'$). Then there is $(i, \mathfrak{H}'_i) \in$ AVAS($\mathfrak{H}'$)\AVAS($\mathfrak{H}$). Then we have $(i, \mathfrak{H}'_i) \in$ AS($\mathfrak{H}'$). With Theorem 3-14-(ii), we also have $(i, \mathfrak{H}'_i)$ = (Dom($\mathfrak{H}$), $\mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H})}$) and hence (Dom($\mathfrak{H}$), $\mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H})}$) $\in$ AS($\mathfrak{H}'$). With Definition 3-1, we then have $\mathfrak{H}' \in$ AF($\mathfrak{H}$). *Ad (ii)*: Suppose AVAP($\mathfrak{H}$) $\subset$ AVAP($\mathfrak{H}'$). With Theorem 2-75, we then have AVAS($\mathfrak{H}'$) $\not\subseteq$ AVAS($\mathfrak{H}$) and thus there is $(i, \mathfrak{H}'_i) \in$ AVAS($\mathfrak{H}'$)\AVAS($\mathfrak{H}$). Then the statement follows in the same way as (i). ■

Theorem 3-17. *AVS, AVAS, AVP and AVAP in transitions without AR*

If $\mathfrak{H} \in$ SEQ and $\mathfrak{H}' \in$ RCE($\mathfrak{H}$)\AF($\mathfrak{H}$), then:

(i) AVS($\mathfrak{H}'$) $\subseteq$ AVS($\mathfrak{H}$) $\cup$ {(Dom($\mathfrak{H}$), $\mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H})}$)},

(ii) AVAS($\mathfrak{H}'$) $\subseteq$ AVAS($\mathfrak{H}$),

(iii) AVP($\mathfrak{H}'$) $\subseteq$ AVP($\mathfrak{H}$) $\cup$ {C($\mathfrak{H}'$)}, and

(iv) AVAP($\mathfrak{H}'$) $\subseteq$ AVAP($\mathfrak{H}$).

Proof: Suppose $\mathfrak{H}' \in$ RCE($\mathfrak{H}$)\AF($\mathfrak{H}$). (i) and (iii) follow with Theorem 3-14-(i) and -(iii). *Ad (ii)*: With $\mathfrak{H}' \in$ RCE($\mathfrak{H}$)\AF($\mathfrak{H}$) and Definition 3-1 to Definition 3-18, we have that (Dom($\mathfrak{H}$), $\mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H})}$) = (Dom($\mathfrak{H}$), ⌜Therefore P($\mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H})}$)⌝) $\notin$ AS($\mathfrak{H}'$) and hence (Dom($\mathfrak{H}$), $\mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H})}$) $\notin$ AVAS($\mathfrak{H}'$). With Theorem 3-14-(ii), we then have AVAS($\mathfrak{H}'$) $\subseteq$ AVAS($\mathfrak{H}$). *Ad (iv)*: (iv) follows with Theorem 2-75 from (ii). ■

Theorem 3-18. *Non-empty AVAS is sufficient for CdI*

If $\mathfrak{H} \in$ SEQ and AVAS($\mathfrak{H}$) $\neq \emptyset$, then $\mathfrak{H} \cup$ {(Dom($\mathfrak{H}$), ⌜Therefore P($\mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))}$) $\rightarrow$ C($\mathfrak{H}$)⌝)} $\in$ CdIF($\mathfrak{H}$).

Proof: Suppose $\mathfrak{H} \in$ SEQ and AVAS($\mathfrak{H}$) $\neq \emptyset$. Then we have (max(Dom(AVAS($\mathfrak{H}$))), $\mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))}$) $\in$ AVAS($\mathfrak{H}$) and P($\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-1}$) = C($\mathfrak{H}$) and there is no l with max(Dom(AVAS($\mathfrak{H}$))) $< l \leq$ Dom($\mathfrak{H}$)-1 such that $(l, \mathfrak{H}_l) \in$ AVAS($\mathfrak{H}$). With Definition 3-2, we then have $\mathfrak{H} \cup$ {(Dom($\mathfrak{H}$), ⌜Therefore P($\mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))}$) $\rightarrow$ C($\mathfrak{H}$)⌝)} $\in$ CdIF($\mathfrak{H}$). ■

Theorem 3-19. *AVS, AVAS, AVP, AVAP and CdI*

If $\mathfrak{H} \in$ SEQ and $\mathfrak{H}' \in$ CdIF($\mathfrak{H}$), then:

(i) $\{(j, \mathfrak{H}'_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}))) \leq j \leq \mathrm{Dom}(\mathfrak{H})\}$ is a CdI-closed segment in $\mathfrak{H}'$,

(ii) $\mathrm{AVS}(\mathfrak{H})\backslash\mathrm{AVS}(\mathfrak{H}') \subseteq \{(j, \mathfrak{H}'_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}))) \leq j < \mathrm{Dom}(\mathfrak{H})\}$,

(iii) $\mathrm{AVS}(\mathfrak{H}') = (\mathrm{AVS}(\mathfrak{H})\backslash\{(j, \mathfrak{H}'_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}))) \leq j < \mathrm{Dom}(\mathfrak{H})\}) \cup \{(\mathrm{Dom}(\mathfrak{H}), \mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H})})\}$,

(iv) $\mathrm{AVAS}(\mathfrak{H})\backslash\mathrm{AVAS}(\mathfrak{H}') = \{(\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}))), \mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))})\}$,

(v) $\mathrm{AVAS}(\mathfrak{H}) = \mathrm{AVAS}(\mathfrak{H}') \cup \{(\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}))), \mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))})\}$,

(vi) $\mathrm{AVP}(\mathfrak{H})\backslash\mathrm{AVP}(\mathfrak{H}') \subseteq \{\mathrm{P}(\mathfrak{H}'_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}))) \leq j < \mathrm{Dom}(\mathfrak{H})\}$,

(vii) $\mathrm{AVP}(\mathfrak{H}) \subseteq \{\mathrm{P}(\mathfrak{H}'_j) \mid j \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}')\upharpoonright\mathrm{Dom}(\mathfrak{H}))\} \cup \{\mathrm{P}(\mathfrak{H}'_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}))) \leq j < \mathrm{Dom}(\mathfrak{H})\}$,

(viii) $\mathrm{AVAP}(\mathfrak{H})\backslash\mathrm{AVAP}(\mathfrak{H}') \subseteq \{\mathrm{P}(\mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))})\}$,

(ix) $\mathrm{AVAP}(\mathfrak{H}) = \mathrm{AVAP}(\mathfrak{H}') \cup \{\mathrm{P}(\mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))})\}$, and

(x) $\mathrm{C}(\mathfrak{H}') = \ulcorner\mathrm{P}(\mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))}) \rightarrow \mathrm{C}(\mathfrak{H})\urcorner$.

Proof: Suppose $\mathfrak{H} \in$ SEQ and $\mathfrak{H}' \in$ CdIF($\mathfrak{H}$). With Definition 3-18, it then holds that $\mathfrak{H}' \in$ RCE($\mathfrak{H}$). With Definition 3-2, we have that there are $\Delta, \Gamma \in$ CFORM and $i \in \mathrm{Dom}(\mathfrak{H})$ such that $\mathrm{P}(\mathfrak{H}_i) = \Delta$ and $(i, \mathfrak{H}_i) \in \mathrm{AVAS}(\mathfrak{H})$ and $\mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-1}) = \Gamma$ and there is no l such that $i < l \leq \mathrm{Dom}(\mathfrak{H})-1$ and $(l, \mathfrak{H}_l) \in \mathrm{AVAS}(\mathfrak{H})$, and $\mathfrak{H}' = \mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \Delta \rightarrow \Gamma\urcorner)\}$. Then we have $\mathfrak{H}' \in$ SEQ and $\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1 = \mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}) = \mathfrak{H}$.

We thus have that $\mathfrak{B} = \{(j, \mathfrak{H}'_j) \mid i \leq j \leq \mathrm{Dom}(\mathfrak{H})\}$ is a segment in $\mathfrak{H}'$ and that $\mathrm{P}(\mathfrak{H}'_i) = \Delta$ and $(i, \mathfrak{H}'_i) \in \mathrm{AVAS}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}))$ and $\mathrm{P}(\mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H})-1}) = \Gamma$ and that there is no l such that $i < l \leq \mathrm{Dom}(\mathfrak{H})-1$ and $(l, \mathfrak{H}'_l) \in \mathrm{AVAS}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}))$, and $\mathrm{P}(\mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H})}) = \ulcorner\Delta \rightarrow \Gamma\urcorner$. With Theorem 2-91, we then have that $\mathfrak{B}$ is a CdI-closed segment and thus a closed segment in $\mathfrak{H}'$.

Since $\max(\mathrm{Dom}(\mathfrak{B})) = \mathrm{Dom}(\mathfrak{H}) = \mathrm{Dom}(\mathfrak{H}')-1$, it follows, with Theorem 2-86, that $\mathrm{AVAS}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1)\backslash\mathrm{AVAS}(\mathfrak{H}') = \{(\min(\mathrm{Dom}(\mathfrak{B})), \mathfrak{H}'_{\min(\mathrm{Dom}(\mathfrak{B}))})\} = \{(\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1))), \mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1)))})\}$. Since $\mathfrak{H} = \mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1$, we thus have $\mathrm{AVAS}(\mathfrak{H})\backslash\mathrm{AVAS}(\mathfrak{H}') = \{(\min(\mathrm{Dom}(\mathfrak{B})), \mathfrak{H}'_{\min(\mathrm{Dom}(\mathfrak{B}))})\} = \{(\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}))), \mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))})\}$. Thus we have $i = \min(\mathrm{Dom}(\mathfrak{B})) = \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))$ and it holds that $\mathfrak{B} = \{(j, \mathfrak{H}'_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}))) \leq j \leq \mathrm{Dom}(\mathfrak{H})\}$. Thus we have (i). We then also have that $\mathrm{P}(\mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))}) = \mathrm{P}(\mathfrak{H}_i) = \Delta$. Because of $\mathrm{C}(\mathfrak{H}) = \Gamma$ and $\mathrm{C}(\mathfrak{H}') = \ulcorner\Delta \rightarrow \Gamma\urcorner$, it then follows that (x) holds. With $\mathrm{AVAS}(\mathfrak{H})\backslash\mathrm{AVAS}(\mathfrak{H}') \neq \emptyset$ and Theorem 2-73, we also have $\mathrm{AVS}(\mathfrak{H})\backslash\mathrm{AVS}(\mathfrak{H}') \neq \emptyset$. With this and with $\mathfrak{H} = \mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1$ and $\mathfrak{B} = \{(j, \mathfrak{H}'_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}))) \leq j \leq$

$\mathrm{Dom}(\mathfrak{H})\}$, the remaining clauses ((ii) to (ix)) follow with Theorem 2-83-(iv) to -(xi) and with the fact that closed segments with the same end are identical (Theorem 2-53). ■

Theorem 3-20. *AVS, AVAS, AVP, AVAP and NI*

If $\mathfrak{H} \in \mathrm{SEQ}$ and $\mathfrak{H}' \in \mathrm{NIF}(\mathfrak{H})$, then:

(i) $\{(j, \mathfrak{H}'_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}))) \leq j \leq \mathrm{Dom}(\mathfrak{H})\}$ is an NI-closed segment in $\mathfrak{H}'$,

(ii) $\mathrm{AVS}(\mathfrak{H})\backslash\mathrm{AVS}(\mathfrak{H}') \subseteq \{(j, \mathfrak{H}'_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}))) \leq j < \mathrm{Dom}(\mathfrak{H})\}$,

(iii) $\mathrm{AVS}(\mathfrak{H}') = (\mathrm{AVS}(\mathfrak{H})\backslash\{(j, \mathfrak{H}'_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}))) \leq j < \mathrm{Dom}(\mathfrak{H})\}) \cup \{(\mathrm{Dom}(\mathfrak{H}), \mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H})})\}$,

(iv) $\mathrm{AVAS}(\mathfrak{H})\backslash\mathrm{AVAS}(\mathfrak{H}') = \{(\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}))), \mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))})\}$,

(v) $\mathrm{AVAS}(\mathfrak{H}) = \mathrm{AVAS}(\mathfrak{H}') \cup \{(\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}))), \mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))})\}$,

(vi) $\mathrm{AVP}(\mathfrak{H})\backslash\mathrm{AVP}(\mathfrak{H}') \subseteq \{\mathrm{P}(\mathfrak{H}'_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}))) \leq j < \mathrm{Dom}(\mathfrak{H})\}$,

(vii) $\mathrm{AVP}(\mathfrak{H}) \subseteq \{\mathrm{P}(\mathfrak{H}'_j) \mid j \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}')\upharpoonright\mathrm{Dom}(\mathfrak{H}))\} \cup \{\mathrm{P}(\mathfrak{H}'_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}))) \leq j < \mathrm{Dom}(\mathfrak{H})\}$,

(viii) $\mathrm{AVAP}(\mathfrak{H})\backslash\mathrm{AVAP}(\mathfrak{H}') \subseteq \{\mathrm{P}(\mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))})\}$,

(ix) $\mathrm{AVAP}(\mathfrak{H}) = \mathrm{AVAP}(\mathfrak{H}') \cup \{\mathrm{P}(\mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))})\}$, and

(x) $\mathrm{C}(\mathfrak{H}') = \ulcorner\neg\mathrm{P}(\mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))})\urcorner$.

Proof: Suppose $\mathfrak{H} \in \mathrm{SEQ}$ and $\mathfrak{H}' \in \mathrm{NIF}(\mathfrak{H})$. With Definition 3-18, it then holds that $\mathfrak{H}' \in \mathrm{RCE}(\mathfrak{H})$. With Definition 3-10, we have that there are $\Delta, \Gamma \in \mathrm{CFORM}$ and $i, j \in \mathrm{Dom}(\mathfrak{H})$ such that $i \leq j$, $\mathrm{P}(\mathfrak{H}_i) = \Delta$ and $(i, \mathfrak{H}_i) \in \mathrm{AVAS}(\mathfrak{H})$, $\mathrm{P}(\mathfrak{H}_j) = \Gamma$ and $\mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-1}) = \ulcorner\neg\Gamma\urcorner$ or $\mathrm{P}(\mathfrak{H}_j) = \ulcorner\neg\Gamma\urcorner$ and $\mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-1}) = \Gamma$ and $(j, \mathfrak{H}_j) \in \mathrm{AVS}(\mathfrak{H})$ and there is no l such that $i < l \leq \mathrm{Dom}(\mathfrak{H})-1$ and $(l, \mathfrak{H}_l) \in \mathrm{AVAS}(\mathfrak{H})$, and $\mathfrak{H}' = \mathfrak{H} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \neg\Delta\urcorner)\}$. Then we have $\mathfrak{H}' \in \mathrm{SEQ}$ and $\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1 = \mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}) = \mathfrak{H}$.

We thus have that $\mathfrak{B} = \{(j, \mathfrak{H}'_j) \mid i \leq j \leq \mathrm{Dom}(\mathfrak{H})\}$ is a segment in $\mathfrak{H}'$ and that $\mathrm{P}(\mathfrak{H}'_i) = \Delta$ and $(i, \mathfrak{H}'_i) \in \mathrm{AVAS}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}))$ and $\mathrm{P}(\mathfrak{H}'_j) = \Gamma$ and $\mathrm{P}(\mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H})-1}) = \ulcorner\neg\Gamma\urcorner$ or $\mathrm{P}(\mathfrak{H}'_j) = \ulcorner\neg\Gamma\urcorner$ and $\mathrm{P}(\mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H})-1}) = \Gamma$ and $(j, \mathfrak{H}'_j) \in \mathrm{AVS}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}))$ and that there is no l such that $i < l \leq \mathrm{Dom}(\mathfrak{H})-1$ and $(l, \mathfrak{H}'_l) \in \mathrm{AVAS}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}))$ and $\mathrm{P}(\mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H})}) = \ulcorner\neg\Delta\urcorner$. With Theorem 2-92, we then have that $\mathfrak{B}$ is an NI-closed segment and thus a closed segment in $\mathfrak{H}'$.

Since $\max(\mathrm{Dom}(\mathfrak{B})) = \mathrm{Dom}(\mathfrak{H}) = \mathrm{Dom}(\mathfrak{H}')-1$, it then follows, with Theorem 2-86, that $\mathrm{AVAS}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1)\backslash\mathrm{AVAS}(\mathfrak{H}') = \{(\min(\mathrm{Dom}(\mathfrak{B})), \mathfrak{H}'_{\min(\mathrm{Dom}(\mathfrak{B}))})\} = \{(\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1))), \mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1)))})\}$. Since $\mathfrak{H} = \mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1$, we thus have $\mathrm{AVAS}(\mathfrak{H})\backslash\mathrm{AVAS}(\mathfrak{H}') = \{(\min(\mathrm{Dom}(\mathfrak{B})), \mathfrak{H}'_{\min(\mathrm{Dom}(\mathfrak{B}))})\} = \{(\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}))), \mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))})\}$. Thus we have $i = \min(\mathrm{Dom}(\mathfrak{B})) =$

max(Dom(AVAS($\mathfrak{H}$))) and it holds that $\mathfrak{B}$ = {(j, $\mathfrak{H}'_j$) | max(Dom(AVAS($\mathfrak{H}$))) ≤ j ≤ Dom($\mathfrak{H}$)}. Thus we have (i). We then also have that P($\mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))}$) = P($\mathfrak{H}_i$) = Δ. Because of C($\mathfrak{H}'$) = ⌜¬Δ⌝, it then follows that (x) holds. With AVAS($\mathfrak{H}$)\AVAS($\mathfrak{H}'$) ≠ ∅ and Theorem 2-73, we also have AVS($\mathfrak{H}$)\AVS($\mathfrak{H}'$) ≠ ∅. With this and with $\mathfrak{H}$ = $\mathfrak{H}'$↾Dom($\mathfrak{H}'$)-1 and $\mathfrak{B}$ = {(j, $\mathfrak{H}'_j$) | max(Dom(AVAS($\mathfrak{H}$))) ≤ j ≤ Dom($\mathfrak{H}$)}, the remaining clauses ((ii) to (ix)) follow with Theorem 2-83-(iv) to -(xi) and with the fact that closed segments with the same end are identical (Theorem 2-53). ■

Theorem 3-21. *AVS, AVAS, AVP, AVAP and PE*
If $\mathfrak{H}$ ∈ SEQ and $\mathfrak{H}'$ ∈ PEF($\mathfrak{H}$), then:

(i) {(j, $\mathfrak{H}'_j$) | max(Dom(AVAS($\mathfrak{H}$))) ≤ j ≤ Dom($\mathfrak{H}$)} is a PE-closed segment in $\mathfrak{H}'$,
(ii) AVS($\mathfrak{H}$)\AVS($\mathfrak{H}'$) ⊆ {(j, $\mathfrak{H}'_j$) | max(Dom(AVAS($\mathfrak{H}$))) ≤ j < Dom($\mathfrak{H}$)},
(iii) AVS($\mathfrak{H}'$) = (AVS($\mathfrak{H}$)\{(j, $\mathfrak{H}'_j$) | max(Dom(AVAS($\mathfrak{H}$))) ≤ j < Dom($\mathfrak{H}$)}) ∪ {(Dom($\mathfrak{H}$), $\mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H})}$)},
(iv) AVAS($\mathfrak{H}$)\AVAS($\mathfrak{H}'$) = {(max(Dom(AVAS($\mathfrak{H}$))), $\mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))}$)},
(v) AVAS($\mathfrak{H}$) = AVAS($\mathfrak{H}'$) ∪ {(max(Dom(AVAS($\mathfrak{H}$))), $\mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))}$)},
(vi) AVP($\mathfrak{H}$)\AVP($\mathfrak{H}'$) ⊆ {P($\mathfrak{H}'_j$) | max(Dom(AVAS($\mathfrak{H}$))) ≤ j < Dom($\mathfrak{H}$)},
(vii) AVP($\mathfrak{H}$) ⊆ {P($\mathfrak{H}'_j$) | j ∈ Dom(AVS($\mathfrak{H}'$)↾Dom($\mathfrak{H}$))} ∪ {P($\mathfrak{H}'_j$) | max(Dom(AVAS($\mathfrak{H}$))) ≤ j < Dom($\mathfrak{H}$)},
(viii) AVAP($\mathfrak{H}$)\AVAP($\mathfrak{H}'$) ⊆ {P($\mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))}$)},
(ix) AVAP($\mathfrak{H}$) = AVAP($\mathfrak{H}'$) ∪ {P($\mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))}$)}, and
(x) C($\mathfrak{H}'$) = C($\mathfrak{H}$).

Proof: Suppose $\mathfrak{H}$ ∈ SEQ and $\mathfrak{H}'$ ∈ PEF($\mathfrak{H}$). With Definition 3-18, we then have $\mathfrak{H}'$ ∈ RCE($\mathfrak{H}$). With Definition 3-15, we have that there are β ∈ PAR, ξ ∈ VAR, Δ ∈ FORM, where FV(Δ) ⊆ {ξ}, Γ ∈ CFORM and i ∈ Dom($\mathfrak{H}$) such that P($\mathfrak{H}_i$) = ⌜⋁ξΔ⌝ and (i, $\mathfrak{H}_i$) ∈ AVS($\mathfrak{H}$), P($\mathfrak{H}_{i+1}$) = [β, ξ, Δ] and (i+1, $\mathfrak{H}_{i+1}$) ∈ AVAS($\mathfrak{H}$), and P($\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-1}$) = Γ, β ∉ STSF({Δ, Γ}), and that there is no j ≤ i such that β ∈ ST($\mathfrak{H}_j$) and that there is no m such that i+1 < m ≤ Dom($\mathfrak{H}$)-1 and (m, $\mathfrak{H}_m$) ∈ AVAS($\mathfrak{H}$), and $\mathfrak{H}'$ = $\mathfrak{H}$ ∪ {(Dom($\mathfrak{H}$), ⌜Therefore Γ⌝)}. Then we have $\mathfrak{H}'$ ∈ SEQ and $\mathfrak{H}'$↾Dom($\mathfrak{H}'$)-1 = $\mathfrak{H}'$↾Dom($\mathfrak{H}$) = $\mathfrak{H}$.

We thus have that $\mathfrak{B}$ = {(j, $\mathfrak{H}'_j$) | i+1 ≤ j ≤ Dom($\mathfrak{H}$)} is a segment in $\mathfrak{H}'$ and that β ∈ PAR, ξ ∈ VAR, Δ ∈ FORM, where FV(Δ) ⊆ {ξ}, Γ ∈ CFORM and P($\mathfrak{H}'_i$) = ⌜⋁ξΔ⌝ and (i, $\mathfrak{H}'_i$) ∈ AVS($\mathfrak{H}'$↾Dom($\mathfrak{H}$)), P($\mathfrak{H}'_{i+1}$) = [β, ξ, Δ] and (i+1, $\mathfrak{H}'_{i+1}$) ∈ AVAS($\mathfrak{H}'$↾Dom($\mathfrak{H}$)-1), and P($\mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H})-1}$) = Γ, β ∉ STSF({Δ, Γ}) and that there is no j ≤ i such that β ∈ ST($\mathfrak{H}'_j$) and that there is no m such that i+1 < m ≤ Dom($\mathfrak{H}$)-1 and (m, $\mathfrak{H}'_m$) ∈ AVAS($\mathfrak{H}'$↾Dom($\mathfrak{H}$)), and that $P(\mathfrak{H}'_{Dom(\mathfrak{H})}) = \Gamma$. With Theorem 2-93, it then holds that $\mathfrak{B}$ is a PE-closed segment and thus a closed segment in $\mathfrak{H}'$.

Since $max(Dom(\mathfrak{B})) = Dom(\mathfrak{H}) = Dom(\mathfrak{H}')-1$, it follows, with Theorem 2-86, that $AVAS(\mathfrak{H}'\upharpoonright Dom(\mathfrak{H}')-1)\backslash AVAS(\mathfrak{H}') = \{(min(Dom(\mathfrak{B})), \mathfrak{H}'_{min(Dom(\mathfrak{B}))})\} = \{(max(Dom(AVAS(\mathfrak{H}'\upharpoonright Dom(\mathfrak{H}')-1))), \mathfrak{H}'_{max(Dom(AVAS(\mathfrak{H}'\upharpoonright Dom(\mathfrak{H}')-1)))})\}$. Since $\mathfrak{H} = \mathfrak{H}'\upharpoonright Dom(\mathfrak{H}')-1$, we thus have $AVAS(\mathfrak{H})\backslash AVAS(\mathfrak{H}') = \{(min(Dom(\mathfrak{B})), \mathfrak{H}'_{min(Dom(\mathfrak{B}))})\} = \{(max(Dom(AVAS(\mathfrak{H}))), \mathfrak{H}'_{max(Dom(AVAS(\mathfrak{H})))})\}$. Thus we have $i = min(Dom(\mathfrak{B})) = max(Dom(AVAS(\mathfrak{H})))$ and it holds that $\mathfrak{B} = \{(j, \mathfrak{H}'_j) \mid max(Dom(AVAS(\mathfrak{H}))) \leq j \leq Dom(\mathfrak{H})\}$. Thus we have (i). We then also have that $C(\mathfrak{H}) = P(\mathfrak{H}'_{Dom(\mathfrak{H})-1}) = \Gamma = C(\mathfrak{H}')$ and thus we have (x). With $AVAS(\mathfrak{H})\backslash AVAS(\mathfrak{H}') \neq \emptyset$ and Theorem 2-73, we also have $AVS(\mathfrak{H})\backslash AVS(\mathfrak{H}') \neq \emptyset$. With this and with $\mathfrak{H} = \mathfrak{H}'\upharpoonright Dom(\mathfrak{H}')-1$ and $\mathfrak{B} = \{(j, \mathfrak{H}'_j) \mid max(Dom(AVAS(\mathfrak{H}))) \leq j \leq Dom(\mathfrak{H})\}$, the remaining clauses ((ii) to (ix)) follow with Theorem 2-83-(iv) to -(xi) and with the fact that closed segments with the same end are identical (Theorem 2-53). ■

Theorem 3-22. *If the proposition assumed last is only once available as an assumption, then it is discharged by CdI, NI and PE*

If $\mathfrak{H} \in SEQ$, $\Delta \in CFORM$ and for all $i \in Dom(AVAS(\mathfrak{H}))$: If $P(\mathfrak{H}_i) = \Delta$, then $i = max(Dom(AVAS(\mathfrak{H})))$, then it holds for all $\mathfrak{H}' \in CdIF(\mathfrak{H}) \cup NIF(\mathfrak{H}) \cup PEF(\mathfrak{H})$ that $AVAP(\mathfrak{H}') \subseteq AVAP(\mathfrak{H})\backslash\{\Delta\}$.

Proof: Suppose $\mathfrak{H} \in SEQ$, $\Delta \in CFORM$ and suppose it holds for all $i \in Dom(AVAS(\mathfrak{H}))$ that if $P(\mathfrak{H}_i) = \Delta$, then $i = max(Dom(AVAS(\mathfrak{H})))$. Now, suppose $\mathfrak{H}' \in CdIF(\mathfrak{H}) \cup NIF(\mathfrak{H}) \cup PEF(\mathfrak{H})$. With Theorem 3-19-(iv), -(v), Theorem 3-20-(iv), -(v) and Theorem 3-21-(iv), -(v), we then have that $AVAS(\mathfrak{H})\backslash AVAS(\mathfrak{H}') = \{(max(Dom(AVAS(\mathfrak{H}))), \mathfrak{H}'_{max(Dom(AVAS(\mathfrak{H})))})\}$ and $AVAS(\mathfrak{H}') \subseteq AVAS(\mathfrak{H})$. With Theorem 2-75, we then have $AVAP(\mathfrak{H}') \subseteq AVAP(\mathfrak{H})$.

Then it holds that $\Delta \notin AVAP(\mathfrak{H}')$. To see this, suppose for contradiction that $\Delta \in AVAP(\mathfrak{H}')$. According to Definition 2-31, there would then be an $i \in Dom(AVAS(\mathfrak{H}'))$ such that $\Delta = P(\mathfrak{H}'_i)$. With $AVAS(\mathfrak{H}') \subseteq AVAS(\mathfrak{H})$, we would then have that $i \in Dom(AVAS(\mathfrak{H}))$ and that $\Delta = P(\mathfrak{H}_i)$. Since, by hypothesis, it holds for all $i \in Dom(AVAS(\mathfrak{H}))$ that if $P(\mathfrak{H}_i) = \Delta$, then $i = max(Dom(AVAS(\mathfrak{H})))$, we would thus have $max(Dom(AVAS(\mathfrak{H}))) = i \in Dom(AVAS(\mathfrak{H}'))$. But with $AVAS(\mathfrak{H})\backslash AVAS(\mathfrak{H}') = \{(max(Dom(AVAS(\mathfrak{H}))), \mathfrak{H}'_{max(Dom(AVAS(\mathfrak{H})))})\}$, we have $max(Dom(AVAS(\mathfrak{H}))) \notin$

Dom(AVAS($\mathfrak{H}$')). Contradiction! Therefore we have $\Delta \notin$ AVAP($\mathfrak{H}$') and thus AVAP($\mathfrak{H}$') $\subseteq$ AVAP($\mathfrak{H}$)\{Δ}. ■

Theorem 3-23. *AVAS-reduction by and only by CdI, NI and PE*
If $\mathfrak{H} \in$ SEQ and $\mathfrak{H}' \in$ RCE($\mathfrak{H}$), then:
AVAS($\mathfrak{H}$') $\subset$ AVAS($\mathfrak{H}$)
iff
AVAS($\mathfrak{H}$)\AVAS($\mathfrak{H}$') = {(max(Dom(AVAS($\mathfrak{H}$))), $\mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))}$)} and $\mathfrak{H}' \in$ CdIF($\mathfrak{H}$) $\cup$ NIF($\mathfrak{H}$) $\cup$ PEF($\mathfrak{H}$).

Proof: Suppose $\mathfrak{H} \in$ SEQ and $\mathfrak{H}' \in$ RCE($\mathfrak{H}$). The right-left-direction follows with clauses (iv) and (v) of Theorem 3-19, Theorem 3-20 and Theorem 3-21.

Now, for the left-right-direction, suppose AVAS($\mathfrak{H}$') $\subset$ AVAS($\mathfrak{H}$). With $\mathfrak{H}' \in$ RCE($\mathfrak{H}$) and with Theorem 3-1, we have $\mathfrak{H}' \in$ SEQ. With Theorem 3-5, we have $\mathfrak{H}'\upharpoonright$Dom($\mathfrak{H}$')-1 = $\mathfrak{H}$ and thus Dom($\mathfrak{H}$) = Dom($\mathfrak{H}$')-1. Because of AVAS($\mathfrak{H}$') $\subset$ AVAS($\mathfrak{H}$) and with Theorem 2-85, we thus have that there is a closed segment $\mathfrak{A}$ in $\mathfrak{H}$' such that min(Dom($\mathfrak{A}$)) $\leq$ Dom($\mathfrak{H}$')-2 = Dom($\mathfrak{H}$)-1 and max(Dom($\mathfrak{A}$)) = Dom($\mathfrak{H}$')-1 = Dom($\mathfrak{H}$) and AVAS($\mathfrak{H}$)\AVAS($\mathfrak{H}$') = {(min(Dom($\mathfrak{A}$)), $\mathfrak{H}'_{\min(\mathrm{Dom}(\mathfrak{A}))}$)} = {(max(Dom(AVAS($\mathfrak{H}$))), $\mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))}$)}. Now, we have to show that $\mathfrak{H}' \in$ CdIF($\mathfrak{H}$) $\cup$ NIF($\mathfrak{H}$) $\cup$ PEF($\mathfrak{H}$). It holds that

AVAS($\mathfrak{H}'\upharpoonright$max(Dom($\mathfrak{A}$))) = AVAS($\mathfrak{H}'\upharpoonright$Dom($\mathfrak{H}$)) = AVAS($\mathfrak{H}$).

With Theorem 2-61, we have that $\mathfrak{A}$ is a CdI- or NI- or PE-closed segment in $\mathfrak{H}$'. Now, suppose $\mathfrak{A}$ is a CdI-closed segment in $\mathfrak{H}$'. With Theorem 2-91, it then holds that

a) (min(Dom($\mathfrak{A}$)), $\mathfrak{H}'_{\min(\mathrm{Dom}(\mathfrak{A}))}$) = (min(Dom($\mathfrak{A}$)), $\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}))}$) $\in$ AVAS($\mathfrak{H}$),
b) P($\mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H})-1}$) = P($\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-1}$) = C($\mathfrak{H}$),
c) There is no r such that min(Dom($\mathfrak{A}$)) < $r \leq$ Dom($\mathfrak{H}$)-1 and (r, $\mathfrak{H}'_r$) = (r, $\mathfrak{H}_r$) $\in$ AVAS($\mathfrak{H}$), and
d) $\mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H})}$ = ⌜Therefore P($\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}))}$) $\rightarrow$ C($\mathfrak{H}$)⌝.

According to Definition 3-2, we then have $\mathfrak{H}' \in$ CdIF($\mathfrak{H}$). Now, suppose $\mathfrak{A}$ is an NI-closed segment in $\mathfrak{H}$'. With Theorem 2-92, it then holds that there are $i \in$ Dom($\mathfrak{H}$') and $\Gamma \in$ CFORM such that

a) min(Dom($\mathfrak{A}$)) $\leq i <$ Dom($\mathfrak{H}$),
b) (min(Dom($\mathfrak{A}$)), $\mathfrak{H}'_{\min(\mathrm{Dom}(\mathfrak{A}))}$) = (min(Dom($\mathfrak{A}$)), $\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{A}))}$) $\in$ AVAS($\mathfrak{H}$),

c) $P(\mathfrak{H}'_i) = P(\mathfrak{H}_i) = \Gamma$ and $P(\mathfrak{H}'_{Dom(\mathfrak{H})-1}) = P(\mathfrak{H}_{Dom(\mathfrak{H})-1}) = \ulcorner\neg\Gamma\urcorner$
or
$P(\mathfrak{H}'_i) = P(\mathfrak{H}_i) = \ulcorner\neg\Gamma\urcorner$ and $P(\mathfrak{H}'_{Dom(\mathfrak{H})-1}) = P(\mathfrak{H}_{Dom(\mathfrak{H})-1}) = \Gamma$,

d) $(i, \mathfrak{H}_i) \in AVS(\mathfrak{H})$,

e) There is no r such that $\min(Dom(\mathfrak{A})) < r \leq Dom(\mathfrak{H})\text{-}1$ and $(r, \mathfrak{H}'_r) = (r, \mathfrak{H}_r) \in AVAS(\mathfrak{H})$, and

f) $\mathfrak{H}'_{Dom(\mathfrak{H})} = \ulcorner$Therefore $\neg P(\mathfrak{H}'_{\min(Dom(\mathfrak{A}))})\urcorner = \ulcorner$Therefore $\neg P(\mathfrak{H}_{\min(Dom(\mathfrak{A}))})\urcorner$.

According to Definition 3-10, we then have $\mathfrak{H}' \in NIF(\mathfrak{H})$. Now, suppose $\mathfrak{A}$ is a PE-closed segment in $\mathfrak{H}'$. With Theorem 2-93, it then holds that there are $\xi \in VAR$, $\beta \in PAR$, $\Delta \in FORM$, where $FV(\Delta) \subseteq \{\xi\}$, $\Gamma \in CFORM$ and $\mathfrak{B} \in SG(\mathfrak{H}')$ such that:

a) $P(\mathfrak{H}'_{\min(Dom(\mathfrak{B}))}) = \ulcorner\bigvee\xi\Delta\urcorner$ and $(\min(Dom(\mathfrak{B})), \mathfrak{H}'_{\min(Dom(\mathfrak{B}))}) \in AVS(\mathfrak{H})$,

b) $P(\mathfrak{H}'_{\min(Dom(\mathfrak{B}))+1}) = [\beta, \xi, \Delta]$ and $(\min(Dom(\mathfrak{B}))+1, \mathfrak{H}'_{\min(Dom(\mathfrak{B}))+1}) \in AVAS(\mathfrak{H})$,

c) $P(\mathfrak{H}'_{\max(Dom(\mathfrak{B}))-1}) = \Gamma$,

d) $\mathfrak{H}'_{\max(Dom(\mathfrak{B}))} = \ulcorner$Therefore $\Gamma\urcorner$,

e) $\beta \notin STSF(\{\Delta, \Gamma\})$,

f) There is no $j \leq \min(Dom(\mathfrak{B}))$ such that $\beta \in ST(\mathfrak{H}'_j)$,

g) $\mathfrak{A} = \mathfrak{B}\setminus\{(\min(Dom(\mathfrak{B})), \mathfrak{H}'_{\min(Dom(\mathfrak{B}))})\}$ and

h) There is no r such that $\min(Dom(\mathfrak{A})) < r \leq Dom(\mathfrak{H})\text{-}1$ and $(r, \mathfrak{H}'_r) \in AVAS(\mathfrak{H})$.

With g), we have $\min(Dom(\mathfrak{A})) = \min(Dom(\mathfrak{B}))+1$ and $Dom(\mathfrak{H}) = \max(Dom(\mathfrak{A})) = \max(Dom(\mathfrak{B}))$. It then follows that $\min(Dom(\mathfrak{B})) < \min(Dom(\mathfrak{A})) \leq Dom(\mathfrak{H})\text{-}1$ and therefore we have $\min(Dom(\mathfrak{B})), \min(Dom(\mathfrak{B}))+1 \in Dom(\mathfrak{H})$ and $\max(Dom(\mathfrak{B}))-1 = Dom(\mathfrak{H})\text{-}1$. It then follows that

a') $P(\mathfrak{H}_{\min(Dom(\mathfrak{B}))}) = \ulcorner\bigvee\xi\Delta\urcorner$ and $(\min(Dom(\mathfrak{B})), \mathfrak{H}_{\min(Dom(\mathfrak{B}))}) \in AVS(\mathfrak{H})$,

b') $P(\mathfrak{H}_{\min(Dom(\mathfrak{B}))+1}) = [\beta, \xi, \Delta]$ and $(\min(Dom(\mathfrak{B}))+1, \mathfrak{H}_{\min(Dom(\mathfrak{B}))+1}) \in AVAS(\mathfrak{H})$,

c') $P(\mathfrak{H}_{Dom(\mathfrak{H})-1}) = \Gamma$,

d') $\mathfrak{H}'_{Dom(\mathfrak{H})} = \ulcorner$Therefore $\Gamma\urcorner$,

e') $\beta \notin STSF(\{\Delta, \Gamma\})$,

f') There is no $j \leq \min(Dom(\mathfrak{B}))$ such that $\beta \in ST(\mathfrak{H}_j)$,

h') There is no r such that $\min(Dom(\mathfrak{B}))+1 < r \leq Dom(\mathfrak{H})\text{-}1$ and $(r, \mathfrak{H}_r) \in AVAS(\mathfrak{H})$.

According to Definition 3-15, we then have $\mathfrak{H}' \in PEF(\mathfrak{H})$. Hence we have in all three cases that $\mathfrak{H}' \in CdIF(\mathfrak{H}) \cup NIF(\mathfrak{H}) \cup PEF(\mathfrak{H})$. ■

Theorem 3-24. *AVS-reduction by and only by CdI, NI and PE*
If $\mathfrak{H} \in$ SEQ and $\mathfrak{H}' \in$ RCE($\mathfrak{H}$), then:
AVS($\mathfrak{H}$) $\not\subseteq$ AVS($\mathfrak{H}'$)
iff
$\{(j, \mathfrak{H}'_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}))) \leq j \leq \mathrm{Dom}(\mathfrak{H})\}$ is a CdI- or NI- or PE-closed segment in $\mathfrak{H}'$ and $\mathfrak{H}' \in$ CdIF($\mathfrak{H}$) $\cup$ NIF($\mathfrak{H}$) $\cup$ PEF($\mathfrak{H}$).

Proof: Suppose $\mathfrak{H} \in$ SEQ and $\mathfrak{H}' \in$ RCE($\mathfrak{H}$). The right-left-direction follows with clause (iv) of Theorem 3-19, Theorem 3-20 and Theorem 3-21, and with Theorem 2-72. Now, for the left-right-direction, suppose AVS($\mathfrak{H}$) $\not\subseteq$ AVS($\mathfrak{H}'$). Then we have AVS($\mathfrak{H}$)\AVS($\mathfrak{H}'$) $\neq \emptyset$. With $\mathfrak{H}' \in$ RCE($\mathfrak{H}$) and Theorem 3-1, we have $\mathfrak{H}' \in$ SEQ and, with Theorem 3-5, $\mathfrak{H}' \upharpoonright \mathrm{Dom}(\mathfrak{H}')\text{-}1 = \mathfrak{H}$. With Theorem 2-83-(vi) and -(vii), it then follows that AVAS($\mathfrak{H}'$) $\subset$ AVAS($\mathfrak{H}$). With Theorem 3-23, it then holds that $\mathfrak{H}' \in$ CdIF($\mathfrak{H}$) $\cup$ NIF($\mathfrak{H}$) $\cup$ PEF($\mathfrak{H}$). With Theorem 3-19-(i), Theorem 3-20-(i) and Theorem 3-21-(i), it then follows that $\{(j, \mathfrak{H}'_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}))) \leq j \leq \mathrm{Dom}(\mathfrak{H})\}$ is a CdI- or NI- or PE-closed segment in $\mathfrak{H}'$. ■

Theorem 3-25. *AVS if CdI, NI and PE are excluded*
If $\mathfrak{H} \in$ SEQ and $\mathfrak{H}' \in$ RCE($\mathfrak{H}$)\(CdIF($\mathfrak{H}$) $\cup$ NIF($\mathfrak{H}$) $\cup$ PEF($\mathfrak{H}$)), then:
AVS($\mathfrak{H}'$) = AVS($\mathfrak{H}$) $\cup$ $\{(\mathrm{Dom}(\mathfrak{H}), \mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H})})\}$.

Proof: Let $\mathfrak{H} \in$ SEQ and $\mathfrak{H}' \in$ RCE($\mathfrak{H}$)\(CdIF($\mathfrak{H}$) $\cup$ NIF($\mathfrak{H}$) $\cup$ PEF($\mathfrak{H}$)). Because of Theorem 3-14-(i), we have AVS($\mathfrak{H}'$) $\subseteq$ AVS($\mathfrak{H}$) $\cup$ $\{(\mathrm{Dom}(\mathfrak{H}), \mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H})})\}$. With Theorem 2-82, we have that C($\mathfrak{H}'$) = P($\mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H}')\text{-}1}$) is available in $\mathfrak{H}'$ at Dom($\mathfrak{H}'$)-1. With Theorem 3-4, we have Dom($\mathfrak{H}'$)-1 = Dom($\mathfrak{H}$). Therefore $(\mathrm{Dom}(\mathfrak{H}), \mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H})}) \in$ AVS($\mathfrak{H}'$). If AVS($\mathfrak{H}$) $\not\subseteq$ AVS($\mathfrak{H}'$), then we would have, with Theorem 3-24, that $\mathfrak{H}' \in$ CdIF($\mathfrak{H}$) $\cup$ NIF($\mathfrak{H}$) $\cup$ PEF($\mathfrak{H}$), which contradicts the hypothesis. Therefore we have AVS($\mathfrak{H}$) $\subseteq$ AVS($\mathfrak{H}'$). Hence we also have AVS($\mathfrak{H}$) $\cup$ $\{(\mathrm{Dom}(\mathfrak{H}), \mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H})})\}$ $\subseteq$ AVS($\mathfrak{H}'$). ■

Theorem 3-26. *AVS, AVAS, AVP, AVAP and CI, BI, DI, UI, PI, II*
If $\mathfrak{H} \in$ SEQ and $\mathfrak{H}' \in$ CIF($\mathfrak{H}$) $\cup$ BIF($\mathfrak{H}$) $\cup$ DIF($\mathfrak{H}$) $\cup$ UIF($\mathfrak{H}$) $\cup$ PIF($\mathfrak{H}$) $\cup$ IIF($\mathfrak{H}$), then:

(i) AVS($\mathfrak{H}'$) $\subseteq$ AVS($\mathfrak{H}$) $\cup$ $\{(\mathrm{Dom}(\mathfrak{H}), \mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H})})\}$,
(ii) AVAS($\mathfrak{H}'$) $\subseteq$ AVAS($\mathfrak{H}$),
(iii) If AVAS($\mathfrak{H}'$) $\subset$ AVAS($\mathfrak{H}$), then $\mathfrak{H}' \in$ PEF($\mathfrak{H}$),
(iv) AVP($\mathfrak{H}'$) $\subseteq$ AVP($\mathfrak{H}$) $\cup$ $\{$C($\mathfrak{H}'$)$\}$,

(v) $\text{AVAP}(\mathfrak{H}') \subseteq \text{AVAP}(\mathfrak{H})$, and
(vi) If $\text{AVAP}(\mathfrak{H}') \subset \text{AVAP}(\mathfrak{H})$, then $\mathfrak{H}' \in \text{PEF}(\mathfrak{H})$.

Proof: Suppose $\mathfrak{H} \in \text{SEQ}$ and $\mathfrak{H}' \in \text{CIF}(\mathfrak{H}) \cup \text{BIF}(\mathfrak{H}) \cup \text{DIF}(\mathfrak{H}) \cup \text{UIF}(\mathfrak{H}) \cup \text{PIF}(\mathfrak{H}) \cup \text{IIF}(\mathfrak{H})$. With Definition 3-18, we then have $\mathfrak{H}' \in \text{RCE}(\mathfrak{H})$. With Definition 3-4, Definition 3-6, Definition 3-8, Definition 3-12, Definition 3-14 and Definition 3-16, we have that there are $A, B \in \text{CFORM}$ and $\theta \in \text{CTERM}$ and $\beta \in \text{PAR}$ and $\xi \in \text{VAR}$ and $\Delta \in \text{FORM}$, where $\text{FV}(\Delta) \subseteq \{\xi\}$ such that $\mathfrak{H}' = \mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } A \wedge B\urcorner)\}$ or $\mathfrak{H}' = \mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } A \leftrightarrow B\urcorner)\}$ or $\mathfrak{H}' = \mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } A \vee B\urcorner)\}$ or $\mathfrak{H}' = \mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \bigwedge\xi\Delta\urcorner)\}$ or $\mathfrak{H}' = \mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \bigvee\xi\Delta\urcorner)\}$ or $\mathfrak{H}' = \mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \theta = \theta\urcorner)\}$. With the theorems on unique readability (Theorem 1-10, Theorem 1-11 and Theorem 1-12), we then have $(\text{Dom}(\mathfrak{H}), \mathfrak{H}'_{\text{Dom}(\mathfrak{H})}) \notin \text{AS}(\mathfrak{H}')$ and thus, with Definition 3-1, that $\mathfrak{H}' \notin \text{AF}(\mathfrak{H})$. Then (i), (ii), (iv) and (v) follow with Theorem 3-17-(i), -(ii), -(iii) and -(iv). With Theorem 3-19-(x), Theorem 3-20-(x) and unique readability, it follows that $\mathfrak{H}' \notin \text{CdIF}(\mathfrak{H}) \cup \text{NIF}(\mathfrak{H})$. With Theorem 3-23, it then follows that if $\text{AVAS}(\mathfrak{H}') \subset \text{AVAS}(\mathfrak{H})$, then $\mathfrak{H}' \in \text{PEF}(\mathfrak{H})$ and hence we have (iii). Now, suppose for (vi) that $\text{AVAP}(\mathfrak{H}') \subset \text{AVAP}(\mathfrak{H})$. Then we have $\text{AVAP}(\mathfrak{H}) \not\subseteq \text{AVAP}(\mathfrak{H}')$ and thus, with Theorem 2-75, $\text{AVAS}(\mathfrak{H}) \not\subseteq \text{AVAS}(\mathfrak{H}')$. With (ii), we then have $\text{AVAS}(\mathfrak{H}') \subset \text{AVAS}(\mathfrak{H})$ and thus, with (iii), that $\mathfrak{H}' \in \text{PEF}(\mathfrak{H})$. ■

Theorem 3-27. *AVS, AVAS, AVP, AVAP and CdE, CE, BE, DE, NE, UE, IE*
If $\mathfrak{H} \in \text{SEQ}$ and $\mathfrak{H}' \in \text{CdEF}(\mathfrak{H}) \cup \text{CEF}(\mathfrak{H}) \cup \text{BEF}(\mathfrak{H}) \cup \text{DEF}(\mathfrak{H}) \cup \text{NEF}(\mathfrak{H}) \cup \text{UEF}(\mathfrak{H}) \cup \text{IEF}(\mathfrak{H})$, then:

(i) $\text{AVS}(\mathfrak{H}') \subseteq \text{AVS}(\mathfrak{H}) \cup \{(\text{Dom}(\mathfrak{H}), \mathfrak{H}'_{\text{Dom}(\mathfrak{H})})\}$,
(ii) $\text{AVAS}(\mathfrak{H}') \subseteq \text{AVAS}(\mathfrak{H})$,
(iii) If $\text{AVAS}(\mathfrak{H}') \subset \text{AVAS}(\mathfrak{H})$, then $\mathfrak{H}' \in \text{CdIF}(\mathfrak{H}) \cup \text{NIF}(\mathfrak{H}) \cup \text{PEF}(\mathfrak{H})$,
(iv) $\text{AVP}(\mathfrak{H}') \subseteq \text{AVP}(\mathfrak{H}) \cup \{\text{C}(\mathfrak{H}')\}$,
(v) $\text{AVAP}(\mathfrak{H}') \subseteq \text{AVAP}(\mathfrak{H})$, and
(vi) If $\text{AVAP}(\mathfrak{H}') \subset \text{AVAP}(\mathfrak{H})$, then $\mathfrak{H}' \in \text{CdIF}(\mathfrak{H}) \cup \text{NIF}(\mathfrak{H}) \cup \text{PEF}(\mathfrak{H})$.

Proof: Suppose $\mathfrak{H} \in \text{SEQ}$ and $\mathfrak{H}' \in \text{CdEF}(\mathfrak{H}) \cup \text{CEF}(\mathfrak{H}) \cup \text{BEF}(\mathfrak{H}) \cup \text{DEF}(\mathfrak{H}) \cup \text{NEF}(\mathfrak{H}) \cup \text{UEF}(\mathfrak{H}) \cup \text{IEF}(\mathfrak{H})$. With Definition 3-18, we then have $\mathfrak{H}' \in \text{RCE}(\mathfrak{H})$. With Definition 3-3, Definition 3-5, Definition 3-7, Definition 3-9, Definition 3-11, Definition 3-13 and Definition 3-17, we have $\mathfrak{H}' = \mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore P}(\mathfrak{H}'_{\text{Dom}(\mathfrak{H})})\urcorner)\}$. Then we have $(\text{Dom}(\mathfrak{H}), \mathfrak{H}'_{\text{Dom}(\mathfrak{H})}) \notin \text{AS}(\mathfrak{H}')$ and thus $(\text{Dom}(\mathfrak{H}), \mathfrak{H}'_{\text{Dom}(\mathfrak{H})}) \notin \text{AVAS}(\mathfrak{H}')$ and $\mathfrak{H}' \notin \text{AF}(\mathfrak{H})$.

Then, with Theorem 3-14-(i), -(ii) and -(iii), we have (i), (ii), (iv) and (v). Clause (iii) follows with Theorem 3-23. Now, suppose for (vi) that $\mathrm{AVAP}(\mathfrak{H}') \subset \mathrm{AVAP}(\mathfrak{H})$. Then we have $\mathrm{AVAP}(\mathfrak{H}) \not\subseteq \mathrm{AVAP}(\mathfrak{H}')$ and thus, with Theorem 2-75, $\mathrm{AVAS}(\mathfrak{H}) \not\subseteq \mathrm{AVAS}(\mathfrak{H}')$. With (ii), we then have $\mathrm{AVAS}(\mathfrak{H}') \subset \mathrm{AVAS}(\mathfrak{H})$ and thus, with (iii), that $\mathfrak{H}' \in \mathrm{CdIF}(\mathfrak{H}) \cup \mathrm{NIF}(\mathfrak{H}) \cup \mathrm{PEF}(\mathfrak{H})$. ■

Theorem 3-28. *Without AR, CdI, NI or PE there is no AVAP-change*
If $\mathfrak{H} \in \mathrm{RCS}$ and $\mathfrak{H} \notin \mathrm{AF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) \cup \mathrm{CdIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) \cup \mathrm{NIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) \cup \mathrm{PEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$, then $\mathrm{AVAP}(\mathfrak{H}) = \mathrm{AVAP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$.

Proof: Suppose $\mathfrak{H} \in \mathrm{RCS}$ and $\mathfrak{H} \notin \mathrm{AF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) \cup \mathrm{CdIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) \cup \mathrm{NIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) \cup \mathrm{PEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. We have $\mathfrak{H} = \emptyset$ or $\mathfrak{H} \neq \emptyset$. In the first case, we have $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1 \subseteq \mathfrak{H} = \emptyset$ and the theorem holds. Now, suppose $\mathfrak{H} \neq \emptyset$. According to Theorem 3-6 and Definition 3-18, it then follows that *first* $\mathfrak{H} \in \mathrm{CIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \mathrm{BIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \mathrm{DIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \mathrm{UIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \mathrm{PIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \mathrm{IIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or *second* $\mathfrak{H} \in \mathrm{CdEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \mathrm{CEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \mathrm{BEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \mathrm{DEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \mathrm{NEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \mathrm{UEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \mathrm{IEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. In the *first* six cases, $\mathrm{AVAP}(\mathfrak{H}) = \mathrm{AVAP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ follows from Theorem 3-26-(v) and -(vi). In the *remaining* cases $\mathrm{AVAP}(\mathfrak{H}) = \mathrm{AVAP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ follows from Theorem 3-27-(v) and -(vi). ■

Theorem 3-29. *AVS, AVAS, AVP and AVAP of restrictions whose conclusion stays available remain intact in the unrestricted sentence sequence.*
If $\mathfrak{H} \in \mathrm{RCS}$ and Γ is available in $\mathfrak{H}$ at i, then:

(i) $\mathrm{AVS}(\mathfrak{H}\upharpoonright i+1) \subseteq \mathrm{AVS}(\mathfrak{H})$,
(ii) $\mathrm{AVAS}(\mathfrak{H}\upharpoonright i+1) \subseteq \mathrm{AVAS}(\mathfrak{H})$,
(iii) $\mathrm{AVP}(\mathfrak{H}\upharpoonright i+1) \subseteq \mathrm{AVP}(\mathfrak{H})$, and
(iv) $\mathrm{AVAP}(\mathfrak{H}\upharpoonright i+1) \subseteq \mathrm{AVAP}(\mathfrak{H})$.

Proof: Suppose $\mathfrak{H} \in \mathrm{RCS}$ and Γ is available in $\mathfrak{H}$ at i. According to Definition 2-26, we then have $i \in \mathrm{Dom}(\mathfrak{H})$ and $\Gamma = \mathrm{P}(\mathfrak{H}_i)$ and there is no closed segment $\mathfrak{A}$ in $\mathfrak{H}$ such that $\min(\mathrm{Dom}(\mathfrak{A})) \leq i < \max(\mathrm{Dom}(\mathfrak{A}))$.

Ad (i): To show $\mathrm{AVS}(\mathfrak{H}\upharpoonright i+1) \subseteq \mathrm{AVS}(\mathfrak{H})$, suppose $(j, \Sigma) \in \mathrm{AVS}(\mathfrak{H}\upharpoonright i+1)$. With Definition 2-28, we then have $j \in \mathrm{Dom}(\mathfrak{H}\upharpoonright i+1)$ and $(\mathfrak{H}\upharpoonright i+1)_j = \Sigma$ and $\mathrm{P}(\Sigma)$ is available in

$\mathfrak{H}\upharpoonright i+1$ at j. According to Definition 2-26, there is thus no closed segment $\mathfrak{A}$ in $\mathfrak{H}\upharpoonright i+1$ such that $\min(\mathrm{Dom}(\mathfrak{A})) \leq j < \max(\mathrm{Dom}(\mathfrak{A}))$. Now, suppose for contradiction, that $(j, \Sigma) \notin \mathrm{AVS}(\mathfrak{H})$. Then we would have $j \notin \mathrm{Dom}(\mathfrak{H})$ or $\mathfrak{H}_j \neq \Sigma$ or $\mathrm{P}(\Sigma)$ is not available in $\mathfrak{H}$ at j. Since $\mathfrak{H}\upharpoonright i+1$ is a restriction of $\mathfrak{H}$ and $j \in \mathrm{Dom}(\mathfrak{H}\upharpoonright i+1)$, the first two cases are excluded. Thus, we would have $j \in \mathrm{Dom}(\mathfrak{H})$ and $\mathfrak{H}_j = \Sigma$ and $\mathrm{P}(\Sigma)$ is not available in $\mathfrak{H}$ at j. According to Definition 2-26, there is thus a closed segment $\mathfrak{A}$ in $\mathfrak{H}$ such that $\min(\mathrm{Dom}(\mathfrak{A})) \leq j < \max(\mathrm{Dom}(\mathfrak{A}))$. According to Theorem 2-64-(viii), $\mathfrak{A}$ is also a closed segment in $\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A}))+1$. If $i < \max(\mathrm{Dom}(\mathfrak{A}))$, then we would have, because of $j \in \mathrm{Dom}(\mathfrak{H}\upharpoonright i+1)$ and thus $j \leq i$, that also $\min(\mathrm{Dom}(\mathfrak{A})) \leq i < \max(\mathrm{Dom}(\mathfrak{A}))$. Thus we would have that $\mathrm{P}(\mathfrak{H}_i) = \Gamma$ is not available in $\mathfrak{H}$ at i, which contradicts the hypothesis. Therefore we have $\max(\mathrm{Dom}(\mathfrak{A})) \leq i$ and thus $\max(\mathrm{Dom}(\mathfrak{A}))+1 \leq i+1$. Therefore we have $\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathfrak{A}))+1 \subseteq \mathfrak{H}\upharpoonright i+1$. With Theorem 2-62-(viii), $\mathfrak{A}$ is then also a closed segment in $\mathfrak{H}\upharpoonright i+1$. Therefore there is a closed segment $\mathfrak{A}$ in $\mathfrak{H}\upharpoonright i+1$ such that $\min(\mathrm{Dom}(\mathfrak{A})) \leq j < \max(\mathrm{Dom}(\mathfrak{A}))$. Contradiction! Therefore $(j, \Sigma) \in \mathrm{AVS}(\mathfrak{H})$.

Ad (ii), (iii) and (iv): With Theorem 2-72, (ii) follows from (i). With Theorem 2-74, (iii) follows from (i). With Theorem 2-75, (iv) follows from (ii). ■

Theorem 3-30. *AVS, AVAS, AVP and AVAP in derivations*

If $\mathfrak{H} \in \mathrm{SEQ}$, then:

$\mathfrak{H} \in \mathrm{RCS}$

iff

for all $i \in \mathrm{Dom}(\mathfrak{H})$:

(i) $\mathfrak{H}\upharpoonright i+1 \in \mathrm{AF}(\mathfrak{H}\upharpoonright i)$ and

- a) $\mathrm{AVS}(\mathfrak{H}\upharpoonright i+1)\setminus\mathrm{AVS}(\mathfrak{H}\upharpoonright i) = \{(i, \mathfrak{H}_i)\}$,
- b) $\mathrm{AVS}(\mathfrak{H}\upharpoonright i+1) = \mathrm{AVS}(\mathfrak{H}\upharpoonright i) \cup \{(i, \mathfrak{H}_i)\}$,
- c) $\mathrm{AVAS}(\mathfrak{H} i+1)\setminus\mathrm{AVAS}(\mathfrak{H}\upharpoonright i) = \{(i, \mathfrak{H}_i)\}$,
- d) $\mathrm{AVAS}(\mathfrak{H}\upharpoonright i+1) = \mathrm{AVAS}(\mathfrak{H}\upharpoonright i) \cup \{(i, \mathfrak{H}_i)\}$,
- e) $\mathrm{AVP}(\mathfrak{H}\upharpoonright i+1)\setminus\mathrm{AVP}(\mathfrak{H}\upharpoonright i) \subseteq \{\mathrm{P}(\mathfrak{H}_i)\}$,
- f) $\mathrm{AVP}(\mathfrak{H}\upharpoonright i+1) = \mathrm{AVP}(\mathfrak{H}\upharpoonright i) \cup \{\mathrm{P}(\mathfrak{H}_i)\}$,
- g) $\mathrm{AVAP}(\mathfrak{H}\upharpoonright i+1)\setminus\mathrm{AVAP}(\mathfrak{H}\upharpoonright i) \subseteq \{\mathrm{P}(\mathfrak{H}_i)\}$, and
- h) $\mathrm{AVAP}(\mathfrak{H}\upharpoonright i+1) = \mathrm{AVAP}(\mathfrak{H}\upharpoonright i) \cup \{\mathrm{P}(\mathfrak{H}_i)\}$

or

(ii) $\mathfrak{H}\upharpoonright i+1 \in \mathrm{CdIF}(\mathfrak{H}\upharpoonright i)$ and

- a) $\{(j, \mathfrak{H}_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i))) \leq j \leq i\}$ is a CdI-closed segment in $\mathfrak{H}\upharpoonright i+1$,
- b) $\mathrm{AVS}(\mathfrak{H}\upharpoonright i)\setminus\mathrm{AVS}(\mathfrak{H}\upharpoonright i+1) \subseteq \{(j, \mathfrak{H}_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i))) \leq j < i\}$,

c) $\mathrm{AVS}(\mathfrak{H}\upharpoonright i{+}1) =$
$(\mathrm{AVS}(\mathfrak{H}\upharpoonright i)\backslash\{(j, \mathfrak{H}_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i))) \leq j < i\}) \cup \{(i, \mathfrak{H}_i)\}$,

d) $\mathrm{AVAS}(\mathfrak{H}\upharpoonright i)\backslash\mathrm{AVAS}(\mathfrak{H}\upharpoonright i{+}1) = \{(\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i))), \mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i)))})\}$,

e) $\mathrm{AVAS}(\mathfrak{H}\upharpoonright i) =$
$\mathrm{AVAS}(\mathfrak{H}\upharpoonright i{+}1) \cup \{(\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i))), \mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i)))})\}$,

f) $\mathrm{AVP}(\mathfrak{H}\upharpoonright i)\backslash\mathrm{AVP}(\mathfrak{H}\upharpoonright i{+}1) \subseteq \{\mathrm{P}(\mathfrak{H}_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i))) \leq j < i\}$,

g) $\mathrm{AVP}(\mathfrak{H}\upharpoonright i) \subseteq \{\mathrm{P}(\mathfrak{H}_j) \mid j \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}\upharpoonright i{+}1)\upharpoonright i)\} \cup$
$\{\mathrm{P}(\mathfrak{H}_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i))) \leq j < i\}$,

h) $\mathrm{AVAP}(\mathfrak{H}\upharpoonright i)\backslash\mathrm{AVAP}(\mathfrak{H}\upharpoonright i{+}1) \subseteq \{\mathrm{P}(\mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i)))})\}$,

i) $\mathrm{AVAP}(\mathfrak{H}\upharpoonright i) = \mathrm{AVAP}(\mathfrak{H}\upharpoonright i{+}1) \cup \{\mathrm{P}(\mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i)))})\}$, and

j) $\mathrm{P}(\mathfrak{H}_i) = \ulcorner\mathrm{P}(\mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i)))}) \rightarrow \mathrm{P}(\mathfrak{H}_{i-1})\urcorner$

or

(iii) $\mathfrak{H}\upharpoonright i{+}1 \in \mathrm{NIF}(\mathfrak{H}\upharpoonright i)$ and

a) $\{(j, \mathfrak{H}_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i))) \leq j \leq i\}$ is an NI-closed segment in $\mathfrak{H}\upharpoonright i{+}1$,

b) $\mathrm{AVS}(\mathfrak{H}\upharpoonright i)\backslash\mathrm{AVS}(\mathfrak{H}\upharpoonright i{+}1) \subseteq \{(j, \mathfrak{H}_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i))) \leq j < i\}$,

c) $\mathrm{AVS}(\mathfrak{H}\upharpoonright i{+}1) =$
$(\mathrm{AVS}(\mathfrak{H}\upharpoonright i)\backslash\{(j, \mathfrak{H}_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i))) \leq j < i\}) \cup \{(i, \mathfrak{H}_i)\}$,

d) $\mathrm{AVAS}(\mathfrak{H}\upharpoonright i)\backslash\mathrm{AVAS}(\mathfrak{H}\upharpoonright i{+}1) = \{(\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i))), \mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i)))})\}$,

e) $\mathrm{AVAS}(\mathfrak{H}\upharpoonright i) =$
$\mathrm{AVAS}(\mathfrak{H}\upharpoonright i{+}1) \cup \{(\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i))), \mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i)))})\}$,

f) $\mathrm{AVP}(\mathfrak{H}\upharpoonright i)\backslash\mathrm{AVP}(\mathfrak{H}\upharpoonright i{+}1) \subseteq \{\mathrm{P}(\mathfrak{H}_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i))) \leq j < i\}$,

g) $\mathrm{AVP}(\mathfrak{H}\upharpoonright i) \subseteq \{\mathrm{P}(\mathfrak{H}_j) \mid j \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}\upharpoonright i{+}1)\upharpoonright i)\} \cup$
$\{\mathrm{P}(\mathfrak{H}_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i))) \leq j < i\}$,

h) $\mathrm{AVAP}(\mathfrak{H}\upharpoonright i)\backslash\mathrm{AVAP}(\mathfrak{H}\upharpoonright i{+}1) \subseteq \{\mathrm{P}(\mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i)))})\}$,

i) $\mathrm{AVAP}(\mathfrak{H}\upharpoonright i) = \mathrm{AVAP}(\mathfrak{H}\upharpoonright i{+}1) \cup \{\mathrm{P}(\mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i)))})\}$, and

j) $\mathrm{P}(\mathfrak{H}_i) = \ulcorner\neg\mathrm{P}(\mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i)))})\urcorner$

or

(iv) $\mathfrak{H}\upharpoonright i{+}1 \in \mathrm{PEF}(\mathfrak{H}\upharpoonright i)$ and

a) $\{(j, \mathfrak{H}_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i))) \leq j \leq i\}$ is a PE-closed segment in $\mathfrak{H}\upharpoonright i{+}1$,

b) $\mathrm{AVS}(\mathfrak{H}\upharpoonright i)\backslash\mathrm{AVS}(\mathfrak{H}\upharpoonright i{+}1) \subseteq \{(j, \mathfrak{H}_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i))) \leq j < i\}$,

c) $\mathrm{AVS}(\mathfrak{H}\upharpoonright i{+}1) =$
$(\mathrm{AVS}(\mathfrak{H}\upharpoonright i)\backslash\{(j, \mathfrak{H}_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i))) \leq j < i\}) \cup \{(i, \mathfrak{H}_i)\}$,

d) $\mathrm{AVAS}(\mathfrak{H}\upharpoonright i)\backslash\mathrm{AVAS}(\mathfrak{H}\upharpoonright i{+}1) = \{(\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i))), \mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i)))})\}$,

e) $\mathrm{AVAS}(\mathfrak{H}\upharpoonright i) =$
$\mathrm{AVAS}(\mathfrak{H}\upharpoonright i{+}1) \cup \{(\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i))), \mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i)))})\}$,

f) $\mathrm{AVP}(\mathfrak{H}\upharpoonright i)\backslash\mathrm{AVP}(\mathfrak{H}\upharpoonright i{+}1) \subseteq \{\mathrm{P}(\mathfrak{H}_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i))) \leq j < i\}$,

g) $\mathrm{AVP}(\mathfrak{H}\upharpoonright i) \subseteq \{\mathrm{P}(\mathfrak{H}_j) \mid j \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}\upharpoonright i{+}1)\upharpoonright i)\} \cup$
$\{\mathrm{P}(\mathfrak{H}_j) \mid \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright i))) \leq j < i\}$,

h) $\mathrm{AVAP}(\mathfrak{H}\restriction i)\backslash\mathrm{AVAP}(\mathfrak{H}\restriction i+1) \subseteq \{\mathrm{P}(\mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\restriction i)))})\}$,

i) $\mathrm{AVAP}(\mathfrak{H}\restriction i) = \mathrm{AVAP}(\mathfrak{H}\restriction i+1) \cup \{\mathrm{P}(\mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\restriction i)))})\}$, and

j) $\mathrm{P}(\mathfrak{H}_i) = \ulcorner\mathrm{P}(\mathfrak{H}_{i-1})\urcorner$

or

(v) $\mathfrak{H}\restriction i+1 \in \mathrm{CIF}(\mathfrak{H}\restriction i) \cup \mathrm{BIF}(\mathfrak{H}\restriction i) \cup \mathrm{DIF}(\mathfrak{H}\restriction i) \cup \mathrm{UIF}(\mathfrak{H}\restriction i) \cup \mathrm{PIF}(\mathfrak{H}\restriction i) \cup \mathrm{IIF}(\mathfrak{H}\restriction i)$ and

a) $\mathrm{AVS}(\mathfrak{H}\restriction i+1) \subseteq \mathrm{AVS}(\mathfrak{H}\restriction i) \cup \{(i, \mathfrak{H}_i)\}$,

b) $\mathrm{AVAS}(\mathfrak{H}\restriction i+1) \subseteq \mathrm{AVAS}(\mathfrak{H}\restriction i)$,

c) If $\mathrm{AVAS}(\mathfrak{H}\restriction i+1) \subset \mathrm{AVAS}(\mathfrak{H}\restriction i)$, then $\mathfrak{H}\restriction i+1 \in \mathrm{PEF}(\mathfrak{H}\restriction i)$,

d) $\mathrm{AVP}(\mathfrak{H}\restriction i+1) \subseteq \mathrm{AVP}(\mathfrak{H}\restriction i) \cup \{\mathrm{P}(\mathfrak{H}_i)\}$,

e) $\mathrm{AVAP}(\mathfrak{H}\restriction i+1) \subseteq \mathrm{AVAP}(\mathfrak{H}\restriction i)$, and

f) If $\mathrm{AVAP}(\mathfrak{H}\restriction i+1) \subset \mathrm{AVAP}(\mathfrak{H}\restriction i)$, then $\mathfrak{H}\restriction i+1 \in \mathrm{PEF}(\mathfrak{H}\restriction i)$

or

(vi) $\mathfrak{H}\restriction i+1 \in \mathrm{CdEF}(\mathfrak{H}\restriction i) \cup \mathrm{CEF}(\mathfrak{H}\restriction i) \cup \mathrm{BEF}(\mathfrak{H}\restriction i) \cup \mathrm{DEF}(\mathfrak{H}\restriction i) \cup \mathrm{NEF}(\mathfrak{H}\restriction i) \cup \mathrm{UEF}(\mathfrak{H}\restriction i) \cup \mathrm{IEF}(\mathfrak{H}\restriction i)$ and

a) $\mathrm{AVS}(\mathfrak{H}\restriction i+1) \subseteq \mathrm{AVS}(\mathfrak{H}\restriction i) \cup \{(i, \mathfrak{H}_i)\}$,

b) $\mathrm{AVAS}(\mathfrak{H}\restriction i+1) \subseteq \mathrm{AVAS}(\mathfrak{H}\restriction i)$,

c) If $\mathrm{AVAS}(\mathfrak{H}\restriction i+1) \subset \mathrm{AVAS}(\mathfrak{H}\restriction i)$, then $\mathfrak{H}\restriction i+1 \in \mathrm{CdIF}(\mathfrak{H}\restriction i) \cup \mathrm{NIF}(\mathfrak{H}\restriction i) \cup \mathrm{PEF}(\mathfrak{H}\restriction i)$,

d) $\mathrm{AVP}(\mathfrak{H}\restriction i+1) \subseteq \mathrm{AVP}(\mathfrak{H}\restriction i) \cup \{\mathrm{P}(\mathfrak{H}_i)\}$,

e) $\mathrm{AVAP}(\mathfrak{H}\restriction i+1) \subseteq \mathrm{AVAP}(\mathfrak{H}\restriction i)$, and

f) If $\mathrm{AVAP}(\mathfrak{H}\restriction i+1) \subset \mathrm{AVAP}(\mathfrak{H}\restriction i)$, then $\mathfrak{H}\restriction(i+1) \in \mathrm{CdIF}(\mathfrak{H}\restriction i) \cup \mathrm{NIF}(\mathfrak{H}\restriction i) \cup \mathrm{PEF}(\mathfrak{H}\restriction i)$.

Proof: Suppose $\mathfrak{H} \in \mathrm{SEQ}$. (*L-R*): Suppose $\mathfrak{H} \in \mathrm{RCS}$. Then it holds, with Definition 3-19, for all $i \in \mathrm{Dom}(\mathfrak{H})$: $\mathfrak{H}\restriction i+1 \in \mathrm{RCE}(\mathfrak{H}\restriction i)$. With Definition 3-18, it then holds for all $i \in \mathrm{Dom}(\mathfrak{H})$ that $\mathfrak{H}\restriction i+1 \in \mathrm{AF}(\mathfrak{H}\restriction i) \cup \mathrm{CdIF}(\mathfrak{H}\restriction i) \cup \mathrm{NIF}(\mathfrak{H}\restriction i) \cup \mathrm{PEF}(\mathfrak{H}\restriction i) \cup \mathrm{CIF}(\mathfrak{H}\restriction i) \cup \mathrm{BIF}(\mathfrak{H}\restriction i) \cup \mathrm{DIF}(\mathfrak{H}\restriction i) \cup \mathrm{UIF}(\mathfrak{H}\restriction i) \cup \mathrm{PIF}(\mathfrak{H}\restriction i) \cup \mathrm{IIF}(\mathfrak{H}\restriction i) \cup \mathrm{CdEF}(\mathfrak{H}\restriction i) \cup \mathrm{CEF}(\mathfrak{H}\restriction i) \cup \mathrm{BEF}(\mathfrak{H}\restriction i) \cup \mathrm{DEF}(\mathfrak{H}\restriction i) \cup \mathrm{NEF}(\mathfrak{H}\restriction i) \cup \mathrm{UEF}(\mathfrak{H}\restriction i) \cup \mathrm{IEF}(\mathfrak{H}\restriction i)$. It then follows for $\mathfrak{H}\restriction i+1 \in \mathrm{AF}(\mathfrak{H}\restriction i)$, with Theorem 3-15, that (i) holds, for $\mathfrak{H}\restriction i+1 \in \mathrm{CdIF}(\mathfrak{H}\restriction i)$, with Theorem 3-19, that (ii) holds, for $\mathfrak{H}\restriction i+1 \in \mathrm{NIF}(\mathfrak{H}\restriction i)$, with Theorem 3-20. that (iii) holds, for $\mathfrak{H}\restriction i+1 \in \mathrm{PEF}(\mathfrak{H}\restriction i)$, with Theorem 3-21, that (iv) holds, for $\mathfrak{H}\restriction i+1 \in \mathrm{CIF}(\mathfrak{H}\restriction i) \cup \mathrm{BIF}(\mathfrak{H}\restriction i) \cup \mathrm{DIF}(\mathfrak{H}\restriction i) \cup \mathrm{UIF}(\mathfrak{H}\restriction i) \cup \mathrm{PIF}(\mathfrak{H}\restriction i) \cup \mathrm{IIF}(\mathfrak{H}\restriction i)$, with Theorem 3-26, that (v) holds, and, last, for $\mathfrak{H}\restriction i+1 \in \mathrm{CdEF}(\mathfrak{H}\restriction i) \cup \mathrm{CEF}(\mathfrak{H}\restriction i) \cup \mathrm{BEF}(\mathfrak{H}\restriction i) \cup \mathrm{DEF}(\mathfrak{H}\restriction i) \cup \mathrm{NEF}(\mathfrak{H}\restriction i) \cup \mathrm{UEF}(\mathfrak{H}\restriction i) \cup \mathrm{IEF}(\mathfrak{H}\restriction i)$, with Theorem 3-27, that (v) holds.

(*R-L*): Now, suppose for all $i \in \mathrm{Dom}(\mathfrak{H})$ holds one of the cases (i) to (vi). With Definition 3-18, it then holds for all $i \in \mathrm{Dom}(\mathfrak{H})$ that $\mathfrak{H}{\restriction}i{+}1 \in \mathrm{RCE}(\mathfrak{H}{\restriction}i)$. With Definition 3-19, we have $\mathfrak{H} \in \mathrm{RCS}$. ■

4 Theorems about the Deductive Consequence Relation

In the following, we will prove theorems about the deductive consequence relation that show that usual properties such as reflexivity, monotony, closure under introduction and elimination of the logical operators and transitivity hold for this relation, and that serve at the same time to prepare the proof of completeness in ch. 6.2. To do this, we first have to do some preparatory work (4.1). Subsequently, we will show that the deductive consequence relation has the desired properties (4.2).

4.1 Preparations

First, we will pave the way for showing that the deductive consequence relation is closed under CdI. To do this, we first show that for every derivation $\mathfrak{H}$ there is a derivation $\mathfrak{H}$* with AVAP($\mathfrak{H}$*) $\subseteq$ AVAP($\mathfrak{H}$) and C($\mathfrak{H}$*) = C($\mathfrak{H}$) in which none of the assumed propositions is available at two positions (Theorem 4-1). Theorem 4-2 then shows that for every derivation $\mathfrak{H}$ and every $\Gamma \in$ CFORM there is a derivation $\mathfrak{H}$* with AVAP($\mathfrak{H}$*) $\subseteq$ AVAP($\mathfrak{H}$) and C($\mathfrak{H}$*) = C($\mathfrak{H}$) such that Γ is available as an assumption only if it is available as the last assumption. This theorem provides the basis for the closure under CdI.

The remaining theorems aim at the closure under introductions and eliminations for which the antecedents of the closure clauses (cf. Theorem 4-18) have the form $X_0 \vdash A_0$, …, $X_{n-1} \vdash A_{n-1}$. Here, we cannot simply concatenate derivations because the emergence of closed segments or the violation of parameter conditions can cause problems. Therefore, we have to show that derivations can be manipulated by adding blocking members, substitution of parameters and the multiple application of UI and UE, so that the desired concatenations can be carried out.

To do this, we first show that derivations that do not have common parameters can be concatenated (Theorem 4-4) if we interpose an assumption that blocks the emergence of closed segments (Theorem 4-3) and that can then be eliminated (Theorem 4-7). Then, we will show that the substitution of a new parameter for a parameter (that may already be used) is RCS-preserving (Theorem 4-8). The proof of this theorem serves as a model for the proof of the next theorem (Theorem 4-9), which on its part prepares the generalisation theorem (Theorem 4-24). Then, we show that the simultanous substitution of several new and pairwise different parameters for pairwise different parameters is also RCS-preserving (Theorem 4-10). Then, we establish some properties of UI- and UE-extensions of derivations, until, eventually, we prove Theorem 4-14, which assures us that two arbitrary derivations can be joined in such a way that, on the one hand, no further available assumptions have to be added, and that, on the other hand, the conclusions of both derivations are still available.

Theorem 4-1. *Non-redundant AVAS*
If $\mathfrak{H} \in$ RCS\{∅} then there is an $\mathfrak{H}^*$ ∈ RCS\{∅} such that
(i) AVAP($\mathfrak{H}^*$) ⊆ AVAP($\mathfrak{H}$)
(ii) C($\mathfrak{H}^*$) = C($\mathfrak{H}$), and
(iii) |AVAS($\mathfrak{H}^*$)| = |AVAP($\mathfrak{H}^*$)|.

Proof: Suppose $\mathfrak{H}$ ∈ RCS\{∅}. The proof is carried out by induction on |AVAS($\mathfrak{H}$)|. Suppose |AVAS($\mathfrak{H}$)| = 0. Obviously, we have AVAP($\mathfrak{H}$) ⊆ AVAP($\mathfrak{H}$) and C($\mathfrak{H}$) = C($\mathfrak{H}$) and, with Theorem 2-77, we also have |AVAP($\mathfrak{H}$)| = 0.

Now, suppose |AVAS($\mathfrak{H}$)| = $k \neq 0$. Suppose the statement holds for all $\mathfrak{H}'$ ∈ RCS\{∅} with |AVAS($\mathfrak{H}'$)| < k. With Theorem 2-76, we then have |AVAP($\mathfrak{H}$)| ≤ |AVAS($\mathfrak{H}$)|. Now, suppose |AVAP($\mathfrak{H}$)| ≠ |AVAS($\mathfrak{H}$)|. Then we have |AVAP($\mathfrak{H}$)| < |AVAS($\mathfrak{H}$)|. Also, it holds that AVAS($\mathfrak{H}$) ≠ ∅. With Theorem 3-18, we thus have $\mathfrak{H}^1 = \mathfrak{H} \cup$ {(Dom($\mathfrak{H}$), ⌜Therefore P($\mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))}$) → C($\mathfrak{H}$)⌝)} ∈ CdIF($\mathfrak{H}$). With Theorem 3-19-(ix), we then have AVAP($\mathfrak{H}^1$) ⊆ AVAP($\mathfrak{H}$) and with Theorem 3-19-(iv) and -(v) follows |AVAS($\mathfrak{H}^1$)| < k. According to the I.H., there is then $\mathfrak{H}^2$ ∈ RCS\{∅} such that AVAP($\mathfrak{H}^2$) ⊆ AVAP($\mathfrak{H}^1$), C($\mathfrak{H}^2$) = C($\mathfrak{H}^1$) and |AVAS($\mathfrak{H}^2$)| = |AVAP($\mathfrak{H}^2$)|. Then we have AVAP($\mathfrak{H}^2$) ⊆ AVAP($\mathfrak{H}^1$) ⊆ AVAP($\mathfrak{H}$) and C($\mathfrak{H}^2$) = C($\mathfrak{H}^1$) = ⌜P($\mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))}$) → C($\mathfrak{H}$)⌝. We have P($\mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))}$) ∈ AVAP($\mathfrak{H}^2$) or P($\mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))}$) ∉ AVAP($\mathfrak{H}^2$).

Suppose P($\mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})))}$) ∈ AVAP($\mathfrak{H}^2$). Then we have $\mathfrak{H}^3 = \mathfrak{H}^{2\frown}${(0, ⌜Therefore C($\mathfrak{H}$)⌝)} ∈ CdEF($\mathfrak{H}^2$) and, with Theorem 3-27-(v), it holds that AVAP($\mathfrak{H}^3$) ⊆ AVAP($\mathfrak{H}^2$) ⊆ AVAP($\mathfrak{H}^1$) ⊆ AVAP($\mathfrak{H}$), and we have C($\mathfrak{H}^3$) = C($\mathfrak{H}$) and |AVAS($\mathfrak{H}^3$)| = |AVAP($\mathfrak{H}^3$)|. The latter one results as follows:

Suppose for contradiction that |AVAS($\mathfrak{H}^3$)| > |AVAP($\mathfrak{H}^3$)|. Then there would be i, j ∈ Dom($\mathfrak{H}^3$) with $i \neq j$ and A ∈ CFORM such that (i, ⌜Suppose A⌝) ∈ AVAS($\mathfrak{H}^3$) and (j, ⌜Suppose A⌝) ∈ AVAS($\mathfrak{H}^3$). Since, with Theorem 3-27-(ii), we have AVAS($\mathfrak{H}^3$) ⊆ AVAS($\mathfrak{H}^2$) there would thus be i, j ∈ Dom($\mathfrak{H}^2$) with $i \neq j$ and A ∈ CFORM such that (i,

$\ulcorner$Suppose A$\urcorner$) $\in$ AVAS($\mathfrak{H}^2$) and (j, $\ulcorner$Suppose A$\urcorner$) $\in$ AVAS($\mathfrak{H}^2$). But then we would also have $|$AVAS($\mathfrak{H}^2$)$|$ > $|$AVAP($\mathfrak{H}^2$)$|$. Therefore we have $|$AVAS($\mathfrak{H}^3$)$| \leq |$AVAP($\mathfrak{H}^3$)$|$ and thus, with Theorem 2-76, $|$AVAS($\mathfrak{H}^3$)$|$ = $|$AVAP($\mathfrak{H}^3$)$|$.

Now, suppose P($\mathfrak{H}_{max(Dom(AVAS(\mathfrak{H})))}$) $\notin$ AVAP($\mathfrak{H}^2$). Now, let $\mathfrak{H}^4 = \mathfrak{H}^{2\frown}$\{(0, $\ulcorner$Suppose P($\mathfrak{H}_{max(Dom(AVAS(\mathfrak{H})))}$)$\urcorner$)\}. Then we have $\mathfrak{H}^4 \in$ AF($\mathfrak{H}^2$). With Theorem 3-15-(viii), we then have AVAP($\mathfrak{H}^4$) = AVAP($\mathfrak{H}^2$) $\cup$ \{P($\mathfrak{H}_{max(Dom(AVAS(\mathfrak{H})))}$)\} $\subseteq$ AVAP($\mathfrak{H}$), and we have C($\mathfrak{H}^4$) = P($\mathfrak{H}_{max(Dom(AVAS(\mathfrak{H})))}$) and $|$AVAS($\mathfrak{H}^4$)$|$ = $|$AVAP($\mathfrak{H}^4$)$|$. The latter is shown as follows:

First, we have $|$AVAP($\mathfrak{H}^2$)$|$ = $|$AVAS($\mathfrak{H}^2$)$|$ and $|$\{P($\mathfrak{H}_{max(Dom(AVAS(\mathfrak{H})))}$)\}$|$ = $|$\{(Dom($\mathfrak{H}^2$), $\ulcorner$Suppose P($\mathfrak{H}_{max(Dom(AVAS(\mathfrak{H})))}$)$\urcorner$)\}$|$. Furthermore, we have AVAS($\mathfrak{H}^2$) $\cap$ \{(Dom($\mathfrak{H}^2$), $\ulcorner$Suppose P($\mathfrak{H}_{max(Dom(AVAS(\mathfrak{H})))}$)$\urcorner$)\} = $\emptyset$ and AVAP($\mathfrak{H}^2$) $\cap$ \{P($\mathfrak{H}_{max(Dom(AVAS(\mathfrak{H})))}$)\} = $\emptyset$. With Theorem 3-15-(iv) and -(viii), we thus have:

$|$AVAS($\mathfrak{H}^4$)$|$ = $|$AVAS($\mathfrak{H}^2$) $\cup$ \{(Dom($\mathfrak{H}^2$), $\ulcorner$Suppose P($\mathfrak{H}_{max(Dom(AVAS(\mathfrak{H})))}$)$\urcorner$)\}$|$

= $|$AVAS($\mathfrak{H}^2$)$|$+$|$\{(Dom($\mathfrak{H}^2$), $\ulcorner$Suppose P($\mathfrak{H}_{max(Dom(AVAS(\mathfrak{H})))}$)$\urcorner$)\}$|$

= $|$AVAP($\mathfrak{H}^2$)$|$+$|$\{P($\mathfrak{H}_{max(Dom(AVAS(\mathfrak{H})))}$)\}$|$

= $|$AVAP($\mathfrak{H}^2$) $\cup$ \{P($\mathfrak{H}_{max(Dom(AVAS(\mathfrak{H})))}$)\}$|$

= $|$AVAP($\mathfrak{H}^4$)$|$.

With Theorem 3-15-(vi), we also have that \{P($\mathfrak{H}_{max(Dom(AVAS(\mathfrak{H})))}$), $\ulcorner$P($\mathfrak{H}_{max(Dom(AVAS(\mathfrak{H})))}$) $\rightarrow$ C($\mathfrak{H}$)$\urcorner$\} $\subseteq$ AVP($\mathfrak{H}^4$). Thus we have $\mathfrak{H}^5 = \mathfrak{H}^{4\frown}$\{(0, $\ulcorner$Therefore C($\mathfrak{H}$)$\urcorner$)\} $\in$ CdEF($\mathfrak{H}^4$) and, with Theorem 3-27-(v), we then have AVAP($\mathfrak{H}^5$) $\subseteq$ AVAP($\mathfrak{H}^4$) $\subseteq$ AVAP($\mathfrak{H}$) and C($\mathfrak{H}^5$) = C($\mathfrak{H}$) and $|$AVAS($\mathfrak{H}^5$)$|$ = $|$AVAP($\mathfrak{H}^5$)$|$. The latter results from $|$AVAS($\mathfrak{H}^4$)$|$ = $|$AVAP($\mathfrak{H}^4$)$|$ in the same way in which we have shown above that $|$AVAS($\mathfrak{H}^3$)$|$ = $|$AVAP($\mathfrak{H}^3$)$|$. ■

The following theorem serves especially to prepare the closure under CdI (Theorem 4-18-(i)).

Theorem 4-2. *CdI-preparation theorem*

If $\mathfrak{H} \in$ RCS\\{$\emptyset$\} and $\Gamma \in$ CFORM, then there is an $\mathfrak{H}^* \in$ RCS\\{$\emptyset$\} such that

(i) AVAP($\mathfrak{H}^*$) $\subseteq$ AVAP($\mathfrak{H}$),

(ii) C($\mathfrak{H}^*$) = C($\mathfrak{H}$), and

(iii) For all $i \in$ Dom(AVAS($\mathfrak{H}^*$)): If P($\mathfrak{H}^*_i$) = Γ, then i = max(Dom(AVAS($\mathfrak{H}^*$))).

Proof: Suppose $\mathfrak{H} \in$ RCS\\{$\emptyset$\} and $\Gamma \in$ CFORM. Then we have $\Gamma \notin$ AVAP($\mathfrak{H}$) or $\Gamma \in$ AVAP($\mathfrak{H}$). If $\Gamma \notin$ AVAP($\mathfrak{H}$), then $\mathfrak{H}$ itself is an $\mathfrak{H}^* \in$ RCS\\{$\emptyset$\} such that (i), (ii) and (iii) hold trivially. Now, suppose $\Gamma \in$ AVAP($\mathfrak{H}$). The proof is carried out by induction on

$|\mathrm{AVAS}(\mathfrak{H})|$. Suppose $|\mathrm{AVAS}(\mathfrak{H})| = 0$. With Theorem 2-77, it follows that $|\mathrm{AVAP}(\mathfrak{H})| = 0$, whereas, according to the hypothesis, $|\mathrm{AVAS}(\mathfrak{H})| \neq 0$. Thus the statement holds trivially for $|\mathrm{AVAS}(\mathfrak{H})| = 0$.

Now, suppose $|\mathrm{AVAS}(\mathfrak{H})| = k \neq 0$. Suppose the statement holds for all $\mathfrak{H}' \in \mathrm{RCS}\setminus\{\emptyset\}$ with $|\mathrm{AVAS}(\mathfrak{H}')| < k$. With Theorem 4-1, there is an $\mathfrak{H}^1 \in \mathrm{RCS}\setminus\{\emptyset\}$ such that $\mathrm{AVAP}(\mathfrak{H}^1) \subseteq \mathrm{AVAP}(\mathfrak{H})$, $\mathrm{C}(\mathfrak{H}^1) = \mathrm{C}(\mathfrak{H})$ and $|\mathrm{AVAS}(\mathfrak{H}^1)| = |\mathrm{AVAP}(\mathfrak{H}^1)| \leq |\mathrm{AVAP}(\mathfrak{H})| \leq |\mathrm{AVAS}(\mathfrak{H})|$. We also have, with $|\mathrm{AVAS}(\mathfrak{H}^1)| = |\mathrm{AVAP}(\mathfrak{H}^1)|$, that it holds for all $\mathrm{B} \in \mathrm{AVAP}(\mathfrak{H}^1)$ that there is exactly one $i \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1))$ such that $\mathrm{B} = \mathrm{P}(\mathfrak{H}_i)$. Suppose, for all $i \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1))$: If $\mathrm{P}(\mathfrak{H}^1{}_i) = \Gamma$, then $i = \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))$. Then we have that $\mathfrak{H}^1$ is the desired element of $\mathrm{RCS}\setminus\{\emptyset\}$.

Now, suppose not for all $i \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1))$: If $\mathrm{P}(\mathfrak{H}^1{}_i) = \Gamma$, then $i = \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))$. Then there is an $i \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1))$ such that $\mathrm{P}(\mathfrak{H}^1{}_i) = \Gamma$ and $i \neq \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))$. Then we have $\mathrm{AVAS}(\mathfrak{H}^1) \neq \emptyset$ and $\Gamma \in \mathrm{AVAP}(\mathfrak{H}^1)$, and it holds for all $j \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1))$: If $\mathrm{P}(\mathfrak{H}_j) = \Gamma$, then $j = i$ and thus also $j \neq \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))$. Thus we have $\mathrm{P}(\mathfrak{H}^1{}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))}) \neq \Gamma$. We also have, with $\mathrm{AVAS}(\mathfrak{H}^1) \neq \emptyset$, Theorem 3-18 and $\mathrm{C}(\mathfrak{H}^1) = \mathrm{C}(\mathfrak{H})$: $\mathfrak{H}^2 = \mathfrak{H}^{1\frown}\{(0, \ulcorner\text{Therefore } \mathrm{P}(\mathfrak{H}^1{}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))}) \rightarrow \mathrm{C}(\mathfrak{H})\urcorner)\} \in \mathrm{CdIF}(\mathfrak{H}^1)$. Then it holds, with Theorem 3-22, that $\mathrm{AVAP}(\mathfrak{H}^2) \subseteq \mathrm{AVAP}(\mathfrak{H}^1)\setminus\{\mathrm{P}(\mathfrak{H}^1{}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))})\} \subseteq \mathrm{AVAP}(\mathfrak{H})$. With Theorem 3-19-(iv) and -(v), it holds that $|\mathrm{AVAS}(\mathfrak{H}^2)| < |\mathrm{AVAS}(\mathfrak{H}^1)| \leq |\mathrm{AVAS}(\mathfrak{H})|$ and that $|\mathrm{AVAS}(\mathfrak{H}^2)| = |\mathrm{AVAP}(\mathfrak{H}^2)|$. The latter is shown as follows:

> Suppose for contradiction that $|\mathrm{AVAS}(\mathfrak{H}^2)| > |\mathrm{AVAP}(\mathfrak{H}^2)|$. Then there would be $i, j \in \mathrm{Dom}(\mathfrak{H}^2)$ with $i \neq j$ and $\mathrm{A} \in \mathrm{CFORM}$ such that $(i, \ulcorner\text{Suppose A}\urcorner) \in \mathrm{AVAS}(\mathfrak{H}^2)$ and $(j, \ulcorner\text{Suppose A}\urcorner) \in \mathrm{AVAS}(\mathfrak{H}^2)$. Since, with Theorem 3-19-(v), $\mathrm{AVAS}(\mathfrak{H}^2) \subseteq \mathrm{AVAS}(\mathfrak{H}^1)$, there would thus be $i, j \in \mathrm{Dom}(\mathfrak{H}^1)$ with $i \neq j$ and $\mathrm{A} \in \mathrm{CFORM}$ such that $(i, \ulcorner\text{Suppose A}\urcorner) \in \mathrm{AVAS}(\mathfrak{H}^1)$ and $(j, \ulcorner\text{Suppose A}\urcorner) \in \mathrm{AVAS}(\mathfrak{H}^1)$. But then we would also have $|\mathrm{AVAS}(\mathfrak{H}^1)| > |\mathrm{AVAP}(\mathfrak{H}^1)|$. Therefore we have $|\mathrm{AVAS}(\mathfrak{H}^2)| \leq |\mathrm{AVAP}(\mathfrak{H}^2)|$ and thus, with Theorem 2-76, that $|\mathrm{AVAS}(\mathfrak{H}^2)| = |\mathrm{AVAP}(\mathfrak{H}^2)|$.

We have $|\mathrm{AVAS}(\mathfrak{H}^2)| < |\mathrm{AVAS}(\mathfrak{H}^1)| \leq |\mathrm{AVAS}(\mathfrak{H})| = k$. According to the I.H., there is thus an $\mathfrak{H}^3 \in \mathrm{RCS}\setminus\{\emptyset\}$ such that $\mathrm{AVAP}(\mathfrak{H}^3) \subseteq \mathrm{AVAP}(\mathfrak{H}^2)$ and $\mathrm{C}(\mathfrak{H}^3) = \mathrm{C}(\mathfrak{H}^2)$ and for all $i \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^3))$: If $\mathrm{P}(\mathfrak{H}^3{}_i) = \Gamma$, then $i = \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^3)))$. Then we have $\mathrm{AVAP}(\mathfrak{H}^3) \subseteq \mathrm{AVAP}(\mathfrak{H}^2) \subseteq \mathrm{AVAP}(\mathfrak{H}^1) \subseteq \mathrm{AVAP}(\mathfrak{H})$, $\mathrm{P}(\mathfrak{H}^1{}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))}) \notin \mathrm{AVAP}(\mathfrak{H}^3)$ and $\mathrm{C}(\mathfrak{H}^3) = \ulcorner\mathrm{P}(\mathfrak{H}^1{}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))}) \rightarrow \mathrm{C}(\mathfrak{H})\urcorner$. With $\Gamma \in \mathrm{AVAP}(\mathfrak{H}^3)$ or $\Gamma \notin \mathrm{AVAP}(\mathfrak{H}^3)$, we can then distinguish *two* cases.

First case: $\Gamma \in \mathrm{AVAP}(\mathfrak{H}^3)$. Then we have $\Gamma = \mathrm{P}(\mathfrak{H}^3_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^3)))})$ and for all $i \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^3))$: If $\Gamma = \mathrm{P}(\mathfrak{H}_i)$, then $i = \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^3)))$. With Theorem 3-18, we then have that $\mathfrak{H}^4 = \mathfrak{H}^{3\frown}\{(0, \ulcorner\text{Therefore } \Gamma \rightarrow (\mathrm{P}(\mathfrak{H}^1_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))}) \rightarrow \mathrm{C}(\mathfrak{H}))\urcorner\} \in \mathrm{CdIF}(\mathfrak{H}^3)$. With Theorem 3-22, it then follows that $\mathrm{AVAP}(\mathfrak{H}^4) \subseteq \mathrm{AVAP}(\mathfrak{H}^3)\backslash\{\Gamma\} \subseteq \mathrm{AVAP}(\mathfrak{H})$. Thus we have $\Gamma \notin \mathrm{AVAP}(\mathfrak{H}^4)$ and thus that for all $i \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^4))$: $\mathrm{P}(\mathfrak{H}^4_i) \neq \Gamma$.

Now, let $\mathfrak{H}^5 = \mathfrak{H}^{4\frown}\{(0, \ulcorner\text{Suppose } \mathrm{P}(\mathfrak{H}^1_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))})\urcorner), (1, \ulcorner\text{Suppose } \Gamma\urcorner)\}$. Then we have $\mathfrak{H}^5 \in \mathrm{AF}(\mathfrak{H}^{4\frown}\{(0, \ulcorner\text{Suppose } \mathrm{P}(\mathfrak{H}^1_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))})\urcorner)\})$ and $\mathfrak{H}^{4\frown}\{(0, \ulcorner\text{Suppose } \mathrm{P}(\mathfrak{H}^1_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))})\urcorner)\} \in \mathrm{AF}(\mathfrak{H}^4)$. Because of $\mathrm{P}(\mathfrak{H}^4_i) \neq \Gamma$ for all $i \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^4))$ and $\Gamma \neq \mathrm{P}(\mathfrak{H}^1_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))})$, we have, with Theorem 3-15-(iv), that for all $i \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^5))$: $\mathrm{P}(\mathfrak{H}^5_i) = \Gamma$ iff $i = \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^5)))$. With Theorem 3-15-(viii), we have $\mathrm{AVAP}(\mathfrak{H}^5) \subseteq \mathrm{AVAP}(\mathfrak{H}^4) \cup \{\Gamma, \mathrm{P}(\mathfrak{H}^1_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))})\} \subseteq \mathrm{AVAP}(\mathfrak{H})$. With Theorem 3-15-(vi), we have $\{\Gamma, \mathrm{P}(\mathfrak{H}^1_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))}), \ulcorner\Gamma \rightarrow (\mathrm{P}(\mathfrak{H}^1_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))}) \rightarrow \mathrm{C}(\mathfrak{H}))\urcorner\} \subseteq \mathrm{AVP}(\mathfrak{H}^5)$, and with Theorem 3-15-(iv) we have that $(\mathrm{Dom}(\mathfrak{H}^4), \ulcorner\text{Suppose } \mathrm{P}(\mathfrak{H}^1_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))})\urcorner) \in \mathrm{AVAS}(\mathfrak{H}^5)$.

Then we have that $\mathfrak{H}^6 = \mathfrak{H}^{5\frown}\{(0, \ulcorner\text{Therefore } \mathrm{P}(\mathfrak{H}^1_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))}) \rightarrow \mathrm{C}(\mathfrak{H})\urcorner)\} \in \mathrm{CdEF}(\mathfrak{H}^5)$, and, with Theorem 3-27-(v), it holds that $\mathrm{AVAP}(\mathfrak{H}^6) \subseteq \mathrm{AVAP}(\mathfrak{H}^5) \subseteq \mathrm{AVAP}(\mathfrak{H})$. Also, we have for all $i \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^6))$: If $\mathrm{P}(\mathfrak{H}^6_i) = \Gamma$, then $i = \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^6)))$. The latter results as follows:

> Suppose for contradiction that there is an $i \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^6))$ such that $\mathrm{P}(\mathfrak{H}^6_i) = \Gamma$ and $i \neq \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^6)))$. With Theorem 3-27-(ii), it then follows that $i \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^5))$. Then we have $i = \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^5))) = \mathrm{Dom}(\mathfrak{H}^4)+1$. However, according to the construction of $\mathfrak{H}^6$, we have $\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^6))) \leq \mathrm{Dom}(\mathfrak{H}^4)+1 = i$. With $i \neq \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^6)))$, we would thus have $\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^6))) < i$. But, with $i \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^6))$, we have $i \leq \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^6)))$. Contradiction!

We have $\ulcorner\mathrm{P}(\mathfrak{H}^1_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))}) \rightarrow \mathrm{C}(\mathfrak{H})\urcorner = \mathrm{C}(\mathfrak{H}^6) \in \mathrm{AVP}(\mathfrak{H}^6)$. Now, suppose for contradiction that $\mathrm{P}(\mathfrak{H}^1_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))}) \notin \mathrm{AVP}(\mathfrak{H}^6)$. Then we would have $(\mathrm{Dom}(\mathfrak{H}^4), \ulcorner\text{Suppose } \mathrm{P}(\mathfrak{H}^1_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))})\urcorner) \notin \mathrm{AVAS}(\mathfrak{H}^6)$ and thus $(\mathrm{Dom}(\mathfrak{H}^4), \ulcorner\text{Suppose } \mathrm{P}(\mathfrak{H}^1_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))})\urcorner) \in \mathrm{AVAS}(\mathfrak{H}^5)\backslash\mathrm{AVAS}(\mathfrak{H}^6)$. With Theorem 2-85, we would then have $\mathrm{AVAS}(\mathfrak{H}^5)\backslash\mathrm{AVAS}(\mathfrak{H}^6) = \{(\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^5))), \mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^5)))})\} = \{(\mathrm{Dom}(\mathfrak{H}^4)+1, \ulcorner\text{Suppose } \Gamma\urcorner)\}$ and therefore $\mathrm{Dom}(\mathfrak{H}^4) = \mathrm{Dom}(\mathfrak{H}^4)+1$. Contradiction!

Thus we have that $\mathfrak{H}^7 = \mathfrak{H}^{6\frown}\{(0, \ulcorner\text{Therefore } \mathrm{C}(\mathfrak{H})\urcorner)\} \in \mathrm{CdEF}(\mathfrak{H}^6)$ and, with Theorem 3-27-(v), it holds that $\mathrm{AVAP}(\mathfrak{H}^7) \subseteq \mathrm{AVAP}(\mathfrak{H}^6) \subseteq \mathrm{AVAP}(\mathfrak{H})$. We also have, with

Theorem 3-27-(ii), for all $i \in$ Dom(AVAS($\mathfrak{H}^7$)): If P($\mathfrak{H}^7{}_i$) = Γ, then i = max(Dom(AVAS($\mathfrak{H}^7$))). Thus we have that $\mathfrak{H}^7$ is the desired element of RCS\{∅}.

Second case: Γ ∉ AVAP($\mathfrak{H}^3$). Now, let $\mathfrak{H}^8 = \mathfrak{H}^{3\frown}$\{(0, ⌜Suppose P($\mathfrak{H}^1{}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))}$)⌝)\}. Then we have $\mathfrak{H}^8 \in$ AF($\mathfrak{H}^3$). With Theorem 3-15-(viii), we have AVAP($\mathfrak{H}^8$) = AVAP($\mathfrak{H}^3$) ∪ \{P($\mathfrak{H}^1{}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))}$)\} ⊆ AVAP($\mathfrak{H}$). With Theorem 3-15-(vi), we have \{P($\mathfrak{H}^1{}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))}$), ⌜P($\mathfrak{H}^1{}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))}$) → C($\mathfrak{H}$)⌝\} ⊆ AVP($\mathfrak{H}^8$). With Γ ∉ AVAP($\mathfrak{H}^3$) and P($\mathfrak{H}^1{}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^1)))}$) ≠ Γ, we also have Γ ∉ AVAP($\mathfrak{H}^8$) and thus for all $i \in$ Dom(AVAS($\mathfrak{H}^8$)): P($\mathfrak{H}^8{}_i$) ≠ Γ. Then we have trivially for all $i \in$ Dom(AVAS($\mathfrak{H}^8$)): If P($\mathfrak{H}^8{}_i$) = Γ, then i = max(Dom(AVAS($\mathfrak{H}^8$))). Then we have $\mathfrak{H}^9 = \mathfrak{H}^{8\frown}$\{(0, ⌜Therefore C($\mathfrak{H}$)⌝)\} ∈ CdEF($\mathfrak{H}^8$) and, with Theorem 3-27-(v), it holds that AVAP($\mathfrak{H}^9$) ⊆ AVAP($\mathfrak{H}^8$) ⊆ AVAP($\mathfrak{H}$). Furthermore, we have again trivially for all $i \in$ Dom(AVAS($\mathfrak{H}^9$)): If P($\mathfrak{H}^9{}_i$) = Γ, then i = max(Dom(AVAS($\mathfrak{H}^9$))). Thus we have that $\mathfrak{H}^9$ is the desired element of RCS\{∅}. ■

Theorem 4-3. *Blocking assumptions*

If $\mathfrak{A}$ is a closed segment in $\mathfrak{H}$, $i \in$ Dom($\mathfrak{A}$) ∩ Dom(AS($\mathfrak{H}$)), Δ = P($\mathfrak{H}_i$) and PAR ∩ ST(Δ) = ∅, then there is a $j \in$ Dom($\mathfrak{H}$) such that $i \neq j$ and Δ ∈ SE($\mathfrak{H}_j$).

Proof: Suppose $\mathfrak{A}$ is a closed segment in $\mathfrak{H}$, $i \in$ Dom($\mathfrak{A}$) ∩ Dom(AS($\mathfrak{H}$)), Δ = P($\mathfrak{H}_i$) and PAR ∩ ST(Δ) = ∅. With Theorem 2-47, it then follows that there is a closed segment $\mathfrak{B}$ in $\mathfrak{H}$ with $\mathfrak{B} \subseteq \mathfrak{A}$ such that i = min(Dom($\mathfrak{B}$)). With Theorem 2-42, $\mathfrak{B}$ is then a CdI- or NI- or RA-like segment in $\mathfrak{H}$. Suppose $\mathfrak{B}$ is a CdI- or an NI-like segment in $\mathfrak{H}$. Then it holds, with Definition 2-11 and Definition 2-12, that max(Dom($\mathfrak{B}$)) ∈ Dom($\mathfrak{H}$), max(Dom($\mathfrak{B}$)) ≠ i and Δ ∈ SE($\mathfrak{H}_{\max(\mathrm{Dom}(\mathfrak{B}))}$). Now, suppose $\mathfrak{B}$ is an RA-like segment in $\mathfrak{H}$. With Definition 2-13, it then holds that min(Dom($\mathfrak{B}$))-1 ∈ Dom($\mathfrak{H}$) and min(Dom($\mathfrak{B}$))-1 ≠ i. Moreover, there are then ξ ∈ VAR, Δ^+ ∈ FORM, where FV(Δ^+) ⊆ \{ξ\} and β ∈ PAR such that P($\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))-1}$) = ⌜⋁ξ$\Delta^+$⌝ and Δ = P($\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))}$) = [β, ξ, Δ^+]. By hypothesis, we have PAR ∩ ST(Δ) = ∅, and thus we have β ∉ ST([β, ξ, Δ^+]). With Theorem 1-14-(ii), we then have Δ = [β, ξ, Δ^+] = Δ^+. Thus we have P($\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))-1}$) = ⌜⋁ξΔ⌝ and hence Δ ∈ SE($\mathfrak{H}_{\min(\mathrm{Dom}(\mathfrak{B}))-1}$) and the statement holds. ■

Theorem 4-4. *Concatenation of RCS-elements that do not have any parameters in common, where the concatenation includes an interposed blocking assumption*

If $\mathfrak{H}$, $\mathfrak{H}' \in$ RCS, PAR $\cap$ STSEQ($\mathfrak{H}$) $\cap$ STSEQ($\mathfrak{H}'$) = ∅ and $\alpha \in$ CONST\(STSEQ($\mathfrak{H}$) $\cup$ STSEQ($\mathfrak{H}'$)), then there is an $\mathfrak{H}^* \in$ RCS\{∅} such that

(i) Dom($\mathfrak{H}^*$) = Dom($\mathfrak{H}$)+1+Dom($\mathfrak{H}'$),

(ii) $\mathfrak{H}^*\restriction$Dom($\mathfrak{H}$) = $\mathfrak{H}$,

(iii) $\mathfrak{H}^*_{\mathrm{Dom}(\mathfrak{H})}$ = ⌜Suppose α = α⌝,

(iv) For all $i \in$ Dom($\mathfrak{H}'$) it holds that $\mathfrak{H}'_i = \mathfrak{H}^*_{\mathrm{Dom}(\mathfrak{H})+1+i}$,

(v) Dom(AVS($\mathfrak{H}^*$)) =
Dom(AVS($\mathfrak{H}$)) $\cup$ {Dom($\mathfrak{H}$)} $\cup$ {Dom($\mathfrak{H}$)+1+l | $l \in$ Dom(AVS($\mathfrak{H}'$))},

(vi) AVP($\mathfrak{H}^*$) = AVP($\mathfrak{H}$) $\cup$ {⌜α = α⌝} $\cup$ AVP($\mathfrak{H}'$), and

(vii) AVAP($\mathfrak{H}^*$) = AVAP($\mathfrak{H}$) $\cup$ {⌜α = α⌝} $\cup$ AVAP($\mathfrak{H}'$).

Proof: We show by induction on Dom($\mathfrak{H}'$) that under the specified conditions there is always an $\mathfrak{H}^*$ such that clauses (i) to (v) are satisified. (vi) and (vii) then follow from the preceding clauses. First, we have from (i) to (v) and Definition 2-30:

B $\in$ AVP($\mathfrak{H}^*$)
iff
there is an $i \in$ Dom(AVS($\mathfrak{H}^*$)) such that B = P($\mathfrak{H}^*_i$)
iff
there is an $i \in$ Dom(AVS($\mathfrak{H}$)) $\cup$ {Dom($\mathfrak{H}$)} $\cup$ {Dom($\mathfrak{H}$)+1+l | $l \in$ Dom(AVS($\mathfrak{H}'$))} such that B = P($\mathfrak{H}^*_i$)
iff
B $\in$ AVP($\mathfrak{H}$) $\cup$ {⌜α = α⌝} $\cup$ AVP($\mathfrak{H}'$).

Second, (vii) results from (i) to (v) and Definition 2-31 as follows:

B $\in$ AVAP($\mathfrak{H}^*$)
iff
there is an $i \in$ Dom(AVAS($\mathfrak{H}^*$)) such that B = P($\mathfrak{H}^*_i$)
iff
there is an $i \in$ Dom(AVS($\mathfrak{H}^*$)) $\cap$ Dom(AS($\mathfrak{H}^*$)) such that B = P($\mathfrak{H}^*_i$)
iff
there is an $i \in$ (Dom(AVS($\mathfrak{H}$)) $\cup$ {Dom($\mathfrak{H}$)} $\cup$ {Dom($\mathfrak{H}$)+1+l | $l \in$ Dom(AVS($\mathfrak{H}'$))}) $\cap$ Dom(AS($\mathfrak{H}^*$)) such that B = P($\mathfrak{H}^*_i$)
iff
there is an $i \in$ (Dom(AVS($\mathfrak{H}$)) $\cap$ Dom(AS($\mathfrak{H}^*$))) $\cup$ ({Dom($\mathfrak{H}$)} $\cap$ Dom(AS($\mathfrak{H}^*$))) $\cup$ ({Dom($\mathfrak{H}$)+1+l | $l \in$ Dom(AVS($\mathfrak{H}'$))} $\cap$ Dom(AS($\mathfrak{H}^*$))) such that B = P($\mathfrak{H}^*_i$)
iff

there is an $i \in (\mathrm{Dom}(\mathrm{AVS}(\mathfrak{H})) \cap \mathrm{Dom}(\mathrm{AS}(\mathfrak{H}))) \cup (\{\mathrm{Dom}(\mathfrak{H})\} \cap \mathrm{Dom}(\mathrm{AS}(\mathfrak{H}^*))) \cup (\{\mathrm{Dom}(\mathfrak{H})+1+l \mid l \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}'))\} \cap (\{\mathrm{Dom}(\mathfrak{H})+1+l \mid l \in \mathrm{Dom}(\mathrm{AS}(\mathfrak{H}'))\})$ such that $B = P(\mathfrak{H}^*_i)$
iff
there is an $i \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H})) \cup \{\mathrm{Dom}(\mathfrak{H})\} \cup (\{\mathrm{Dom}(\mathfrak{H})+1+l \mid l \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}'))\}$ such that $B = P(\mathfrak{H}^*_i)$
iff
$B \in \mathrm{AVAP}(\mathfrak{H}) \cup \{\ulcorner\alpha = \alpha\urcorner\} \cup \mathrm{AVAP}(\mathfrak{H}')$.

Now for the proof by induction: Suppose the statement holds for $k < \mathrm{Dom}(\mathfrak{H}')$ and suppose $\mathfrak{H}$, $\mathfrak{H}'$ are as required and suppose $\alpha \in \mathrm{CONST}\setminus(\mathrm{STSEQ}(\mathfrak{H}) \cup \mathrm{STSEQ}(\mathfrak{H}'))$. Suppose $\mathrm{Dom}(\mathfrak{H}') = 0$. Then we have $\mathfrak{H}' = \emptyset$ and with $\mathfrak{H}^* = \mathfrak{H}^\frown\{(0, \ulcorner\text{Suppose } \alpha = \alpha\urcorner)\}$ and Theorem 3-15-(ii) the statement holds. Now, suppose $\mathrm{Dom}(\mathfrak{H}') > 0$. Then we have $\mathfrak{H}' \in \mathrm{RCS}\setminus\{\emptyset\}$. With Theorem 3-6, we then have $\mathfrak{H}' \in \mathrm{RCE}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1)$ and $\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1 \in \mathrm{RCS}$. With $\mathrm{PAR} \cap \mathrm{STSEQ}(\mathfrak{H}) \cap \mathrm{STSEQ}(\mathfrak{H}') = \emptyset$, we also have $\mathrm{PAR} \cap \mathrm{STSEQ}(\mathfrak{H}) \cap \mathrm{STSEQ}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1) = \emptyset$ and with $\alpha \in \mathrm{CONST}\setminus(\mathrm{STSEQ}(\mathfrak{H}) \cup \mathrm{STSEQ}(\mathfrak{H}'))$ it also holds that $\alpha \in \mathrm{CONST}\setminus(\mathrm{STSEQ}(\mathfrak{H}) \cup \mathrm{STSEQ}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1))$. According to the I.H., there is then for $\mathfrak{H}$, $\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1$ and α an $\mathfrak{H}^* \in \mathrm{RCS}$ for which (i) to (v) hold. Then it holds that:

i') $\mathrm{Dom}(\mathfrak{H}^*) = \mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')\text{-}1 = \mathrm{Dom}(\mathfrak{H})+\mathrm{Dom}(\mathfrak{H}')$,
ii') $\mathfrak{H}^*\upharpoonright\mathrm{Dom}(\mathfrak{H}) = \mathfrak{H}$,
iii') $\mathfrak{H}^*_{\mathrm{Dom}(\mathfrak{H})} = \ulcorner\text{Suppose } \alpha = \alpha\urcorner$,
iv') For all $i \in \mathrm{Dom}(\mathfrak{H}')\text{-}1$ it holds that $\mathfrak{H}'_i = (\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1)_i = \mathfrak{H}^*_{\mathrm{Dom}(\mathfrak{H})+1+i}$,
v') $\mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^*)) =$
$\mathrm{Dom}(\mathrm{AVS}(\mathfrak{H})) \cup \{\mathrm{Dom}(\mathfrak{H})\} \cup \{(\mathrm{Dom}(\mathfrak{H})+1+l \mid l \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1))\}$.

From $\mathfrak{H}' \in \mathrm{RCE}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1)$ it follows, with Definition 3-18, that $\mathfrak{H}' \in \mathrm{AF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1)$ or $\mathfrak{H}' \in \mathrm{CdIF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1)$ or $\mathfrak{H}' \in \mathrm{CdEF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1)$ or $\mathfrak{H}' \in \mathrm{CIF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1)$ or $\mathfrak{H}' \in \mathrm{CEF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1)$ or $\mathfrak{H}' \in \mathrm{BIF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1)$ or $\mathfrak{H}' \in \mathrm{BEF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1)$ or $\mathfrak{H}' \in \mathrm{DIF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1)$ or $\mathfrak{H}' \in \mathrm{DEF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1)$ or $\mathfrak{H}' \in \mathrm{NIF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1)$ or $\mathfrak{H}' \in \mathrm{NEF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1)$ or $\mathfrak{H}' \in \mathrm{UIF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1)$ or $\mathfrak{H}' \in \mathrm{UEF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1)$ or $\mathfrak{H}' \in \mathrm{PIF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1)$ or $\mathfrak{H}' \in \mathrm{PEF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1)$ or $\mathfrak{H}' \in \mathrm{IIF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1)$ or $\mathfrak{H}' \in \mathrm{IEF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1)$. Now let

vi') $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')\text{-}1, \mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H}')\text{-}1})\}$.

Then we already have that $\mathfrak{H}^+ \neq \emptyset$ and clauses (i) to (iv) hold for $\mathfrak{H}^+$. Now, we will show that for each of the cases AF … IEF we have that $\mathfrak{H}^+ \in$ RCS\{∅} and that (v) holds, with which we have that $\mathfrak{H}^+$ is in each case the desired RCS-element. First, we note that, because of $\alpha \in$ CONST\(STSEQ($\mathfrak{H}$) ∪ STSEQ($\mathfrak{H}$')), there is no $l \in$ Dom($\mathfrak{H}$*) ⊆ Dom($\mathfrak{H}^+$) such that $l \neq$ Dom($\mathfrak{H}$) and $\ulcorner\alpha = \alpha\urcorner \in$ SE($\mathfrak{H}^+{}_l$). With $\mathfrak{H}^*{}_{\mathrm{Dom}(\mathfrak{H})} = \mathfrak{H}^+{}_{\mathrm{Dom}(\mathfrak{H})} = \ulcorner$Suppose $\alpha = \alpha\urcorner$ and Theorem 4-3, it thus holds:

vii') There is no closed segment $\mathfrak{A}$ in $\mathfrak{H}^+$ and there is no closed segment $\mathfrak{A}$ in $\mathfrak{H}$* such that min(Dom($\mathfrak{A}$)) ≤ Dom($\mathfrak{H}$) < max(Dom($\mathfrak{A}$)).

Thus it also follows that:

viii') Dom($\mathfrak{H}$) ∈ Dom(AVAS($\mathfrak{H}^+$)), Dom($\mathfrak{H}$) ∈ Dom(AVAS($\mathfrak{H}$*)) and Dom($\mathfrak{H}$) ≤ max(Dom(AVAS($\mathfrak{H}$*))).

To simplifiy the treatment of CdEF, CIF, CEF, BIF, BEF, DIF, DEF, NEF, UIF, UEF, PIF, IIF and IEF, we will now show in preparation of the main part of the proof that:

ix') If $\mathfrak{H}^+ \in$ CdIF($\mathfrak{H}$*) ∪ NIF($\mathfrak{H}$*) ∪ PEF($\mathfrak{H}$*), then $\mathfrak{H}' \in$ CdIF($\mathfrak{H}'\upharpoonright$Dom($\mathfrak{H}$')-1) ∪ NIF($\mathfrak{H}'\upharpoonright$Dom($\mathfrak{H}$')-1) ∪ PEF($\mathfrak{H}'\upharpoonright$Dom($\mathfrak{H}$')-1).

Preparatory part: First, suppose $\mathfrak{H}^+ \in$ CdIF($\mathfrak{H}$*). According to Definition 3-2, Theorem 3-19-(i) and vii') and viii'), there is then Dom($\mathfrak{H}$)+1+i ∈ Dom(AVAS($\mathfrak{H}$*)) such that, with i') and iv'), $P(\mathfrak{H}^*{}_{\mathrm{Dom}(\mathfrak{H})+1+i}) = P(\mathfrak{H}'_i)$ and $C(\mathfrak{H}^*) = P(\mathfrak{H}^*{}_{\mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')-2}) = P(\mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H}')-2}) = C(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1)$ and there is no l such that Dom($\mathfrak{H}$)+1+i < l ≤ Dom($\mathfrak{H}$)+1+Dom($\mathfrak{H}$')-2 and l ∈ Dom(AVAS($\mathfrak{H}$*)), and $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')-1, \ulcorner\mathrm{Therefore}\ P(\mathfrak{H}^*{}_{\mathrm{Dom}(\mathfrak{H})+1+i}) \rightarrow C(\mathfrak{H}^*)\urcorner\} = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')-1, \ulcorner\mathrm{Therefore}\ P(\mathfrak{H}'_i) \rightarrow C(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1)\urcorner\}$. Then it holds with i'), iv') and v'): i ∈ Dom(AVAS($\mathfrak{H}'\upharpoonright$Dom($\mathfrak{H}$')-1)) and there is no l such that $i < l \leq$ Dom($\mathfrak{H}$')-2 and l ∈ Dom(AVAS($\mathfrak{H}'\upharpoonright$Dom($\mathfrak{H}$')-1)). Also, with vi'), we have $\mathfrak{H}' = \mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1 \cup \{(\mathrm{Dom}(\mathfrak{H}')-1, \ulcorner\mathrm{Therefore}\ P(\mathfrak{H}'_i) \rightarrow C(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1)\urcorner\}$. Hence we have $\mathfrak{H}' \in$ CdIF($\mathfrak{H}'\upharpoonright$Dom($\mathfrak{H}$')-1). In the case that $\mathfrak{H}^+ \in$ NIF($\mathfrak{H}$*), one shows analogously that then also $\mathfrak{H}' \in$ NIF($\mathfrak{H}'\upharpoonright$Dom($\mathfrak{H}$')-1).

Now, suppose $\mathfrak{H}^+ \in$ PEF($\mathfrak{H}$*). With Definition 3-15, Theorem 3-21-(i), $P(\mathfrak{H}^*{}_{\mathrm{Dom}(\mathfrak{H})}) = \ulcorner\alpha = \alpha\urcorner$ and vii') and viii'), there are then $\beta \in$ PAR, $\xi \in$ VAR, $\Delta \in$ FORM, where FV(Δ) ⊆ {ξ}, and Dom($\mathfrak{H}$)+1+i ∈ Dom(AVS($\mathfrak{H}$*)) such that, with i') and iv'), $\ulcorner\bigvee\xi\Delta\urcorner = P(\mathfrak{H}^*{}_{\mathrm{Dom}(\mathfrak{H})+1+i}) = P(\mathfrak{H}'_i)$ and $[\beta, \xi, \Delta] = P(\mathfrak{H}^*{}_{\mathrm{Dom}(\mathfrak{H})+2+i}) = P(\mathfrak{H}'_{i+1})$, where Dom($\mathfrak{H}$)+2+$i$ ∈ Dom(AVAS($\mathfrak{H}$*)) and $C(\mathfrak{H}^*) = P(\mathfrak{H}^*{}_{\mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')-2}) = P(\mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H}')-2}) = C(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1)$

and $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')-1, \ulcorner\text{Therefore } C(\mathfrak{H}^*)\urcorner\} = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')-1, \ulcorner\text{Therefore } C(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1)\urcorner\}$ and $\beta \notin \mathrm{STSF}(\{\Delta, C(\mathfrak{H}^*)\})$ and there is no $j \leq \mathrm{Dom}(\mathfrak{H})+1+i$ such that $\beta \in \mathrm{ST}(\mathfrak{H}^*_j)$ and there is no l such that $\mathrm{Dom}(\mathfrak{H})+2+i < l \leq \mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')-2$ and $l \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^*))$. It then holds with i'), iv') and v'): $i \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1))$ and $i+1 \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1))$ and $\beta \notin \mathrm{STSF}(\{\Delta, C(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1)\})$ and there is no $j \leq i$ such that $\beta \in \mathrm{ST}(\mathfrak{H}'_j)$, and there is no l such that $i+1 < l \leq \mathrm{Dom}(\mathfrak{H}')-2$ and $l \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1)$. Also, with vi'), we have $\mathfrak{H}' = \mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1 \cup \{(\mathrm{Dom}(\mathfrak{H}')-1, \ulcorner\text{Therefore } C(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1)\urcorner\}$ and hence we have $\mathfrak{H}' \in \mathrm{PEF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1)$.

Main part: Now, we will show that for each of the cases AF … IEF it holds that $\mathfrak{H}^+ \in \mathrm{RCS}\setminus\{\emptyset\}$ and that v) holds:

(*AF*): Suppose $\mathfrak{H}' \in \mathrm{AF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1)$. According to Definition 3-1, we then have $\mathfrak{H}' = \mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1 \cup \{(\mathrm{Dom}(\mathfrak{H}')-1, \ulcorner\text{Suppose } P(\mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H}')-1})\urcorner)\}$. With vi'), we then have $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')-1, \ulcorner\text{Suppose } P(\mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H}')-1})\urcorner)\} \in \mathrm{AF}(\mathfrak{H}^*) \subseteq \mathrm{RCS}\setminus\{\emptyset\}$. With Theorem 3-15-(ii), it then follows that $\mathrm{AVS}(\mathfrak{H}') = \mathrm{AVS}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1) \cup \{(\mathrm{Dom}(\mathfrak{H}')-1, \ulcorner\text{Suppose } P(\mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H}')-1})\urcorner)\}$ and $\mathrm{AVS}(\mathfrak{H}^+) = \mathrm{AVS}(\mathfrak{H}^*) \cup \{(\mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')-1, \ulcorner\text{Suppose } P(\mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H}')-1})\urcorner)\}$. With v'), it then follows that:

$i \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^+))$
iff
$i \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^*)) \cup \{\mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')-1\}$
iff
$i \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H})) \cup \{\mathrm{Dom}(\mathfrak{H})\} \cup \{(\mathrm{Dom}(\mathfrak{H})+1+l \mid l \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1))\} \cup \{\mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')-1\}$
iff
$i \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H})) \cup \{\mathrm{Dom}(\mathfrak{H})\} \cup \{(\mathrm{Dom}(\mathfrak{H})+1+l \mid l \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}'))\}$

and thus that $\mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^+)) = \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H})) \cup \{\mathrm{Dom}(\mathfrak{H})\} \cup \{(\mathrm{Dom}(\mathfrak{H})+1+l \mid l \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}'))\}$ and hence that (v) holds.

(*CdIF, NIF*): Now, suppose $\mathfrak{H}' \in \mathrm{CdIF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1)$. According to Definition 3-2, there is then an $i \in \mathrm{Dom}(\mathfrak{H}')-1$ such that, with iv'), $P(\mathfrak{H}'_i) = P(\mathfrak{H}^*_{\mathrm{Dom}(\mathfrak{H})+1+i})$ and $i \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1))$ and $C(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1) = P(\mathfrak{H}^*_{\mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')-2}) = C(\mathfrak{H}^*)$ and there is no l such that $i < l \leq \mathrm{Dom}(\mathfrak{H}')-2$ and $l \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1))$ and $\mathfrak{H}' = \mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')-1 \cup \{(\mathrm{Dom}(\mathfrak{H}')-1, \ulcorner\text{Therefore } P(\mathfrak{H}_i) \rightarrow C(\mathfrak{H}^*)\urcorner)\}$. With vi'), we then have $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')-1, \ulcorner\text{Therefore } P(\mathfrak{H}_i) \rightarrow C(\mathfrak{H}^*)\urcorner)\}$. With iv') and v'), we then have $\mathrm{Dom}(\mathfrak{H})+1+i \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^*))$ and there is no l such that $\mathrm{Dom}(\mathfrak{H})+1+i < l \leq$

$\mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')-2$ and $l \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^*))$. Thus we then have $\mathfrak{H}^+ \in \mathrm{CdIF}(\mathfrak{H}^*) \subseteq \mathrm{RCS}\setminus\{\emptyset\}$. With Theorem 3-19-(iii), it then holds that $\mathrm{AVS}(\mathfrak{H}') = (\mathrm{AVS}(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1)\setminus\{(j, \mathfrak{H}'_j) \mid i \leq j < \mathrm{Dom}(\mathfrak{H}')-1\}) \cup \{(\mathrm{Dom}(\mathfrak{H}')-1, \mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H}')-1})\}$ and $\mathrm{AVS}(\mathfrak{H}^+) = (\mathrm{AVS}(\mathfrak{H}^*)\setminus\{(r, \mathfrak{H}^+_r) \mid \mathrm{Dom}(\mathfrak{H})+1+i \leq r < \mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')-1\}) \cup \{(\mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')-1, \mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H}')-1})\}$. With v'), it then follows that:

$k \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^+))$
iff
$k \in (\mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^*))\setminus\{r \mid \mathrm{Dom}(\mathfrak{H})+1+i \leq r < \mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')-1\}) \cup \{\mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')-1\}$
iff
$k \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^*))$ and $k < \mathrm{Dom}(\mathfrak{H})+1+i$ or $k = \mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')-1$
iff
$k \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H})) \cup \{\mathrm{Dom}(\mathfrak{H})\}$ or $k \in \{(\mathrm{Dom}(\mathfrak{H})+1+l \mid l \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1))\}$ and $k < \mathrm{Dom}(\mathfrak{H})+1+i$ or $k = \mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')-1$
iff
$k < \mathrm{Dom}(\mathfrak{H})+1$ and $k \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H})) \cup \{\mathrm{Dom}(\mathfrak{H})\}$ or $k \geq \mathrm{Dom}(\mathfrak{H})+1$ and $k\text{-}\mathrm{Dom}(\mathfrak{H})+1 \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1))$ and $k\text{-}\mathrm{Dom}(\mathfrak{H})+1 < i$ or $k\text{-}\mathrm{Dom}(\mathfrak{H})+1 = \mathrm{Dom}(\mathfrak{H}')-1$
iff
$k < \mathrm{Dom}(\mathfrak{H})+1$ and $k \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H})) \cup \{\mathrm{Dom}(\mathfrak{H})\}$ or $k \geq \mathrm{Dom}(\mathfrak{H})+1$ and $k\text{-}\mathrm{Dom}(\mathfrak{H})+1 \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1))\setminus\{j \mid i \leq j < \mathrm{Dom}(\mathfrak{H}')-1\}$ or $k\text{-}\mathrm{Dom}(\mathfrak{H})+1 = \mathrm{Dom}(\mathfrak{H}')-1$
iff
$k < \mathrm{Dom}(\mathfrak{H})+1$ and $k \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H})) \cup \{\mathrm{Dom}(\mathfrak{H})\}$ or $k \geq \mathrm{Dom}(\mathfrak{H})+1$ and $k\text{-}\mathrm{Dom}(\mathfrak{H})+1 \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}'))$

and thus that $\mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^+)) = \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H})) \cup \{\mathrm{Dom}(\mathfrak{H})\} \cup \{(\mathrm{Dom}(\mathfrak{H})+1+l \mid l \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}'))\}$ and thus v) holds. In the case that $\mathfrak{H}' \in \mathrm{NIF}(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1)$, one shows analogously that then also $\mathfrak{H}^+ \in \mathrm{NIF}(\mathfrak{H}^*) \subseteq \mathrm{RCS}\setminus\{\emptyset\}$ and (v) holds.

(*PEF*): Now, suppose $\mathfrak{H}' \in \mathrm{PEF}(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1)$. According to Definition 3-15, there are then $\beta \in \mathrm{PAR}$, $\xi \in \mathrm{VAR}$, $\Delta \in \mathrm{FORM}$, where $\mathrm{FV}(\Delta) \subseteq \{\xi\}$, and $i \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1))$ such that, with iv'), $\ulcorner\forall\xi\Delta\urcorner = \mathrm{P}(\mathfrak{H}'_i) = \mathrm{P}(\mathfrak{H}^*_{\mathrm{Dom}(\mathfrak{H})+1+i})$ and $[\beta, \xi, \Delta] = \mathrm{P}(\mathfrak{H}'_{i+1}) = \mathrm{P}(\mathfrak{H}^*_{\mathrm{Dom}(\mathfrak{H})+2+i})$, where $i \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1))$ and $\mathrm{C}(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1) = \mathrm{P}(\mathfrak{H}'_{\mathrm{Dom}(\mathfrak{H}')-2}) = \mathrm{P}(\mathfrak{H}^*_{\mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')-2}) = \mathrm{C}(\mathfrak{H}^*)$ and $\mathfrak{H}' = \mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1 \cup \{(\mathrm{Dom}(\mathfrak{H}')-1, \ulcorner\text{Therefore } \mathrm{C}(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1)\}$ and $\beta \notin \mathrm{STSF}(\{\Delta, \mathrm{C}(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1)\})$ and there is no $j \leq i$ such that $\beta \in \mathrm{ST}(\mathfrak{H}'_j)$, and there is no l such that $i+1 < l \leq \mathrm{Dom}(\mathfrak{H}')-2$ and $l \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1)$.

With iv') and v'), we then have: $\mathrm{Dom}(\mathfrak{H})+1+i \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^*))$ and $\mathrm{Dom}(\mathfrak{H})+2+i \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^*))$ and there is no l such that $\mathrm{Dom}(\mathfrak{H})+2+i < l \leq \mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')-2$

and $l \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^*))$. With vi'), we also have that $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')-1, \ulcorner\text{Therefore } C(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1)\urcorner\} = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')-1, \ulcorner\text{Therefore } C(\mathfrak{H}^*)\urcorner\}$.

We have that $\xi \in FV(\Delta)$ or $\xi \notin FV(\Delta)$. Suppose $\xi \in FV(\Delta)$. Then we have $\beta \in ST([\beta, \xi, \Delta]) \subseteq STSEQ(\mathfrak{H}')$. Since, according to the hypothesis, $PAR \cap STSEQ(\mathfrak{H}) \cap STSEQ(\mathfrak{H}') = \emptyset$, we thus have $\beta \notin STSEQ(\mathfrak{H})$. With i') to iv'), $\beta \notin STSF(\{\Delta, C(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1)\})$ and that there is no $j \leq i$ such that $\beta \in ST(\mathfrak{H}'_j)$, it then follows that $\beta \notin STSF(\{\Delta, C(\mathfrak{H}^*)\})$ and that there is no $j \leq \mathrm{Dom}(\mathfrak{H})+1+i$ such that $\beta \in ST(\mathfrak{H}^*_j)$. Thus we have $\mathfrak{H}^+ \in PEF(\mathfrak{H}^*)$. Now, suppose $\xi \notin FV(\Delta)$. Then we have $\beta \notin ST([\beta, \xi, \Delta])$. We have that there is a $\beta^* \in PAR\setminus(STSEQ(\mathfrak{H}) \cup STSEQ(\mathfrak{H}'))$. With Theorem 1-14-(ii), we then have $[\beta^*, \xi, \Delta] = \Delta = [\beta, \xi, \Delta] = P(\mathfrak{H}'_{i+1}) = P(\mathfrak{H}^*_{\mathrm{Dom}(\mathfrak{H})+2+i})$. Also, we have that $\beta^* \notin STSF(\{\Delta, C(\mathfrak{H}^*)\})$ and that there is no $j \leq \mathrm{Dom}(\mathfrak{H})+1+i$ such that $\beta^* \in ST(\mathfrak{H}^*_j)$. Thus we then have again $\mathfrak{H}^+ \in PEF(\mathfrak{H}^*)$. Hence we have in both cases that $\mathfrak{H}^+ \in PEF(\mathfrak{H}^*) \subseteq RCS\setminus\{\emptyset\}$. That (v) holds, then follows, with v') and Theorem 3-21-(iii), in the same way as it did for CdIF and NIF.

(*CdEF*, *CIF*, *CEF*, *BIF*, *BEF*, *DIF*, *DEF*, *NEF*, *UEF*, *PIF*, *IIF*, *IEF*): Now, suppose $\mathfrak{H}' \in CdEF(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1)$. According to Definition 3-3, there are then $\Delta, \Gamma \in CFORM$ such that $\Delta, \ulcorner\Delta \rightarrow \Gamma\urcorner \in AVP(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1)$ and $\mathfrak{H}' = \mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1 \cup \{(\mathrm{Dom}(\mathfrak{H}')-1, \ulcorner\text{Therefore } \Gamma\urcorner)\}$. With vi'), it then holds that $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')-1, \ulcorner\text{Therefore } \Gamma\urcorner)\}$. With $\Delta, \ulcorner\Delta \rightarrow \Gamma\urcorner \in AVP(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1)$, Definition 2-30 and iv'), we have that there are $i, j \in \mathrm{Dom}(AVS(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1))$ such that $\Delta = P(\mathfrak{H}'_i) = P(\mathfrak{H}^*_{\mathrm{Dom}(\mathfrak{H})+1+i})$ and $\ulcorner\Delta \rightarrow \Gamma\urcorner = P(\mathfrak{H}'_j) = P(\mathfrak{H}^*_{\mathrm{Dom}(\mathfrak{H})+1+j})$. With v'), we then have that $\mathrm{Dom}(\mathfrak{H})+1+i, \mathrm{Dom}(\mathfrak{H})+1+j \in \mathrm{Dom}(AVS(\mathfrak{H}^*))$. Hence we have $\mathfrak{H}^+ \in CdEF(\mathfrak{H}^*) \subseteq RCS\setminus\{\emptyset\}$.

We have $\mathfrak{H}' \in CdIF(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1) \cup NIF(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1) \cup PEF(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1)$ or $\mathfrak{H}' \notin CdIF(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1) \cup NIF(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1) \cup PEF(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1)$. In the first case, v) is shown in the same way as for the respective subcases. Now, suppose $\mathfrak{H}' \notin CdIF(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1) \cup NIF(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1) \cup PEF(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1)$. With ix'), it then holds that $\mathfrak{H}^+ \notin CdIF(\mathfrak{H}^*) \cup NIF(\mathfrak{H}^*) \cup PEF(\mathfrak{H}^*)$. With Theorem 3-25, it then holds that $AVS(\mathfrak{H}') = AVS(\mathfrak{H}'\restriction\mathrm{Dom}(\mathfrak{H}')-1) \cup \{(\mathrm{Dom}(\mathfrak{H})-1, \ulcorner\text{Therefore } \Gamma\urcorner)\}$ and $AVS(\mathfrak{H}^+) = AVS(\mathfrak{H}^*) \cup \{(\mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')-1, \ulcorner\text{Therefore } \Gamma\urcorner)\}$. With v'), it then follows in the same way as for AF that $AVS(\mathfrak{H}^+) = \mathrm{Dom}(AVS(\mathfrak{H})) \cup \{\mathrm{Dom}(\mathfrak{H})\} \cup \{(\mathrm{Dom}(\mathfrak{H})+1+l \mid l \in \mathrm{Dom}(AVS(\mathfrak{H}'))\}$ and thus that (v) holds.

If $\mathfrak{H}' \in \mathrm{CIF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1) \cup \mathrm{CEF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1) \cup \mathrm{BIF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1) \cup \mathrm{BEF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1) \cup \mathrm{DIF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1) \cup \mathrm{DEF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1) \cup \mathrm{NEF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1) \cup \mathrm{UEF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1) \cup \mathrm{PIF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1) \cup \mathrm{IIF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1) \cup \mathrm{IEF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1)$, one shows analogously that then also $\mathfrak{H}^{+} \in \mathrm{CIF}(\mathfrak{H}^*) \cup \mathrm{CEF}(\mathfrak{H}^*) \cup \mathrm{BIF}(\mathfrak{H}^*) \cup \mathrm{BEF}(\mathfrak{H}^*) \cup \mathrm{DIF}(\mathfrak{H}^*) \cup \mathrm{DEF}(\mathfrak{H}^*) \cup \mathrm{NEF}(\mathfrak{H}^*) \cup \mathrm{UEF}(\mathfrak{H}^*) \cup \mathrm{PIF}(\mathfrak{H}^*) \cup \mathrm{IIF}(\mathfrak{H}^*) \cup \mathrm{IEF}(\mathfrak{H}^*) \subseteq \mathrm{RCS}\backslash\{\emptyset\}$ and that v) holds in each case.

(*UIF*): Now, suppose $\mathfrak{H}' \in \mathrm{UIF}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1)$. According to Definition 3-12, there are then $\beta \in \mathrm{PAR}$, $\xi \in \mathrm{VAR}$ and $\Delta \in \mathrm{FORM}$, where $\mathrm{FV}(\Delta) \subseteq \{\xi\}$, such that $[\beta, \xi, \Delta] \in \mathrm{AVP}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1)$, $\beta \notin \mathrm{STSF}(\{\Delta\} \cup \mathrm{AVAP}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1))$ and $\mathfrak{H}' = \mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1 \cup \{(\mathrm{Dom}(\mathfrak{H}')\text{-}1, \ulcorner\text{Therefore } \bigwedge\xi\Delta\urcorner)\}$. With vi'), we then have $\mathfrak{H}^{+} = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H})+1+\mathrm{Dom}(\mathfrak{H}')\text{-}1, \ulcorner\text{Therefore } \bigwedge\xi\Delta\urcorner)\}$. With $[\beta, \xi, \Delta] \in \mathrm{AVP}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1)$, Definition 2-30 and iv'), we have that there is $i \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1))$ such that $[\beta, \xi, \Delta] = \mathrm{P}(\mathfrak{H}'_i) = \mathrm{P}(\mathfrak{H}^*_{\mathrm{Dom}(\mathfrak{H})+1+i})$. We then have with v') that $\mathrm{Dom}(\mathfrak{H})+1+i \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^*))$.

We have that $\xi \in \mathrm{FV}(\Delta)$ or $\xi \notin \mathrm{FV}(\Delta)$. Now, suppose $\xi \in \mathrm{FV}(\Delta)$. Then we have $\beta \in \mathrm{ST}([\beta, \xi, \Delta]) \subseteq \mathrm{STSEQ}(\mathfrak{H}')$. Since, according to the hypothesis, $\mathrm{PAR} \cap \mathrm{STSEQ}(\mathfrak{H}) \cap \mathrm{STSEQ}(\mathfrak{H}') = \emptyset$, we thus have $\beta \notin \mathrm{STSEQ}(\mathfrak{H})$. It thus follows with i') to v') and $\beta \notin \mathrm{STSF}(\{\Delta\} \cup \mathrm{AVAP}(\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}')\text{-}1))$, that $\beta \notin \mathrm{STSF}(\{\Delta\} \cup \mathrm{AVAP}(\mathfrak{H}^*))$. Thus we have $\mathfrak{H}^{+} \in \mathrm{UIF}(\mathfrak{H}^*)$. Now, suppose $\xi \notin \mathrm{FV}(\Delta)$. Then we have $\beta \notin \mathrm{ST}([\beta, \xi, \Delta])$. Now, let $\beta^* \in \mathrm{PAR}\backslash(\mathrm{STSEQ}(\mathfrak{H}) \cup \mathrm{STSEQ}(\mathfrak{H}'))$. With Theorem 1-14-(ii), we then have $[\beta^*, \xi, \Delta] = \Delta = [\beta, \xi, \Delta] = \mathrm{P}(\mathfrak{H}'_i) = \mathrm{P}(\mathfrak{H}^*_{\mathrm{Dom}(\mathfrak{H})+1+i})$. Also, we have that $\beta^* \notin \mathrm{STSF}(\{\Delta\} \cup \mathrm{AVAP}(\mathfrak{H}^*))$. Thus we have again $\mathfrak{H}^{+} \in \mathrm{UIF}(\mathfrak{H}^*)$. Hence we have that $\mathfrak{H}^{+} \in \mathrm{UIF}(\mathfrak{H}^*) \subseteq \mathrm{RCS}\backslash\{\emptyset\}$. v) follows in the same way as for CdEF … IEF. ■

Theorem 4-5. *Successful CE-extension*

If $\mathfrak{H} \in \mathrm{RCS}\backslash\{\emptyset\}$ and $\ulcorner A \wedge B\urcorner \in \mathrm{AVP}(\mathfrak{H})$, then there is an $\mathfrak{H}^* \in \mathrm{RCS}\backslash\{\emptyset\}$ such that

(i) $\mathrm{AVAP}(\mathfrak{H}^*) = \mathrm{AVAP}(\mathfrak{H})$,

(ii) $A, B \in \mathrm{AVP}(\mathfrak{H}^*)$, and

(iii) $\mathrm{C}(\mathfrak{H}^*) = B$.

Proof: Suppose $\mathfrak{H} \in \mathrm{RCS}\backslash\{\emptyset\}$ and $\ulcorner A \wedge B\urcorner \in \mathrm{AVP}(\mathfrak{H})$. Then there is an $i \in \mathrm{Dom}(\mathfrak{H})$ such that $\mathrm{P}(\mathfrak{H}_i) = \ulcorner A \wedge B\urcorner$ and $(i, \mathfrak{H}_i) \in \mathrm{AVS}(\mathfrak{H})$. Let the following sentence sequences be defined, where $\alpha \in \mathrm{CONST}\backslash\mathrm{STSEQ}(\mathfrak{H})$:

$$\begin{array}{lllll} \mathfrak{H}^1 & = \mathfrak{H} & \cup & \{(\mathrm{Dom}(\mathfrak{H}), & \ulcorner\text{Therefore } \alpha = \alpha\urcorner)\} \\ \mathfrak{H}^2 & = \mathfrak{H}^1 & \cup & \{(\mathrm{Dom}(\mathfrak{H}^1), & \ulcorner\text{Therefore A}\urcorner)\} \\ \mathfrak{H}^3 & = \mathfrak{H}^2 & \cup & \{(\mathrm{Dom}(\mathfrak{H}^2), & \ulcorner\text{Therefore } \alpha = \alpha\urcorner)\} \\ \mathfrak{H}^4 & = \mathfrak{H}^3 & \cup & \{(\mathrm{Dom}(\mathfrak{H}^3), & \ulcorner\text{Therefore B}\urcorner)\}. \end{array}$$

With Theorem 1-10 and Theorem 1-11, we have that $C(\mathfrak{H}^1)$ and $C(\mathfrak{H}^3)$ are neither negations nor conditionals, and neither identical to $C(\mathfrak{H})$ nor to $C(\mathfrak{H}^2)$, because otherwise $\alpha \in \mathrm{STSEQ}(\mathfrak{H})$ or $\alpha \in \mathrm{ST}(\mathfrak{H}_i) \subseteq \mathrm{STSEQ}(\mathfrak{H})$. Therefore $\mathfrak{H}^1 \notin \mathrm{CdIF}(\mathfrak{H}) \cup \mathrm{NIF}(\mathfrak{H}) \cup \mathrm{PEF}(\mathfrak{H})$ and $\mathfrak{H}^3 \notin \mathrm{CdIF}(\mathfrak{H}^2) \cup \mathrm{NIF}(\mathfrak{H}^2) \cup \mathrm{PEF}(\mathfrak{H}^2)$. If $\ulcorner\alpha = \alpha\urcorner \in \mathrm{SF}(A) \cup \mathrm{SF}(B)$, then we would have $\alpha \in \mathrm{ST}(\mathfrak{H}_i) \subseteq \mathrm{STSEQ}(\mathfrak{H})$. Therefore we have $\ulcorner\alpha = \alpha\urcorner \notin \mathrm{SF}(A)$ and $\ulcorner\alpha = \alpha\urcorner \notin \mathrm{SF}(B)$ and thus $\mathfrak{H}^2 \notin \mathrm{CdIF}(\mathfrak{H}^1) \cup \mathrm{PEF}(\mathfrak{H}^1)$ and $\mathfrak{H}^4 \notin \mathrm{CdIF}(\mathfrak{H}^3) \cup \mathrm{PEF}(\mathfrak{H}^3)$. Suppose for contradiction that $\mathfrak{H}^2 \in \mathrm{NIF}(\mathfrak{H}^1)$ or $\mathfrak{H}^4 \in \mathrm{NIF}(\mathfrak{H}^2)$. Then there would be a $j \in \mathrm{Dom}(\mathfrak{H}^3)$ such that $P(\mathfrak{H}_j) = \ulcorner\neg\alpha = \alpha\urcorner$. With Theorem 1-10 and Theorem 1-11, we have $j \notin \{\mathrm{Dom}(\mathfrak{H}^3)\text{-}1, \mathrm{Dom}(\mathfrak{H}^3)\text{-}3\}$. Because of $\ulcorner\alpha = \alpha\urcorner \notin \mathrm{SF}(A)$, we have $j \neq \mathrm{Dom}(\mathfrak{H}^3)\text{-}2$. Therefore we would have $j \in \mathrm{Dom}(\mathfrak{H}^3)\setminus\{\mathrm{Dom}(\mathfrak{H}^3)\text{-}1, \mathrm{Dom}(\mathfrak{H}^3)\text{-}2, \mathrm{Dom}(\mathfrak{H}^3)\text{-}3\} = \mathrm{Dom}(\mathfrak{H})$. With $\alpha \in \mathrm{ST}(\mathfrak{H}^3{}_j) = \mathrm{ST}(\mathfrak{H}_j)$, we would then have $\alpha \in \mathrm{STSEQ}(\mathfrak{H})$. Contradiction! Therefore $\mathfrak{H}^2 \notin \mathrm{NIF}(\mathfrak{H}^1)$ and $\mathfrak{H}^4 \notin \mathrm{NIF}(\mathfrak{H}^3)$.

On the other hand, we have, *first*, with Definition 3-16, that $\mathfrak{H}^1 \in \mathrm{IIF}(\mathfrak{H})$, thus $\mathfrak{H}^1 \in \mathrm{RCS}\setminus\{\emptyset\}$, and with Theorem 3-25, $\mathrm{AVS}(\mathfrak{H}^1) = \mathrm{AVS}(\mathfrak{H}) \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \alpha = \alpha\urcorner)\}$. Thus we have $\mathrm{AVAS}(\mathfrak{H}^1) = \mathrm{AVAS}(\mathfrak{H})$ and $\ulcorner A \wedge B\urcorner \in \mathrm{AVP}(\mathfrak{H}) \subseteq \mathrm{AVP}(\mathfrak{H}^1)$. Therefore we have, *second*, with Definition 3-5, that $\mathfrak{H}^2 \in \mathrm{CEF}(\mathfrak{H}^1) \subseteq \mathrm{RCS}\setminus\{\emptyset\}$ and, with Theorem 3-25, $\mathrm{AVS}(\mathfrak{H}^2) = \mathrm{AVS}(\mathfrak{H}^1) \cup \{(\mathrm{Dom}(\mathfrak{H}^1), \ulcorner\text{Therefore A}\urcorner)\}$. Thus we have $\mathrm{AVAS}(\mathfrak{H}^2) = \mathrm{AVAS}(\mathfrak{H}^1)$, $\ulcorner A \wedge B\urcorner \in \mathrm{AVP}(\mathfrak{H}^1) \subseteq \mathrm{AVP}(\mathfrak{H}^2)$ and $A \in \mathrm{AVP}(\mathfrak{H}^2)$. *Third*, with Definition 3-16, we have $\mathfrak{H}^3 \in \mathrm{IIF}(\mathfrak{H}^2)$, $\mathfrak{H}^3 \in \mathrm{RCS}\setminus\{\emptyset\}$ and, with Theorem 3-25, $\mathrm{AVS}(\mathfrak{H}^3) = \mathrm{AVS}(\mathfrak{H}^2) \cup \{(\mathrm{Dom}(\mathfrak{H}^2), \ulcorner\text{Therefore } \alpha = \alpha\urcorner)\}$. Thus we have $\mathrm{AVAS}(\mathfrak{H}^3) = \mathrm{AVAS}(\mathfrak{H}^2)$ and A, $\ulcorner A \wedge B\urcorner \in \mathrm{AVP}(\mathfrak{H}^2) \subseteq \mathrm{AVP}(\mathfrak{H}^3)$. *Fourth*, with Definition 3-5, we then have $\mathfrak{H}^4 \in \mathrm{CEF}(\mathfrak{H}^3) \subseteq \mathrm{RCS}\setminus\{\emptyset\}$ and, with Theorem 3-25, $\mathrm{AVS}(\mathfrak{H}^4) = \mathrm{AVS}(\mathfrak{H}^3) \cup \{(\mathrm{Dom}(\mathfrak{H}^3), \ulcorner\text{Therefore B}\urcorner)\}$. Thus we have $\mathrm{AVAS}(\mathfrak{H}^4) = \mathrm{AVAS}(\mathfrak{H}^3)$, $A \in \mathrm{AVP}(\mathfrak{H}^3) \subseteq \mathrm{AVP}(\mathfrak{H}^4)$ and $B \in \mathrm{AVP}(\mathfrak{H}^4)$. Hence we have $\mathfrak{H}^4 \in \mathrm{RCS}\setminus\{\emptyset\}$, $\mathrm{AVAP}(\mathfrak{H}^4) = \mathrm{AVAP}(\mathfrak{H}^3) = \mathrm{AVAP}(\mathfrak{H}^2) = \mathrm{AVAP}(\mathfrak{H}^1) = \mathrm{AVAP}(\mathfrak{H})$, A, $B \in \mathrm{AVP}(\mathfrak{H}^4)$ and $C(\mathfrak{H}^4) = B$. ■

Theorem 4-6. *Available propositions as conclusions*
If $\mathfrak{H} \in \text{RCS}\backslash\{\emptyset\}$ and $A \in \text{AVP}(\mathfrak{H})$, then there is an $\mathfrak{H}^* \in \text{RCS}\backslash\{\emptyset\}$ such that

(i) $\text{AVAP}(\mathfrak{H}^*) = \text{AVAP}(\mathfrak{H})$,

(ii) $\text{AVP}(\mathfrak{H}) \subseteq \text{AVP}(\mathfrak{H}^*)$, and

(iii) $\text{C}(\mathfrak{H}^*) = A$.

Proof: Suppose $\mathfrak{H} \in \text{RCS}\backslash\{\emptyset\}$ and $A \in \text{AVP}(\mathfrak{H})$. Then there is an $i \in \text{Dom}(\mathfrak{H})$ such that $\text{P}(\mathfrak{H}_i) = A$ and $(i, \mathfrak{H}_i) \in \text{AVS}(\mathfrak{H})$. Let the following sentence sequences be defined, where $\alpha \in \text{CONST}\backslash\text{STSEQ}(\mathfrak{H})$:

$$\begin{aligned}
\mathfrak{H}^1 &= \mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \alpha = \alpha\urcorner)\}\\
\mathfrak{H}^2 &= \mathfrak{H}^1 \cup \{(\text{Dom}(\mathfrak{H}^1), \ulcorner\text{Therefore } A \wedge A\urcorner)\}\\
\mathfrak{H}^3 &= \mathfrak{H}^2 \cup \{(\text{Dom}(\mathfrak{H}^2), \ulcorner\text{Therefore } \alpha = \alpha\urcorner)\}\\
\mathfrak{H}^4 &= \mathfrak{H}^3 \cup \{(\text{Dom}(\mathfrak{H}^3), \ulcorner\text{Therefore } A\urcorner)\}.
\end{aligned}$$

With Theorem 1-10 and Theorem 1-11, $\text{C}(\mathfrak{H}^1)$, $\text{C}(\mathfrak{H}^2)$ and $\text{C}(\mathfrak{H}^3)$ are neither negations nor conditionals. Moreover, $\text{C}(\mathfrak{H}^1)$ and $\text{C}(\mathfrak{H}^3)$ are neither identical to $\text{C}(\mathfrak{H})$ nor to $\text{C}(\mathfrak{H}^2)$. With Theorem 1-10-(vi) $\text{C}(\mathfrak{H})$ is not identical to $\text{C}(\mathfrak{H}^1)$. Therefore $\mathfrak{H}^1 \notin \text{CdIF}(\mathfrak{H}) \cup \text{NIF}(\mathfrak{H}) \cup \text{PEF}(\mathfrak{H})$, $\mathfrak{H}^2 \notin \text{CdIF}(\mathfrak{H}^1) \cup \text{NIF}(\mathfrak{H}^1) \cup \text{PEF}(\mathfrak{H}^1)$, and $\mathfrak{H}^3 \notin \text{CdIF}(\mathfrak{H}^2) \cup \text{NIF}(\mathfrak{H}^2) \cup \text{PEF}(\mathfrak{H}^2)$. If $\ulcorner\alpha = \alpha\urcorner \in \text{SF}(A)$, then we would have $\alpha \in \text{ST}(\mathfrak{H}_i) \subseteq \text{STSEQ}(\mathfrak{H})$. Therefore we have $\ulcorner\alpha = \alpha\urcorner \notin \text{SF}(A)$ and thus $\mathfrak{H}^4 \notin \text{CdIF}(\mathfrak{H}^3) \cup \text{PEF}(\mathfrak{H}^3)$. Now, suppose for contradiction that $\mathfrak{H}^4 \in \text{NIF}(\mathfrak{H}^3)$. Then there would be a $j \in \text{Dom}(\mathfrak{H}^3)$ such that $\text{P}(\mathfrak{H}_j) = \ulcorner\neg\alpha = \alpha\urcorner$. With Theorem 1-10 and Theorem 1-11, we have $j \notin \{\text{Dom}(\mathfrak{H}^3)\text{-}1, \text{Dom}(\mathfrak{H}^3)\text{-}2, \text{Dom}(\mathfrak{H}^3)\text{-}3\}$. Therefore $j \in \text{Dom}(\mathfrak{H}^3)\backslash\{\text{Dom}(\mathfrak{H}^3)\text{-}1, \text{Dom}(\mathfrak{H}^3)\text{-}2, \text{Dom}(\mathfrak{H}^3)\text{-}3\} = \text{Dom}(\mathfrak{H})$. With $\alpha \in \text{ST}(\mathfrak{H}^3{}_j) = \text{ST}(\mathfrak{H}_j)$, we would then have $\alpha \in \text{STSEQ}(\mathfrak{H})$. Contradiction! Therefore $\mathfrak{H}^4 \notin \text{NIF}(\mathfrak{H}^3)$.

On the other hand, we have, *first*, with Definition 3-16, that $\mathfrak{H}^1 \in \text{IIF}(\mathfrak{H})$, thus $\mathfrak{H}^1 \in \text{RCS}\backslash\{\emptyset\}$ and, with Theorem 3-25, $\text{AVS}(\mathfrak{H}^1) = \text{AVS}(\mathfrak{H}) \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \alpha = \alpha\urcorner)\}$. Thus we have $\text{AVAS}(\mathfrak{H}^1) = \text{AVAS}(\mathfrak{H})$ and $A \in \text{AVP}(\mathfrak{H}) \subseteq \text{AVP}(\mathfrak{H}^1)$. Therefore we have, *second*, with Definition 3-4, $\mathfrak{H}^2 \in \text{CIF}(\mathfrak{H}^1) \subseteq \text{RCS}\backslash\{\emptyset\}$ and, with Theorem 3-25, $\text{AVS}(\mathfrak{H}^2) = \text{AVS}(\mathfrak{H}^1) \cup \{(\text{Dom}(\mathfrak{H}^1), \ulcorner\text{Therefore } A \wedge A\urcorner)\}$. Thus we have $\text{AVAS}(\mathfrak{H}^2) = \text{AVAS}(\mathfrak{H}^1)$, $\text{AVP}(\mathfrak{H}^1) \subseteq \text{AVP}(\mathfrak{H}^2)$ and $\ulcorner A \wedge A\urcorner \in \text{AVP}(\mathfrak{H}^2)$. Then we have, *third*, with Definition 3-16, $\mathfrak{H}^3 \in \text{IIF}(\mathfrak{H}^2) \subseteq \text{RCS}\backslash\{\emptyset\}$ and, with Theorem 3-25, $\text{AVS}(\mathfrak{H}^3) = \text{AVS}(\mathfrak{H}^2) \cup \{(\text{Dom}(\mathfrak{H}^2), \ulcorner\text{Therefore } \alpha = \alpha\urcorner)\}$. Thus we have $\text{AVAS}(\mathfrak{H}^3) = \text{AVAS}(\mathfrak{H}^2)$ and $\ulcorner A \wedge A\urcorner \in \text{AVP}(\mathfrak{H}^2) \subseteq \text{AVP}(\mathfrak{H}^3)$. *Fourth*, with Definition 3-5, we thus have $\mathfrak{H}^4 \in \text{CEF}(\mathfrak{H}^3) \subseteq$

$\text{RCS}\backslash\{\emptyset\}$ and, with Theorem 3-25, $\text{AVS}(\mathfrak{H}^4) = \text{AVS}(\mathfrak{H}^3) \cup \{(\text{Dom}(\mathfrak{H}^3), \ulcorner\text{Therefore A}\urcorner)\}$. Thus we have $\text{AVAS}(\mathfrak{H}^4) = \text{AVAS}(\mathfrak{H}^3)$ and $\text{AVP}(\mathfrak{H}^3) \subseteq \text{AVP}(\mathfrak{H}^4)$. Hence we have $\mathfrak{H}^4 \in \text{RCS}\backslash\{\emptyset\}$, $\text{AVAP}(\mathfrak{H}^4) = \text{AVAP}(\mathfrak{H}^3) = \text{AVAP}(\mathfrak{H}^2) = \text{AVAP}(\mathfrak{H}^1) = \text{AVAP}(\mathfrak{H})$, $\text{AVP}(\mathfrak{H}) \subseteq \text{AVP}(\mathfrak{H}^4)$ and $\text{C}(\mathfrak{H}^4) = \text{A}$. ■

Theorem 4-7. *Eliminability of an assumption of* $\ulcorner\alpha = \alpha\urcorner$

If $\mathfrak{H} \in \text{RCS}\backslash\{\emptyset\}$, $\alpha \in \text{CONST}$ and $\text{A}, \text{B} \in \text{AVP}(\mathfrak{H})$, then there is a $\mathfrak{H}^* \in \text{RCS}\backslash\{\emptyset\}$ such that

(i) $\text{AVAP}(\mathfrak{H}^*) \subseteq \text{AVAP}(\mathfrak{H})\backslash\{\ulcorner\alpha = \alpha\urcorner\}$,

(ii) $\text{A}, \text{B} \in \text{AVP}(\mathfrak{H}^*)$, and

(iii) $\text{C}(\mathfrak{H}^*) = \text{B}$.

Proof: Let $\mathfrak{H} \in \text{RCS}\backslash\{\emptyset\}$, $\alpha \in \text{CONST}$ and $\text{A}, \text{B} \in \text{AVP}(\mathfrak{H})$. Suppose $\ulcorner\alpha = \alpha\urcorner \notin \text{AVAP}(\mathfrak{H})$. Then we have $\text{AVAP}(\mathfrak{H}) \subseteq \text{AVAP}(\mathfrak{H})\backslash\{\ulcorner\alpha = \alpha\urcorner\}$. With Theorem 4-6, there is then an $\mathfrak{H}^* \in \text{RCS}\backslash\{\emptyset\}$ such that $\text{AVAP}(\mathfrak{H}^*) = \text{AVAP}(\mathfrak{H}) \subseteq \text{AVAP}(\mathfrak{H})\backslash\{\ulcorner\alpha = \alpha\urcorner\}$, $\text{A}, \text{B} \in \text{AVP}(\mathfrak{H}) \subseteq \text{AVP}(\mathfrak{H}^*)$ and $\text{C}(\mathfrak{H}^*) = \text{B}$.

Now, suppose $\ulcorner\alpha = \alpha\urcorner \in \text{AVAP}(\mathfrak{H})$. Then we have $\mathfrak{H}^1 = \mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore A} \wedge \text{B}\urcorner)\} \in \text{CIF}(\mathfrak{H})$. Then we have $\mathfrak{H}^1 \in \text{RCS}\backslash\{\emptyset\}$ and $\ulcorner\text{A} \wedge \text{B}\urcorner \in \text{AVP}(\mathfrak{H}^1)$ and, with Theorem 3-26-(v), $\text{AVAP}(\mathfrak{H}^1) \subseteq \text{AVAP}(\mathfrak{H})$. According to Theorem 4-2, there is then an $\mathfrak{H}^+ \in \text{RCS}\backslash\{\emptyset\}$ such that $\text{AVAP}(\mathfrak{H}^+) \subseteq \text{AVAP}(\mathfrak{H}^1) \subseteq \text{AVAP}(\mathfrak{H})$, $\text{C}(\mathfrak{H}^+) = \text{C}(\mathfrak{H}^1) = \ulcorner\text{A} \wedge \text{B}\urcorner$ and for all $k \in \text{Dom}(\text{AVAS}(\mathfrak{H}^+))$: If $\text{P}(\mathfrak{H}^+_k) = \ulcorner\alpha = \alpha\urcorner$, then $k = \max(\text{Dom}(\text{AVAS}(\mathfrak{H}^+)))$. Then we have $\ulcorner\alpha = \alpha\urcorner \in \text{AVAP}(\mathfrak{H}^+)$ or $\ulcorner\alpha = \alpha\urcorner \notin \text{AVAP}(\mathfrak{H}^+)$.

First case: Suppose $\ulcorner\alpha = \alpha\urcorner \in \text{AVAP}(\mathfrak{H}^+)$. Then we have $\text{P}(\mathfrak{H}^+_{\max(\text{Dom}(\text{AVAS}(\mathfrak{H}^+)))}) = \ulcorner\alpha = \alpha\urcorner$ and for all $k \in \text{Dom}(\text{AVAS}(\mathfrak{H}^+))$: If $\text{P}(\mathfrak{H}^+_k) = \ulcorner\alpha = \alpha\urcorner$, then $k = \max(\text{Dom}(\text{AVAS}(\mathfrak{H}^+)))$. Now, let the following sentence sequences be defined:

$$\mathfrak{H}^2 = \mathfrak{H}^+ \cup \{(\text{Dom}(\mathfrak{H}^+), \ulcorner\text{Therefore } \alpha = \alpha \rightarrow (\text{A} \wedge \text{B})\urcorner)\}$$
$$\mathfrak{H}^3 = \mathfrak{H}^2 \cup \{(\text{Dom}(\mathfrak{H}^2), \ulcorner\text{Therefore } \alpha = \alpha\urcorner)\}$$
$$\mathfrak{H}^4 = \mathfrak{H}^3 \cup \{(\text{Dom}(\mathfrak{H}^3), \ulcorner\text{Therefore A} \wedge \text{B}\urcorner)\}.$$

According to Definition 3-2, we have $\mathfrak{H}^2 \in \text{CdIF}(\mathfrak{H}^+)$, thus $\mathfrak{H}^2 \in \text{RCS}\backslash\{\emptyset\}$ and, with Theorem 3-19-(ix), $\text{AVAP}(\mathfrak{H}^2) \subseteq \text{AVAP}(\mathfrak{H}^+) \subseteq \text{AVAP}(\mathfrak{H})$. With Theorem 3-22, we have that $\ulcorner\alpha = \alpha\urcorner \notin \text{AVAP}(\mathfrak{H}^2)$ and thus $\text{AVAP}(\mathfrak{H}^2) \subseteq \text{AVAP}(\mathfrak{H})\backslash\{\ulcorner\alpha = \alpha\urcorner\}$. We also have $\ulcorner\alpha = \alpha \rightarrow (\text{A} \wedge \text{B})\urcorner \in \text{AVP}(\mathfrak{H}^2)$.

With Theorem 1-10 and Theorem 1-11, $\text{C}(\mathfrak{H}^3)$ and $\text{C}(\mathfrak{H}^4)$ are neither negations nor conditionals and also $\text{C}(\mathfrak{H}^3)$ is not identical to $\text{C}(\mathfrak{H}^2)$ and $\text{C}(\mathfrak{H}^4)$ is not identical to $\text{C}(\mathfrak{H}^3)$.

Therefore we have $\mathfrak{H}^3 \notin \text{CdIF}(\mathfrak{H}^2) \cup \text{NIF}(\mathfrak{H}^2) \cup \text{PEF}(\mathfrak{H}^2)$ and $\mathfrak{H}^4 \notin \text{CdIF}(\mathfrak{H}^3) \cup \text{NIF}(\mathfrak{H}^3) \cup \text{PEF}(\mathfrak{H}^3)$. According to Definition 3-16, we have $\mathfrak{H}^3 \in \text{IIF}(\mathfrak{H}^1) \subseteq \text{RCS}\backslash\{\emptyset\}$ and, with Theorem 3-25, $\text{AVS}(\mathfrak{H}^3) = \text{AVS}(\mathfrak{H}^2) \cup \{(\text{Dom}(\mathfrak{H}^2), \ulcorner\text{Therefore } \alpha = \alpha\urcorner)\}$. Thus we have $\text{AVAS}(\mathfrak{H}^3) = \text{AVAS}(\mathfrak{H}^1)$, $\ulcorner\alpha = \alpha \rightarrow (\text{A} \wedge \text{B})\urcorner \in \text{AVP}(\mathfrak{H}^2) \subseteq \text{AVP}(\mathfrak{H}^3)$ and $\ulcorner\alpha = \alpha\urcorner \in \text{AVP}(\mathfrak{H}^3)$. According to Definition 3-3, we therefore have $\mathfrak{H}^4 \in \text{CdEF}(\mathfrak{H}^3) \subseteq \text{RCS}\backslash\{\emptyset\}$ and, with Theorem 3-25, $\text{AVS}(\mathfrak{H}^4) = \text{AVS}(\mathfrak{H}^3) \cup \{(\text{Dom}(\mathfrak{H}^3), \ulcorner\text{Therefore A} \wedge \text{B}\urcorner)\}$. Thus we have $\text{AVAS}(\mathfrak{H}^4) = \text{AVAS}(\mathfrak{H}^3)$. Thus we have $\mathfrak{H}^4 \in \text{RCS}\backslash\{\emptyset\}$, $\text{AVAP}(\mathfrak{H}^4) = \text{AVAP}(\mathfrak{H}^3) = \text{AVAP}(\mathfrak{H}^2) \subseteq \text{AVAP}(\mathfrak{H})\backslash\{\ulcorner\alpha = \alpha\urcorner\}$ and $\ulcorner\text{A} \wedge \text{B}\urcorner \in \text{AVP}(\mathfrak{H}^4)$. With Theorem 4-5, there is then an $\mathfrak{H}^* \in \text{RCS}\backslash\{\emptyset\}$ such that $\text{AVAP}(\mathfrak{H}^*) = \text{AVAP}(\mathfrak{H}^4) \subseteq \text{AVAP}(\mathfrak{H})\backslash\{\ulcorner\alpha = \alpha\urcorner\}$ and $\text{A, B} \in \text{AVP}(\mathfrak{H}^*)$ and $\text{C}(\mathfrak{H}^*) = \text{B}$.

Second case: Suppose $\ulcorner\alpha = \alpha\urcorner \notin \text{AVAP}(\mathfrak{H}^+)$ and thus $\text{AVAP}(\mathfrak{H}^+) \subseteq \text{AVAP}(\mathfrak{H})\backslash\{\ulcorner\alpha = \alpha\urcorner\}$. We have $\ulcorner\text{A} \wedge \text{B}\urcorner = \text{C}(\mathfrak{H}^+) \in \text{AVP}(\mathfrak{H}^+)$. With Theorem 4-5 there is then an $\mathfrak{H}^* \in \text{RCS}\backslash\{\emptyset\}$ such that $\text{AVAP}(\mathfrak{H}^*) = \text{AVAP}(\mathfrak{H}^+) \subseteq \text{AVAP}(\mathfrak{H})\backslash\{\ulcorner\alpha = \alpha\urcorner\}$ and $\text{A, B} \in \text{AVP}(\mathfrak{H}^*)$ and $\text{C}(\mathfrak{H}^*) = \text{B}$. ■

Theorem 4-8. *Substitution of a new parameter for a parameter is RCS-preserving*
If $\mathfrak{H} \in \text{RCS}$, and $\beta^* \in \text{PAR}\backslash\text{STSEQ}(\mathfrak{H})$ and $\beta \in \text{PAR}\backslash\{\beta^*\}$, then $[\beta^*, \beta, \mathfrak{H}] \in \text{RCS}$ and $\text{Dom}(\text{AVS}([\beta^*, \beta, \mathfrak{H}])) = \text{Dom}(\text{AVS}(\mathfrak{H}))$.

Proof: By induction on $\text{Dom}(\mathfrak{H})$. Suppose $\mathfrak{H} \in \text{RCS}$, and $\beta^* \in \text{PAR}\backslash\text{STSEQ}(\mathfrak{H})$ and $\beta \in \text{PAR}\backslash\{\beta^*\}$ and that the statement holds for all $k < \text{Dom}(\mathfrak{H})$. Suppose $\text{Dom}(\mathfrak{H}) = 0$. Then we have $\mathfrak{H} = \emptyset = [\beta^*, \beta, \mathfrak{H}]$ and thus $[\beta^*, \beta, \mathfrak{H}] \in \text{RCS}$ and $\text{Dom}(\text{AVS}([\beta^*, \beta, \mathfrak{H}])) = \emptyset = \text{Dom}(\text{AVS}(\mathfrak{H}))$. Now, suppose $0 < \text{Dom}(\mathfrak{H})$. Then we have $\mathfrak{H} \in \text{RCS}\backslash\{\emptyset\}$. With Theorem 3-6, we then have $\mathfrak{H} \in \text{RCE}(\mathfrak{H}\restriction\text{Dom}(\mathfrak{H})\text{-}1)$. According to the I.H., we then have:

a) $\mathfrak{H}^* = [\beta^*, \beta, \mathfrak{H}\restriction\text{Dom}(\mathfrak{H})\text{-}1] \in \text{RCS}$ and $\text{Dom}(\text{AVS}(\mathfrak{H}^*)) = \text{Dom}(\text{AVS}(\mathfrak{H}\restriction\text{Dom}(\mathfrak{H})\text{-}1))$.

With $\mathfrak{H} \in \text{RCE}(\mathfrak{H}\restriction\text{Dom}(\mathfrak{H})\text{-}1)$ and Definition 3-18, we have that $\mathfrak{H} \in \text{AF}(\mathfrak{H}\restriction\text{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \text{CdIF}(\mathfrak{H}\restriction\text{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \text{CdEF}(\mathfrak{H}\restriction\text{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \text{CIF}(\mathfrak{H}\restriction\text{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \text{CEF}(\mathfrak{H}\restriction\text{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \text{BIF}(\mathfrak{H}\restriction\text{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \text{BEF}(\mathfrak{H}\restriction\text{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \text{DIF}(\mathfrak{H}\restriction\text{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \text{DEF}(\mathfrak{H}\restriction\text{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \text{NIF}(\mathfrak{H}\restriction\text{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \text{NEF}(\mathfrak{H}\restriction\text{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \text{UIF}(\mathfrak{H}\restriction\text{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \text{UEF}(\mathfrak{H}\restriction\text{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \text{PIF}(\mathfrak{H}\restriction\text{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \text{PEF}(\mathfrak{H}\restriction\text{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \text{IIF}(\mathfrak{H}\restriction\text{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \text{IEF}(\mathfrak{H}\restriction\text{Dom}(\mathfrak{H})\text{-}1)$.

Since operators are not affected by substitution, we first have:

b) For all $i \in \mathrm{Dom}(\mathfrak{H})$-1: $\mathrm{P}(\mathfrak{H}^*_i) = [\beta^*, \beta, \mathrm{P}(\mathfrak{H}_i)]$ and $\mathfrak{H}^*_i = \ulcorner\Xi\ [\beta^*, \beta, \mathrm{P}(\mathfrak{H}_i)]\urcorner$, where $\mathfrak{H}_i = \ulcorner\Xi\ \mathrm{P}(\mathfrak{H}_i)\urcorner$ for a $\Xi \in$ PERF.

With $\beta^* \in \mathrm{PAR}\backslash\mathrm{STSEQ}(\mathfrak{H})$ and $\beta \in \mathrm{PAR}\backslash\{\beta^*\}$, we have:

c) For every $i \in \mathrm{Dom}(\mathfrak{H})$: $\beta^* \notin \mathrm{ST}(\mathrm{P}(\mathfrak{H}_i))$ and $\beta \notin \mathrm{ST}([\beta^*, \beta, \mathrm{P}(\mathfrak{H}_i)])$,

if not, we would have $\beta^* \in \mathrm{STSEQ}(\mathfrak{H})$ or $\beta = \beta^*$, which both contradict the hypothesis. Now, let:

d) $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H})\text{-}1, [\beta^*, \beta, \mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-1}])\}$.

Then we have that $\mathfrak{H}^+ = [\beta^*, \beta, \mathfrak{H}]$. Now we will show that in each of the cases AF … IEF we have that $\mathfrak{H}^+ \in$ RCS and $\mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^+)) = \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}))$, with which we prove that the statement holds for $[\beta^*, \beta, \mathfrak{H}]$.

To simplify the treatment of CdEF, CIF, CEF, BIF, BEF, DIF, DEF, NEF, UIF, UEF, PIF, IIF and IEF, we will now show in preparation of the main part of the proof that

e) If $\mathfrak{H}^+ \in \mathrm{CdIF}(\mathfrak{H}^*) \cup \mathrm{NIF}(\mathfrak{H}^*) \cup \mathrm{PEF}(\mathfrak{H}^*)$, then $\mathfrak{H} \in \mathrm{CdIF}(\mathfrak{H}{\restriction}\mathrm{Dom}(\mathfrak{H})\text{-}1) \cup \mathrm{NIF}(\mathfrak{H}{\restriction}\mathrm{Dom}(\mathfrak{H}\text{-}1) \cup \mathrm{PEF}(\mathfrak{H}{\restriction}\mathrm{Dom}(\mathfrak{H}\text{-}1)$.

Preparatory part: Suppose $\mathfrak{H}^+ \in \mathrm{CdIF}(\mathfrak{H}^*)$. According to Definition 3-2, there is then an $i \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^*))$ such that, with b) and d), $\mathrm{P}(\mathfrak{H}^*_i) = [\beta^*, \beta, \mathrm{P}(\mathfrak{H}_i)]$ and $\mathrm{C}(\mathfrak{H}^*) = [\beta^*, \beta, \mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-2})]$ and there is no l such that $i < l \leq \mathrm{Dom}(\mathfrak{H})\text{-}2$ and $l \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^*))$, and $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H})\text{-}1, \ulcorner\text{Therefore } \mathrm{P}(\mathfrak{H}^*_i) \rightarrow \mathrm{P}(\mathfrak{H}^*_{\mathrm{Dom}(\mathfrak{H})-2})\urcorner)\} = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H})\text{-}1, \ulcorner\text{Therefore } [\beta^*, \beta, \mathrm{P}(\mathfrak{H}_i)] \rightarrow [\beta^*, \beta, \mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-2})]\urcorner)\}$. With d), we have $\ulcorner\text{Therefore } [\beta^*, \beta, \mathrm{P}(\mathfrak{H}_i)] \rightarrow [\beta^*, \beta, \mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-2})]\urcorner = [\beta^*, \beta, \ulcorner\text{Therefore } \mathrm{P}(\mathfrak{H}_i) \rightarrow \mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-2})\urcorner] = [\beta^*, \beta, \mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-1}]$. With Theorem 1-21, we then have $\ulcorner\text{Therefore } \mathrm{P}(\mathfrak{H}_i) \rightarrow \mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-2})\urcorner = \mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-1}$ and thus $\mathfrak{H} = \mathfrak{H}{\restriction}\mathrm{Dom}(\mathfrak{H})\text{-}1 \cup \{(\mathrm{Dom}(\mathfrak{H})\text{-}1, \ulcorner\text{Therefore } \mathrm{P}(\mathfrak{H}_i) \rightarrow \mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-2})\urcorner)\}$. We also have with a) and b): $i \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}{\restriction}\mathrm{Dom}(\mathfrak{H})\text{-}1))$ and there is no l such that $i < l \leq \mathrm{Dom}(\mathfrak{H})\text{-}2$ and $l \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}{\restriction}\mathrm{Dom}(\mathfrak{H})\text{-}1))$. Hence we have $\mathfrak{H} \in \mathrm{CdIF}(\mathfrak{H}{\restriction}\mathrm{Dom}(\mathfrak{H})\text{-}1)$. In the case that $\mathfrak{H}^+ \in \mathrm{NIF}(\mathfrak{H}^*)$, one shows analogously that then also $\mathfrak{H} \in \mathrm{NIF}(\mathfrak{H}{\restriction}\mathrm{Dom}(\mathfrak{H})\text{-}1)$.

Now, suppose $\mathfrak{H}^+ \in \mathrm{PEF}(\mathfrak{H}^*)$. According to Definition 3-15 and with b) and d), there are then $\beta^+ \in$ PAR, $\zeta \in$ VAR, $\Delta \in$ FORM, where $\mathrm{FV}(\Delta) \subseteq \{\zeta\}$, and $i \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^*))$ such that $\mathrm{P}(\mathfrak{H}^*_i) = \ulcorner\bigvee\zeta\Delta\urcorner = [\beta^*, \beta, \mathrm{P}(\mathfrak{H}_i)]$ and $\mathrm{P}(\mathfrak{H}^*_{i+1}) = [\beta^+, \zeta, \Delta] = [\beta^*, \beta, \mathrm{P}(\mathfrak{H}_{i+1})]$, where $i+1 \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^*))$, $[\beta^*, \beta, \mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-2})] = \mathrm{C}(\mathfrak{H}^*)$, $\beta^+ \notin \mathrm{STSF}(\{\Delta, [\beta^*, \beta,$

$P(\mathfrak{H}_{Dom(\mathfrak{H})-2})]\})$, there is no $j \leq i$ such that $\beta^+ \in ST(\mathfrak{H}^*_j)$, there is no l such that $i+1 < l \leq Dom(\mathfrak{H})-2$ and $l \in Dom(AVAS(\mathfrak{H}^*))$, and $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(Dom(\mathfrak{H})-1, \ulcorner\text{Therefore } C(\mathfrak{H}^*)\urcorner)\} = \mathfrak{H}^* \cup \{(Dom(\mathfrak{H})-1, \ulcorner\text{Therefore } [\beta^*, \beta, P(\mathfrak{H}_{Dom(\mathfrak{H})-2})]\urcorner)\} = \mathfrak{H}^* \cup \{(Dom(\mathfrak{H})-1, [\beta^*, \beta, \ulcorner\text{Therefore } P(\mathfrak{H}_{Dom(\mathfrak{H})-2})\urcorner])\}$. With d), we have $[\beta^*, \beta, \ulcorner\text{Therefore } P(\mathfrak{H}_{Dom(\mathfrak{H})-2})\urcorner] = [\beta^*, \beta, \mathfrak{H}_{Dom(\mathfrak{H})-1}]$. With Theorem 1-21, we then have $\ulcorner\text{Therefore } P(\mathfrak{H}_{Dom(\mathfrak{H})-2})\urcorner = \mathfrak{H}_{Dom(\mathfrak{H})-1}$ and thus $\mathfrak{H} = \mathfrak{H}\restriction Dom(\mathfrak{H})-1 \cup \{(Dom(\mathfrak{H})-1, \ulcorner\text{Therefore } P(\mathfrak{H}_{Dom(\mathfrak{H})-2})\urcorner)\}$.

Then we have, with a) and b): $i \in Dom(AVS(\mathfrak{H}\restriction Dom(\mathfrak{H})-1))$, $i+1 \in Dom(AVAS(\mathfrak{H}\restriction Dom(\mathfrak{H})-1))$ and there is no l such that $i+1 < l \leq Dom(\mathfrak{H})-2$ such that $l \in Dom(AVAS(\mathfrak{H}\restriction Dom(\mathfrak{H})-1))$. Now, we have to show that $P(\mathfrak{H}_i)$, $P(\mathfrak{H}_{i+1})$ and $P(\mathfrak{H}_{Dom(\mathfrak{H})-2})$ satisfy the conditions for $\mathfrak{H} \in PEF(\mathfrak{H}\restriction Dom(\mathfrak{H})-1)$.

We have $[\beta^*, \beta, P(\mathfrak{H}_i)] = P(\mathfrak{H}^*_i) = \ulcorner\bigvee\zeta\Delta\urcorner$ and $[\beta^*, \beta, P(\mathfrak{H}_{i+1})] = P(\mathfrak{H}^*_{i+1}) = [\beta^+, \zeta, \Delta]$. Since operators are not affected by substitution, we thus have, because of $[\beta^*, \beta, P(\mathfrak{H}_i)] = \ulcorner\bigvee\zeta\Delta\urcorner$, that $P(\mathfrak{H}_i) = \ulcorner\bigvee\zeta\Delta^{+}\urcorner$ for a $\Delta^+ \in FORM$, where $\beta^* \notin ST(\Delta^+)$ and $FV(\Delta^+) \subseteq \{\zeta\}$. Thus we have $\ulcorner\bigvee\zeta\Delta\urcorner = [\beta^*, \beta, P(\mathfrak{H}_i)] = [\beta^*, \beta, \ulcorner\bigvee\zeta\Delta^{+}\urcorner] = \ulcorner\bigvee\zeta[\beta^*, \beta, \Delta^+]\urcorner$ and hence $\Delta = [\beta^*, \beta, \Delta^+]$. Thus we have: $[\beta^*, \beta, P(\mathfrak{H}_{i+1})] = [\beta^+, \zeta, \Delta] = [\beta^+, \zeta, [\beta^*, \beta, \Delta^+]]$ and $\beta^+ \notin ST([\beta^*, \beta, \Delta^+])$. Also, we have $\beta^* = \beta^+$ or $\beta^* \neq \beta^+$.

First case: Suppose $\beta^* = \beta^+$. Then we have $\beta^* \notin ST([\beta^*, \beta, \Delta^+])$ and thus $\beta \notin ST(\Delta^+)$. Then we have $\Delta = [\beta^*, \beta, \Delta^+] = \Delta^+$ and, because of $\beta^* = \beta^+$, we then have $[\beta^*, \beta, P(\mathfrak{H}_{i+1})] = [\beta^+, \zeta, \Delta] = [\beta^*, \zeta, \Delta^+]$. We have $\beta^* \notin ST(\Delta^+)$ and $\beta^* \notin ST(P(\mathfrak{H}_{i+1}))$. It thus holds with Theorem 1-23, because of $[\beta^*, \beta, P(\mathfrak{H}_{i+1})] = [\beta^*, \zeta, \Delta^+]$, that $P(\mathfrak{H}_{i+1}) = [\beta, \zeta, \Delta^+]$. Now, suppose for contradiction that $\beta \in STSF(\{\Delta^+, P(\mathfrak{H}_{Dom(\mathfrak{H})-2})\})$ or that there is a $j \leq i$ such that $\beta \in ST(\mathfrak{H}_j)$. Then we would have, with b) and $\beta^* = \beta^+$, that $\beta^+ \in STSF(\{[\beta^*, \beta, \Delta^+], [\beta^*, \beta, P(\mathfrak{H}_{Dom(\mathfrak{H})-2})]\})$ or that there is $j \leq i$ such that $\beta^+ \in ST(\mathfrak{H}^*_j)$. Contradiction! Hence we have $P(\mathfrak{H}_i) = \ulcorner\bigvee\zeta\Delta^{+}\urcorner$ and $P(\mathfrak{H}_{i+1}) = [\beta, \zeta, \Delta^+]$ and $\beta \notin STSF(\{\Delta^+, P(\mathfrak{H}_{Dom(\mathfrak{H})-2})\})$ and there is no $j \leq i$ such that $\beta \in ST(\mathfrak{H}_j)$ and thus we have $\mathfrak{H} \in PEF(\mathfrak{H}\restriction Dom(\mathfrak{H})-1)$.

Second case: Suppose $\beta^* \neq \beta^+$. With $\beta^+ \in ST([\beta^*, \beta, P(\mathfrak{H}_{i+1})])$ and $\beta^+ \notin ST([\beta^*, \beta, P(\mathfrak{H}_{i+1})])$, we can distinguish two subcases. *First subcase*: Suppose $\beta^+ \in ST([\beta^*, \beta, P(\mathfrak{H}_{i+1})])$. Then we have $\beta^+ \neq \beta$ and thus $\beta \notin ST(\beta^+)$. Then, with $\Delta = [\beta^*, \beta, \Delta^+]$ and Theorem 1-25-(ii): $[\beta^*, \beta, P(\mathfrak{H}_{i+1})] = [\beta^+, \zeta, \Delta] = [\beta^+, \zeta, [\beta^*, \beta, \Delta^+]] = [\beta^*, \beta, [\beta^+, \zeta, \Delta^+]]$. We also have $\beta^* \notin ST(P(\mathfrak{H}_{i+1}))$ and, because of $\beta^* \neq \beta^+$ and $\beta^* \notin ST(\Delta^+)$, we also have $\beta^* \notin ST([\beta^+, \zeta, \Delta^+])$. With Theorem 1-20, we thus have $P(\mathfrak{H}_{i+1}) = [\beta^+, \zeta, \Delta^+]$. Now, suppose for contradiction that $\beta^+ \in STSF(\{\Delta^+, P(\mathfrak{H}_{Dom(\mathfrak{H})-2})\})$ or that there is a $j \leq i$ such that $\beta^+ \in ST(\mathfrak{H}_j)$. Because of $\beta^+ \neq \beta$ and with b), we would then also have $\beta^+ \in STSF(\{[\beta^*, \beta, \Delta^+],$

$[\beta^*, \beta, P(\mathfrak{H}_{Dom(\mathfrak{H})-2})]\})$ or there would be a $j \leq i$ such that $\beta^+ \in ST(\mathfrak{H}^*_j)$. Contradiction! Hence the parameter condition for β^+ is satisified in $\mathfrak{H}\upharpoonright Dom(\mathfrak{H})$-1 and thus we have for the first subcase again that $\mathfrak{H} \in PEF(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})-1)$.

Second subcase: Now, suppose $\beta^+ \notin ST([\beta^*, \beta, P(\mathfrak{H}_{i+1})])$. Then it holds, with $[\beta^*, \beta, P(\mathfrak{H}_{i+1})] = [\beta^+, \zeta, [\beta^*, \beta, \Delta^+]]$, that $\zeta \notin FV([\beta^*, \beta, \Delta^+])$. Then we have $[\beta^*, \beta, P(\mathfrak{H}_{i+1})] = [\beta^+, \zeta, [\beta^*, \beta, \Delta^+]] = [\beta^*, \beta, \Delta^+]$ and thus, with $\beta^* \notin ST(P(\mathfrak{H}_{i+1})) \cup ST(\Delta^+)$ and with Theorem 1-20, $P(\mathfrak{H}_{i+1}) = \Delta^+$, where, with $\zeta \notin FV([\beta^*, \beta, \Delta^+])$, also $\zeta \notin FV(\Delta^+)$. Now, let $\beta^§ \in PAR \setminus STSEQ(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})-1)$. Then it holds, with $\zeta \notin FV(\Delta^+)$, that $P(\mathfrak{H}_{i+1}) = \Delta^+ = [\beta^§, \zeta, \Delta^+]$ and we have that $\beta^§ \notin STSF(\{\Delta^+, P(\mathfrak{H}_{Dom(\mathfrak{H})-2})\})$ and that there is no $j \leq i$ such that $\beta^§ \in ST(\mathfrak{H}_j)$. Thus we then also have $\mathfrak{H} \in PEF(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})-1)$. Hence we have in both subcases and thus in both cases that $\mathfrak{H} \in PEF(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})-1)$.

Main part: Now we will show that for each of the cases AF … IEF it holds that $\mathfrak{H}^+ \in$ RCS and $Dom(AVS(\mathfrak{H}^+)) = Dom(AVS(\mathfrak{H}))$. First, we will deal with CdIF, NIF and PEF. Then we can make an exclusion assumption that allows us to determine $Dom(AVS(\mathfrak{H}^+))$ for all other cases just with a), e) and Theorem 3-25.

(*CdIF*, *NIF*): Suppose $\mathfrak{H} \in CdIF(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})-1)$. According to Definition 3-2, there is then an $i \in Dom(AVAS(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})-1))$ such that there is no $l \in Dom(AVAS(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})-1))$ with $i < l \leq Dom(\mathfrak{H})-2$, and $\mathfrak{H} = \mathfrak{H}\upharpoonright Dom(\mathfrak{H})-1 \cup \{(Dom(\mathfrak{H})-1, \ulcorner\text{Therefore } P(\mathfrak{H}_i) \rightarrow C(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})-1)\urcorner)\}$. Then it holds with a), b) and d): $i \in Dom(AVAS(\mathfrak{H}^*))$ and there is no l such that $i < l \leq Dom(\mathfrak{H})-2$ and $l \in Dom(AVAS(\mathfrak{H}^*))$, and $P(\mathfrak{H}^*_i) = [\beta^*, \beta, P(\mathfrak{H}_i)]$ and $C(\mathfrak{H}^*) = [\beta^*, \beta, C(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})-1)]$ and $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(Dom(\mathfrak{H})-1, [\beta^*, \beta, \ulcorner\text{Therefore } P(\mathfrak{H}_i) \rightarrow C(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})-1)\urcorner])\} = \mathfrak{H}^* \cup \{(Dom(\mathfrak{H})-1, \ulcorner\text{Therefore } P(\mathfrak{H}^*_i) \rightarrow C(\mathfrak{H}^*)\urcorner)\}$. Thus we have $\mathfrak{H}^+ \in CdIF(\mathfrak{H}^*)$ and thus $\mathfrak{H}^+ \in$ RCS.

With Theorem 3-19-(iii), we then have $AVS(\mathfrak{H}) = AVS(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})-1) \setminus \{(j, \mathfrak{H}_j) \mid i \leq j < Dom(\mathfrak{H})-1\} \cup \{(Dom(\mathfrak{H})-1, \ulcorner\text{Therefore } P(\mathfrak{H}_i) \rightarrow C(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})-1)\urcorner)\}$ and that $AVS(\mathfrak{H}^+) = AVS(\mathfrak{H}^*) \setminus \{(j, \mathfrak{H}^+_j) \mid i \leq j < Dom(\mathfrak{H})-1\} \cup \{(Dom(\mathfrak{H})-1, \ulcorner\text{Therefore } [\beta^*, \beta, P(\mathfrak{H}_i)] \rightarrow [\beta^*, \beta, C(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})-1)]\urcorner)\}$. With $Dom(AVS(\mathfrak{H}^*)) = Dom(AVS(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})-1))$, it then follows that also $Dom(AVS(\mathfrak{H}^+)) = Dom(AVS(\mathfrak{H}))$. In the case that $\mathfrak{H} \in NIF(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})-1)$, one shows analogously that then also $\mathfrak{H}^+ \in NIF(\mathfrak{H}^*) \subseteq$ RCS and $Dom(AVS(\mathfrak{H}^+)) = Dom(AVS(\mathfrak{H}))$.

(*PEF*): Now, suppose $\mathfrak{H} \in PEF(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})-1)$. According to Definition 3-15, there are then $\beta^+ \in PAR$, $\zeta \in VAR$, $\Delta \in FORM$, where $FV(\Delta) \subseteq \{\zeta\}$, and $i \in$

$\mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1))$ such that $\mathrm{P}(\mathfrak{H}_i) = \ulcorner\bigvee\zeta\Delta\urcorner$, $\mathrm{P}(\mathfrak{H}_{i+1}) = [\beta^+, \zeta, \Delta]$, where $i+1 \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1))$, $\beta^+ \notin \mathrm{STSF}(\{\Delta, \mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})\text{-}2})\})$, there is no $j \leq i$ such that $\beta^+ \in \mathrm{ST}(\mathfrak{H}_j)$, there is no l such that $i+1 < l \leq \mathrm{Dom}(\mathfrak{H})\text{-}2$ and $l \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1))$, and $\mathfrak{H} = \mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1 \cup \{(\mathrm{Dom}(\mathfrak{H})\text{-}1, \ulcorner\text{Therefore } \mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})\text{-}2})\urcorner)\}$.

Then it follows, with a), b) and d), that $i \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^*))$ and $\mathrm{P}(\mathfrak{H}^*_i) = [\beta^*, \beta, \mathrm{P}(\mathfrak{H}_i)] = [\beta^*, \beta, \ulcorner\bigvee\zeta\Delta\urcorner] = \ulcorner\bigvee\zeta[\beta^*, \beta, \Delta]\urcorner$, $i+1 \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^*))$ and $\mathrm{P}(\mathfrak{H}^*_{i+1}) = [\beta^*, \beta, \mathrm{P}(\mathfrak{H}_{i+1})] = [\beta^*, \beta, [\beta^+, \zeta, \Delta]]$, $\mathrm{C}(\mathfrak{H}^*) = \mathrm{P}(\mathfrak{H}^*_{\mathrm{Dom}(\mathfrak{H})\text{-}2}) = [\beta^*, \beta, \mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})\text{-}2})]$ and $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H})\text{-}1, [\beta^*, \beta, \ulcorner\text{Therefore } \mathrm{C}(\mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1)\urcorner])\} = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H})\text{-}1, \ulcorner\text{Therefore } \mathrm{C}(\mathfrak{H}^*)]\urcorner)\}$ and there is no l such that $i+1 < l \leq \mathrm{Dom}(\mathfrak{H})\text{-}2$ and $l \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^*))$. With $\beta^+ = \beta$ and $\beta^+ \neq \beta$, we can distinguish two cases.

First case: Suppose $\beta^+ = \beta$. Then we have $\mathrm{P}(\mathfrak{H}_{i+1}) = [\beta^*, \beta, [\beta^+, \zeta, \Delta]] = [\beta^*, \beta, [\beta, \zeta, \Delta]]$ and, with $\beta^+ \notin \mathrm{ST}(\Delta)$, also $\beta \notin \mathrm{ST}(\Delta)$ and hence, with Theorem 1-24-(ii), $\mathrm{P}(\mathfrak{H}_{i+1}) = [\beta^*, \beta, [\beta, \zeta, \Delta]] = [\beta^*, \zeta, \Delta]$. With $\beta \notin \mathrm{ST}(\Delta)$, we then have $[\beta^*, \beta, \Delta] = \Delta$ and thus $\mathrm{P}(\mathfrak{H}^*_i) = \ulcorner\bigvee\zeta[\beta^*, \beta, \Delta]\urcorner = \ulcorner\bigvee\zeta\Delta\urcorner$. With $\beta = \beta^+$ and $\beta^* \notin \mathrm{STSEQ}(\mathfrak{H})$, we also have $\beta, \beta^* \notin \mathrm{STSF}(\{\Delta, \mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})\text{-}2})\})$ and thus also $\beta^* \notin \mathrm{STSF}(\{\Delta, [\beta^*, \beta, \mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})\text{-}2})]\})$. Now, suppose for contradiction that there is a $j \leq i$ such that $\beta^* \in \mathrm{ST}(\mathfrak{H}^*_j)$. With b), we would then have $\beta^* \in \mathrm{ST}(\mathfrak{H}^*_j) = [\beta^*, \beta, \mathfrak{H}_j]$. With $\beta^* \notin \mathrm{STSEQ}(\mathfrak{H})$, it also holds that $\beta^* \notin \mathrm{ST}(\mathfrak{H}_j)$. But then we have, with $\beta^* \in \mathrm{ST}(\mathfrak{H}^*_j)$, that $\beta \in \mathrm{ST}(\mathfrak{H}_j)$, while, on the other hand, we have, by hypothesis, that $\beta = \beta^+ \notin \mathrm{ST}(\mathfrak{H}_j)$. Contradiction! Therefore we have that there is no $j \leq i$ such that $\beta^* \in \mathrm{ST}(\mathfrak{H}^*_j)$. Hence, altogether, we have $\mathfrak{H}^+ \in \mathrm{PEF}(\mathfrak{H}^*)$.

Second case: Now, suppose $\beta^+ \neq \beta$. With $\beta^+ \neq \beta^*$ and $\beta^+ = \beta^*$, we can then distinguish two subcases. *First subcase*: Suppose $\beta^+ \neq \beta^*$. With Theorem 1-25-(ii) and $\beta^+ \neq \beta$, we then have $\mathrm{P}(\mathfrak{H}^*_{i+1}) = [\beta^*, \beta, [\beta^+, \zeta, \Delta]] = [\beta^+, \zeta, [\beta^*, \beta, \Delta]]$. We also have $\mathrm{P}(\mathfrak{H}^*_i) = \ulcorner\bigvee\zeta[\beta^*, \beta, \Delta]\urcorner$. If $\beta^+ \in \mathrm{STSF}(\{[\beta^*, \beta, \Delta], [\beta^*, \beta, \mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})\text{-}2})]\})$ or if there was a $j \leq i$ such that $\beta^+ \in \mathrm{ST}(\mathfrak{H}^*_j)$, then it would hold, because of $\beta^+ \neq \beta^*$ and with b), that $\beta^+ \in \mathrm{STSF}(\{\Delta, \mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})\text{-}2})\})$ or that there is a $j \leq i$ such that $\beta^+ \in \mathrm{ST}(\mathfrak{H}_j)$, which contradicts the assumption about β^+. Therefore we have $\beta^+ \notin \mathrm{STSF}(\{[\beta^*, \beta, \Delta], [\beta^*, \beta, \mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})\text{-}2})]\})$ and there is no $j \leq i$ such that $\beta^+ \in \mathrm{ST}(\mathfrak{H}^*_j)$ and hence we have again $\mathfrak{H}^+ \in \mathrm{PEF}(\mathfrak{H}^*)$.

Second subcase: Now, suppose $\beta^+ = \beta^*$. Then we have $\zeta \notin \mathrm{FV}(\Delta)$, because, if not, we would have $\beta^* \in \mathrm{ST}([\beta^+, \zeta, \Delta]) \subseteq \mathrm{STSEQ}(\mathfrak{H})$. We then have $[\beta^+, \zeta, \Delta] = \Delta$ and thus $\mathrm{P}(\mathfrak{H}^*_{i+1}) = [\beta^*, \beta, [\beta^+, \zeta, \Delta]] = [\beta^*, \beta, \Delta]$ and we have $\mathrm{P}(\mathfrak{H}^*_i) = \ulcorner\bigvee\zeta[\beta^*, \beta, \Delta]\urcorner$. Now, let $\beta^\S \in \mathrm{PAR}\setminus\mathrm{STSEQ}(\mathfrak{H}^*)$. With $\zeta \notin \mathrm{FV}(\Delta)$, we also have $\zeta \notin \mathrm{FV}([\beta^*, \beta, \Delta])$ and thus $\mathrm{P}(\mathfrak{H}^*_{i+1}) = [\beta^*, \beta, \Delta] = [\beta^\S, \zeta, [\beta^*, \beta, \Delta]]$ and it holds that $\beta^\S \notin \mathrm{STSF}(\{[\beta^*, \beta, \Delta], [\beta^*, \beta,$

$P(\mathfrak{H}_{Dom(\mathfrak{H})-2})]\})$ and there is no $j \leq i$ such that $\beta^{\S} \in ST(\mathfrak{H}^*_j)$. Thus we have again $\mathfrak{H}^+ \in PEF(\mathfrak{H}^*)$. Thus we have in both subcases and hence in both cases that $\mathfrak{H}^+ \in PEF(\mathfrak{H}^*)$ and thus $\mathfrak{H}^+ \in RCS$.

It then follows, with Theorem 3-21-(iii), that $AVS(\mathfrak{H}) = AVS(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1)\setminus\{(j, \mathfrak{H}_j) \mid i+1 \leq j < Dom(\mathfrak{H})\text{-}1\} \cup \{(Dom(\mathfrak{H})\text{-}1, \ulcorner\text{Therefore } P(\mathfrak{H}_{Dom(\mathfrak{H})-2})\urcorner)\}$ and that $AVS(\mathfrak{H}^+) = AVS(\mathfrak{H}^*)\setminus\{(j, \mathfrak{H}^+_j) \mid i+1 \leq j < Dom(\mathfrak{H})\text{-}1\} \cup \{(Dom(\mathfrak{H})\text{-}1, \ulcorner\text{Therefore } [\beta^*, \beta, P(\mathfrak{H}_{Dom(\mathfrak{H})-2})]\urcorner)\}$. With $Dom(AVS(\mathfrak{H}^*)) = Dom(AVS(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1))$, it then follows that $Dom(AVS(\mathfrak{H}^+)) = Dom(AVS(\mathfrak{H}))$.

Exclusion assumption: For the remaining steps, suppose $\mathfrak{H} \notin CdIF(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1) \cup NIF(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1) \cup PEF(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1)$. With e), we then have $\mathfrak{H}^+ \notin CdIF(\mathfrak{H}^*) \cup NIF(\mathfrak{H}^*) \cup PEF(\mathfrak{H}^*)$. With Theorem 3-25, we then have for all of he following cases that $AVS(\mathfrak{H}) = AVS(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1) \cup \{(Dom(\mathfrak{H})\text{-}1, C(\mathfrak{H}))\}$ and that $AVS(\mathfrak{H}^+) = AVS(\mathfrak{H}^*) \cup \{(Dom(\mathfrak{H})\text{-}1, C(\mathfrak{H}^+))\}$. With $Dom(AVS(\mathfrak{H}^*)) = Dom(AVS(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1))$, it then follows that $Dom(AVS(\mathfrak{H}^+)) = Dom(AVS(\mathfrak{H}))$ for all remaining cases.

(*AF*): Suppose $\mathfrak{H} \in AF(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1)$. With Definition 3-1, we then have $\mathfrak{H} = \mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1 \cup \{(Dom(\mathfrak{H})\text{-}1, \ulcorner\text{Suppose } P(\mathfrak{H}_{Dom(\mathfrak{H})-1})\urcorner)$. With d), we then have $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(Dom(\mathfrak{H})\text{-}1, \ulcorner\text{Suppose } [\beta^*, \beta, P(\mathfrak{H}_{Dom(\mathfrak{H})-1})]\urcorner)\} \in AF(\mathfrak{H}^*)$ and thus $\mathfrak{H}^+ \in RCS$.

(*CdEF*, *CIF*, *CEF*, *BIF*, *BEF*, *DIF*, *DEF*, *NEF*): Now, suppose $\mathfrak{H} \in CdEF(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1)$. With Definition 3-3, there are then $A, B \in CFORM$ such that A, $\ulcorner A \rightarrow B\urcorner \in AVP(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1)$ and $\mathfrak{H} = \mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1 \cup \{(Dom(\mathfrak{H})\text{-}1, \ulcorner\text{Therefore } B\urcorner)\}$. With d), it then follows that $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(Dom(\mathfrak{H})\text{-}1, \ulcorner\text{Therefore } [\beta^*, \beta, B]\urcorner)\}$. Since A, $\ulcorner A \rightarrow B\urcorner \in AVP(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1)$, we then have, with Definition 2-30, that there are $i, j \in Dom(AVS(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1))$ such that $P(\mathfrak{H}_i) = A$ and $P(\mathfrak{H}_j) = \ulcorner A \rightarrow B\urcorner$. With a) and b), it then follows that $i, j \in Dom(AVS(\mathfrak{H}^*))$ and $P(\mathfrak{H}^*_i) = [\beta^*, \beta, A]$ and $P(\mathfrak{H}^*_j) = \ulcorner[\beta^*, \beta, A] \rightarrow [\beta^*, \beta, B]\urcorner$. With d), we then have $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(Dom(\mathfrak{H})\text{-}1, \ulcorner\text{Therefore } [\beta^*, \beta, B]\urcorner)\} \in CdEF(\mathfrak{H}^*)$ and thus $\mathfrak{H}^+ \in RCS$. For CIF, CEF, BIF, BEF, DIF, DEF and NEF the proof is carried out analogously.

(*UIF*): Now, suppose $\mathfrak{H} \in UIF(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1)$. According to Definition 3-12, there are then $\beta^+ \in PAR$, $\zeta \in VAR$, $\Delta \in FORM$, where $FV(\Delta) \subseteq \{\zeta\}$, such that $[\beta^+, \zeta, \Delta] \in AVP(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1)$, $\beta^+ \notin STSF(\{\Delta\} \cup AVAP(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1))$, and $\mathfrak{H} = \mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1 \cup \{(Dom(\mathfrak{H})\text{-}1, \ulcorner\text{Therefore } \bigwedge\zeta\Delta\urcorner)\}$. With d), we then have $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(Dom(\mathfrak{H})\text{-}1, [\beta^*, \beta, \ulcorner\text{Therefore } \bigwedge\zeta\Delta\urcorner])\} = \mathfrak{H}^* \cup \{(Dom(\mathfrak{H})\text{-}1, \ulcorner\text{Therefore } \bigwedge\zeta[\beta^*, \beta, \Delta]\urcorner)\}$. With $[\beta^+, \zeta, \Delta] \in AVP(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1)$ and Definition 2-30, we then have that there is an $i \in$

$\mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1)$ such that $[\beta^+, \zeta, \Delta] = \mathrm{P}(\mathfrak{H}_i)$. With a) and b), it then follows that $i \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^*))$ and $\mathrm{P}(\mathfrak{H}^*_i) = [\beta^*, \beta, \mathrm{P}(\mathfrak{H}_i)] = [\beta^*, \beta, [\beta^+, \zeta, \Delta]]$. With $\beta^+ = \beta$ and $\beta^+ \neq \beta$ we can then distinguish two cases.

First case: Suppose $\beta^+ = \beta$. Then we have $\mathrm{P}(\mathfrak{H}^*_i) = [\beta^*, \beta, [\beta^+, \zeta, \Delta]] = [\beta^*, \beta, [\beta, \zeta, \Delta]]$ and, with $\beta^+ \notin \mathrm{ST}(\Delta)$, we also have $\beta \notin \mathrm{ST}(\Delta)$ and thus we have, with Theorem 1-24-(ii), that $\mathrm{P}(\mathfrak{H}^*_i) = [\beta^*, \beta, [\beta, \zeta, \Delta]] = [\beta^*, \zeta, \Delta]$. With $\beta \notin \mathrm{ST}(\Delta)$, we then have $[\beta^*, \beta, \Delta] = \Delta$ and thus $\mathrm{C}(\mathfrak{H}^+) = \ulcorner\bigwedge\zeta[\beta^*, \beta, \Delta]\urcorner = \ulcorner\bigwedge\zeta\Delta\urcorner$. With $\beta^+ = \beta$ and $\beta^* \notin \mathrm{STSEQ}(\mathfrak{H})$, we also have $\beta, \beta^* \notin \mathrm{STSF}(\{\Delta\} \cup \mathrm{AVAP}(\mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1))$ and thus, with a) and b), also $\beta^* \notin \mathrm{STSF}(\{\Delta\} \cup \mathrm{AVAP}(\mathfrak{H}^*))$. To see this, suppose for contradiction that $\beta^* \in \mathrm{STSF}(\{\Delta\} \cup \mathrm{AVAP}(\mathfrak{H}^*))$. Then we have $\beta^* \notin \mathrm{ST}(\Delta)$, because, if not, we would have $\beta^* \in \mathrm{ST}(\Delta) \subseteq \mathrm{ST}(\ulcorner\bigwedge\zeta\Delta\urcorner) = \mathrm{ST}(\mathrm{C}(\mathfrak{H})) \subseteq \mathrm{STSEQ}(\mathfrak{H})$, which contradicts $\beta^* \notin \mathrm{STSEQ}(\mathfrak{H})$. Therefore there would be a $\mathrm{B} \in \mathrm{AVAP}(\mathfrak{H}^*)$ such that $\beta^* \in \mathrm{ST}(\mathrm{B})$. With Definition 2-31, there would then be a $j \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^*))$ such that $\beta^* \in \mathrm{ST}(\mathrm{P}(\mathfrak{H}^*_j))$. With b), we then have $\mathrm{P}(\mathfrak{H}^*_j) = [\beta^*, \beta, \mathrm{P}(\mathfrak{H}_j)]$. Since $\beta^* \notin \mathrm{STSEQ}(\mathfrak{H})$, we also have $\beta^* \notin \mathrm{ST}(\mathrm{P}(\mathfrak{H}_j))$. But then we have, with $\beta^* \in \mathrm{ST}(\mathrm{P}(\mathfrak{H}^*_j))$ and $\mathrm{P}(\mathfrak{H}^*_j) = [\beta^*, \beta, \mathrm{P}(\mathfrak{H}_j)]$, that $\beta \in \mathrm{ST}(\mathrm{P}(\mathfrak{H}_j))$. Moreover, with a) and b), it follows from $j \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^*))$ that $j \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1))$ and hence that $\mathrm{P}(\mathfrak{H}_j) \in \mathrm{AVAP}(\mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1)$. But then we would have $\beta \in \mathrm{STSF}(\mathrm{AVAP}(\mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1))$, whereas, by hypothesis, we have $\beta = \beta^+ \notin \mathrm{STSF}(\mathrm{AVAP}(\mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1))$. Contradiction! Therefore we have $\beta^* \notin \mathrm{STSF}(\{\Delta\} \cup \mathrm{AVAP}(\mathfrak{H}^*))$. Since we have $\mathrm{P}(\mathfrak{H}^*_i) = [\beta^*, \zeta, \Delta]$, $i \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^*))$ and $\mathrm{C}(\mathfrak{H}^+) = \ulcorner\bigwedge\zeta\Delta\urcorner$, we thus have $\mathfrak{H}^+ \in \mathrm{UIF}(\mathfrak{H}^*)$.

Second case: Now, suppose $\beta^+ \neq \beta$. With $\beta^+ \neq \beta^*$ and $\beta^+ = \beta^*$, we can then distinguish two subcases. *First subcase*: Suppose $\beta^+ \neq \beta^*$. With Theorem 1-25-(ii) and $\beta^+ \neq \beta$, we then have $\mathrm{P}(\mathfrak{H}^*_i) = [\beta^*, \beta, [\beta^+, \zeta, \Delta]] = [\beta^+, \zeta, [\beta^*, \beta, \Delta]]$. Also, we have $\mathrm{C}(\mathfrak{H}^+) = \ulcorner\bigwedge\zeta[\beta^*, \beta, \Delta]\urcorner$. Now, suppose for contradiction that $\beta^+ \in \mathrm{STSF}(\{[\beta^*, \beta, \Delta]\} \cup \mathrm{AVAP}(\mathfrak{H}^*))$. Since $\beta^+ \neq \beta^*$ and $\beta^+ \notin \mathrm{ST}(\Delta)$, we have $\beta^+ \notin \mathrm{ST}([\beta^*, \beta, \Delta])$. Therefore we would have $\beta^+ \in \mathrm{STSF}(\mathrm{AVAP}(\mathfrak{H}^*))$ and thus there would be, with Definition 2-31, a $j \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^*))$ such that $\beta^+ \in \mathrm{ST}(\mathrm{P}(\mathfrak{H}^*_j))$. Since, with b), $\mathrm{P}(\mathfrak{H}^*_j) = [\beta^*, \beta, \mathrm{P}(\mathfrak{H}_j)]$ and since $\beta^+ \neq \beta^*$, we would thus have that $\beta^+ \in \mathrm{ST}(\mathrm{P}(\mathfrak{H}_j))$. With a) and b), it follows from $j \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^*))$ that $j \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1))$, and thus we would have $\mathrm{P}(\mathfrak{H}_j) \in \mathrm{AVAP}(\mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1))$ and thus $\beta^+ \in \mathrm{STSF}(\mathrm{AVAP}(\mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1))$, wheras, by hypothesis, we have $\beta^+ \notin \mathrm{STSF}(\mathrm{AVAP}(\mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1))$. Contradiction! Therefore we have $\beta^+ \notin \mathrm{STSF}(\{[\beta^*, \beta, \Delta]\} \cup \mathrm{AVAP}(\mathfrak{H}^*))$ and hence again $\mathfrak{H}^+ \in \mathrm{UIF}(\mathfrak{H}^*)$.

Second subcase: Now, suppose $\beta^+ = \beta^*$. Then we have $\zeta \notin \mathrm{FV}(\Delta)$, because, if not, we would have $\beta^* \in \mathrm{ST}([\beta^+, \zeta, \Delta]) \subseteq \mathrm{STSEQ}(\mathfrak{H})$. Thus we then have $[\beta^+, \zeta, \Delta] = \Delta$ and thus $\mathrm{P}(\mathfrak{H}^*_i) = [\beta^*, \beta, [\beta^+, \zeta, \Delta]] = [\beta^*, \beta, \Delta]$, and we have $\mathrm{C}(\mathfrak{H}^+) = \ulcorner\bigwedge\zeta[\beta^*, \beta, \Delta]\urcorner$. Now, let $\beta^\S \in \mathrm{PAR}\backslash\mathrm{STSEQ}(\mathfrak{H}^*)$. With $\zeta \notin \mathrm{FV}(\Delta)$, we also have $\zeta \notin \mathrm{FV}([\beta^*, \beta, \Delta])$, and thus $\mathrm{P}(\mathfrak{H}^*_i) = [\beta^*, \beta, \Delta] = [\beta^\S, \zeta, [\beta^*, \beta, \Delta]]$, and it holds that $\beta^\S \notin \mathrm{STSF}(\{[\beta^*, \beta, \Delta]\} \cup \mathrm{AVAP}(\mathfrak{H}^*))$ and thus again $\mathfrak{H}^+ \in \mathrm{UIF}(\mathfrak{H}^*)$. Thus we have in both subcases and hence in both cases that $\mathfrak{H}^+ \in \mathrm{UIF}(\mathfrak{H}^*) \subseteq \mathrm{RCS}$.

(*UEF*): Now, suppose $\mathfrak{H} \in \mathrm{UEF}(\mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1)$. According to Definition 3-13, there are then $\theta \in \mathrm{CTERM}$, $\zeta \in \mathrm{VAR}$, $\Delta \in \mathrm{FORM}$, where $\mathrm{FV}(\Delta) \subseteq \{\zeta\}$, such that $\ulcorner\bigwedge\zeta\Delta\urcorner \in \mathrm{AVP}(\mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1)$, and $\mathfrak{H} = \mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1 \cup \{(\mathrm{Dom}(\mathfrak{H})\text{-}1, \ulcorner\text{Therefore } [\theta, \zeta, \Delta]\urcorner)\}$. With d), we then have $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H})\text{-}1, [\beta^*, \beta, \ulcorner\text{Therefore } [\theta, \zeta, \Delta]\urcorner])\} = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H})\text{-}1, \ulcorner\text{Therefore } [\beta^*, \beta, [\theta, \zeta, \Delta]]\urcorner)\}$. With $\ulcorner\bigwedge\zeta\Delta\urcorner \in \mathrm{AVP}(\mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1)$ and Definition 2-30, there is then an $i \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1))$ such that $\mathrm{P}(\mathfrak{H}_i) = \ulcorner\bigwedge\zeta\Delta\urcorner$. With a) and b), we then have $i \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^*))$ and $\mathrm{P}(\mathfrak{H}^*_i) = [\beta^*, \beta, \ulcorner\bigwedge\zeta\Delta\urcorner] = \ulcorner\bigwedge\zeta[\beta^*, \beta, \Delta]\urcorner$. With Theorem 1-26-(ii), we have $\mathrm{C}(\mathfrak{H}^+) = [\beta^*, \beta, [\theta, \zeta, \Delta]] = [[\beta^*, \beta, \theta], \zeta, [\beta^*, \beta, \Delta]]$, where, with $\theta \in \mathrm{CTERM}$, also $[\beta^*, \beta, \theta] \in \mathrm{CTERM}$, and, with $\mathrm{FV}(\Delta) \subseteq \{\zeta\}$, also $\mathrm{FV}([\beta^*, \beta, \Delta]) \subseteq \{\zeta\}$. Hence we have $\mathfrak{H}^+ \in \mathrm{UEF}(\mathfrak{H}^*) \subseteq \mathrm{RCS}$.

(*PIF*): Now, suppose $\mathfrak{H} \in \mathrm{PIF}(\mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1)$. According to Definition 3-14, there are then $\theta \in \mathrm{CTERM}$, $\zeta \in \mathrm{VAR}$, $\Delta \in \mathrm{FORM}$, where $\mathrm{FV}(\Delta) \subseteq \{\zeta\}$, such that $[\theta, \zeta, \Delta] \in \mathrm{AVP}(\mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1)$, and $\mathfrak{H} = \mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1 \cup \{(\mathrm{Dom}(\mathfrak{H})\text{-}1, \ulcorner\text{Therefore } \bigvee\zeta\Delta\urcorner)\}$. With d), we then have $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H})\text{-}1, [\beta^*, \beta, \ulcorner\text{Therefore } \bigvee\zeta\Delta\urcorner])\} = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H})\text{-}1, \ulcorner\text{Therefore } \bigvee\zeta[\beta^*, \beta, \Delta]\urcorner)\}$. With $[\theta, \zeta, \Delta] \in \mathrm{AVP}(\mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1)$ and Definition 2-30, there is an $i \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1))$ such that $\mathrm{P}(\mathfrak{H}_i) = [\theta, \zeta, \Delta]$. With a) and b), we then have $i \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^*))$ and $\mathrm{P}(\mathfrak{H}^*_i) = [\beta^*, \beta, \mathrm{P}(\mathfrak{H}_i)]$. With Theorem 1-26-(ii), we then have $\mathrm{P}(\mathfrak{H}^*_i) = [\beta^*, \beta, \mathrm{P}(\mathfrak{H}_i)] = [\beta^*, \beta, [\theta, \zeta, \Delta]] = [[\beta^*, \beta, \theta], \zeta, [\beta^*, \beta, \Delta]]$, where, with $\theta \in \mathrm{CTERM}$, also $[\beta^*, \beta, \theta] \in \mathrm{CTERM}$, and, with $\mathrm{FV}(\Delta) \subseteq \{\zeta\}$, also $\mathrm{FV}([\beta^*, \beta, \Delta]) \subseteq \{\zeta\}$. Hence we have $\mathfrak{H}^+ \in \mathrm{PIF}(\mathfrak{H}^*) \subseteq \mathrm{RCS}$.

(*IIF*): Now, suppose $\mathfrak{H} \in \mathrm{IIF}(\mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1)$. With Definition 3-16, there is then $\theta \in \mathrm{CTERM}$ such that $\mathfrak{H} = \mathfrak{H}\restriction\mathrm{Dom}(\mathfrak{H})\text{-}1 \cup \{(\mathrm{Dom}(\mathfrak{H})\text{-}1, \ulcorner\text{Therefore } \theta = \theta\urcorner)\}$. With d), we then have $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H})\text{-}1, [\beta^*, \beta, \ulcorner\text{Therefore } \theta = \theta\urcorner])\} = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H})\text{-}1, \ulcorner\text{Therefore } [\beta^*, \beta, \theta] = [\beta^*, \beta, \theta]\urcorner)\}$, where, with $\theta \in \mathrm{CTERM}$, also $[\beta^*, \beta, \theta] \in \mathrm{CTERM}$. Hence we have $\mathfrak{H}^+ \in \mathrm{IIF}(\mathfrak{H}^*) \subseteq \mathrm{RCS}$.

(*IEF*): Now, suppose $\mathfrak{H} \in$ IEF($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1). With Definition 3-17, there are then θ_0, θ_1 $\in$ CTERM, $\zeta \in$ VAR and $\Delta \in$ FORM, where FV(Δ) $\subseteq$ {ζ}, such that $\ulcorner\theta_0 = \theta_1\urcorner$, [$\theta_0$, ζ, Δ] $\in$ AVP($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1), and $\mathfrak{H}$ = $\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1 $\cup$ {(Dom($\mathfrak{H}$)-1, $\ulcorner$Therefore [θ_1, ζ, Δ]$\urcorner$)}. With d), we then have $\mathfrak{H}^+$ = $\mathfrak{H}^*$ $\cup$ {(Dom($\mathfrak{H}$)-1, [β^*, β, $\ulcorner$Therefore [θ_1, ζ, Δ]$\urcorner$])} = $\mathfrak{H}^*$ $\cup$ {(Dom($\mathfrak{H}$)-1, $\ulcorner$Therefore [β^*, β, [θ_1, ζ, Δ]]$\urcorner$)}. With $\ulcorner\theta_0 = \theta_1\urcorner$, [$\theta_0$, ζ, Δ] $\in$ AVP($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1) and Definition 2-30, there are then i, j $\in$ Dom(AVS($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1)) such that P($\mathfrak{H}_i$) = $\ulcorner\theta_0 = \theta_1\urcorner$ and P($\mathfrak{H}_j$) = [θ_0, ζ, Δ]. With a) and b), it then holds that i, j $\in$ Dom(AVS($\mathfrak{H}^*$)) and P($\mathfrak{H}^*_i$) = [β^*, β, P($\mathfrak{H}_i$)] = [β^*, β, $\ulcorner\theta_0 = \theta_1\urcorner$] = $\ulcorner$[β^*, β, θ_0] = [β^*, β, θ_1]$\urcorner$ and P($\mathfrak{H}^*_j$) = [β^*, β, P($\mathfrak{H}_j$)]. With Theorem 1-26-(ii), we then have P($\mathfrak{H}^*_j$) = [β^*, β, P($\mathfrak{H}_j$)] = [β^*, β, [θ_0, ζ, Δ]] = [[β^*, β, θ_0], ζ, [β^*, β, Δ]] and C($\mathfrak{H}^+$) = [β^*, β, [θ_1, ζ, Δ]] = [[β^*, β, θ_1], ζ, [β^*, β, Δ]], where, with θ_0, θ_1 $\in$ CTERM, also [β^*, β, θ_0], [β^*, β, θ_1] $\in$ CTERM, and, with FV(Δ) $\subseteq$ {ζ}, also FV([β^*, β, Δ]) $\subseteq$ {ζ}. Hence it follows that $\mathfrak{H}^+$ $\in$ IEF($\mathfrak{H}^*$) $\subseteq$ RCS. ■

The following theorem prepares the generalisation theorem (Theorem 4-24). The proof resembles the proof of Theorem 4-8.

Theorem 4-9. *Substitution of a new parameter for an individual constant is RCS-preserving*
If $\mathfrak{H} \in$ RCS, $\alpha \in$ CONST and $\beta \in$ PAR\STSEQ($\mathfrak{H}$), then there is an $\mathfrak{H}^+$ $\in$ RCS\{∅} such that

(i) $\alpha \notin$ STSEQ($\mathfrak{H}^+$),
(ii) STSEQ($\mathfrak{H}^+$) $\subseteq$ STSEQ($\mathfrak{H}$) $\cup$ {β},
(iii) AVAP($\mathfrak{H}$) = {[α, β, B] | B $\in$ AVAP($\mathfrak{H}^+$)}, and
(iv) If $\mathfrak{H} \neq$ ∅, then C($\mathfrak{H}$) = [α, β, C($\mathfrak{H}^+$)].

Proof: Suppose $\mathfrak{H} \in$ RCS, $\alpha \in$ CONST and $\beta \in$ PAR\STSEQ($\mathfrak{H}$). Let $\mathfrak{H}^+$ be defined as follows:

a) $\mathfrak{H}^+$ = {(0, $\ulcorner$Therefore $\beta = \beta\urcorner$)}$^\frown$[β, α, $\mathfrak{H}$].

Then clauses (i) and (ii) already hold and we also have $\mathfrak{H}^+ \neq$ ∅. For $\mathfrak{H}^+$, we will will now show by induction on Dom($\mathfrak{H}$) that $\mathfrak{H}^+$ $\in$ RCS and

b) Dom(AVS($\mathfrak{H}^+$)) = {(l+1 | l $\in$ Dom(AVS($\mathfrak{H}$))} $\cup$ {0}.

Clauses (iii) and (iv) then follow with a) and b). *Ad (iii)*: Suppose $\Delta \in$ AVAP($\mathfrak{H}$). Then there is an i $\in$ Dom(AVS($\mathfrak{H}$)) such that $\mathfrak{H}_i$ = $\ulcorner$Suppose $\Delta\urcorner$. Therefore, with b), i+1 $\in$ Dom(AVS($\mathfrak{H}^+$)) and, with a), $\mathfrak{H}^+_{i+1}$ = $\ulcorner$Suppose [β, α, Δ]$\urcorner$. Therefore we have [β, α, Δ] $\in$

AVAP($\mathfrak{H}^+$) and thus [α, β, [β, α, Δ]] ∈ {[α, β, B] | B ∈ AVAP($\mathfrak{H}^+$)}. We have β ∉ STSEQ($\mathfrak{H}$) and thus β ∉ ST(Δ) and thus, with Theorem 1-24-(ii), [α, β, [β, α, Δ]] = [α, α, Δ] = Δ. Therefore Δ ∈ {[α, β, B] | B ∈ AVAP($\mathfrak{H}^+$)}. Now, suppose Δ ∈ {[α, β, B] | B ∈ AVAP($\mathfrak{H}^+$)}. Then there is a Δ* ∈ AVAP($\mathfrak{H}^+$) such that Δ = [α, β, Δ*]. Because of Δ* ∈ AVAP($\mathfrak{H}^+$), there is then, with a), an i+1 ∈ Dom(AVS($\mathfrak{H}^+$)) with $\mathfrak{H}^+_{i+1}$ = ⌜Suppose Δ*⌝. With b), we then have i ∈ Dom(AVS($\mathfrak{H}$)) and, with a), $\mathfrak{H}^+_{i+1}$ = [β, α, $\mathfrak{H}_i$]. Thus we have [β, α, $\mathfrak{H}_i$] = ⌜Suppose Δ*⌝, and thus [α, β, [β, α, $\mathfrak{H}_i$]] = [α, β, ⌜Suppose Δ*⌝] = ⌜Suppose [α, β, Δ*]⌝ = ⌜Suppose Δ⌝. With Theorem 1-24-(iii) and β ∉ STSEQ($\mathfrak{H}$), we then have [α, β, [β, α, $\mathfrak{H}_i$]] = [α, α, $\mathfrak{H}_i$] = $\mathfrak{H}_i$ and thus $\mathfrak{H}_i$ = ⌜Suppose Δ⌝ and P($\mathfrak{H}_i$) = Δ. Thus we have Δ ∈ AVAP($\mathfrak{H}$). Hence we have (iii).

Ad (iv): Suppose $\mathfrak{H}$ ≠ ∅. Because of β ∉ STSEQ($\mathfrak{H}$) and a) and Theorem 1-24-(ii), we have [α, β, C($\mathfrak{H}^+$)] = [α, β, P($\mathfrak{H}^+_{\mathrm{Dom}(\mathfrak{H}^+)-1}$)] = [α, β, [β, α, P($\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H}^+)-2}$)]] = [α, α, P($\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H}^+)-2}$)] = P($\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H}^+)-2}$). We have Dom($\mathfrak{H}^+$) = Dom($\mathfrak{H}$)+1. Hence we have [α, β, C($\mathfrak{H}^+$)] = P($\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H}^+)-2}$) = P($\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-1}$) = C($\mathfrak{H}$).

Now for the proof by induction: Suppose $\mathfrak{H}^+$ ∈ RCS and b) hold for all k < Dom($\mathfrak{H}$). Suppose Dom($\mathfrak{H}$) = 0. Then we have $\mathfrak{H}$ = ∅ = {(l+1 | l ∈ Dom(AVS($\mathfrak{H}$))}. With a) and Definition 3-16, we have $\mathfrak{H}^+$ = {(0, ⌜Therefore β = β⌝)} ∈ IIF(∅) ⊆ RCS. Obviously, we have Dom(AVS($\mathfrak{H}^+$)) = {0} = {(l+1 | l ∈ Dom(AVS($\mathfrak{H}$))} ∪ {0}. Now, suppose 0 < Dom($\mathfrak{H}$). Then we have $\mathfrak{H}$ ∈ RCS\{∅}. With Theorem 3-6, we then have $\mathfrak{H}$ ∈ RCE($\mathfrak{H}$↾Dom($\mathfrak{H}$)-1). According to the I.H., we then have

c) $\mathfrak{H}^*$ = {(0, ⌜Therefore β = β⌝)}⁀[β, α, $\mathfrak{H}$↾Dom($\mathfrak{H}$)-1] ∈ RCS and Dom(AVS($\mathfrak{H}^*$)) = {l+1 | l ∈ Dom(AVS($\mathfrak{H}$↾Dom($\mathfrak{H}$)-1))} ∪ {0}.

With $\mathfrak{H}$ ∈ RCE($\mathfrak{H}$↾Dom($\mathfrak{H}$)-1) and Definition 3-18, we have that $\mathfrak{H}$ ∈ AF($\mathfrak{H}$↾Dom($\mathfrak{H}$)-1) or $\mathfrak{H}$ ∈ CdIF($\mathfrak{H}$↾Dom($\mathfrak{H}$)-1) or $\mathfrak{H}$ ∈ CdEF($\mathfrak{H}$↾Dom($\mathfrak{H}$)-1) or $\mathfrak{H}$ ∈ CIF($\mathfrak{H}$↾Dom($\mathfrak{H}$)-1) or $\mathfrak{H}$ ∈ CEF($\mathfrak{H}$↾Dom($\mathfrak{H}$)-1) or $\mathfrak{H}$ ∈ BIF($\mathfrak{H}$↾Dom($\mathfrak{H}$)-1) or $\mathfrak{H}$ ∈ BEF($\mathfrak{H}$↾Dom($\mathfrak{H}$)-1) or $\mathfrak{H}$ ∈ DIF($\mathfrak{H}$↾Dom($\mathfrak{H}$)-1) or $\mathfrak{H}$ ∈ DEF($\mathfrak{H}$↾Dom($\mathfrak{H}$)-1) or $\mathfrak{H}$ ∈ NIF($\mathfrak{H}$↾Dom($\mathfrak{H}$)-1) or $\mathfrak{H}$ ∈ NEF($\mathfrak{H}$↾Dom($\mathfrak{H}$)-1) or $\mathfrak{H}$ ∈ UIF($\mathfrak{H}$↾Dom($\mathfrak{H}$)-1) or $\mathfrak{H}$ ∈ UEF($\mathfrak{H}$↾Dom($\mathfrak{H}$)-1) or $\mathfrak{H}$ ∈ PIF($\mathfrak{H}$↾Dom($\mathfrak{H}$)-1) or $\mathfrak{H}$ ∈ PEF($\mathfrak{H}$↾Dom($\mathfrak{H}$)-1) or $\mathfrak{H}$ ∈ IIF($\mathfrak{H}$↾Dom($\mathfrak{H}$)-1) or $\mathfrak{H}$ ∈ IEF($\mathfrak{H}$↾Dom($\mathfrak{H}$)-1).

Since operators are not affected by substitution, we have

d) For all i ∈ Dom($\mathfrak{H}$)-1: P($\mathfrak{H}^*_{i+1}$) = [β, α, P($\mathfrak{H}_i$)] and $\mathfrak{H}^*_{i+1}$ = ⌜Ξ [β, α, P($\mathfrak{H}_i$)]⌝, where $\mathfrak{H}_i$ = ⌜Ξ P($\mathfrak{H}_i$)⌝ for a Ξ ∈ PERF.

With $\beta \in$ PAR\STSEQ($\mathfrak{H}$) and $\alpha \in$ CONST, we also have

e) For all $i \in$ Dom($\mathfrak{H}$): $\beta \notin$ ST(P($\mathfrak{H}_i$)) and $\alpha \notin$ ST([β, α, P($\mathfrak{H}_i$)]),

because, if not, we would have $\beta \in$ STSEQ($\mathfrak{H}$) or $\alpha = \beta$, which contradicts the hypothesis and Postulate 1-1 respectively. With a), it holds that

f) $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(\text{Dom}(\mathfrak{H}^*), \mathfrak{H}^+_{\text{Dom}(\mathfrak{H}^*)})\} = \mathfrak{H}^* \cup \{(\text{Dom}(\mathfrak{H}), [\beta, \alpha, \mathfrak{H}_{\text{Dom}(\mathfrak{H})-1}])\}$.

Now, we will show that in each of the cases AF … IEF it holds that $\mathfrak{H}^+ \in$ RCS and that b), with which $\mathfrak{H}^+$ is then in each case the desired RCS-element. In order to ease the treatment of CdEF, CIF, CEF, BIF, BEF, DIF, DEF, NEF, UIF, UEF, PIF, IIF and IEF, we will now first show that

g) If $\mathfrak{H}^+ \in$ CdIF($\mathfrak{H}^*$) $\cup$ NIF($\mathfrak{H}^*$) $\cup$ PEF($\mathfrak{H}^*$), then $\mathfrak{H} \in$ CdIF($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1) $\cup$ NIF($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1) $\cup$ PEF($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1).

Preparatory part: Suppose $\mathfrak{H}^+ \in$ CdIF($\mathfrak{H}^*$). According to Definition 3-2 and with c) and f), there is then an $i \in$ Dom(AVAS($\mathfrak{H}^*$)) such that there is no l such that $i < l \leq$ Dom($\mathfrak{H}$)-1 and $l \in$ Dom(AVAS($\mathfrak{H}^*$)), and $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore P}(\mathfrak{H}^*_i) \rightarrow \text{C}(\mathfrak{H}^*)\urcorner)\}$. We have $\mathfrak{H}^*_0 = \ulcorner$Therefore $\beta = \beta\urcorner \notin$ AVAS($\mathfrak{H}^*$). Therefore we have $i \neq 0$. With d), we have P($\mathfrak{H}^*_i$) = [β, α, P($\mathfrak{H}_{i-1}$)] and C($\mathfrak{H}^*$) = [β, α, P($\mathfrak{H}_{\text{Dom}(\mathfrak{H})-2}$)]. Therefore we have $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } [\beta, \alpha, \text{P}(\mathfrak{H}_{i-1})] \rightarrow [\beta, \alpha, \text{P}(\mathfrak{H}_{\text{Dom}(\mathfrak{H})-2})]\urcorner)\}$. With f), it holds that $\ulcorner$Therefore [β, α, P($\mathfrak{H}_{i-1}$)] $\rightarrow$ [β, α, P($\mathfrak{H}_{\text{Dom}(\mathfrak{H})-2}$)]$\urcorner$ = [β, α, $\ulcorner$Therefore P($\mathfrak{H}_{i-1}$) $\rightarrow$ P($\mathfrak{H}_{\text{Dom}(\mathfrak{H})-2}$)$\urcorner$] = [$\beta$, α, $\mathfrak{H}_{\text{Dom}(\mathfrak{H})-1}$]. Theorem 1-21 then yields $\ulcorner$Therefore P($\mathfrak{H}_{i-1}$) $\rightarrow$ P($\mathfrak{H}_{\text{Dom}(\mathfrak{H})-2}$)$\urcorner$ = $\mathfrak{H}_{\text{Dom}(\mathfrak{H})-1}$ and thus we have $\mathfrak{H} = \mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1 $\cup$ {(Dom($\mathfrak{H}$)-1, $\ulcorner$Therefore P($\mathfrak{H}_{i-1}$) $\rightarrow$ P($\mathfrak{H}_{\text{Dom}(\mathfrak{H})-2}$)$\urcorner$)}. With c), d) and $i \neq 0$, we also have i-1 $\in$ Dom(AVAS($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1)) and there is no l such that i-1 $< l \leq$ Dom($\mathfrak{H}$)-2 and $l \in$ Dom(AVAS($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1)). Hence we have $\mathfrak{H} \in$ CdIF($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1). In the case that $\mathfrak{H}^+ \in$ NIF($\mathfrak{H}^*$), one shows analogously that then also $\mathfrak{H} \in$ NIF($\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1).

Now, suppose $\mathfrak{H}^+ \in$ PEF($\mathfrak{H}^*$). According to Definition 3-15 and with c), d) and f), there are then $\beta^* \in$ PAR, $\zeta \in$ VAR, $\Delta \in$ FORM, where FV(Δ) $\subseteq \{\zeta\}$, and $i \in$ Dom(AVS($\mathfrak{H}^*$)) such that P($\mathfrak{H}^*_i$) = $\ulcorner\bigvee\zeta\Delta\urcorner$ and P($\mathfrak{H}^*_{i+1}$) = [β^*, ζ, Δ] = [β, α, P($\mathfrak{H}_i$)], where i+1 $\in$ Dom(AVAS($\mathfrak{H}^*$)), [β, α, P($\mathfrak{H}_{\text{Dom}(\mathfrak{H})-2}$)] = C($\mathfrak{H}^*$), $\beta^* \notin$ STSF({Δ, [β, α, P($\mathfrak{H}_{\text{Dom}(\mathfrak{H})-2}$)]}), there is no $j \leq i$ such that $\beta^* \in$ ST($\mathfrak{H}^*_j$), there is no l such that i+1 $< l \leq$ Dom($\mathfrak{H}$)-1 and $l \in$ Dom(AVAS($\mathfrak{H}^*$)), and $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore C}(\mathfrak{H}^*)\urcorner)\}$ = $\mathfrak{H}^* \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } [\beta, \alpha, \text{P}(\mathfrak{H}_{\text{Dom}(\mathfrak{H})-2})]\urcorner)\}$ = $\mathfrak{H}^* \cup$ {(Dom($\mathfrak{H}$), [β, α, $\ulcorner$Therefore

$P(\mathfrak{H}_{Dom(\mathfrak{H})-2})\urcorner])\}$. With f), we have $[\beta, \alpha, \ulcorner\text{Therefore } P(\mathfrak{H}_{Dom(\mathfrak{H})-2})\urcorner] = [\beta, \alpha, \mathfrak{H}_{Dom(\mathfrak{H})-1}]$. Theorem 1-21 then yields $\ulcorner\text{Therefore } P(\mathfrak{H}_{Dom(\mathfrak{H})-2})\urcorner = \mathfrak{H}_{Dom(\mathfrak{H})-1}$ and thus $\mathfrak{H} = \mathfrak{H}\restriction Dom(\mathfrak{H})\text{-}1 \cup \{(Dom(\mathfrak{H})\text{-}1, \ulcorner\text{Therefore } P(\mathfrak{H}_{Dom(\mathfrak{H})-2})\urcorner)\}$. With $P(\mathfrak{H}^*_i) = \ulcorner\bigvee\zeta\Delta\urcorner \neq \ulcorner\beta = \beta\urcorner = P(\mathfrak{H}^*_0)$, it holds that $i \neq 0$ and thus that $P(\mathfrak{H}^*_i) = \ulcorner\bigvee\zeta\Delta\urcorner = [\beta, \alpha, P(\mathfrak{H}_{i-1})]$.

With c), d) and $i \neq 0$, we have $i\text{-}1 \in Dom(AVS(\mathfrak{H}\restriction Dom(\mathfrak{H})\text{-}1))$, $i \in Dom(AVAS(\mathfrak{H}\restriction Dom(\mathfrak{H})\text{-}1))$ and there is no l such that $i < l \leq Dom(\mathfrak{H})\text{-}2$ and $l \in Dom(AVAS(\mathfrak{H}\restriction Dom(\mathfrak{H})\text{-}1))$. Now, we have to show that $P(\mathfrak{H}_{i-1})$, $P(\mathfrak{H}_i)$ and $P(\mathfrak{H}_{Dom(\mathfrak{H})-2})$ satisfy the requirements for $\mathfrak{H} \in PEF(\mathfrak{H}\restriction Dom(\mathfrak{H})\text{-}1)$.

We have $[\beta, \alpha, P(\mathfrak{H}_{i-1})] = P(\mathfrak{H}^*_i) = \ulcorner\bigvee\zeta\Delta\urcorner$ and $[\beta, \alpha, P(\mathfrak{H}_i)] = P(\mathfrak{H}^*_{i+1}) = [\beta^*, \zeta, \Delta]$. Since operators are not affected by substitution, we thus have because of $[\beta, \alpha, P(\mathfrak{H}_{i-1})] = \ulcorner\bigvee\zeta\Delta\urcorner$: $P(\mathfrak{H}_{i-1}) = \ulcorner\bigvee\zeta\Delta^+\urcorner$ for a $\Delta^+ \in FORM$, where $\beta \notin ST(\Delta^+)$ and $FV(\Delta^+) \subseteq \{\zeta\}$. Thus we have $\ulcorner\bigvee\zeta\Delta\urcorner = [\beta, \alpha, P(\mathfrak{H}_{i-1})] = [\beta, \alpha, \ulcorner\bigvee\zeta\Delta^+\urcorner] = \ulcorner\bigvee\zeta[\beta, \alpha, \Delta^+]\urcorner$ and hence $\Delta = [\beta, \alpha, \Delta^+]$. Thus we have $[\beta, \alpha, P(\mathfrak{H}_i)] = [\beta^*, \zeta, \Delta] = [\beta^*, \zeta, [\beta, \alpha, \Delta^+]]$ and $\beta^* \notin ST([\beta, \alpha, \Delta^+])$. Also, we have $\beta = \beta^*$ or $\beta \neq \beta^*$. If $\beta = \beta^*$, then there would be no $j \leq i$ such that $\beta \in ST(\mathfrak{H}^*_j)$. However, we have $\beta \in ST(\ulcorner\text{Therefore } \beta = \beta\urcorner) = ST(\mathfrak{H}^*_0)$ and $0 \leq i$. Therefore we have $\beta \neq \beta^*$. With $\beta^* \in ST([\beta, \alpha, P(\mathfrak{H}_i)])$ and $\beta^* \notin ST([\beta, \alpha, P(\mathfrak{H}_i)])$, we can then distinguish two cases.

First case: Suppose $\beta^* \in ST([\beta, \alpha, P(\mathfrak{H}_i)])$. With $\Delta = [\beta, \alpha, \Delta^+]$ and Theorem 1-25-(ii), we have $[\beta, \alpha, P(\mathfrak{H}_i)] = [\beta^*, \zeta, \Delta] = [\beta^*, \zeta, [\beta, \alpha, \Delta^+]] = [\beta, \alpha, [\beta^*, \zeta, \Delta^+]]$. We have that $\beta \notin ST(P(\mathfrak{H}_i))$ and, because of $\beta \neq \beta^*$ and $\beta \notin ST(\Delta^+)$, also $\beta \notin ST([\beta^*, \zeta, \Delta^+])$ and thus, with Theorem 1-20, $P(\mathfrak{H}_i) = [\beta^*, \zeta, \Delta^+]$. Now, suppose for contradiction that $\beta^* \in STSF(\{\Delta^+, P(\mathfrak{H}_{Dom(\mathfrak{H})-2})\})$ or that there is a $j \leq i\text{-}1$ such that $\beta^* \in ST(\mathfrak{H}_j)$. Because of $\beta^* \neq \alpha$ and with d), we would then also have $\beta^* \in STSF(\{[\beta, \alpha, \Delta^+], [\beta, \alpha, P(\mathfrak{H}_{Dom(\mathfrak{H})-2})]\})$ or there would be a $j \leq i$ such that $\beta^* \in ST(\mathfrak{H}^*_j)$. Contradiction! Thus the parameter conditions for β^* are also satisfied in $\mathfrak{H}\restriction Dom(\mathfrak{H})\text{-}1$ and hence we have $\mathfrak{H} \in PEF(\mathfrak{H}\restriction Dom(\mathfrak{H})\text{-}1)$.

Second case: Now, suppose $\beta^* \notin ST([\beta, \alpha, P(\mathfrak{H}_i)])$. With $[\beta, \alpha, P(\mathfrak{H}_i)] = [\beta^*, \zeta, [\beta, \alpha, \Delta^+]]$, we then have $\zeta \notin FV([\beta, \alpha, \Delta^+])$. Then we have $[\beta, \alpha, P(\mathfrak{H}_i)] = [\beta^*, \zeta, [\beta, \alpha, \Delta^+]] = [\beta, \alpha, \Delta^+]$ and thus, with $\beta \notin ST(P(\mathfrak{H}_i)) \cup ST(\Delta^+)$ and Theorem 1-20, $P(\mathfrak{H}_i) = \Delta^+$, where, with $\zeta \notin FV([\beta, \alpha, \Delta^+])$, also $\zeta \notin FV(\Delta^+)$. Now, let $\beta^+ \in PAR\backslash STSEQ(\mathfrak{H}\restriction Dom(\mathfrak{H})\text{-}1)$. With $\zeta \notin FV(\Delta^+)$, we then have $P(\mathfrak{H}_i) = \Delta^+ = [\beta^+, \zeta, \Delta^+]$ and it holds that $\beta^+ \notin STSF(\{\Delta^+, P(\mathfrak{H}_{Dom(\mathfrak{H})-2})\})$ and that there is no $j \leq i$ such that $\beta^+ \in ST(\mathfrak{H}_j)$. Hence we have again $\mathfrak{H} \in PEF(\mathfrak{H}\restriction Dom(\mathfrak{H})\text{-}1)$. Therefore we have in both cases $\mathfrak{H} \in PEF(\mathfrak{H}\restriction Dom(\mathfrak{H})\text{-}1)$.

Main part: Now we will show that in each of the cases AF … IEF it holds that $\mathfrak{H}^+ \in \mathrm{RCS}$ and $\mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^+)) = \{l+1 \mid l \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}))\} \cup \{0\}$. First we will deal with CdIF, NIF and PEF. Then we can make an exclusion assumption that allows us to determine $\mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^+))$ for all other cases just with c), g) and Theorem 3-25.

(*CdIF*, *NIF*): Suppose $\mathfrak{H} \in \mathrm{CdIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. According to Definition 3-2, there is then an $i \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1))$ such that there is no $l \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1))$ such that $i < l \leq \mathrm{Dom}(\mathfrak{H})\text{-}2$, and $\mathfrak{H} = \mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1 \cup \{(\mathrm{Dom}(\mathfrak{H})\text{-}1, \ulcorner\text{Therefore } \mathrm{P}(\mathfrak{H}_i) \rightarrow \mathrm{C}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\urcorner)\}$. With a), d) and f), it then holds that $i+1 \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^*))$ and that there is no l such that $i+1 < l \leq \mathrm{Dom}(\mathfrak{H})\text{-}1 = \mathrm{Dom}(\mathfrak{H}^*)\text{-}1$ and $l \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^*))$, and $\mathrm{P}(\mathfrak{H}^*_{i+1}) = [\beta, \alpha, \mathrm{P}(\mathfrak{H}_i)]$ and $\mathrm{C}(\mathfrak{H}^*) = [\beta, \alpha, \mathrm{C}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)]$ and $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H}), [\beta, \alpha, \ulcorner\text{Therefore } \mathrm{P}(\mathfrak{H}_i) \rightarrow \mathrm{C}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\urcorner])\} = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } [\beta, \alpha, \mathrm{P}(\mathfrak{H}_i)] \rightarrow [\beta, \alpha, \mathrm{C}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)]\urcorner)\} = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \mathrm{P}(\mathfrak{H}^*_{i+1}) \rightarrow \mathrm{C}(\mathfrak{H}^*)\urcorner)\}$. Hence we have $\mathfrak{H}^+ \in \mathrm{CdIF}(\mathfrak{H}^*)$ and thus $\mathfrak{H}^+ \in \mathrm{RCS}$.

With Theorem 3-19-(iii), we then have $\mathrm{AVS}(\mathfrak{H}) = \mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\setminus\{(j, \mathfrak{H}_j) \mid i \leq j < \mathrm{Dom}(\mathfrak{H})\text{-}1\} \cup \{(\mathrm{Dom}(\mathfrak{H})\text{-}1, \ulcorner\text{Therefore } \mathrm{P}(\mathfrak{H}_i) \rightarrow \mathrm{C}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\urcorner)\}$ and $\mathrm{AVS}(\mathfrak{H}^+) = \mathrm{AVS}(\mathfrak{H}^*)\setminus\{(j, \mathfrak{H}^+_j) \mid i+1 \leq j < \mathrm{Dom}(\mathfrak{H})\} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } [\beta, \alpha, \mathrm{P}(\mathfrak{H}_i)] \rightarrow [\beta, \alpha, \mathrm{C}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)]\urcorner)\}$. With $\mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^*)) = \{l+1 \mid l \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1))\} \cup \{0\}$ it then follows that also $\mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^+)) = \{l+1 \mid l \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}))\} \cup \{0\}$. In the case that $\mathfrak{H} \in \mathrm{NIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$, one shows analogously that then also $\mathfrak{H}^+ \in \mathrm{NIF}(\mathfrak{H}^*) \subseteq \mathrm{RCS}$ and $\mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^+)) = \{l+1 \mid l \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}))\} \cup \{0\}$.

(*PEF*): Now, suppose $\mathfrak{H} \in \mathrm{PEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. According to Definition 3-15, there are then $\beta^* \in \mathrm{PAR}$, $\zeta \in \mathrm{VAR}$, $\Delta \in \mathrm{FORM}$, where $\mathrm{FV}(\Delta) \subseteq \{\zeta\}$, and $i \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1))$ such that $\mathrm{P}(\mathfrak{H}_i) = \ulcorner\bigvee\zeta\Delta\urcorner$, $\mathrm{P}(\mathfrak{H}_{i+1}) = [\beta^*, \zeta, \Delta]$, where $i+1 \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1))$, $\beta^* \notin \mathrm{STSF}(\{\Delta, \mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})\text{-}2})\})$, there is no $j \leq i$ such that $\beta^* \in \mathrm{ST}(\mathfrak{H}_j)$, there is no l such that $i+1 < l \leq \mathrm{Dom}(\mathfrak{H})\text{-}2$ and $l \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1))$, and $\mathfrak{H} = \mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1 \cup \{(\mathrm{Dom}(\mathfrak{H})\text{-}1, \ulcorner\text{Therefore } \mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})\text{-}2})\urcorner)\}$.

With c), d) and f), it then follows that $i+1 \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^*))$ and $\mathrm{P}(\mathfrak{H}^*_{i+1}) = [\beta, \alpha, \mathrm{P}(\mathfrak{H}_i)] = [\beta, \alpha, \ulcorner\bigvee\zeta\Delta\urcorner] = \ulcorner\bigvee\zeta[\beta, \alpha, \Delta]\urcorner$, $i+2 \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^*))$ and $\mathrm{P}(\mathfrak{H}^*_{i+2}) = [\beta, \alpha, \mathrm{P}(\mathfrak{H}_{i+1})] = [\beta, \alpha, [\beta^*, \zeta, \Delta]]$, $\mathrm{C}(\mathfrak{H}^*) = \mathrm{P}(\mathfrak{H}^*_{\mathrm{Dom}(\mathfrak{H})\text{-}1}) = [\beta, \alpha, \mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})\text{-}2})]$ and $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H}), [\beta, \alpha, \ulcorner\text{Therefore } \mathrm{C}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\urcorner])\} = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } [\beta, \alpha, \mathrm{C}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)]\urcorner)\} = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \mathrm{C}(\mathfrak{H}^*)\urcorner)\}$, and that there is no l such that $i+2 < l \leq \mathrm{Dom}(\mathfrak{H})-1 = \mathrm{Dom}(\mathfrak{H}^*)-1$ and $l \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^*))$. With $\beta^* \neq \beta$ and $\beta^* = \beta$, we can distinguish two cases.

First case: Suppose $\beta^* \neq \beta$. With Theorem 1-25-(ii), we have $\mathrm{P}(\mathfrak{H}^*_{i+2}) = [\beta, \alpha, [\beta^*, \zeta, \Delta]] = [\beta^*, \zeta, [\beta, \alpha, \Delta]]$. Also, we have $\mathrm{P}(\mathfrak{H}^*_{i+1}) = \ulcorner\bigvee\zeta[\beta, \alpha, \Delta]\urcorner$. If $\beta^* \in \mathrm{STSF}(\{[\beta, \alpha, \Delta], [\beta, \alpha, \mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-2})]\})$ or if there was a $j \leq i+1$ such that $\beta^* \in \mathrm{ST}(\mathfrak{H}^*_j)$, then we would have, because of $\beta^* \neq \beta$ and with d), also $\beta^* \in \mathrm{STSF}(\{\Delta, \mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-2})\})$ or there would be a $j \leq i$ such that $\beta^* \in \mathrm{ST}(\mathfrak{H}_j)$. Contradiction! Therefore we have $\beta^* \notin \mathrm{STSF}(\{[\beta, \alpha, \Delta], [\beta, \alpha, \mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-2})]\})$ and there is no $j \leq i+1$ such that $\beta^* \in \mathrm{ST}(\mathfrak{H}^*_j)$ and hence we have that $\mathfrak{H}^+ \in \mathrm{PEF}(\mathfrak{H}^*)$ and thus $\mathfrak{H}^+ \in \mathrm{RCS}$.

Second case: Now, suppose $\beta^* = \beta$. Then we have $\zeta \notin \mathrm{FV}(\Delta)$, because, if not, we would have $\beta \in \mathrm{ST}([\beta^*, \zeta, \Delta]) \subseteq \mathrm{STSEQ}(\mathfrak{H})$. Then we have $[\beta^*, \zeta, \Delta] = \Delta$ and thus $\mathrm{P}(\mathfrak{H}^*_{i+2}) = [\beta, \alpha, [\beta^*, \zeta, \Delta]] = [\beta, \alpha, \Delta]$ and we have $\mathrm{P}(\mathfrak{H}^*_{i+1}) = \ulcorner\bigvee\zeta[\beta, \alpha, \Delta]\urcorner$. Now, let $\beta^+ \in \mathrm{PAR}\setminus\mathrm{STSEQ}(\mathfrak{H}^*)$. Then with $\zeta \notin \mathrm{FV}(\Delta)$ also $\zeta \notin \mathrm{FV}([\beta, \alpha, \Delta])$ and thus $\mathrm{P}(\mathfrak{H}^*_{i+2}) = [\beta, \alpha, \Delta] = [\beta^+, \zeta, [\beta, \alpha, \Delta]]$ and it holds that $\beta^+ \notin \mathrm{STSF}(\{[\beta, \alpha, \Delta], [\beta, \alpha, \mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-2})]\})$ and that there is no $j \leq i+1$ such that $\beta^+ \in \mathrm{ST}(\mathfrak{H}^*_j)$. Hence we have again $\mathfrak{H}^+ \in \mathrm{PEF}(\mathfrak{H}^*)$ and thus $\mathfrak{H}^+ \in \mathrm{RCS}$. Thus we have in both cases $\mathfrak{H}^+ \in \mathrm{PEF}(\mathfrak{H}^*)$ and thus $\mathfrak{H}^+ \in \mathrm{RCS}$.

With Theorem 3-21-(iii), we have that $\mathrm{AVS}(\mathfrak{H}) = \mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})-1)\setminus\{(j, \mathfrak{H}_j) \mid i+1 \leq j < \mathrm{Dom}(\mathfrak{H})-1\} \cup \{(\mathrm{Dom}(\mathfrak{H})-1, \ulcorner\text{Therefore } \mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-2})\urcorner)\}$ and that $\mathrm{AVS}(\mathfrak{H}^+) = \mathrm{AVS}(\mathfrak{H}^*)\setminus\{(j, \mathfrak{H}^+_j) \mid i+2 \leq j < \mathrm{Dom}(\mathfrak{H})\} \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } [\beta, \alpha, \mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-2})]\urcorner)\}$. With $\mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^*)) = \{l+1 \mid l \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})-1))\} \cup \{0\}$, it then follows that $\mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^+)) = \{l+1 \mid l \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}))\} \cup \{0\}$.

Exclusion assumption: For the remaining cases suppose $\mathfrak{H} \notin \mathrm{CdIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})-1) \cup \mathrm{NIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})-1) \cup \mathrm{PEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})-1)$. With g), we then have $\mathfrak{H}^+ \notin \mathrm{CdIF}(\mathfrak{H}^*) \cup \mathrm{NIF}(\mathfrak{H}^*) \cup \mathrm{PEF}(\mathfrak{H}^*)$. With Theorem 3-25, we thus have for all of the following cases that $\mathrm{AVS}(\mathfrak{H}) = \mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})-1) \cup \{(\mathrm{Dom}(\mathfrak{H})-1, \mathrm{C}(\mathfrak{H}))\}$ and that $\mathrm{AVS}(\mathfrak{H}^+) = \mathrm{AVS}(\mathfrak{H}^*) \cup \{(\mathrm{Dom}(\mathfrak{H}), \mathrm{C}(\mathfrak{H}^*))\}$. With $\mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^*)) = \{l+1 \mid l \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})-1))\} \cup \{0\}$ it then holds for all remaining cases that $\mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^+)) = \{l+1 \mid l \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}))\} \cup \{0\}$.

(*AF*): Suppose $\mathfrak{H} \in \mathrm{AF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})-1)$. According to Definition 3-1, we then have $\mathfrak{H} = \mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})-1 \cup \{(\mathrm{Dom}(\mathfrak{H})-1, \ulcorner\text{Suppose } \mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-1})\urcorner)$. With f), we then have $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Suppose } [\beta, \alpha, \mathrm{P}(\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H})-1})]\urcorner)\} \in \mathrm{AF}(\mathfrak{H}^*)$ and thus $\mathfrak{H}^+ \in \mathrm{RCS}$.

(*CdEF*, *CIF*, *CEF*, *BIF*, *BEF*, *DIF*, *DEF*, *NEF*): Now, suppose $\mathfrak{H} \in \mathrm{CdEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})-1)$. According to Definition 3-3, there are then $\mathrm{A}, \mathrm{B} \in \mathrm{CFORM}$ such that A, $\ulcorner A \rightarrow B\urcorner \in AVP(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1)$ and $\mathfrak{H} = \mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1 \cup \{(Dom(\mathfrak{H})\text{-}1, \ulcorner \text{Therefore } B\urcorner)\}$. With f), it then follows that: $\mathfrak{H}^{+} = \mathfrak{H}^{*} \cup \{(Dom(\mathfrak{H}), \ulcorner \text{Therefore } [\beta, \alpha, B]\urcorner)\}$. With A, $\ulcorner A \rightarrow B\urcorner \in AVP(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1)$ and Definition 2-30, there are $i, j \in Dom(AVS(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1))$ such that $P(\mathfrak{H}_i) = A$ and $P(\mathfrak{H}_j) = \ulcorner A \rightarrow B\urcorner$. With c) and d), it then follows that $i+1, j+1 \in Dom(AVS(\mathfrak{H}^{*}))$ and $P(\mathfrak{H}^{*}_{i+1}) = [\beta, \alpha, A]$ and $P(\mathfrak{H}^{*}_{j+1}) = \ulcorner[\beta, \alpha, A] \rightarrow [\beta, \alpha, B]\urcorner$. Thus we have $\mathfrak{H}^{+} = \mathfrak{H}^{*} \cup \{(Dom(\mathfrak{H}), \ulcorner \text{Therefore } [\beta, \alpha, B]\urcorner)\} \in CdEF(\mathfrak{H}^{*})$ and thus $\mathfrak{H}^{+} \in RCS$. CIF, CEF, BIF, BEF, DIF, DEF and NEF are treated analogously.

(*UIF*): Now, suppose $\mathfrak{H} \in UIF(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1)$. According to Definition 3-12, there are then $\beta^{*} \in PAR$, $\zeta \in VAR$, $\Delta \in FORM$, where $FV(\Delta) \subseteq \{\zeta\}$, such that $[\beta^{*}, \zeta, \Delta] \in AVP(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1)$, $\beta^{*} \notin STSF(\{\Delta\} \cup AVAP(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1))$ and $\mathfrak{H} = \mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1 \cup \{(Dom(\mathfrak{H})\text{-}1, \ulcorner \text{Therefore } \bigwedge\zeta\Delta\urcorner)\}$. With f), we then have $\mathfrak{H}^{+} = \mathfrak{H}^{*} \cup \{(Dom(\mathfrak{H}), [\beta, \alpha, \ulcorner \text{Therefore } \bigwedge\zeta\Delta\urcorner])\} = \mathfrak{H}^{*} \cup \{(Dom(\mathfrak{H})\text{-}1, \ulcorner \text{Therefore } \bigwedge\zeta[\beta, \alpha, \Delta]\urcorner)\}$. With $[\beta^{*}, \zeta, \Delta] \in AVP(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1)$ and Definition 2-30, there is an $i \in Dom(AVS(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1))$ such that $[\beta^{*}, \zeta, \Delta] = P(\mathfrak{H}_i)$. With a) and d), it then follows that $i+1 \in Dom(AVS(\mathfrak{H}^{*}))$ and that $P(\mathfrak{H}^{*}_{i+1}) = [\beta, \alpha, P(\mathfrak{H}_i)] = [\beta, \alpha, [\beta^{*}, \zeta, \Delta]]$. With $\beta^{*} \neq \beta$ and $\beta^{*} = \beta$, we can distinguish two cases.

First case: Suppose $\beta^{*} \neq \beta$. With Theorem 1-25-(ii), we have $P(\mathfrak{H}^{*}_{i+1}) = [\beta, \alpha, [\beta^{*}, \zeta, \Delta]] = [\beta^{*}, \zeta, [\beta, \alpha, \Delta]]$. We have $C(\mathfrak{H}^{+}) = \ulcorner\bigwedge\zeta[\beta, \alpha, \Delta]\urcorner$. Now, suppose for contradiction that $\beta^{*} \in STSF(\{[\beta, \alpha, \Delta]\} \cup AVAP(\mathfrak{H}^{*}))$. Since $\beta^{*} \neq \beta$ and $\beta^{*} \notin ST(\Delta)$, we have $\beta^{*} \notin ST([\beta, \alpha, \Delta])$. Thus we would have $\beta^{*} \in STSF(AVAP(\mathfrak{H}^{*}))$. With Definition 2-31, there would then be a $j \in Dom(AVAS(\mathfrak{H}^{*}))$ such that $\beta^{*} \in ST(P(\mathfrak{H}^{*}_j))$. With $\mathfrak{H}^{*}_0 \in ISENT$, we have $j \neq 0$. But with d), we would then have $P(\mathfrak{H}^{*}_j) = [\beta, \alpha, P(\mathfrak{H}_{j\text{-}1})]$ and since $\beta^{*} \neq \beta$, we would then have $\beta^{*} \in ST(P(\mathfrak{H}_{j\text{-}1}))$. With c) and d) and $j \in Dom(AVAS(\mathfrak{H}^{*}))$, we would also have that $j\text{-}1 \in Dom(AVAS(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1))$. Thus we would have $P(\mathfrak{H}_{j\text{-}1}) \in AVAP(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1)$ and $\beta^{*} \in STSF(AVAP(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1))$, whereas, by hypothesis, we have $\beta^{*} \notin STSF(AVAP(\mathfrak{H}\upharpoonright Dom(\mathfrak{H})\text{-}1))$. Contradiction! Therefore we have $\beta^{*} \notin STSF(\{[\beta, \alpha, \Delta]\} \cup AVAP(\mathfrak{H}^{*}))$ and hence $\mathfrak{H}^{+} \in UIF(\mathfrak{H}^{*})$.

Second case: Now, suppose $\beta^{*} = \beta$. Then we have $\zeta \notin FV(\Delta)$, because, if not, we would have $\beta \in ST([\beta^{*}, \zeta, \Delta]) \subseteq STSEQ(\mathfrak{H})$. Thus we have $[\beta^{*}, \zeta, \Delta] = \Delta$ and thus $P(\mathfrak{H}^{*}_{i+1}) = [\beta, \alpha, [\beta^{*}, \zeta, \Delta]] = [\beta, \alpha, \Delta]$ and we have $C(\mathfrak{H}^{+}) = \ulcorner\bigwedge\zeta[\beta, \alpha, \Delta]\urcorner$. Now, let $\beta^{+} \in PAR\backslash STSEQ(\mathfrak{H}^{*})$. Then with $\zeta \notin FV(\Delta)$ also $\zeta \notin FV([\beta, \alpha, \Delta])$ and thus $P(\mathfrak{H}^{*}_{i+1}) = [\beta, \alpha,$

$\Delta] = [\beta^+, \zeta, [\beta, \alpha, \Delta]]$ and it holds that $\beta^+ \notin \mathrm{STSF}(\{[\beta, \alpha, \Delta]\} \cup \mathrm{AVAP}(\mathfrak{H}^*))$. Hence we have again $\mathfrak{H}^+ \in \mathrm{UIF}(\mathfrak{H}^*)$. Thus we have in both cases that $\mathfrak{H}^+ \in \mathrm{UIF}(\mathfrak{H}^*) \subseteq \mathrm{RCS}$.

(*UEF*): Now, suppose $\mathfrak{H} \in \mathrm{UEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. According to Definition 3-13, there are then $\theta \in \mathrm{CTERM}$, $\zeta \in \mathrm{VAR}$, $\Delta \in \mathrm{FORM}$, where $\mathrm{FV}(\Delta) \subseteq \{\zeta\}$, such that $\ulcorner\bigwedge\zeta\Delta\urcorner \in \mathrm{AVP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ and $\mathfrak{H} = \mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1 \cup \{(\mathrm{Dom}(\mathfrak{H})\text{-}1, \ulcorner\text{Therefore } [\theta, \zeta, \Delta]\urcorner)\}$. With f), we then have $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H}), [\beta, \alpha, \ulcorner\text{Therefore } [\theta, \zeta, \Delta]\urcorner])\} = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } [\beta, \alpha, [\theta, \zeta, \Delta]]\urcorner)\}$. With $\ulcorner\bigwedge\zeta\Delta\urcorner \in \mathrm{AVP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ and Definition 2-30, there is an $i \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1))$ such that $\mathrm{P}(\mathfrak{H}_i) = \ulcorner\bigwedge\zeta\Delta\urcorner$. With c) and d), we then have $i{+}1 \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^*))$ and $\mathrm{P}(\mathfrak{H}^*_{i+1}) = [\beta, \alpha, \ulcorner\bigwedge\zeta\Delta\urcorner] = \ulcorner\bigwedge\zeta[\beta, \alpha, \Delta]\urcorner$. With Theorem 1-26-(ii), we have $\mathrm{C}(\mathfrak{H}^+) = [\beta, \alpha, [\theta, \zeta, \Delta]] = [[\beta, \alpha, \theta], \zeta, [\beta, \alpha, \Delta]]$, where, with $\theta \in \mathrm{CTERM}$, also $[\beta, \alpha, \theta] \in \mathrm{CTERM}$ and, with $\mathrm{FV}(\Delta) \subseteq \{\zeta\}$, also $\mathrm{FV}([\beta, \alpha, \Delta]) \subseteq \{\zeta\}$. Hence we have $\mathfrak{H}^+ \in \mathrm{UEF}(\mathfrak{H}^*) \subseteq \mathrm{RCS}$.

(*PIF*): Now, suppose $\mathfrak{H} \in \mathrm{PIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. According to Definition 3-14, there are then $\theta \in \mathrm{CTERM}$, $\zeta \in \mathrm{VAR}$, $\Delta \in \mathrm{FORM}$, where $\mathrm{FV}(\Delta) \subseteq \{\zeta\}$, such that $[\theta, \zeta, \Delta] \in \mathrm{AVP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ and $\mathfrak{H} = \mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1 \cup \{(\mathrm{Dom}(\mathfrak{H})\text{-}1, \ulcorner\text{Therefore } \bigvee\zeta\Delta\urcorner)\}$. With f), we then have $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H}), [\beta, \alpha, \ulcorner\text{Therefore } \bigvee\zeta\Delta\urcorner])\} = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \bigvee\zeta[\beta, \alpha, \Delta]\urcorner)\}$. With $[\theta, \zeta, \Delta] \in \mathrm{AVP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ and Definition 2-30, there is an $i \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1))$ such that $\mathrm{P}(\mathfrak{H}_i) = [\theta, \zeta, \Delta]$. With c) and d), we then have $i{+}1 \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}^*))$ and $\mathrm{P}(\mathfrak{H}^*_{i+1}) = [\beta, \alpha, \mathrm{P}(\mathfrak{H}_i)]$. With Theorem 1-26-(ii), we then have $\mathrm{P}(\mathfrak{H}^*_{i+1}) = [\beta, \alpha, \mathrm{P}(\mathfrak{H}_i)] = [\beta, \alpha, [\theta, \zeta, \Delta]] = [[\beta, \alpha, \theta], \zeta, [\beta, \alpha, \Delta]]$, where, with $\theta \in \mathrm{CTERM}$, also $[\beta, \alpha, \theta] \in \mathrm{CTERM}$ and, with $\mathrm{FV}(\Delta) \subseteq \{\zeta\}$, also $\mathrm{FV}([\beta, \alpha, \Delta]) \subseteq \{\zeta\}$. Hence we have $\mathfrak{H}^+ \in \mathrm{PIF}(\mathfrak{H}^*) \subseteq \mathrm{RCS}$.

(*IIF*): Now, suppose $\mathfrak{H} \in \mathrm{IIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. According to Definition 3-16, there is then $\theta \in \mathrm{CTERM}$ such that $\mathfrak{H} = \mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1 \cup \{(\mathrm{Dom}(\mathfrak{H})\text{-}1, \ulcorner\text{Therefore } \theta = \theta\urcorner)\}$. With f), we then have $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H}), [\beta, \alpha, \ulcorner\text{Therefore } \theta = \theta\urcorner])\} = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } [\beta, \alpha, \theta] = [\beta, \alpha, \theta]\urcorner)\}$, where with $\theta \in \mathrm{CTERM}$ also $[\beta, \alpha, \theta] \in \mathrm{CTERM}$. Hence we have $\mathfrak{H}^+ \in \mathrm{IIF}(\mathfrak{H}^*) \subseteq \mathrm{RCS}$.

(*IEF*): Now, suppose $\mathfrak{H} \in \mathrm{IEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. According to Definition 3-17, there are then $\theta_0, \theta_1 \in \mathrm{CTERM}$, $\zeta \in \mathrm{VAR}$ and $\Delta \in \mathrm{FORM}$, where $\mathrm{FV}(\Delta) \subseteq \{\zeta\}$, such that $\ulcorner\theta_0 = \theta_1\urcorner$, $[\theta_0, \zeta, \Delta] \in \mathrm{AVP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ and $\mathfrak{H} = \mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1 \cup \{(\mathrm{Dom}(\mathfrak{H})\text{-}1, \ulcorner\text{Therefore } [\theta_1, \zeta, \Delta]\urcorner)\}$. With f), we then have $\mathfrak{H}^+ = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H}), [\beta, \alpha, \ulcorner\text{Therefore } [\theta_1, \zeta, \Delta]\urcorner])\} = \mathfrak{H}^* \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } [\beta, \alpha, [\theta_1, \zeta, \Delta]]\urcorner)\}$. With $\ulcorner\theta_0 = \theta_1\urcorner$, $[\theta_0, \zeta, \Delta] \in \mathrm{AVP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ and Definition 2-30, there are $i, j \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1))$ such that $P(\mathfrak{H}_i) = \ulcorner\theta_0 = \theta_1\urcorner$ and $P(\mathfrak{H}_j) = [\theta_0, \zeta, \Delta]$. With c) and d), it then holds that i+1, j+1 $\in$ Dom(AVS($\mathfrak{H}$*)) and $P(\mathfrak{H}^*_{i+1}) = [\beta, \alpha, P(\mathfrak{H}_i)] = [\beta, \alpha, \ulcorner\theta_0 = \theta_1\urcorner] = \ulcorner[\beta, \alpha, \theta_0] = [\beta, \alpha, \theta_1]\urcorner$ and $P(\mathfrak{H}^*_{j+1}) = [\beta, \alpha, P(\mathfrak{H}_j)]$. With Theorem 1-26-(ii), we then have $P(\mathfrak{H}^*_{j+1}) = [\beta, \alpha, P(\mathfrak{H}_j)] = [\beta, \alpha, [\theta_0, \zeta, \Delta]] = [[\beta, \alpha, \theta_0], \zeta, [\beta, \alpha, \Delta]]$ and $C(\mathfrak{H}^+) = [\beta, \alpha, [\theta_1, \zeta, \Delta]] = [[\beta, \alpha, \theta_1], \zeta, [\beta, \alpha, \Delta]]$, where with $\theta_0, \theta_1 \in$ CTERM also $[\beta, \alpha, \theta_0], [\beta, \alpha, \theta_1] \in$ CTERM and with FV(Δ) $\subseteq \{\zeta\}$ also FV($[\beta, \alpha, \Delta]$) $\subseteq \{\zeta\}$. Hence we have $\mathfrak{H}^+ \in$ IEF($\mathfrak{H}$*) $\subseteq$ RCS. ■

In the proof of the following theorem, Theorem 4-8 provides the induction basis and is used in the induction step. The theorem prepares the RCS-preserving concatenation of two RCS-elements that share common paramateres.

Theorem 4-10. *Multiple substitution of new and pairwise different parameters for pairwise different parameters is RCS-preserving*
If $\mathfrak{H} \in$ RCS, $k \in \mathbb{N}\backslash\{0\}$ and $\{\beta^*_0, \ldots, \beta^*_{k-1}\} \subseteq$ PAR\STSEQ($\mathfrak{H}$), where for all $i, j < k$ with $i \neq j$ it holds that $\beta^*_i \neq \beta^*_j$, and $\{\beta_0, \ldots, \beta_{k-1}\} \subseteq$ PAR\$\{\beta^*_0, \ldots, \beta^*_{k-1}\}$, where for all $i, j < k$ with $i \neq j$ it holds that $\beta_i \neq \beta_j$, then $[\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \mathfrak{H}] \in$ RCS and Dom(AVS($[\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \mathfrak{H}]$)) = Dom(AVS($\mathfrak{H}$)).

Proof: By induction on k. With Theorem 4-8, the statement holds for k = 1. Now, suppose the statement holds for k. Now, suppose $\mathfrak{H} \in$ RCS, k+1 $\in \mathbb{N}\backslash\{0\}$ and $\{\beta^*_0, \ldots, \beta^*_k\} \subseteq$ PAR\STSEQ($\mathfrak{H}$), where for all $i, j < k$+1 with $i \neq j$ it holds that $\beta^*_i \neq \beta^*_j$, and $\{\beta_0, \ldots, \beta_k\} \subseteq$ PAR\$\{\beta^*_0, \ldots, \beta^*_k\}$, where for all $i, j < k$+1 with $i \neq j$ it holds that $\beta_i \neq \beta_j$. According to the I.H., we then have $[\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \mathfrak{H}] \in$ RCS and Dom(AVS($[\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \mathfrak{H}]$)) = Dom(AVS($\mathfrak{H}$)). With Theorem 1-27-(iv), we have $[\beta^*_k, \beta_k, [\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \mathfrak{H}]] = [\langle\beta^*_0, \ldots, \beta^*_k\rangle, \langle\beta_0, \ldots, \beta_k\rangle, \mathfrak{H}]$. With Theorem 4-8, we thus have $[\langle\beta^*_0, \ldots, \beta^*_k\rangle, \langle\beta_0, \ldots, \beta_k\rangle, \mathfrak{H}] \in$ RCS and Dom(AVS($[\langle\beta^*_0, \ldots, \beta^*_k\rangle, \langle\beta_0, \ldots, \beta_k\rangle, \mathfrak{H}]$)) = Dom(AVS($[\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \mathfrak{H}]$)) = Dom(AVS($\mathfrak{H}$)). ■

Theorem 4-11. *UI-extension of a sentence sequence*

If $\mathfrak{H} \in \text{RCS}\backslash\{\emptyset\}$, $k \in \mathbb{N}\backslash\{0\}$, $\{\xi_0, \ldots, \xi_{k-1}\} \subseteq \text{VAR}$, where for all $i, j < k$ with $i \neq j$ it holds that $\xi_i \neq \xi_j$, $\Delta \in \text{FORM}$, where $\text{FV}(\Delta) \subseteq \{\xi_0, \ldots, \xi_{k-1}\}$, and $\{\beta_0, \ldots, \beta_{k-1}\} \subseteq \text{PAR}\backslash\text{STSF}(\{\Delta\} \cup \text{AVAP}(\mathfrak{H}))$, where for all $i, j < k$ with $i \neq j$ it holds that $\beta_i \neq \beta_j$, and $\text{C}(\mathfrak{H}) = [\langle\beta_0, \ldots, \beta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta]$, then there is an $\mathfrak{H}^* \in \text{RCS}\backslash\{\emptyset\}$ such that

(i) $\text{PAR} \cap \text{STSEQ}(\mathfrak{H}^*) = \text{PAR} \cap \text{STSEQ}(\mathfrak{H})$,

(ii) $\text{AVAP}(\mathfrak{H}^*) \subseteq \text{AVAP}(\mathfrak{H})$, and

(iii) $\text{C}(\mathfrak{H}^*) = \ulcorner\bigwedge\xi_0\ldots\bigwedge\xi_{k-1}\Delta\urcorner$.

Proof: By induction on k. Suppose $k = 1$ and $\mathfrak{H} \in \text{RCS}\backslash\{\emptyset\}$, suppose $\xi \in \text{VAR}$, $\Delta \in \text{FORM}$, where $\text{FV}(\Delta) \subseteq \{\xi\}$, and $\beta \in \text{PAR}\backslash\text{STSF}(\{\Delta\} \cup \text{AVAP}(\mathfrak{H}))$ and $\text{C}(\mathfrak{H}) = [\beta, \xi, \Delta]$. With Theorem 2-82, we have $[\beta, \xi, \Delta] = \text{C}(\mathfrak{H}) \in \text{AVP}(\mathfrak{H})$, and thus, according to Definition 3-12, $\mathfrak{H}^* = \mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \bigwedge\xi\Delta\urcorner)\} \in \text{UIF}(\mathfrak{H}) \subseteq \text{RCS}\backslash\{\emptyset\}$ and $\text{C}(\mathfrak{H}^*) = \ulcorner\bigwedge\xi\Delta\urcorner$. We also have that $\text{PAR} \cap \text{STSEQ}(\mathfrak{H}^*) = (\text{PAR} \cap \text{STSEQ}(\mathfrak{H})) \cup (\text{PAR} \cap \text{ST}(\ulcorner\bigwedge\xi\Delta\urcorner)) = \text{PAR} \cap \text{STSEQ}(\mathfrak{H})$, and, with Theorem 3-26-(v), we have $\text{AVAP}(\mathfrak{H}^*) \subseteq \text{AVAP}(\mathfrak{H})$.

Now, suppose the statement holds for k and suppose $\mathfrak{H} \in \text{RCS}\backslash\{\emptyset\}$, $\{\xi_0, \ldots, \xi_k\} \subseteq \text{VAR}$, where for all $i, j < k+1$ with $i \neq j$ it holds that $\xi_i \neq \xi_j$, $\Delta \in \text{FORM}$, where $\text{FV}(\Delta) \subseteq \{\xi_0, \ldots, \xi_k\}$, and $\{\beta_0, \ldots, \beta_k\} \subseteq \text{PAR}\backslash\text{STSF}(\{\Delta\} \cup \text{AVAP}(\mathfrak{H}))$, where for all $i, j < k+1$ with $i \neq j$ it holds that $\beta_i \neq \beta_j$, and $\text{C}(\mathfrak{H}) = [\langle\beta_0, \ldots, \beta_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \Delta]$. With Theorem 1-28-(ii), we then have $\text{C}(\mathfrak{H}) = [\langle\beta_0, \ldots, \beta_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \Delta] = [\beta_k, \xi_k, [\langle\beta_0, \ldots, \beta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta]]$. With $\text{FV}(\Delta) \subseteq \{\xi_0, \ldots, \xi_k\}$ we then have $\text{FV}\langle[\langle\beta_0, \ldots, \beta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta]) \subseteq \{\xi_k\}$. Since β_i are pairwise different and $\{\beta_0, \ldots, \beta_k\} \subseteq \text{PAR}\backslash\text{STSF}(\{\Delta\} \cup \text{AVAP}(\mathfrak{H}))$, we then have $\beta_k \in \text{PAR}\backslash\text{STSF}(\{[\langle\beta_0, \ldots, \beta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta]\} \cup \text{AVAP}(\mathfrak{H}))$. Since $[\beta_k, \xi_k, [\langle\beta_0, \ldots, \beta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta]] = \text{C}(\mathfrak{H}) \in \text{AVP}(\mathfrak{H})$, we then have, according to Definition 3-12, $\mathfrak{H}' = \mathfrak{H} \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Therefore } \bigwedge\xi_k[\langle\beta_0, \ldots, \beta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta]\urcorner)\} \in \text{UIF}(\mathfrak{H}) \subseteq \text{RCS}\backslash\{\emptyset\}$ and $\text{C}(\mathfrak{H}') = \ulcorner\bigwedge\xi_k[\langle\beta_0, \ldots, \beta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta]\urcorner$ and $\text{PAR} \cap \text{STSEQ}(\mathfrak{H}') = (\text{PAR} \cap \text{STSEQ}(\mathfrak{H})) \cup (\text{PAR} \cap \text{ST}(\ulcorner\bigwedge\xi_k[\langle\beta_0, \ldots, \beta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta]\urcorner)) = \text{PAR} \cap \text{STSEQ}(\mathfrak{H})$ and, with Theorem 3-26-(v), we have $\text{AVAP}(\mathfrak{H}') \subseteq \text{AVAP}(\mathfrak{H})$. Since the ξ_i are pairwise different, we have for all $i < k$: $\xi_i \neq \xi_k$. Thus we then have $\text{C}(\mathfrak{H}) = \ulcorner\bigwedge\xi_k[\langle\beta_0, \ldots, \beta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta]\urcorner = [\langle\beta_0, \ldots, \beta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \ulcorner\bigwedge\xi_k\Delta\urcorner]$. With $\text{FV}(\Delta) \subseteq \{\xi_0, \ldots, \xi_k\}$, we then have $\text{FV}(\ulcorner\bigwedge\xi_k\Delta\urcorner) \subseteq \{\xi_0, \ldots, \xi_{k-1}\}$, where the ξ_i with $i < k$ are pairwise different. With $\{\beta_0, \ldots, \beta_k\} \subseteq \text{PAR}\backslash\text{STSF}(\{\Delta\} \cup \text{AVAP}(\mathfrak{H}))$, we have $\{\beta_0, \ldots, \beta_{k-1}\} \subseteq \text{PAR}\backslash\text{STSF}(\{\ulcorner\bigwedge\xi_k\Delta\urcorner\} \cup \text{AVAP}(\mathfrak{H}))$, where the β_i with $i < k$ are also pairwise different. According to the I.H., there is thus, with $C(\mathfrak{H}') = [\langle\beta_0, \ldots, \beta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \ulcorner\bigwedge\xi_k\Delta\urcorner]$, an $\mathfrak{H}^* \in RCS\backslash\{\emptyset\}$ such that $PAR \cap STSEQ(\mathfrak{H}^*) = PAR \cap STSEQ(\mathfrak{H}') = PAR \cap STSEQ(\mathfrak{H})$, $AVAP(\mathfrak{H}^*) \subseteq AVAP(\mathfrak{H}') \subseteq AVAP(\mathfrak{H})$ and $C(\mathfrak{H}^*) = \ulcorner\bigwedge\xi_0\ldots\bigwedge\xi_k\Delta\urcorner$. ■

Theorem 4-12. *UE-extension of a sentence sequence*

If $\mathfrak{H} \in RCS\backslash\{\emptyset\}$, $k \in \mathbb{N}\backslash\{0\}$, $\{\theta_0, \ldots, \theta_{k-1}\} \subseteq CTERM$, $\{\xi_0, \ldots, \xi_{k-1}\} \subseteq VAR$, where for all $i, j < k$ with $i \neq j$ it holds that $\xi_i \neq \xi_j$, $\Delta \in FORM$, where $FV(\Delta) \subseteq \{\xi_0, \ldots, \xi_{k-1}\}$, and $\ulcorner\bigwedge\xi_0\ldots\bigwedge\xi_{k-1}\Delta\urcorner \in AVP(\mathfrak{H})$, then there is an $\mathfrak{H}^* \in RCS\backslash\{\emptyset\}$ such that

(i) $Dom(\mathfrak{H}^*) = Dom(\mathfrak{H})+k$,

(ii) $\mathfrak{H}^*\upharpoonright Dom(\mathfrak{H}) = \mathfrak{H}$,

(iii) $AVAP(\mathfrak{H}^*) \subseteq AVAP(\mathfrak{H})$,

(iv) For all $i < k$-1: $C(\mathfrak{H}^*\upharpoonright Dom(\mathfrak{H})+i+1) = \ulcorner\bigwedge\xi_{i+1}\ldots\bigwedge\xi_{k-1}[\langle\theta_0, \ldots, \theta_i\rangle, \langle\xi_0, \ldots, \xi_i\rangle, \Delta]\urcorner$, and

(v) $C(\mathfrak{H}^*) = [\langle\theta_0, \ldots, \theta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta]$.

Proof: By induction on k: Suppose $k = 1$. Suppose $\mathfrak{H} \in RCS\backslash\{\emptyset\}$, $\theta \in CTERM$, $\xi \in VAR$, $\Delta \in FORM$, where $FV(\Delta) \subseteq \{\xi\}$, and $\ulcorner\bigwedge\xi\Delta\urcorner \in AVP(\mathfrak{H})$. With Definition 3-13, it then holds that $\mathfrak{H}^* = \mathfrak{H}^\frown\{(0, \ulcorner\text{Therefore } [\theta, \xi, \Delta]\urcorner)\} \in UEF(\mathfrak{H}) \subseteq RCS\backslash\{\emptyset\}$, and it holds that $Dom(\mathfrak{H}^*) = Dom(\mathfrak{H})+1$ and $\mathfrak{H}^*\upharpoonright Dom(\mathfrak{H}) = \mathfrak{H}$ and, with Theorem 3-27-(v), that $AVAP(\mathfrak{H}^*) \subseteq AVAP(\mathfrak{H})$. Because of $k = 1$, clause (iv) is satisfied trivially and we have $C(\mathfrak{H}') = [\theta, \xi, \Delta]$.

Now, suppose the statement holds for k and suppose $\mathfrak{H} \in RCS\backslash\{\emptyset\}$, $\{\theta_0, \ldots, \theta_k\} \subseteq CTERM$, $\{\xi_0, \ldots, \xi_k\} \subseteq VAR$, where for all $i, j < k+1$ with $i \neq j$ it holds that $\xi_i \neq \xi_j$, $\Delta \in FORM$, where $FV(\Delta) \subseteq \{\xi_0, \ldots, \xi_k\}$, and $\ulcorner\bigwedge\xi_0\ldots\bigwedge\xi_k\Delta\urcorner \in AVP(\mathfrak{H})$. With $FV(\Delta) \subseteq \{\xi_0, \ldots, \xi_k\}$, we then have $FV(\bigwedge\xi_1\ldots\bigwedge\xi_k\Delta) \subseteq \{\xi_0\}$ and, with $\theta_0 \in CTERM$ and $\ulcorner\bigwedge\xi_0\ldots\bigwedge\xi_k\Delta\urcorner \in AVP(\mathfrak{H})$ and Definition 3-13, we have $\mathfrak{H}' = \mathfrak{H}^\frown\{(0, \ulcorner\text{Therefore } [\theta_0, \xi_0, \bigwedge\xi_1\ldots\bigwedge\xi_k\Delta]\urcorner)\} \in UEF(\mathfrak{H}) \subseteq RCS\backslash\{\emptyset\}$. Then we have $Dom(\mathfrak{H}') = Dom(\mathfrak{H})+1$ and $\mathfrak{H}'\upharpoonright Dom(\mathfrak{H}) = \mathfrak{H}$ and, with Theorem 3-27-(v), we have $AVAP(\mathfrak{H}') \subseteq AVAP(\mathfrak{H})$. Since the ξ_i are pairwise different, we have for all i with $0 < i \leq k$: $\xi_0 \neq \xi_i$. Thus we then have $C(\mathfrak{H}') = [\theta_0, \xi_0, \ulcorner\bigwedge\xi_1\ldots\bigwedge\xi_k\Delta\urcorner] = \ulcorner\bigwedge\xi_1\ldots\bigwedge\xi_k[\theta_0, \xi_0, \Delta]\urcorner$.

Now, let $\zeta_i = \xi_{i+1}$ and $\theta'_i = \theta_{i+1}$ for all $i \in k$. Then we have $\{\theta'_0, \ldots, \theta'_{k-1}\} \subseteq CTERM$, $\{\zeta_0, \ldots, \zeta_{k-1}\} \subseteq VAR$, where for all $i, j < k$ with $i \neq j$ $\zeta_i \neq \zeta_j$, $[\theta_0, \xi_0, \Delta] \in FORM$, where, with $FV(\Delta) \subseteq \{\xi_0, \ldots, \xi_k\}$ and $\theta_0 \in CTERM$, it holds that $FV([\theta_0, \xi_0, \Delta]) \subseteq \{\xi_1, \ldots, \xi_k\} = \{\zeta_0, \ldots, \zeta_{k-1}\}$, and, with Theorem 2-82, it holds that $\ulcorner\bigwedge\zeta_0\ldots\bigwedge\zeta_{k-1}[\theta_0, \xi_0, \Delta]\urcorner = \ulcorner\bigwedge\xi_1\ldots\bigwedge\xi_k[\theta_0, \xi_0, \Delta]\urcorner = C(\mathfrak{H}') \in AVP(\mathfrak{H}')$. According to the I.H., there is then an $\mathfrak{H}^* \in RCS\backslash\{\emptyset\}$ such that:

a) $\mathrm{Dom}(\mathfrak{H}^*) = \mathrm{Dom}(\mathfrak{H}')+k$,

b) $\mathfrak{H}^*\upharpoonright\mathrm{Dom}(\mathfrak{H}') = \mathfrak{H}'$

c) $\mathrm{AVAP}(\mathfrak{H}^*) \subseteq \mathrm{AVAP}(\mathfrak{H}')$,

d) For all $i < k$-1: $C(\mathfrak{H}^*\upharpoonright\mathrm{Dom}(\mathfrak{H}')+i+1) = \ulcorner\Lambda\zeta_{i+1}\ldots\Lambda\zeta_{k-1}[\langle\theta'_0, \ldots, \theta'_i\rangle, \langle\zeta_0, \ldots, \zeta_i\rangle, [\theta_0, \xi_0, \Delta]]\urcorner$, and

e) $C(\mathfrak{H}^*) = [\langle\theta'_0, \ldots, \theta'_{k-1}\rangle, \langle\zeta_0, \ldots, \zeta_{k-1}\rangle, [\theta_0, \xi_0, \Delta]]$.

With a) and because of $\mathrm{Dom}(\mathfrak{H}') = \mathrm{Dom}(\mathfrak{H})+1$, we then have $\mathrm{Dom}(\mathfrak{H}^*) = \mathrm{Dom}(\mathfrak{H})+k+1$. With b) and because of $\mathfrak{H}'\upharpoonright\mathrm{Dom}(\mathfrak{H}) = \mathfrak{H}$, we also have $\mathfrak{H}^*\upharpoonright\mathrm{Dom}(\mathfrak{H}) = \mathfrak{H}$. With c) and because of $\mathrm{AVAP}(\mathfrak{H}') \subseteq \mathrm{AVAP}(\mathfrak{H})$, we have that $\mathrm{AVAP}(\mathfrak{H}^*) \subseteq \mathrm{AVAP}(\mathfrak{H})$. Thus we have that $\mathfrak{H}^* \in \mathrm{RCS}\setminus\{\emptyset\}$ and that clauses (i) to (iii) hold for $\mathfrak{H}^*$. With d) and $\zeta_i = \xi_{i+1}$ and $\theta'_i = \theta_{i+1}$ we also have

For all $i < k$-1: $C(\mathfrak{H}^*\upharpoonright\mathrm{Dom}(\mathfrak{H}')+i+1) = \ulcorner\Lambda\xi_{i+2}\ldots\Lambda\xi_k[\langle\theta_1, \ldots, \theta_{i+1}\rangle, \langle\xi_1, \ldots, \xi_{i+1}\rangle, [\theta_0, \xi_0, \Delta]]\urcorner$.

With $\mathrm{Dom}(\mathfrak{H}') = \mathrm{Dom}(\mathfrak{H})+1$ we thus have

f) For all $i < k$-1: $C(\mathfrak{H}^*\upharpoonright\mathrm{Dom}(\mathfrak{H})+i+1+1) = \ulcorner\Lambda\xi_{i+2}\ldots\Lambda\xi_k[\langle\theta_1, \ldots, \theta_{i+1}\rangle, \langle\xi_1, \ldots, \xi_{i+1}\rangle, [\theta_0, \xi_0, \Delta]]\urcorner$.

Thus we have

g) For all i with $0 < i < k$: $C(\mathfrak{H}^*\upharpoonright\mathrm{Dom}(\mathfrak{H})+i+1) = \ulcorner\Lambda\xi_{i+1}\ldots\Lambda\xi_k[\langle\theta_1, \ldots, \theta_i\rangle, \langle\xi_1, \ldots, \xi_i\rangle, [\theta_0, \xi_0, \Delta]]\urcorner$.

We also have

h) For all i with $0 < i < k+1$: $[\langle\theta_1, \ldots, \theta_i\rangle, \langle\xi_1, \ldots, \xi_i\rangle, [\theta_0, \xi_0, \Delta]] = [\langle\theta_0, \ldots, \theta_i\rangle, \langle\xi_0, \ldots, \xi_i\rangle, \Delta]$.

h) can be shown by induction on i. First, we have, with Theorem 1-28-(ii), that $[\theta_1, \xi_1, [\theta_0, \xi_0, \Delta]] = [\langle\theta_0, \theta_1\rangle, \langle\xi_0, \xi_1\rangle, \Delta]$. Now, suppose for i it holds that if $0 < i < k+1$, then $[\langle\theta_1, \ldots, \theta_i\rangle, \langle\xi_1, \ldots, \xi_i\rangle, [\theta_0, \xi_0, \Delta]] = [\langle\theta_0, \ldots, \theta_i\rangle, \langle\xi_0, \ldots, \xi_i\rangle, \Delta]$. Now, suppose $0 < i+1 < k+1$. Then we have $i = 0$ or $0 < i$. For $i = 0$, the statement follows in the same way as the induction basis. Now, suppose $0 < i$. With Theorem 1-28-(ii), we first have $[\langle\theta_1, \ldots, \theta_{i+1}\rangle, \langle\xi_1, \ldots, \xi_{i+1}\rangle, [\theta_0, \xi_0, \Delta]] = [\theta_{i+1}, \xi_{i+1}, [\langle\theta_1, \ldots, \theta_i\rangle, \langle\xi_1, \ldots, \xi_i\rangle, [\theta_0, \xi_0, \Delta]]]$. With the I.H., it then holds that $[\theta_{i+1}, \xi_{i+1}, [\langle\theta_1, \ldots, \theta_i\rangle, \langle\xi_1, \ldots, \xi_i\rangle, [\theta_0, \xi_0, \Delta]]] = [\theta_{i+1}, \xi_{i+1}, [\langle\theta_0, \ldots, \theta_i\rangle, \langle\xi_0, \ldots, \xi_i\rangle, \Delta]]$. Again with Theorem 1-28-(ii), we then have $[\theta_{i+1}, \xi_{i+1}, [\langle\theta_0, \ldots, \theta_i\rangle, \langle\xi_0, \ldots, \xi_i\rangle, \Delta]]$

$= [\langle\theta_0, \ldots, \theta_{i+1}\rangle, \langle\xi_0, \ldots, \xi_{i+1}\rangle, \Delta]$ and hence $[\langle\theta_1, \ldots, \theta_{i+1}\rangle, \langle\xi_1, \ldots, \xi_{i+1}\rangle, [\theta_0, \xi_0, \Delta]] = [\langle\theta_0, \ldots, \theta_{i+1}\rangle, \langle\xi_0, \ldots, \xi_{i+1}\rangle, \Delta]$. Therefore we have h).

With Dom($\mathfrak{H}$') = Dom($\mathfrak{H}$)+1 and C($\mathfrak{H}$*↾Dom($\mathfrak{H}$')) = C($\mathfrak{H}$') = ⌜$\bigwedge\xi_1\ldots\bigwedge\xi_k[\theta_0, \xi_0, \Delta]$⌝, we have C($\mathfrak{H}$*↾Dom($\mathfrak{H}$)+0+1) = ⌜$\bigwedge\xi_1\ldots\bigwedge\xi_k[\theta_0, \xi_0, \Delta]$⌝. With g) and h), we thus get that clause (iv) holds:

For all $i < k$: C($\mathfrak{H}$*↾Dom($\mathfrak{H}$)+i+1) = ⌜$\bigwedge\xi_{i+1}\ldots\bigwedge\xi_k[\langle\theta_0, \ldots, \theta_i\rangle, \langle\xi_0, \ldots, \xi_i\rangle, \Delta]$⌝.

Last, it holds, with e), h) and $\theta'_i = \theta_{i+1}$ and $\zeta_i = \xi_{i+1}$ that

C($\mathfrak{H}$*) = $[\langle\theta'_0, \ldots, \theta'_{k-1}\rangle, \langle\zeta_0, \ldots, \zeta_{k-1}\rangle, [\theta_0, \xi_0, \Delta]]$
=
$[\langle\theta_1, \ldots, \theta_k\rangle, \langle\xi_1, \ldots, \xi_k\rangle, [\theta_0, \xi_0, \Delta]]$
=
$[\langle\theta_0, \ldots, \theta_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \Delta]$.

Thus clause (v) holds as well, and hence the theorem holds for k+1. ■

Theorem 4-13. *Induction basis for Theorem 4-14*
If $\mathfrak{H}$, $\mathfrak{H}$' ∈ RCS\{∅} and AVAS($\mathfrak{H}$') = ∅, then there is an $\mathfrak{H}$* ∈ RCS\{∅} such that
(i) C($\mathfrak{H}$), C($\mathfrak{H}$') ∈ AVP($\mathfrak{H}$*) and
(ii) AVAP($\mathfrak{H}$*) ⊆ AVAP($\mathfrak{H}$).

Proof: Suppose $\mathfrak{H}$, $\mathfrak{H}$' ∈ RCS\{∅} and suppose AVAS($\mathfrak{H}$') = ∅. If C($\mathfrak{H}$) = C($\mathfrak{H}$'), we can choose $\mathfrak{H}$ as well as $\mathfrak{H}$' for $\mathfrak{H}$*. Now, suppose C($\mathfrak{H}$) ≠ C($\mathfrak{H}$'). With PAR ∩ STSEQ($\mathfrak{H}$) ∩ STSEQ($\mathfrak{H}$') = ∅ and PAR ∩ STSEQ($\mathfrak{H}$) ∩ STSEQ($\mathfrak{H}$') ≠ ∅, we can then distinguish two cases.

First case: Suppose PAR ∩ STSEQ($\mathfrak{H}$) ∩ STSEQ($\mathfrak{H}$') = ∅. There is an α ∈ CONST\(STSEQ($\mathfrak{H}$) ∪ STSEQ($\mathfrak{H}$')). With Theorem 4-4, there is then an $\mathfrak{H}^+$ ∈ RCS\{∅} such that AVP($\mathfrak{H}$) ∪ AVP($\mathfrak{H}$') ⊆ AVP($\mathfrak{H}^+$) and AVAP($\mathfrak{H}^+$) = AVAP($\mathfrak{H}$) ∪ {⌜α = α⌝} ∪ AVAP($\mathfrak{H}$'). With Theorem 2-82, we have C($\mathfrak{H}$) ∈ AVP($\mathfrak{H}$) and C($\mathfrak{H}$') ∈ AVP($\mathfrak{H}$') and thus we have C($\mathfrak{H}$), C($\mathfrak{H}$') ∈ AVP($\mathfrak{H}^+$). With Theorem 4-7, there is then an $\mathfrak{H}$* ∈ RCS\{∅} such that AVAP($\mathfrak{H}$*) ⊆ AVAP($\mathfrak{H}^+$)\{⌜α = α⌝} = (AVAP($\mathfrak{H}$) ∪ {⌜α = α⌝} ∪ AVAP($\mathfrak{H}$'))\{⌜α = α⌝} ⊆ AVAP($\mathfrak{H}$) ∪ AVAP($\mathfrak{H}$') and C($\mathfrak{H}$), C($\mathfrak{H}$') ∈ AVP($\mathfrak{H}$*), with which $\mathfrak{H}$* is the desired RCS-element.

Second case: Now, suppose PAR ∩ STSEQ($\mathfrak{H}$) ∩ STSEQ($\mathfrak{H}$') ≠ ∅. Then there occur k pairwise different parameters in $\mathfrak{H}$' for a $k \in \mathbb{N}\setminus\{0\}$. Now, let $\{\beta_0, \ldots, \beta_{k-1}\}$ = PAR ∩

STSEQ($\mathfrak{H}$'), where for all $i, j < k$ with $i \neq j$ it holds that $\beta_i \neq \beta_j$. There are $\beta^*_0, \ldots, \beta^*_{k-1} \in$ PAR\(STSEQ($\mathfrak{H}$) ∪ STSEQ($\mathfrak{H}$')), where for all $i, j < k$ it holds that if $i \neq j$, then $\beta^*_i \neq \beta^*_j$. Also, there are $\xi_0, \ldots, \xi_{k-1} \in$ VAR\(STSEQ($\mathfrak{H}$) ∪ STSEQ($\mathfrak{H}$')), where for all $i, j < k$: If $i \neq j$, then $\xi_i \neq \xi_j$.

With Theorem 2-77 and AVAS($\mathfrak{H}$') = ∅, we also have AVAP($\mathfrak{H}$') = ∅. With Theorem 1-16, there is a $\Delta \in$ FORM, where FV(Δ) ⊆ $\{\xi_0, \ldots, \xi_{k-1}\}$ ∪ FV(C($\mathfrak{H}$')) = $\{\xi_0, \ldots, \xi_{k-1}\}$ and ST(Δ) ∩ $\{\beta_0, \ldots, \beta_{k-1}\}$ = ∅, such that C($\mathfrak{H}$') = $[\langle\beta_0, \ldots, \beta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta]$. With Theorem 4-11, it then follows that there is $\mathfrak{H}^1 \in$ RCS\{∅} such that PAR ∩ STSEQ($\mathfrak{H}^1$) = PAR ∩ STSEQ($\mathfrak{H}$'), AVAP($\mathfrak{H}^1$) ⊆ AVAP($\mathfrak{H}$') = ∅ and thus also AVAS($\mathfrak{H}^1$) = ∅ and C($\mathfrak{H}^1$) = $\ulcorner\bigwedge\xi_0\ldots\bigwedge\xi_{k-1}\Delta\urcorner$. With C($\mathfrak{H}$') = $[\langle\beta_0, \ldots, \beta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta]$, it follows that PAR ∩ ST(Δ) ⊆ PAR ∩ STSEQ($\mathfrak{H}$') = $\{\beta_0, \ldots, \beta_{k-1}\}$ and thus, with ST(Δ) ∩ $\{\beta_0, \ldots, \beta_{k-1}\}$ = ∅, it follows that PAR ∩ ST(Δ) = PAR ∩ ST($\ulcorner\bigwedge\xi_0\ldots\bigwedge\xi_{k-1}\Delta\urcorner$) = PAR ∩ ST(C($\mathfrak{H}^1$)) = ∅.

We also have, with Theorem 4-10, that $\mathfrak{H}^2 = [\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle, \mathfrak{H}^1] \in$ RCS and Dom(AVS($\mathfrak{H}^2$)) = Dom(AVS($\mathfrak{H}^1$)) and thus Dom(AVAS($\mathfrak{H}^2$)) = Dom(AVAS($\mathfrak{H}^1$)) = ∅ and hence also AVAP($\mathfrak{H}^2$) = ∅. Moreover, we have PAR ∩ STSEQ($\mathfrak{H}$) ∩ STSEQ($\mathfrak{H}^2$) ⊆ PAR ∩ STSEQ($\mathfrak{H}$) ∩ $\{\beta^*_0, \ldots, \beta^*_{k-1}\}$ = ∅. Furthermore, we have, because of PAR ∩ ST(C($\mathfrak{H}^1$)) = ∅, that C($\mathfrak{H}^2$) = $[\langle\beta^*_0, \ldots, \beta^*_{k-1}\rangle, \langle\beta_0, \ldots, \beta_{k-1}\rangle$, C($\mathfrak{H}^1$)] = C($\mathfrak{H}^1$) = $\ulcorner\bigwedge\xi_0\ldots\bigwedge\xi_{k-1}\Delta\urcorner$. There is an $\alpha \in$ CONST\(ST($\mathfrak{H}$) ∪ ST($\mathfrak{H}^2$)). With Theorem 4-4, there is then, because of PAR ∩ STSEQ($\mathfrak{H}$) ∩ STSEQ($\mathfrak{H}^2$) = ∅, an $\mathfrak{H}^3 \in$ RCS\{∅} such that:

a) Dom($\mathfrak{H}^3$) = Dom($\mathfrak{H}$)+1+Dom($\mathfrak{H}^2$),
b) $\mathfrak{H}^3$↾Dom($\mathfrak{H}$) = $\mathfrak{H}$,
c) $\mathfrak{H}^3_{\mathrm{Dom}(\mathfrak{H})}$ = ⌜Suppose $\alpha = \alpha$⌝,
d) For all $i \in$ Dom($\mathfrak{H}^2$) it holds that $\mathfrak{H}^2_i = \mathfrak{H}^3_{\mathrm{Dom}(\mathfrak{H})+1+i}$,
e) Dom(AVS($\mathfrak{H}^3$)) = Dom(AVS($\mathfrak{H}$)) ∪ {Dom($\mathfrak{H}$)} ∪ {Dom($\mathfrak{H}$)+1+l | $l \in$ Dom(AVS($\mathfrak{H}^2$))},
f) AVP($\mathfrak{H}^3$) = AVP($\mathfrak{H}$) ∪ {⌜$\alpha = \alpha$⌝} ∪ AVP($\mathfrak{H}^2$), and
g) AVAP($\mathfrak{H}^3$) = AVAP($\mathfrak{H}$) ∪ {⌜$\alpha = \alpha$⌝} ∪ AVAP($\mathfrak{H}^2$) = AVAP($\mathfrak{H}$) ∪ {⌜$\alpha = \alpha$⌝}.

With Theorem 2-82, we have C($\mathfrak{H}$) ∈ AVP($\mathfrak{H}$) and hence, with f), C($\mathfrak{H}$) ∈ AVP($\mathfrak{H}^3$). We have $\ulcorner\bigwedge\xi_0\ldots\bigwedge\xi_{k-1}\Delta\urcorner$ = C($\mathfrak{H}^2$) = C($\mathfrak{H}^3$). With Theorem 4-12, there is then an $\mathfrak{H}^4 \in$ RCS\{∅} such that

h) $\text{Dom}(\mathfrak{H}^4) = \text{Dom}(\mathfrak{H}^3)+k$,
i) $\mathfrak{H}^4\restriction\text{Dom}(\mathfrak{H}^3) = \mathfrak{H}^3$,
j) $\text{AVAP}(\mathfrak{H}^4) \subseteq \text{AVAP}(\mathfrak{H}^3) = \text{AVAP}(\mathfrak{H}) \cup \{\ulcorner\alpha = \alpha\urcorner\}$,
k) For all $i < k$: $C(\mathfrak{H}^4\restriction\text{Dom}(\mathfrak{H}^3)+i+1) = \ulcorner\Lambda\xi_{i+1}\ldots\Lambda\xi_{k-1}[\langle\beta_0, \ldots, \beta_i\rangle, \langle\xi_0, \ldots, \xi_i\rangle, \Delta]\urcorner$, and
l) $C(\mathfrak{H}^4) = [\langle\beta_0, \ldots, \beta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta]$.

Then we have $C(\mathfrak{H}') = [\langle\beta_0, \ldots, \beta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta] = C(\mathfrak{H}^4) \in \text{AVP}(\mathfrak{H}^4)$. We also have: $\mathfrak{H}^4_{\text{Dom}(\mathfrak{H})} = \mathfrak{H}^3_{\text{Dom}(\mathfrak{H})} = \ulcorner\text{Suppose } \alpha = \alpha\urcorner$. Since $\alpha \in \text{CONST}\setminus(\text{ST}(\mathfrak{H}) \cup \text{ST}(\mathfrak{H}^2))$ and thus $\alpha \notin \text{ST}(\Delta)$ and since $\text{PAR} \cap \text{CONST} = \emptyset$, it follows, with a), b), c), d), h), i), k) and l), that for all $l \in \text{Dom}(\mathfrak{H}^4)$ it holds that

$$\alpha \in \text{ST}(\mathfrak{H}^4_l) \text{ iff } l = \text{Dom}(\mathfrak{H}).$$

With $\mathfrak{H}^4_{\text{Dom}(\mathfrak{H})} \in \text{AS}(\mathfrak{H}^4)$ and Theorem 4-3, we then have that there is no closed segment $\mathfrak{A}$ in $\mathfrak{H}^4$ such that $\min(\text{Dom}(\mathfrak{A})) \leq \text{Dom}(\mathfrak{H}) < \max(\text{Dom}(\mathfrak{A}))$. If $\mathfrak{A}$ was a closed segment in $\mathfrak{H}^4$ such that $\min(\text{Dom}(\mathfrak{A})) \leq \text{Dom}(\mathfrak{H})\text{-}1 < \max(\text{Dom}(\mathfrak{A}))$, then we would have $\min(\text{Dom}(\mathfrak{A})) \leq \text{Dom}(\mathfrak{H}) \leq \max(\text{Dom}(\mathfrak{A}))$. Therefore there is no closed segment $\mathfrak{A}$ in $\mathfrak{H}^4$ such that $\min(\text{Dom}(\mathfrak{A})) \leq \text{Dom}(\mathfrak{H})\text{-}1 < \max(\text{Dom}(\mathfrak{A}))$ and thus we have $P(\mathfrak{H}^4_{\text{Dom}(\mathfrak{H})\text{-}1}) = C(\mathfrak{H}) \in \text{AVP}(\mathfrak{H}^4)$. We also have $C(\mathfrak{H}') = C(\mathfrak{H}^4) \in \text{AVP}(\mathfrak{H})$. With Theorem 4-7, there is thus an $\mathfrak{H}^5 \in \text{RCS}\setminus\{\emptyset\}$ such that $\text{AVAP}(\mathfrak{H}^5) \subseteq \text{AVAP}(\mathfrak{H}^4)\setminus\{\ulcorner\alpha = \alpha\urcorner\} \subseteq (\text{AVAP}(\mathfrak{H}) \cup \{\alpha = \alpha\})\setminus\{\ulcorner\alpha = \alpha\urcorner\} \subseteq \text{AVAP}(\mathfrak{H})$ and $C(\mathfrak{H}), C(\mathfrak{H}') \in \text{AVP}(\mathfrak{H}^5)$. ■

Theorem 4-14. *CdE-, CI-, BI-, BE- and IE-preparation theorem*
If $\mathfrak{H}, \mathfrak{H}' \in \text{RCS}\setminus\{\emptyset\}$, then there is an $\mathfrak{H}^* \in \text{RCS}\setminus\{\emptyset\}$ such that
(i) $C(\mathfrak{H}), C(\mathfrak{H}') \in \text{AVP}(\mathfrak{H}^*)$ and
(ii) $\text{AVAP}(\mathfrak{H}^*) \subseteq \text{AVAP}(\mathfrak{H}) \cup \text{AVAP}(\mathfrak{H}')$.

Proof: Proof by induction on $|\text{AVAS}(\mathfrak{H}')|$. For $|\text{AVAS}(\mathfrak{H}')| = 0$ the statement holds with Theorem 4-13. Now, suppose the statement holds for n and suppose $\mathfrak{H}, \mathfrak{H}' \in \text{RCS}\setminus\{\emptyset\}$ and $|\text{AVAS}(\mathfrak{H}')| = n+1$. With Theorem 3-18, we then have $\mathfrak{H}^1 = \mathfrak{H}'^\frown\{(0, \ulcorner\text{Therefore } P(\mathfrak{H}'_{\max(\text{Dom}(\text{AVAS}(\mathfrak{H}')))}) \rightarrow C(\mathfrak{H}')\urcorner)\} \in \text{CdIF}(\mathfrak{H}') \subseteq \text{RCS}\setminus\{\emptyset\}$. With Theorem 3-19-(iv) and (v), we have $|\text{AVAS}(\mathfrak{H}^1)| = n$ and, with Theorem 3-19-(ix), we have $\text{AVAP}(\mathfrak{H}^1) \subseteq \text{AVAP}(\mathfrak{H}')$. With the I.H., it then holds that there is an $\mathfrak{H}^2 \in \text{RCS}\setminus\{\emptyset\}$ such that

a) $C(\mathfrak{H}), C(\mathfrak{H}^1) \in \mathrm{AVP}(\mathfrak{H}^2)$ and

b) $\mathrm{AVAP}(\mathfrak{H}^2) \subseteq \mathrm{AVAP}(\mathfrak{H}) \cup \mathrm{AVAP}(\mathfrak{H}^1) \subseteq \mathrm{AVAP}(\mathfrak{H}) \cup \mathrm{AVAP}(\mathfrak{H}')$.

Now, let the following sentence sequences be defined, where $\alpha \in \mathrm{CONST} \setminus \mathrm{STSEQ}(\mathfrak{H}^2)$:

$$\mathfrak{H}^3 = \mathfrak{H}^2 \cup \{(\mathrm{Dom}(\mathfrak{H}^2), \ulcorner\text{Suppose } P(\mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}')))})\urcorner)\}$$
$$\mathfrak{H}^4 = \mathfrak{H}^3 \cup \{(\mathrm{Dom}(\mathfrak{H}^3), \ulcorner\text{Therefore } \alpha = \alpha\urcorner)\}$$
$$\mathfrak{H}^5 = \mathfrak{H}^4 \cup \{(\mathrm{Dom}(\mathfrak{H}^4), \ulcorner\text{Therefore } C(\mathfrak{H}')\urcorner)\}.$$

With Theorem 1-12, we have $C(\mathfrak{H}^3) \notin \mathrm{ISENT}$ and thus $\mathfrak{H}^3 \notin \mathrm{CdIF}(\mathfrak{H}^2) \cup \mathrm{NIF}(\mathfrak{H}^2) \cup \mathrm{PEF}(\mathfrak{H}^2)$. With Theorem 1-10 and Theorem 1-11, we have that $C(\mathfrak{H}^4)$ is neither a negation nor a conditional and thus we have $\mathfrak{H}^4 \notin \mathrm{CdIF}(\mathfrak{H}^3) \cup \mathrm{NIF}(\mathfrak{H}^3)$. If $P(\mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}')))}) = \ulcorner\alpha = \alpha\urcorner$, then we would have $\alpha \in \mathrm{ST}(P(\mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}')))})) \subseteq \mathrm{ST}(C(\mathfrak{H}^1)) \subseteq \mathrm{STSF}(\mathrm{AVP}(\mathfrak{H}^2)) \subseteq \mathrm{STSEQ}(\mathfrak{H}^2)$ and thus a contradiction. Therefore $\mathfrak{H}^4 \notin \mathrm{CdIF}(\mathfrak{H}^3) \cup \mathrm{NIF}(\mathfrak{H}^3) \cup \mathrm{PEF}(\mathfrak{H}^3)$. If $\mathfrak{H}^5 \in \mathrm{CdIF}(\mathfrak{H}^4) \cup \mathrm{NIF}(\mathfrak{H}^4) \cup \mathrm{PEF}(\mathfrak{H}^4)$, then we would have $\alpha \in \mathrm{ST}(P(\mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}')))})) \cup \mathrm{ST}(C(\mathfrak{H}')) \subseteq \mathrm{ST}(C(\mathfrak{H}^1)) \subseteq \mathrm{STSEQ}(\mathfrak{H}^2)$ and thus again a contradiction. Therefore $\mathfrak{H}^5 \notin \mathrm{CdIF}(\mathfrak{H}^4) \cup \mathrm{NIF}(\mathfrak{H}^4) \cup \mathrm{PEF}(\mathfrak{H}^4)$.

On the other hand, we have that $\mathfrak{H}^3 \in \mathrm{AF}(\mathfrak{H}^2)$ and thus $\mathfrak{H}^3 \in \mathrm{RCS}$ and, with Theorem 3-15-(vi), $C(\mathfrak{H})$, $C(\mathfrak{H}^1)$, $P(\mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}')))}) \in \mathrm{AVP}(\mathfrak{H}^2) \cup \{P(\mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}')))})\} = \mathrm{AVP}(\mathfrak{H}^3)$ and, with Theorem 3-15-(viii), $\mathrm{AVAP}(\mathfrak{H}^3) = \mathrm{AVAP}(\mathfrak{H}^2) \cup \{P(\mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}')))})\} \subseteq \mathrm{AVAP}(\mathfrak{H}) \cup \mathrm{AVAP}(\mathfrak{H}')$. Next, we have $\mathfrak{H}^4 \in \mathrm{IIF}(\mathfrak{H}^3)$ and thus $\mathfrak{H}^4 \in \mathrm{RCS}$ and, with Theorem 3-25, $\mathrm{AVS}(\mathfrak{H}^4) = \mathrm{AVS}(\mathfrak{H}^3) \cup \{(\mathrm{Dom}(\mathfrak{H}^3), \ulcorner\text{Therefore } \alpha = \alpha\urcorner)\}$. Thus we have $\mathrm{AVAP}(\mathfrak{H}^4) = \mathrm{AVAP}(\mathfrak{H}^3) \subseteq \mathrm{AVAP}(\mathfrak{H}) \cup \mathrm{AVAP}(\mathfrak{H}')$ and $C(\mathfrak{H})$, $C(\mathfrak{H}^1)$, $P(\mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}')))}) \in \mathrm{AVP}(\mathfrak{H}^3) \subseteq \mathrm{AVP}(\mathfrak{H}^4)$. Because of $C(\mathfrak{H}^1) = \ulcorner P(\mathfrak{H}'_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}')))}) \rightarrow C(\mathfrak{H}')\urcorner$, we have $\mathfrak{H}^5 \in \mathrm{CdEF}(\mathfrak{H}^4) \subseteq \mathrm{RCS} \setminus \{\emptyset\}$. With Theorem 3-25, we have $\mathrm{AVS}(\mathfrak{H}^5) = \mathrm{AVS}(\mathfrak{H}^4) \cup \{(\mathrm{Dom}(\mathfrak{H}^4), \ulcorner\text{Therefore } C(\mathfrak{H}')\urcorner)\}$. Thus we have $\mathrm{AVAP}(\mathfrak{H}^5) = \mathrm{AVAP}(\mathfrak{H}^4) \subseteq \mathrm{AVAP}(\mathfrak{H}) \cup \mathrm{AVAP}(\mathfrak{H}')$ and $C(\mathfrak{H}) \in \mathrm{AVP}(\mathfrak{H}^4) \subseteq \mathrm{AVP}(\mathfrak{H}^5)$ and, with Theorem 2-82, $C(\mathfrak{H}') = C(\mathfrak{H}^5) \in \mathrm{AVP}(\mathfrak{H}^5)$ and $\mathfrak{H}^5 \in \mathrm{RCS} \setminus \{\emptyset\}$. $\mathfrak{H}^5$ is thus the desired RCS-element. ■

4.2 Properties of the Deductive Consequence Relation

Now, we will establish some usual theorems about the deductive consequence relation. In particular, we will show that the deductive consequence relation is reflexive (Theorem 4-15), monotone (Theorem 4-16), closed under the introduction and elimination of logical operators (Theorem 4-18) and transitive (Theorem 4-19).

Theorem 4-15. *Extended reflexivity (AR)*
If $X \subseteq$ CFORM and A $\in X$, then $X \vdash$ A.

Proof: Suppose $X \subseteq$ CFORM and A $\in X$. Then we have A $\in$ CFORM and, according to Definition 3-1, we have that {(0, ⌜Suppose A⌝)} $\in$ AF(∅) $\subseteq$ RCS\{∅} and we have C({(0, ⌜Suppose A⌝)}) = A and AVAP({(0, ⌜Suppose A⌝)}) = {A} $\subseteq X$. With Theorem 3-12, we thus have $X \vdash$ A. ■

Theorem 4-16. *Monotony*
If $X \vdash$ B and $X \subseteq Y \subseteq$ CFORM, then $Y \vdash$ B.

Proof: Suppose $X \vdash$ B and $X \subseteq Y \subseteq$ CFORM. With Theorem 3-12, there is then an $\mathfrak{H} \in$ RCS\{∅} such that AVAP($\mathfrak{H}$) $\subseteq X$ and C($\mathfrak{H}$) = B. Then we have AVAP($\mathfrak{H}$) $\subseteq Y$ and thus $Y \vdash$ B. ■

Theorem 4-17. *Principium non contradictionis*
If $X \cup \{\Gamma\} \subseteq$ CFORM, then $X \vdash$ ⌜$\neg(\Gamma \wedge \neg\Gamma)$⌝.

Proof: Suppose $X \cup \{\Gamma\} \subseteq$ CFORM. Now, let $\mathfrak{H}$ be the following sentence sequence:

0	Suppose	$\Gamma \wedge \neg\Gamma$
1	Therefore	Γ
2	Therefore	$\neg\Gamma$
3	Therefore	$\neg(\Gamma \wedge \neg\Gamma)$

According to Definition 3-1, we have $\mathfrak{H}\upharpoonright 1 \in$ AF(∅) $\subseteq$ RCS\{∅} and, with Theorem 3-15, we have AVS($\mathfrak{H}\upharpoonright 1$) ={(0, ⌜Suppose $\Gamma \wedge \neg\Gamma$⌝)} = $\mathfrak{H}\upharpoonright 1$ and AVP($\mathfrak{H}\upharpoonright 1$) = {⌜$\Gamma \wedge \neg\Gamma$⌝} and AVAS($\mathfrak{H}\upharpoonright 1$) = {(0, ⌜Suppose $\Gamma \wedge \neg\Gamma$⌝)}und AVAP($\mathfrak{H}\upharpoonright 1$) = {⌜$\Gamma \wedge \neg\Gamma$⌝}. According to Definition 3-5, we then have $\mathfrak{H}\upharpoonright 2 \in$ CEF($\mathfrak{H}\upharpoonright 1$) $\subseteq$ RCS\{∅}. Since, with Theorem 1-8, ⌜Γ

$\wedge \neg\Gamma\urcorner \notin \mathrm{SF}(\Gamma)$, we have, with Definition 3-2, Definition 3-10 and Definition 3-15, that $\mathfrak{H}\upharpoonright 2 \notin \mathrm{CdIF}(\mathfrak{H}\upharpoonright 1) \cup \mathrm{NIF}(\mathfrak{H}\upharpoonright 1) \cup \mathrm{PEF}(\mathfrak{H}\upharpoonright 1)$. With Theorem 3-25, it then follows that $\mathrm{AVS}(\mathfrak{H}\upharpoonright 2) = \mathrm{AVS}(\mathfrak{H}\upharpoonright 1) \cup \{(1, \ulcorner\text{Therefore } \Gamma\urcorner)\} = \mathfrak{H}\upharpoonright 2$. We also have with Theorem 3-27-(ii) and -(iii) that $\mathrm{AVAS}(\mathfrak{H}\upharpoonright 2) = \mathrm{AVAS}(\mathfrak{H}\upharpoonright 1)$ and thus $\mathrm{AVAP}(\mathfrak{H}\upharpoonright 2) = \mathrm{AVAP}(\mathfrak{H}\upharpoonright 1) = \{\ulcorner\Gamma \wedge \neg\Gamma\urcorner\}$.

With Definition 3-5, we then have $\mathfrak{H}\upharpoonright 3 \in \mathrm{CEF}(\mathfrak{H}\upharpoonright 2) \subseteq \mathrm{RCS}\setminus\{\emptyset\}$. Since, with Theorem 1-8, $\ulcorner\Gamma \wedge \neg\Gamma\urcorner \notin \mathrm{SF}(\ulcorner\neg\Gamma\urcorner)$ and $\Gamma \neq \ulcorner\neg\Gamma\urcorner$, we have, with Definition 3-2, Definition 3-10 and Definition 3-15 that $\mathfrak{H}\upharpoonright 3 \notin \mathrm{CdIF}(\mathfrak{H}\upharpoonright 2) \cup \mathrm{NIF}(\mathfrak{H}\upharpoonright 2) \cup \mathrm{PEF}(\mathfrak{H}\upharpoonright 2)$. With Theorem 3-25, it then follows that $\mathrm{AVS}(\mathfrak{H}\upharpoonright 3) = \mathrm{AVS}(\mathfrak{H}\upharpoonright 2) \cup \{1, \ulcorner\text{Therefore } \neg\Gamma\urcorner)\} = \mathfrak{H}\upharpoonright 3$ and, with Theorem 3-27-(ii) and -(iii), that $\mathrm{AVAS}(\mathfrak{H}\upharpoonright 3) = \mathrm{AVAS}(\mathfrak{H}\upharpoonright 2)$ and thus that $\mathrm{AVAP}(\mathfrak{H}\upharpoonright 3) = \mathrm{AVAP}(\mathfrak{H}\upharpoonright 2) = \{\ulcorner\Gamma \wedge \neg\Gamma\urcorner\}$. Then we have $0 = \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright 3)))$ and $1, 2 \in \mathrm{Dom}(\mathrm{AVS}(\mathfrak{H}\upharpoonright 3))$ and $\mathrm{P}(\mathfrak{H}\upharpoonright 3_1) = \Gamma$ and $\mathrm{P}(\mathfrak{H}\upharpoonright 3_2) = \ulcorner\neg\Gamma\urcorner$. According to Definition 3-10, we thus have $\mathfrak{H} \in \mathrm{NIF}(\mathfrak{H}\upharpoonright 3)$. According to Theorem 3-20, we have $\mathrm{AVAS}(\mathfrak{H}) = \mathrm{AVAS}(\mathfrak{H}\upharpoonright 3)\setminus\{(0, \ulcorner\text{Suppose } \Gamma \wedge \neg\Gamma\urcorner)\} = \emptyset$ and thus also $\mathrm{AVAP}(\mathfrak{H}) = \emptyset$. Hence we have $\mathfrak{H} \in \mathrm{RCS}\setminus\{\emptyset\}$ and $\mathrm{AVAP}(\mathfrak{H}) = \emptyset$ and $\mathrm{C}(\mathfrak{H}) = \ulcorner\neg(\Gamma \wedge \neg\Gamma)\urcorner$. With Theorem 3-12, we then have $\emptyset \vdash \ulcorner\neg(\Gamma \wedge \neg\Gamma)\urcorner$ and thus it holds, with Theorem 4-16, that $X \vdash \ulcorner\neg(\Gamma \wedge \neg\Gamma)\urcorner$. ■

Theorem 4-18. *Closure under introduction and elimination*

If $\mathrm{A}, \mathrm{B}, \Gamma \in \mathrm{CFORM}$, $\theta_0, \theta_1 \in \mathrm{CTERM}$, $\xi \in \mathrm{VAR}$ and $\Delta \in \mathrm{FORM}$, where $\mathrm{FV}(\Delta) \subseteq \{\xi\}$, then:

(i)	If $X \vdash \mathrm{B}$ and $\mathrm{A} \in X$, then $X\setminus\{\mathrm{A}\} \vdash \ulcorner\mathrm{A} \rightarrow \mathrm{B}\urcorner$,	(CdI)
(ii)	If $X \vdash \mathrm{A}$ and $Y \vdash \ulcorner\mathrm{A} \rightarrow \mathrm{B}\urcorner$, then $X \cup Y \vdash \mathrm{B}$,	(CdE)
(iii)	If $X \vdash \mathrm{A}$ and $Y \vdash \mathrm{B}$, then $X \cup Y \vdash \ulcorner\mathrm{A} \wedge \mathrm{B}\urcorner$,	(CI)
(iv)	If $X \vdash \ulcorner\mathrm{A} \wedge \mathrm{B}\urcorner$ or $X \vdash \ulcorner\mathrm{B} \wedge \mathrm{A}\urcorner$, then $X \vdash \mathrm{A}$,	(CE)
(v)	If $X \vdash \ulcorner\mathrm{A} \rightarrow \mathrm{B}\urcorner$ and $Y \vdash \ulcorner\mathrm{B} \rightarrow \mathrm{A}\urcorner$, then $X \cup Y \vdash \ulcorner\mathrm{A} \leftrightarrow \mathrm{B}\urcorner$,	(BI)
(vi)	If $X \vdash \mathrm{B}$ and $\mathrm{A} \in X$ and $Y \vdash \mathrm{A}$ and $\mathrm{B} \in Y$, then $(X\setminus\{\mathrm{A}\}) \cup (Y\setminus\{\mathrm{B}\}) \vdash \ulcorner\mathrm{A} \leftrightarrow \mathrm{B}\urcorner$,	(BI*)
(vii)	If $X \vdash \mathrm{A}$ and $Y \vdash \ulcorner\mathrm{A} \leftrightarrow \mathrm{B}\urcorner$ or $Y \vdash \ulcorner\mathrm{B} \leftrightarrow \mathrm{A}\urcorner$, then $X \cup Y \vdash \mathrm{B}$,	(BE)
(viii)	If $X \vdash \mathrm{A}$ or $X \vdash \mathrm{B}$, then $X \vdash \ulcorner\mathrm{A} \vee \mathrm{B}\urcorner$,	(DI)
(ix)	If $X \vdash \ulcorner\mathrm{A} \vee \mathrm{B}\urcorner$ and $Y \vdash \ulcorner\mathrm{A} \rightarrow \Gamma\urcorner$ and $Z \vdash \ulcorner\mathrm{B} \rightarrow \Gamma\urcorner$, then $X \cup Y \cup Z \vdash \Gamma$,	(DE)
(x)	If $X \vdash \ulcorner\mathrm{A} \vee \mathrm{B}\urcorner$ and $Y \vdash \Gamma$ and $\mathrm{A} \in Y$ and $Z \vdash \Gamma$ and $\mathrm{B} \in Z$, then $X \cup (Y\setminus\{\mathrm{A}\}) \cup (Z\setminus\{\mathrm{B}\}) \vdash \Gamma$,	(DE*)
(xi)	If $X \vdash \Gamma$ and $Y \vdash \ulcorner\neg\Gamma\urcorner$ and $\mathrm{A} \in X \cup Y$, then $(X \cup Y)\setminus\{\mathrm{A}\} \vdash \ulcorner\neg\mathrm{A}\urcorner$,	(NI)
(xii)	If $X \vdash \ulcorner\neg\neg\Gamma\urcorner$, then $X \vdash \Gamma$,	(NE)
(xiii)	If $X \vdash [\beta, \xi, \Delta]$ and $\beta \notin \mathrm{STSF}(X \cup \{\Delta\})$, then $X \vdash \ulcorner\bigwedge\xi\Delta\urcorner$,	(UI)

(xiv) If $X \vdash \ulcorner\bigwedge\xi\Delta\urcorner$, then $X \vdash [\theta_0, \xi, \Delta]$, (UE)

(xv) If $X \vdash [\theta_0, \xi, \Delta]$, then $X \vdash \ulcorner\bigvee\xi\Delta\urcorner$, (PI)

(xvi) If $X \vdash \ulcorner\bigvee\xi\Delta\urcorner$ and $Y \vdash \Gamma$ and $[\beta, \xi\ \Delta] \in Y$ and $\beta \notin$ STSF(($Y\backslash\{[\beta, \xi, \Delta]\}) \cup \{\Delta, \Gamma\}$), then $X \cup (Y\backslash\{[\beta, \xi, \Delta]\}) \vdash \Gamma$, (PE)

(xvii) If $X \subseteq$ CFORM, then $X \vdash \ulcorner\theta_0 = \theta_0\urcorner$, and (II)

(xviii) If $X \vdash \ulcorner\theta_0 = \theta_1\urcorner$ and $Y \vdash [\theta_0, \xi, \Delta]$, then $X \cup Y \vdash [\theta_1, \xi, \Delta]$. (IE)

Proof: Suppose A, B, $\Gamma \in$ CFORM, $\theta_0, \theta_1 \in$ CTERM, $\xi \in$ VAR and $\Delta \in$ FORM, where FV(Δ) $\subseteq \{\xi\}$. First, we will deal with case (i), in which the set of premises is reduced. Then we will treat the cases (ii), (iii), (v), (vii) and (xviii), in which two premise sets are joined. In the cases (iv), (viii), (xii), (xiii), (xiv) and (xv), the premise set does not change. The remaining special cases will be dealt with in the order (vi), (ix), (x), (xi), (xvi), (xvii).

Ad (i) (CdI): Suppose $X \vdash$ B and A $\in X$. According to Theorem 3-12, there is then an $\mathfrak{H} \in$ RCS\{∅} such that C($\mathfrak{H}$) = B and AVAP($\mathfrak{H}$) $\subseteq X$. With Theorem 4-2, there is then an $\mathfrak{H}' \in$ RCS\{∅} such that AVAP($\mathfrak{H}'$) $\subseteq$ AVAP($\mathfrak{H}$) and C($\mathfrak{H}'$) = C($\mathfrak{H}$) and for all $i \in$ Dom(AVAS($\mathfrak{H}'$)): If P($\mathfrak{H}'_i$) = A, then i = max(Dom(AVAS($\mathfrak{H}'$))). With Theorem 2-82, we then have B = C($\mathfrak{H}'$) $\in$ AVP($\mathfrak{H}'$). With A $\in$ AVAP($\mathfrak{H}'$) and A $\notin$ AVAP($\mathfrak{H}'$), we can now distinguish two cases.

First case: Suppose A $\in$ AVAP($\mathfrak{H}'$). Then we have AVAS($\mathfrak{H}'$) ≠ ∅ and it holds for all $i \in$ Dom(AVAS($\mathfrak{H}'$)): P($\mathfrak{H}'_i$) = A iff i = max(Dom(AVAS($\mathfrak{H}'$))). With Theorem 3-18, we then have $\mathfrak{H}^+ = \mathfrak{H}'^\frown\{(0, \ulcorner$Therefore A $\rightarrow$ B$\urcorner)\} \in$ CdIF($\mathfrak{H}'$) $\subseteq$ RCS\{∅}. With Theorem 3-22, it then holds that AVAP($\mathfrak{H}^+$) $\subseteq$ AVAP($\mathfrak{H}'$)\{A} $\subseteq$ AVAP($\mathfrak{H}$)\{A} $\subseteq X$\{A}. Hence we have $\mathfrak{H}^+ \in$ RCS\{∅}, C($\mathfrak{H}^+$) = ⌜A → B⌝ and AVAP($\mathfrak{H}^+$) $\subseteq X$\{A} and thus, with Theorem 3-12, X\{A} ⊢ ⌜A → B⌝.

Second case: Now, suppose A $\notin$ AVAP($\mathfrak{H}'$). Then we can extend $\mathfrak{H}'$ as follows to an $\mathfrak{H}^4 \in$ SEQ with $\mathfrak{H}^4\restriction$Dom($\mathfrak{H}'$) = $\mathfrak{H}'$:

$\mathfrak{H}^1$	=	$\mathfrak{H}'$	∪	{(Dom($\mathfrak{H}'$),	⌜Suppose A⌝)}
$\mathfrak{H}^2$	=	$\mathfrak{H}^1$	∪	{(Dom($\mathfrak{H}^1$),	⌜Therefore A ∧ B⌝)}
$\mathfrak{H}^3$	=	$\mathfrak{H}^2$	∪	{(Dom($\mathfrak{H}^2$),	⌜Therefore B⌝)}
$\mathfrak{H}^4$	=	$\mathfrak{H}^3$	∪	{(Dom($\mathfrak{H}^3$),	⌜Therefore A → B⌝)}.

First, we have $\mathfrak{H}^4_{\mathrm{Dom}(\mathfrak{H}')} \in$ ASENT. With Theorem 1-8, Theorem 1-10 and Theorem 1-11, we have C($\mathfrak{H}^1$) ≠ C($\mathfrak{H}^2$) und C($\mathfrak{H}^2$) ≠ C($\mathfrak{H}^3$). We also have that C($\mathfrak{H}^2$) is neither a conditional nor a negation. We further have with Theorem 1-8 that $C(\mathfrak{H}^3) = B \neq \ulcorner A \rightarrow (A \wedge B)\urcorner$ and that $P(\mathfrak{H}^3{}_{Dom(\mathfrak{H}')}) = A \neq \ulcorner\neg(A \wedge B)\urcorner = \ulcorner\neg P(\mathfrak{H}^3{}_{Dom(\mathfrak{H}^1)})\urcorner$. With Theorem 2-42, Definition 2-11, Definition 2-12 and Definition 2-13, we then have that it holds for all k with $1 \leq k \leq 3$ that there is no closed segment $\mathfrak{A}$ in $\mathfrak{H}^k$ such that $\min(Dom(\mathfrak{A})) = Dom(\mathfrak{H}')$. With Theorem 2-47, we thus have for all k with $1 \leq k \leq 3$ that there is no closed segment $\mathfrak{A}$ in $\mathfrak{H}^k$ such that $\min(Dom(\mathfrak{A})) \leq Dom(\mathfrak{H}') \leq \max(Dom(\mathfrak{A}))$. Thus we also get that it holds for all k with $1 \leq k \leq 3$ that $Dom(\mathfrak{H}') = \max(Dom(AVAS(\mathfrak{H}^k)))$. With Theorem 3-19-(i), Theorem 3-20-(i), Theorem 3-21-(i) and Theorem 2-61, we then have for all k with $2 \leq k \leq 3$ that $\mathfrak{H}^k \notin CdIF(\mathfrak{H}^{k-1}) \cup NIF(\mathfrak{H}^{k-1}) \cup PEF(\mathfrak{H}^{k-1})$.

On the other hand, we *first* have, according to Definition 3-1, $\mathfrak{H}^1 \in AF(\mathfrak{H}') \subseteq RCS\setminus\{\emptyset\}$ and, with Theorem 3-15, $AVS(\mathfrak{H}^1) = AVS(\mathfrak{H}') \cup \{(Dom(\mathfrak{H}'), \ulcorner\text{Suppose } A\urcorner)\}$ and $(Dom(\mathfrak{H}'), \ulcorner\text{Suppose } A\urcorner) \in AVAS(\mathfrak{H}') \cup \{(Dom(\mathfrak{H}'), \ulcorner\text{Suppose } A\urcorner)\} = AVAS(\mathfrak{H}^1)$ and $B \in AVP(\mathfrak{H}') \subseteq AVP(\mathfrak{H}^1)$ and $A \in AVP(\mathfrak{H}^1)$. Therefore we have *second*, according to Definition 3-4, $\mathfrak{H}^2 \in CIF(\mathfrak{H}^1) \subseteq RCS\setminus\{\emptyset\}$ and, with Theorem 3-25, $AVS(\mathfrak{H}^2) = AVS(\mathfrak{H}^1) \cup \{(Dom(\mathfrak{H}^1), \ulcorner\text{Therefore } A \wedge B\urcorner)\}$. Thus we have $(Dom(\mathfrak{H}'), \ulcorner\text{Suppose } A\urcorner) \in AVAS(\mathfrak{H}^1) = AVAS(\mathfrak{H}^2)$ and $\ulcorner A \wedge B\urcorner \in AVP(\mathfrak{H}^2)$. Therefore we have *third*, according to Definition 3-5, $\mathfrak{H}^3 \in CEF(\mathfrak{H}^2) \subseteq RCS\setminus\{\emptyset\}$ and, with Theorem 3-25, $AVS(\mathfrak{H}^3) = AVS(\mathfrak{H}^2) \cup \{(Dom(\mathfrak{H}^2), \ulcorner\text{Therefore } B\urcorner)\}$. Thus we have $Dom(\mathfrak{H}') \in Dom(\mathfrak{H}^3)$ and $P(\mathfrak{H}^3{}_{Dom(\mathfrak{H}')}) = A$ and $(Dom(\mathfrak{H}'), \ulcorner\text{Suppose } A\urcorner) \in AVAS(\mathfrak{H}^2) = AVAS(\mathfrak{H}^3)$ and $P(\mathfrak{H}^3{}_{Dom(\mathfrak{H}^3)-1}) = B$ and there is no l such that $Dom(\mathfrak{H}') < l \leq Dom(\mathfrak{H}^3)-1$ and $(l, \mathfrak{H}^3{}_l) \in AVAS(\mathfrak{H}^3)$. According to Definition 3-2, we thus have $\mathfrak{H}^4 \in CdIF(\mathfrak{H}^3) \subseteq RCS\setminus\{\emptyset\}$ and, with Theorem 3-19-(iv) and -(v), $AVAS(\mathfrak{H}^4) = AVAS(\mathfrak{H}^3)\setminus\{(\max(Dom(AVAS(\mathfrak{H}^3))), \mathfrak{H}^4{}_{\max(Dom(AVAS(\mathfrak{H}^3)))})\} = AVAS(\mathfrak{H}^3)\setminus\{(Dom(\mathfrak{H}'), \ulcorner\text{Suppose } A\urcorner)\} = AVAS(\mathfrak{H}^1)\setminus\{(Dom(\mathfrak{H}'), \ulcorner\text{Suppose } A\urcorner)\} = (AVAS(\mathfrak{H}') \cup \{(Dom(\mathfrak{H}'), \ulcorner\text{Suppose } A\urcorner)\})\setminus\{(Dom(\mathfrak{H}'), \ulcorner\text{Suppose } A\urcorner)\} = AVAS(\mathfrak{H}')\setminus\{(Dom(\mathfrak{H}'), \ulcorner\text{Suppose } A\urcorner)\} \subseteq AVAS(\mathfrak{H}')$. With Theorem 2-75, we then have $AVAP(\mathfrak{H}^4) \subseteq AVAP(\mathfrak{H}')$ and, because of $A \notin AVAP(\mathfrak{H}')$ and $AVAP(\mathfrak{H}') \subseteq AVAP(\mathfrak{H}) \subseteq X$, we then also have $AVAP(\mathfrak{H}^4) \subseteq AVAP(\mathfrak{H})\setminus\{A\} \subseteq X\setminus\{A\}$. Since $C(\mathfrak{H}^4) = \ulcorner A \rightarrow B\urcorner$, it holds, with Theorem 3-12, that $X\setminus\{A\} \vdash \ulcorner A \rightarrow B\urcorner$.

Ad (ii) (CdE), (iii) (CI), (v) (BI), (vii) (BE), (xviii) (IE): We prove (ii) exemplarily, clauses (iii), (v), (vii) and (xviii) are shown analogously. Suppose for (ii) that $X \vdash A$ and $Y \vdash \ulcorner A \rightarrow B\urcorner$. According to Theorem 3-12, there are then $\mathfrak{H}, \mathfrak{H}' \in RCS\setminus\{\emptyset\}$ such that $AVAP(\mathfrak{H}) \subseteq X$ and $C(\mathfrak{H}) = A$ and $AVAP(\mathfrak{H}') \subseteq Y$ and $C(\mathfrak{H}') = \ulcorner A \rightarrow B\urcorner$. With Theorem

4-14, there is then an $\mathfrak{H}^* \in \text{RCS}\backslash\{\emptyset\}$ such that A, $\ulcorner\text{A} \rightarrow \text{B}\urcorner \in \text{AVP}(\mathfrak{H}^*)$ and $\text{AVAP}(\mathfrak{H}^*) \subseteq \text{AVAP}(\mathfrak{H}) \cup \text{AVAP}(\mathfrak{H}') \subseteq X \cup Y$. According to Definition 3-3, we then have $\mathfrak{H}^+ = \mathfrak{H}^{*\frown}\{(0, \ulcorner\text{Therefore B}\urcorner)\} \in \text{CdEF}(\mathfrak{H}^*) \subseteq \text{RCS}\backslash\{\emptyset\}$ and, with Theorem 3-27-(v), we have $\text{AVAP}(\mathfrak{H}^+) \subseteq \text{AVAP}(\mathfrak{H}^*) \subseteq X \cup Y$ and we have $\text{C}(\mathfrak{H}^+) = \text{B}$. It then holds, with Theorem 3-12, that $X \cup Y \vdash \text{B}$.

Ad (iv) (CE), (viii) (DI), (xii) (NE), (xiii) (UI), (xiv) (UE), (xv) (PI): We prove (iv) exemplarily, clauses (viii), (xii), (xiii), (xiv) and (xv) are shown analogously. Suppose for (iv) that $X \vdash \ulcorner\text{A} \wedge \text{B}\urcorner$ or $X \vdash \ulcorner\text{B} \wedge \text{A}\urcorner$. Now, suppose $X \vdash \ulcorner\text{A} \wedge \text{B}\urcorner$. According to Theorem 3-12, there is then an $\mathfrak{H} \in \text{RCS}\backslash\{\emptyset\}$ such that $\text{AVAP}(\mathfrak{H}) \subseteq X$ and $\text{C}(\mathfrak{H}) = \ulcorner\text{A} \wedge \text{B}\urcorner$. With Theorem 2-82, we have $\ulcorner\text{A} \wedge \text{B}\urcorner \in \text{AVP}(\mathfrak{H})$ and thus, according to Definition 3-5, $\mathfrak{H}' = \mathfrak{H}^\frown\{(0, \ulcorner\text{Therefore A}\urcorner)\} \in \text{CEF}(\mathfrak{H}) \subseteq \text{RCS}\backslash\{\emptyset\}$ and, with Theorem 3-27-(v), we have $\text{AVAP}(\mathfrak{H}') \subseteq \text{AVAP}(\mathfrak{H}) \subseteq X$ and we have $\text{C}(\mathfrak{H}') = \text{A}$. With Theorem 3-12, we then have $X \vdash \text{A}$. In the case that $X \vdash \ulcorner\text{B} \wedge \text{A}\urcorner$, one shows analogously that $X \vdash \text{A}$ holds as well.

Ad (vi:)(BI)*: Suppose $X \vdash \text{B}$ and $\text{A} \in X$ and $Y \vdash \text{A}$ and $\text{B} \in Y$. With (i), we then have $X\backslash\{\text{A}\} \vdash \ulcorner\text{A} \rightarrow \text{B}\urcorner$ and $Y\backslash\{\text{B}\} \vdash \ulcorner\text{B} \rightarrow \text{A}\urcorner$. With (v), it then holds that $(X\backslash\{\text{A}\}) \cup (Y\backslash\{\text{B}\}) \vdash \ulcorner\text{A} \leftrightarrow \text{B}\urcorner$.

Ad (ix) (DE): Suppose $X \vdash \ulcorner\text{A} \vee \text{B}\urcorner$ and $Y \vdash \ulcorner\text{A} \rightarrow \Gamma\urcorner$ and $Z \vdash \ulcorner\text{B} \rightarrow \Gamma\urcorner$. By double application of (iii), we then get $X \cup Y \cup Z \vdash \ulcorner(\text{A} \vee \text{B}) \wedge ((\text{A} \rightarrow \Gamma) \wedge (\text{B} \rightarrow \Gamma))\urcorner$. With Theorem 3-12, there is then an $\mathfrak{H} \in \text{RCS}\backslash\{\emptyset\}$ such that $\text{AVAP}(\mathfrak{H}) \subseteq X \cup Y \cup Z$ and $\text{C}(\mathfrak{H}) = \ulcorner(\text{A} \vee \text{B}) \wedge ((\text{A} \rightarrow \Gamma) \wedge (\text{B} \rightarrow \Gamma))\urcorner$. There is an $\alpha \in \text{CONST}\backslash\text{STSEQ}(\mathfrak{H})$. Thus we can extend $\mathfrak{H}$ as follows to an $\mathfrak{H}^6 \in \text{SEQ}$ with $\mathfrak{H}^6{\restriction}\text{Dom}(\mathfrak{H}) = \mathfrak{H}$:

$$\begin{array}{lllll}
\mathfrak{H}^1 & = \mathfrak{H} & \cup & \{(\text{Dom}(\mathfrak{H}), & \ulcorner\text{Suppose } \alpha = \alpha\urcorner)\} \\
\mathfrak{H}^2 & = \mathfrak{H}^1 & \cup & \{(\text{Dom}(\mathfrak{H}^1), & \ulcorner\text{Therefore A} \vee \text{B}\urcorner)\} \\
\mathfrak{H}^3 & = \mathfrak{H}^2 & \cup & \{(\text{Dom}(\mathfrak{H}^2), & \ulcorner\text{Therefore } (\text{A} \rightarrow \Gamma) \wedge (\text{B} \rightarrow \Gamma)\urcorner)\} \\
\mathfrak{H}^4 & = \mathfrak{H}^3 & \cup & \{(\text{Dom}(\mathfrak{H}^3), & \ulcorner\text{Therefore A} \rightarrow \Gamma\urcorner)\} \\
\mathfrak{H}^5 & = \mathfrak{H}^4 & \cup & \{(\text{Dom}(\mathfrak{H}^4), & \ulcorner\text{Therefore B} \rightarrow \Gamma\urcorner)\} \\
\mathfrak{H}^6 & = \mathfrak{H}^5 & \cup & \{(\text{Dom}(\mathfrak{H}^5), & \ulcorner\text{Therefore } \Gamma\urcorner)\}.
\end{array}$$

First, we have $\mathfrak{H}^6{}_{\text{Dom}(\mathfrak{H})} \in \text{ASENT}$. With $\alpha \in \text{CONST}\backslash\text{STSEQ}(\mathfrak{H})$, we also have $\alpha \notin \text{STSF}(\{\text{A}, \text{B}, \Gamma\})$ and thus we have for all k with $1 \leq k \leq 6$: If $i \in \text{Dom}(\mathfrak{H}^k)$, then: $\alpha \in$

ST($\mathfrak{H}^k_i$) iff i = Dom($\mathfrak{H}$). Furthermore, it holds for all k with $1 \leq k \leq 6$ that Dom($\mathfrak{H}$) $\in$ Dom(AS($\mathfrak{H}^k$)). With Theorem 4-3, we thus have for all k with $1 \leq k \leq 6$: There is no closed segment $\mathfrak{A}$ in $\mathfrak{H}^k$ such that min(Dom($\mathfrak{A}$)) $\leq$ Dom($\mathfrak{H}$) $\leq$ max(Dom($\mathfrak{A}$)). Thus we also get that for all k with $1 \leq k \leq 6$ it holds that Dom($\mathfrak{H}$) = max(Dom(AVAS($\mathfrak{H}^k$))). With Theorem 3-19-(i), Theorem 3-20-(i), Theorem 3-21-(i) and Theorem 2-61, we then have that for all k with $2 \leq k \leq 6$ it holds that $\mathfrak{H}^k \notin$ CdIF($\mathfrak{H}^{k-1}$) $\cup$ NIF($\mathfrak{H}^{k-1}$) $\cup$ PEF($\mathfrak{H}^{k-1}$).

On the other hand, we have *first*, according to Definition 3-1, $\mathfrak{H}^1 \in$ AF($\mathfrak{H}$) $\subseteq$ RCS\{∅} and, with Theorem 3-15, AVS($\mathfrak{H}^1$) = AVS($\mathfrak{H}$) $\cup$ {(Dom($\mathfrak{H}$), ⌜Suppose α = α⌝)} and AVAS($\mathfrak{H}^1$) = AVAS($\mathfrak{H}$) $\cup$ {(Dom($\mathfrak{H}$), ⌜Suppose α = α⌝)} and ⌜(A ∨ B) ∧ ((A → Γ) ∧ (B → Γ))⌝ $\in$ AVP($\mathfrak{H}$) $\subseteq$ AVP($\mathfrak{H}^1$). Therefore we have *second*, according to Definition 3-5, $\mathfrak{H}^2 \in$ CEF($\mathfrak{H}^1$) $\subseteq$ RCS\{∅} and, with Theorem 3-25, AVS($\mathfrak{H}^2$) = AVS($\mathfrak{H}^1$) $\cup$ {(Dom($\mathfrak{H}^1$), ⌜Therefore A ∨ B⌝)}. Thus we have AVAS($\mathfrak{H}^2$) = AVAS($\mathfrak{H}^1$), ⌜(A ∨ B) ∧ ((A → Γ) ∧ (B → Γ))⌝ $\in$ AVP($\mathfrak{H}^1$) $\subseteq$ AVP($\mathfrak{H}^2$) and ⌜A ∨ B⌝ $\in$ AVP($\mathfrak{H}^2$). Therefore we have *third*, according to Definition 3-5, $\mathfrak{H}^3 \in$ CEF($\mathfrak{H}^2$) $\subseteq$ RCS\{∅} and, with Theorem 3-25, AVS($\mathfrak{H}^3$) = AVS($\mathfrak{H}^2$) $\cup$ {(Dom($\mathfrak{H}^2$), ⌜Therefore (A → Γ) ∧ (B → Γ)⌝)}. Thus we have AVAS($\mathfrak{H}^3$) = AVAS($\mathfrak{H}^2$), ⌜A ∨ B⌝ $\in$ AVP($\mathfrak{H}^2$) $\subseteq$ AVP($\mathfrak{H}^3$) and ⌜(A → Γ) ∧ (B → Γ)⌝ $\in$ AVP($\mathfrak{H}^3$). Therefore we have *fourth*, according to Definition 3-5, $\mathfrak{H}^4 \in$ CEF($\mathfrak{H}^3$) $\subseteq$ RCS\{∅} and, with Theorem 3-25, AVS($\mathfrak{H}^4$) = AVS($\mathfrak{H}^3$) $\cup$ {(Dom($\mathfrak{H}^3$), ⌜Therefore A → Γ⌝)}. Thus we have AVAS($\mathfrak{H}^4$) = AVAS($\mathfrak{H}^3$), ⌜A ∨ B⌝, ⌜(A → Γ) ∧ (B → Γ)⌝ $\in$ AVP($\mathfrak{H}^3$) $\subseteq$ AVP($\mathfrak{H}^4$) and ⌜A → Γ⌝ $\in$ AVP($\mathfrak{H}^4$). Therefore we have *fifth*, according to Definition 3-5, $\mathfrak{H}^5 \in$ CEF($\mathfrak{H}^4$) $\subseteq$ RCS\{∅} and, with Theorem 3-25, AVS($\mathfrak{H}^5$) = AVS($\mathfrak{H}^4$) $\cup$ {(Dom($\mathfrak{H}^4$), ⌜Therefore B → Γ⌝)}. Thus we have AVAS($\mathfrak{H}^5$) = AVAS($\mathfrak{H}^4$), ⌜A ∨ B⌝, ⌜A → Γ⌝ $\in$ AVP($\mathfrak{H}^4$) $\subseteq$ AVP($\mathfrak{H}^5$) and ⌜B → Γ⌝ $\in$ AVP($\mathfrak{H}^5$). Finally, we have *sixth*, according to Definition 3-9, $\mathfrak{H}^6 \in$ DEF($\mathfrak{H}^5$) $\subseteq$ RCS\{∅} and, with Theorem 3-25, AVS($\mathfrak{H}^6$) = AVS($\mathfrak{H}^5$) $\cup$ {(Dom($\mathfrak{H}^5$), ⌜Therefore Γ⌝)}. Thus we have AVAS($\mathfrak{H}^6$) = AVAS($\mathfrak{H}^5$) = AVAS($\mathfrak{H}$) $\cup$ {(Dom($\mathfrak{H}$), ⌜Suppose α = α⌝)}. Thus we have AVAP($\mathfrak{H}^6$) = AVAP($\mathfrak{H}$) $\cup$ {⌜α = α⌝} and we have Γ $\in$ AVP($\mathfrak{H}^6$). With Theorem 4-7, there is then an $\mathfrak{H}^+ \in$ RCS\{∅} such that AVAP($\mathfrak{H}^+$) $\subseteq$ AVAP($\mathfrak{H}^6$)\{⌜α = α⌝} = (AVAP($\mathfrak{H}$) $\cup$ {⌜α = α⌝})\{⌜α = α⌝} = AVAP($\mathfrak{H}$)\{⌜α = α⌝} $\subseteq$ ($X \cup Y \cup Z$)\{⌜α = α⌝} $\subseteq X \cup Y \cup Z$ and C($\mathfrak{H}^+$) = Γ. With Theorem 3-12, we then have $X \cup Y \cup Z \vdash$ Γ.

Ad (x) (DE)*: Suppose $X \vdash \ulcorner A \vee B \urcorner$ and $Y \vdash \Gamma$ and $A \in Y$ and $Z \vdash \Gamma$ and $B \in Z$. Then it holds with (i): $Y \setminus \{A\} \vdash \ulcorner A \rightarrow B \urcorner$ and $Z \setminus \{B\} \vdash \ulcorner B \rightarrow A \urcorner$. Then it holds with (ix): $X \cup (Y \setminus \{A\}) \cup (Z \setminus \{B\}) \vdash \Gamma$.

Ad (xi) (NI): Suppose $X \vdash \Gamma$ and $Y \vdash \ulcorner \neg\Gamma \urcorner$ and $A \in X \cup Y$. If $A = \ulcorner \Delta' \wedge \neg\Delta' \urcorner$ for a $\Delta' \in$ CFORM, then it holds, with Theorem 4-17, that $(X \cup Y) \setminus \{A\} \vdash \ulcorner \neg(\Delta' \wedge \neg\Delta') \urcorner = \ulcorner \neg A \urcorner$. Now, suppose $A \neq \ulcorner \Delta' \wedge \neg\Delta' \urcorner$ for all Δ'. With (iii), it holds that $X \cup Y \vdash \ulcorner \Gamma \wedge \neg\Gamma \urcorner$. Also, we have, again with Theorem 4-17, $X \cup Y \vdash \ulcorner \neg(\Gamma \wedge \neg\Gamma) \urcorner$ and thus we have, with (iii), $X \cup Y \vdash \ulcorner (\Gamma \wedge \neg\Gamma) \wedge \neg(\Gamma \wedge \neg\Gamma) \urcorner$. With (i), it then follows that $(X \cup Y) \setminus \{A\} \vdash \ulcorner A \rightarrow ((\Gamma \wedge \neg\Gamma) \wedge \neg(\Gamma \wedge \neg\Gamma)) \urcorner$. Thus there is, with Theorem 3-12, an $\mathfrak{H} \in \mathrm{RCS} \setminus \{\emptyset\}$ such that $\mathrm{AVAP}(\mathfrak{H}) \subseteq (X \cup Y) \setminus \{A\}$ and $C(\mathfrak{H}) = \ulcorner A \rightarrow ((\Gamma \wedge \neg\Gamma) \wedge \neg(\Gamma \wedge \neg\Gamma)) \urcorner$. Then we can extend $\mathfrak{H}$ as follows to an $\mathfrak{H}^5 \in$ SEQ with $\mathfrak{H}^5 \restriction \mathrm{Dom}(\mathfrak{H}) = \mathfrak{H}$:

$$\begin{array}{lllll}
\mathfrak{H}^1 & = \mathfrak{H} & \cup & \{(\mathrm{Dom}(\mathfrak{H}), & \ulcorner \text{Suppose } A \urcorner)\} \\
\mathfrak{H}^2 & = \mathfrak{H}^1 & \cup & \{(\mathrm{Dom}(\mathfrak{H}^1), & \ulcorner \text{Therefore } (\Gamma \wedge \neg\Gamma) \wedge \neg(\Gamma \wedge \neg\Gamma) \urcorner)\} \\
\mathfrak{H}^3 & = \mathfrak{H}^2 & \cup & \{(\mathrm{Dom}(\mathfrak{H}^2), & \ulcorner \text{Therefore } \Gamma \wedge \neg\Gamma \urcorner)\} \\
\mathfrak{H}^4 & = \mathfrak{H}^3 & \cup & \{(\mathrm{Dom}(\mathfrak{H}^3), & \ulcorner \text{Therefore } \neg(\Gamma \wedge \neg\Gamma) \urcorner)\} \\
\mathfrak{H}^5 & = \mathfrak{H}^4 & \cup & \{(\mathrm{Dom}(\mathfrak{H}^4), & \ulcorner \text{Therefore } \neg A \urcorner)\}.
\end{array}$$

First, we have $\mathfrak{H}^5{}_{\mathrm{Dom}(\mathfrak{H})} \in$ ASENT. By hypothesis, we have $C(\mathfrak{H}^1) = A \neq C(\mathfrak{H}^2)$. With Theorem 1-8, Theorem 1-10 and Theorem 1-11 we have $C(\mathfrak{H}^2) \neq C(\mathfrak{H}^3)$ and $C(\mathfrak{H}^3) \neq C(\mathfrak{H}^4)$. We also have that $C(\mathfrak{H}^2)$ and $C(\mathfrak{H}^3)$ are neither conditionals nor negations and that $C(\mathfrak{H}^4)$ is not a conditional and by hypothesis $C(\mathfrak{H}^4) = \ulcorner \neg(\Gamma \wedge \neg\Gamma) \urcorner \neq \ulcorner \neg A \urcorner$. With Theorem 2-42, Definition 2-11, Definition 2-12 and Definition 2-13, we then have that it holds for all k with $1 \leq k \leq 4$ that there is no closed segment $\mathfrak{A}$ in $\mathfrak{H}^k$ such that $\min(\mathrm{Dom}(\mathfrak{A})) = \mathrm{Dom}(\mathfrak{H})$. With Theorem 2-47, we thus have for all k with $1 \leq k \leq 4$ that there is no closed segment $\mathfrak{A}$ in $\mathfrak{H}^k$ such that $\min(\mathrm{Dom}(\mathfrak{A})) \leq \mathrm{Dom}(\mathfrak{H}) \leq \max(\mathrm{Dom}(\mathfrak{A}))$. Thus we also get that it holds for all k with $1 \leq k \leq 4$ that $\mathrm{Dom}(\mathfrak{H}) = \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^k)))$. With Theorem 3-19-(i), Theorem 3-20-(i), Theorem 3-21-(i) and Theorem 2-61, we thus have for all k with $2 \leq k \leq 4$ that $\mathfrak{H}^k \notin \mathrm{CdIF}(\mathfrak{H}^{k-1}) \cup \mathrm{NIF}(\mathfrak{H}^{k-1}) \cup \mathrm{PEF}(\mathfrak{H}^{k-1})$.

On the other hand, we have *first*, according to Definition 3-1, $\mathfrak{H}^1 \in \mathrm{AF}(\mathfrak{H}) \subseteq \mathrm{RCS} \setminus \{\emptyset\}$ and, with Theorem 3-15, $\mathrm{AVS}(\mathfrak{H}^1) = \mathrm{AVS}(\mathfrak{H}) \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner \text{Suppose } A \urcorner)\}$ and $\mathrm{AVAS}(\mathfrak{H}^1) = \mathrm{AVAS}(\mathfrak{H}) \cup \{(\mathrm{Dom}(\mathfrak{H}), \ulcorner \text{Suppose } A \urcorner)\}$, $\ulcorner A \rightarrow ((\Gamma \wedge \neg\Gamma) \wedge \neg(\Gamma \wedge \neg\Gamma)) \urcorner \in$

AVP($\mathfrak{H}$) ⊆ AVP($\mathfrak{H}^1$) and A ∈ AVP($\mathfrak{H}^1$). Then we have *second*, according to Definition 3-3, $\mathfrak{H}^2$ ∈ CdEF($\mathfrak{H}^1$) ⊆ RCS\{∅} and, with Theorem 3-25, AVS($\mathfrak{H}^2$) = AVS($\mathfrak{H}^1$) ∪ {(Dom($\mathfrak{H}^1$), ⌜Therefore (Γ ∧ ¬Γ) ∧ ¬(Γ ∧ ¬Γ)⌝)}. Thus we have AVAS($\mathfrak{H}^2$) = AVAS($\mathfrak{H}^1$) and ⌜(Γ ∧ ¬Γ) ∧ ¬(Γ ∧ ¬Γ)⌝ ∈ AVP($\mathfrak{H}^2$). Therefore we have *third*, according to Definition 3-5, $\mathfrak{H}^3$ ∈ CEF($\mathfrak{H}^2$) ⊆ RCS\{∅} and, with Theorem 3-25, AVS($\mathfrak{H}^3$) = AVS($\mathfrak{H}^2$) ∪ {(Dom($\mathfrak{H}^2$), ⌜Therefore Γ ∧ ¬Γ⌝)}. Thus we have AVAS($\mathfrak{H}^3$) = AVAS($\mathfrak{H}^2$), ⌜(Γ ∧ ¬Γ) ∧ ¬(Γ ∧ ¬Γ)⌝ ∈ AVP($\mathfrak{H}^2$) ⊆ AVP($\mathfrak{H}^3$) and ⌜Γ ∧ ¬Γ⌝ ∈ AVP($\mathfrak{H}^3$). Then we have *fourth*, according to Definition 3-5, $\mathfrak{H}^4$ ∈ CEF($\mathfrak{H}^3$) ⊆ RCS\{∅} and, with Theorem 3-25, AVS($\mathfrak{H}^4$) = AVS($\mathfrak{H}^3$) ∪ {(Dom($\mathfrak{H}^3$), ⌜Therefore ¬(Γ ∧ ¬Γ)⌝)}. Thus we have AVAS($\mathfrak{H}^4$) = AVAS($\mathfrak{H}^3$) = AVAS($\mathfrak{H}^1$) and (Dom($\mathfrak{H}^2$), ⌜Therefore Γ ∧ ¬Γ⌝), (Dom($\mathfrak{H}^3$), ⌜Therefore ¬(Γ ∧ ¬Γ)⌝) ∈ AVS($\mathfrak{H}^4$) and (Dom($\mathfrak{H}$), ⌜Suppose A⌝) ∈ AVAS($\mathfrak{H}^1$) = AVAS($\mathfrak{H}^4$).

Thus we have Dom($\mathfrak{H}$), Dom($\mathfrak{H}^2$) ∈ Dom($\mathfrak{H}^4$), where Dom($\mathfrak{H}$) ≤ Dom($\mathfrak{H}^2$), P($\mathfrak{H}^4_{\mathrm{Dom}(\mathfrak{H})}$) = A and (Dom($\mathfrak{H}$), $\mathfrak{H}^4_{\mathrm{Dom}(\mathfrak{H})}$) ∈ AVAS($\mathfrak{H}^4$), P($\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H}^2)}$) = ⌜Γ ∧ ¬Γ⌝ and P($\mathfrak{H}^4_{\mathrm{Dom}(\mathfrak{H}^4)\text{-}1}$) = ⌜¬(Γ ∧ ¬Γ)⌝, (Dom($\mathfrak{H}^2$), $\mathfrak{H}_{\mathrm{Dom}(\mathfrak{H}^2)}$) ∈ AVS($\mathfrak{H}^4$) and there is no l such that Dom($\mathfrak{H}$) < l ≤ Dom($\mathfrak{H}^4$)-1 and (l, $\mathfrak{H}^4_l$) ∈ AVAS($\mathfrak{H}^4$). Finally we thus have *fifth*, according to Definition 3-10, $\mathfrak{H}^5$ ∈ NIF($\mathfrak{H}^4$) ⊆ RCS\{∅} and, with Theorem 3-20-(iv) and -(v), AVAS($\mathfrak{H}^5$) = AVAS($\mathfrak{H}^4$)\{(max(Dom(AVAS($\mathfrak{H}^4$))), $\mathfrak{H}^5_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^4)))}$)} = AVAS($\mathfrak{H}^4$)\{(Dom($\mathfrak{H}$), ⌜Suppose A⌝)} = AVAS($\mathfrak{H}^1$)\{(Dom($\mathfrak{H}$), ⌜Suppose A⌝)} = (AVAS($\mathfrak{H}$) ∪ {(Dom($\mathfrak{H}$), ⌜Suppose A⌝)})\{(Dom($\mathfrak{H}$), ⌜Suppose A⌝)} = AVAS($\mathfrak{H}$)\{(Dom($\mathfrak{H}$), ⌜Suppose A⌝)} ⊆ AVAS($\mathfrak{H}$). With Theorem 2-75, we then have AVAP($\mathfrak{H}^5$) ⊆ AVAP($\mathfrak{H}$) ⊆ (X ∪ Y)\{A}. Since C($\mathfrak{H}^5$) = ⌜¬A⌝, it holds, with Theorem 3-12, that (X ∪ Y)\{A} ⊢ ⌜¬A⌝.

Ad (xvi) (PE): Suppose X ⊢ ⌜⋁ξΔ⌝ and Y ⊢ Γ and [β, ξ Δ] ∈ Y and β ∉ STSF((Y\{[β, ξ, Δ]}) ∪ {Δ, Γ}). Then it holds, with (i), that Y\{[β, ξ, Δ]} ⊢ ⌜[β, ξ, Δ] → Γ⌝. We also have with Γ ∈ CFORM: [β, ξ, Γ] = Γ. Thus we have [β, ξ, ⌜Δ → Γ⌝] = ⌜[β, ξ, Δ] → [β, ξ, Γ]⌝ = ⌜[β, ξ, Δ] → Γ⌝ and thus we have Y\{[β, ξ, Δ]} ⊢ [β, ξ, ⌜Δ → Γ⌝]. With β ∉ STSF({Δ, Γ}), we have β ∉ ST(⌜Δ → Γ⌝). With Γ ∈ CFORM and FV(Δ) ⊆ {ξ}, we also have FV(⌜Δ → Γ⌝) ⊆ {ξ}. Since by hypothesis also β ∉ STSF(Y\{[β, ξ, Δ]}), it then follows, with (xv), that Y\{[β, ξ, Δ]} ⊢ ⌜⋀ξ(Δ → Γ)⌝. With (iii), we then have X ∪ (Y\{[β, ξ, Δ]}) ⊢ ⌜⋀ξ(Δ → Γ) ∧ ⋁ξΔ⌝.

According to Theorem 3-12, there is thus an $\mathfrak{H} \in \text{RCS}\backslash\{\emptyset\}$ such that $\text{AVAP}(\mathfrak{H}) \subseteq X \cup (Y\backslash\{[\beta, \xi, \Delta]\})$ and $\text{C}(\mathfrak{H}) = \ulcorner\bigwedge\xi(\Delta \rightarrow \Gamma) \wedge \bigvee\xi\Delta\urcorner$. With Theorem 4-5, there is then an $\mathfrak{H}^* \in \text{RCS}\backslash\{\emptyset\}$ such that $\text{AVAP}(\mathfrak{H}^*) = \text{AVAP}(\mathfrak{H}) \subseteq X \cup (Y\backslash\{[\beta, \xi, \Delta]\})$ and $\ulcorner\bigwedge\xi(\Delta \rightarrow \Gamma)\urcorner$, $\ulcorner\bigvee\xi\Delta\urcorner \in \text{AVP}(\mathfrak{H}^*)$ and $\text{C}(\mathfrak{H}^*) = \ulcorner\bigvee\xi\Delta\urcorner$. With Theorem 2-82, we have more precisely that $(\text{Dom}(\mathfrak{H}^*)\text{-}1, \ulcorner\Xi\ \bigvee\xi\Delta\urcorner) \in \text{AVS}(\mathfrak{H}^*)$ for a $\Xi \in \text{PERF}$. There is a $\beta^* \in \text{PAR}\backslash\text{STSEQ}(\mathfrak{H}^*)$ and an $\alpha \in \text{CONST}\backslash\text{STSEQ}(\mathfrak{H}^*)$. Thus we can extend $\mathfrak{H}^*$ as follows to an $\mathfrak{H}^5 \in \text{SEQ}$ with $\mathfrak{H}^5{\upharpoonright}\text{Dom}(\mathfrak{H}^*) = \mathfrak{H}^*$:

$$\begin{array}{lllll}
\mathfrak{H}^1 & = \mathfrak{H}^* & \cup & \{(\text{Dom}(\mathfrak{H}^*), & \ulcorner\text{Suppose } [\beta^*, \xi, \Delta]\urcorner)\} \\
\mathfrak{H}^2 & = \mathfrak{H}^1 & \cup & \{(\text{Dom}(\mathfrak{H}^1), & \ulcorner\text{Therefore } \alpha = \alpha\urcorner)\} \\
\mathfrak{H}^3 & = \mathfrak{H}^2 & \cup & \{(\text{Dom}(\mathfrak{H}^2), & \ulcorner\text{Therefore } [\beta^*, \xi, \Delta] \rightarrow \Gamma\urcorner)\} \\
\mathfrak{H}^4 & = \mathfrak{H}^3 & \cup & \{(\text{Dom}(\mathfrak{H}^3), & \ulcorner\text{Therefore } \Gamma\urcorner)\} \\
\mathfrak{H}^5 & = \mathfrak{H}^4 & \cup & \{(\text{Dom}(\mathfrak{H}^4), & \ulcorner\text{Therefore } \Gamma\urcorner)\}.
\end{array}$$

First, we have $\mathfrak{H}^5_{\text{Dom}(\mathfrak{H}^*)} \in \text{ASENT}$. We have, with $\alpha \in \text{CONST}\backslash\text{STSEQ}(\mathfrak{H})$, also $\alpha \notin \text{STSF}(\{[\beta^*, \xi, \Delta], \Gamma\})$ and thus $\text{C}(\mathfrak{H}^1) \neq \text{C}(\mathfrak{H}^2)$, $\text{C}(\mathfrak{H}^2) \neq \text{C}(\mathfrak{H}^3)$ and $\text{C}(\mathfrak{H}^3) \neq \ulcorner[\beta^*, \xi, \Delta] \rightarrow \text{C}(\mathfrak{H}^2)\urcorner$. With Theorem 1-8, we also have $\text{C}(\mathfrak{H}^3) \neq \text{C}(\mathfrak{H}^4)$. Furthermore we have, with Theorem 1-10 and Theorem 1-11, that $\text{C}(\mathfrak{H}^2)$ is not a conditional and that $\text{C}(\mathfrak{H}^2)$ and $\text{C}(\mathfrak{H}^3)$ are not negations. In addition we have $\text{C}(\mathfrak{H}^1) = \ulcorner[\beta^*, \xi, \Delta]\urcorner \neq \ulcorner\neg([\beta^*, \xi, \Delta] \rightarrow \Gamma)\urcorner = \ulcorner\neg\text{C}(\mathfrak{H}^3)\urcorner$ and $\text{C}(\mathfrak{H}^1) = \Gamma \neq \ulcorner[\beta^*, \xi, \Delta] \rightarrow ([\beta^*, \xi, \Delta] \rightarrow \Gamma)\urcorner = \ulcorner\text{C}(\mathfrak{H}^1) \rightarrow \text{C}(\mathfrak{H}^3)\urcorner$. With Theorem 2-42, Definition 2-11, Definition 2-12 and Definition 2-13, it then holds for all k with $1 \leq k \leq 4$ that there is no closed segment $\mathfrak{A}$ in $\mathfrak{H}^k$ such that $\min(\text{Dom}(\mathfrak{A})) = \text{Dom}(\mathfrak{H}^*)$. With Theorem 2-47, we thus have for all k with $1 \leq k \leq 4$ that there is no closed segment $\mathfrak{A}$ in $\mathfrak{H}^k$ such that $\min(\text{Dom}(\mathfrak{A})) \leq \text{Dom}(\mathfrak{H}^*) \leq \max(\text{Dom}(\mathfrak{A}))$. Thus we also get that it holds for all k with $1 \leq k \leq 4$ that $\text{Dom}(\mathfrak{H}^*) = \max(\text{Dom}(\text{AVAS}(\mathfrak{H}^k)))$. With Theorem 3-19-(i), Theorem 3-20-(i), Theorem 3-21-(i) and Theorem 2-61, we thus have for all k with $2 \leq k \leq 4$ that $\mathfrak{H}^k \notin \text{CdIF}(\mathfrak{H}^{k\text{-}1}) \cup \text{NIF}(\mathfrak{H}^{k\text{-}1}) \cup \text{PEF}(\mathfrak{H}^{k\text{-}1})$.

On the other hand, we have *first*, according to Definition 3-1, $\mathfrak{H}^1 \in \text{AF}(\mathfrak{H}) \subseteq \text{RCS}\backslash\{\emptyset\}$ and, with Theorem 3-15, $\text{AVS}(\mathfrak{H}^1) = \text{AVS}(\mathfrak{H}^*) \cup \{(\text{Dom}(\mathfrak{H}^*), \ulcorner\text{Suppose } [\beta^*, \xi, \Delta]\urcorner)\}$ and $\text{AVAS}(\mathfrak{H}^1) = \text{AVAS}(\mathfrak{H}^*) \cup \{(\text{Dom}(\mathfrak{H}), \ulcorner\text{Suppose } [\beta^*, \xi, \Delta]\urcorner)\}$, $(\text{Dom}(\mathfrak{H}^*)\text{-}1, \mathfrak{H}^5_{\text{Dom}(\mathfrak{H}^*)\text{-}1}) \in \text{AVS}(\mathfrak{H}^1)$, where $\text{P}(\mathfrak{H}^5_{\text{Dom}(\mathfrak{H}^*)\text{-}1}) = \ulcorner\bigvee\xi\Delta\urcorner$, and $\ulcorner\bigwedge\xi(\Delta \rightarrow \Gamma)\urcorner \in \text{AVP}(\mathfrak{H}^*) \subseteq \text{AVP}(\mathfrak{H}^1)$ and $[\beta^*, \xi, \Delta] \in \text{AVP}(\mathfrak{H}^1)$. Then we have *second*, according to Definition 3-16, $\mathfrak{H}^2 \in \text{IIF}(\mathfrak{H}^1) \subseteq \text{RCS}\backslash\{\emptyset\}$ and, with Theorem 3-25, $\text{AVS}(\mathfrak{H}^2) = \text{AVS}(\mathfrak{H}^1) \cup \{(\text{Dom}(\mathfrak{H}^1), \ulcorner\text{Therefore } \alpha = \alpha\urcorner)\}$. Thus we have $(\text{Dom}(\mathfrak{H}^*), \ulcorner\text{Suppose } [\beta^*, \xi, \Delta]\urcorner) \in \text{AVAS}(\mathfrak{H}^1) =$

AVAS($\mathfrak{H}^2$) and ⌜⋀ξ(Δ → Γ)⌝, [β*, ξ, Δ] ∈ AVP($\mathfrak{H}^1$) ⊆ AVP($\mathfrak{H}^2$) and (Dom($\mathfrak{H}$*)-1, $\mathfrak{H}^5{}_{\mathrm{Dom}(\mathfrak{H}^*)-1}$) ∈ AVS($\mathfrak{H}^2$). Therefore we have *third*, according to Definition 3-13, $\mathfrak{H}^3$ ∈ UEF($\mathfrak{H}^2$) ⊆ RCS\{∅} and, with Theorem 3-25, AVS($\mathfrak{H}^3$) = AVS($\mathfrak{H}^2$) ∪ {(Dom($\mathfrak{H}^2$), ⌜Therefore [β*, ξ, Δ] → Γ⌝)}. Thus we have (Dom($\mathfrak{H}$*), ⌜Suppose [β*, ξ, Δ]⌝) ∈ AVAS($\mathfrak{H}^2$) = AVAS($\mathfrak{H}^3$) and (Dom($\mathfrak{H}$*)-1, $\mathfrak{H}^5{}_{\mathrm{Dom}(\mathfrak{H}^*)-1}$) ∈ AVS($\mathfrak{H}^3$) and [β*, ξ, Δ] ∈ AVP($\mathfrak{H}^2$) ⊆ AVP($\mathfrak{H}^3$) and ⌜[β*, ξ, Δ] → Γ⌝ ∈ AVP($\mathfrak{H}^3$). Therefore we have *fourth*, according to Definition 3-3, $\mathfrak{H}^4$ ∈ CdEF($\mathfrak{H}^3$) ⊆ RCS\{∅} and, with Theorem 3-25, AVS($\mathfrak{H}^4$) = AVS($\mathfrak{H}^3$) ∪ {(Dom($\mathfrak{H}^3$), ⌜Therefore Γ⌝)}. Thus we have (Dom($\mathfrak{H}$*), ⌜Suppose [β*, ξ, Δ]⌝) ∈ AVAS($\mathfrak{H}^3$) = AVAS($\mathfrak{H}^4$) and (Dom($\mathfrak{H}$*)-1, $\mathfrak{H}^5{}_{\mathrm{Dom}(\mathfrak{H}^*)-1}$), (Dom($\mathfrak{H}$*)+3, ⌜Therefore Γ⌝) ∈ AVS($\mathfrak{H}^4$).

Altogether we thus have β* ∈ PAR, ξ ∈ VAR, Δ ∈ FORM, FV(Δ) ⊆ {ξ}, Γ ∈ CFORM Dom($\mathfrak{H}$*)-1 ∈ Dom($\mathfrak{H}^4$), P($\mathfrak{H}^4{}_{\mathrm{Dom}(\mathfrak{H}^*)-1}$) = ⌜⋁ξΔ⌝ and (Dom($\mathfrak{H}$*)-1, $\mathfrak{H}^4{}_{\mathrm{Dom}(\mathfrak{H}^*)-1}$) ∈ AVS($\mathfrak{H}^4$), P($\mathfrak{H}^4{}_{\mathrm{Dom}(\mathfrak{H}^*)}$) = [β*, ξ, Δ] and (Dom($\mathfrak{H}$*), $\mathfrak{H}^4{}_{\mathrm{Dom}(\mathfrak{H}^*)}$) ∈ AVAS($\mathfrak{H}^4$), P($\mathfrak{H}^4{}_{\mathrm{Dom}(\mathfrak{H}^4)-1}$) = Γ, β* ∉ STSF({Δ, Γ}) and there is no j ≤ Dom($\mathfrak{H}$*)-1 such that β* ∈ ST($\mathfrak{H}^4{}_j$) and there is no m such that Dom($\mathfrak{H}$*) < m ≤ Dom($\mathfrak{H}^4$)-1 and (m, $\mathfrak{H}^4{}_m$) ∈ AVAS($\mathfrak{H}^4$). Finally we thus have, according to Definition 3-15, $\mathfrak{H}^5$ ∈ PEF($\mathfrak{H}^4$) ⊆ RCS\{∅} and, with Theorem 3-21-(iv) and -(v), AVAS($\mathfrak{H}^5$) = AVAS($\mathfrak{H}^4$)\{(max(Dom(AVAS($\mathfrak{H}^4$))), $\mathfrak{H}^5{}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}^4)))}$)} = AVAS($\mathfrak{H}^4$)\{(Dom($\mathfrak{H}$*), ⌜Suppose [β*, ξ, Δ]⌝)} = AVAS($\mathfrak{H}^1$)\{(Dom($\mathfrak{H}$*), ⌜Suppose [β*, ξ, Δ]⌝)} = (AVAS($\mathfrak{H}$*) ∪ {(Dom($\mathfrak{H}$*), ⌜Suppose [β*, ξ, Δ]⌝)})\{(Dom($\mathfrak{H}$*), ⌜Suppose [β*, ξ, Δ]⌝)} = AVAS($\mathfrak{H}$*)\{(Dom($\mathfrak{H}$*), ⌜Suppose [β*, ξ, Δ]⌝)} ⊆ AVAS($\mathfrak{H}$*). With Theorem 2-75, we then have AVAP($\mathfrak{H}^5$) ⊆ AVAP($\mathfrak{H}$*) ⊆ X ∪ (Y\{[β, ξ, Δ]}). Since C($\mathfrak{H}^5$) = Γ, it thus holds, with Theorem 3-12, that X ∪ (Y\{[β, ξ, Δ]}) ⊢ Γ.

Ad (xvii) (II): Suppose X ⊆ CFORM. According to Definition 3-16, we then have {(0, ⌜Therefore θ = θ⌝)} ∈ IE(∅) ⊆ RCS\{∅} and we have AVAS({(0, ⌜Therefore $\theta_0 = \theta_0$⌝)}) = ∅ and hence, according to Definition 2-31, AVAP({(0, ⌜Therefore $\theta_0 = \theta_0$⌝)}) = ∅ and we have C({(0, ⌜Therefore $\theta_0 = \theta_0$⌝)}) = ⌜$\theta_0 = \theta_0$⌝ and thus, according to Theorem 3-12, ∅ ⊢ ⌜$\theta_0 = \theta_0$⌝. With Theorem 4-16, we hence have X ⊢ ⌜$\theta_0 = \theta_0$⌝. ■

Theorem 4-19. *Transitivity*
If $X\ _{M}\vdash Y$ and $Y \vdash$ B, then $X \vdash$ B.

Proof: First we show by induction on $|Y|$ that the statement holds for all finite Y: Suppose the statement holds for all $k < |Y| \in \mathbb{N}$. Suppose $|Y| = 0$. Now, suppose $X\ _{M}\vdash Y$ and $Y \vdash$ B. Then we have $Y = \emptyset \subseteq X \subseteq$ CFORM. With Theorem 4-16 follows $X \vdash$ B.

Now, suppose $0 < |Y|$ and suppose $X\ _{M}\vdash Y$ and $Y \vdash$ B. According to Definition 3-25, we then have $X \cup Y \subseteq$ CFORM and for all $\Delta \in Y$: $X \vdash \Delta$. Now, suppose $Y \vdash$ B. Since $|Y| \neq 0$, we have that there is an A $\in Y$. With Theorem 4-18-(i), we then have $Y\setminus\{A\} \vdash$ ⌜A → B⌝. Then we have $|Y\setminus\{A\}| < |Y|$. By the I.H., we thus have $X \vdash$ ⌜A → B⌝, and, since A $\in Y$, we also have $X \vdash$ A. With Theorem 4-18-(ii), we thus have $X \vdash$ B.

As the statement holds for finite Y, it also holds in general: Suppose $X\ _{M}\vdash Y$ and $Y \vdash$ B. According to Definition 3-25, we have $X \cup Y \subseteq$ CFORM and for all $\Delta \in Y$: $X \vdash \Delta$. Now, suppose $Y \vdash$ B. With Theorem 3-12, there is then an $\mathfrak{H} \in$ RCS$\setminus\{\emptyset\}$ such that AVAP($\mathfrak{H}$) $\subseteq Y$ and C($\mathfrak{H}$) = B. According to Theorem 3-9, AVAP($\mathfrak{H}$) is finite and AVAP($\mathfrak{H}$) $\subseteq$ CFORM. According to Theorem 3-12, we have that AVAP($\mathfrak{H}$) $\vdash$ B. We also have with AVAP($\mathfrak{H}$) $\subseteq Y$ that it holds for all $\Gamma \in$ AVAP($\mathfrak{H}$) that $X \vdash \Gamma$ and thus that $X\ _{M}\vdash$ AVAP($\mathfrak{H}$). Thus it then follows that $X \vdash$ B. ■

Theorem 4-20. *Cut*
If $X \cup \{B\} \vdash$ A and $Y \vdash$ B, then $X \cup Y \vdash$ A.

Proof: Suppose $X \cup \{B\} \vdash$ A and $Y \vdash$ B. With Theorem 4-18-(i), we then have $X\setminus\{B\}$ $\vdash$ ⌜B → A⌝ and thus with Theorem 4-16 that $X \vdash$ ⌜B → A⌝. With Theorem 4-18-(ii), it thus holds that $X \cup Y \vdash$ A. ■

Theorem 4-21. *Deduction theorem and its inverse*
$X \cup \{A\} \vdash$ B iff $X \vdash$ ⌜A → B⌝.

Proof: First, suppose $X \cup \{A\} \vdash$ B. Then it holds, with Theorem 4-18-(i), that $X\setminus\{A\} \vdash$ ⌜A → B⌝ and thus, with Theorem 4-16, that $X \vdash$ ⌜A → B⌝. Now, suppose $X \vdash$ ⌜A →

B⌝. According to Definition 3-21 and Theorem 3-9, we then have ⌜A → B⌝ ∈ CFORM and thus also A ∈ CFORM. With Theorem 4-15, we then have {A} ⊢ A and hence, with Theorem 4-18-(ii), X ∪ {A} ⊢ B. ■

Theorem 4-22. *Inconsistence and derivability*
X ⊢ A iff X ∪ {⌜¬A⌝} is inconsistent.

Proof: (*L-R*): First, suppose X ⊢ A. With Definition 3-21 and Theorem 3-9, we then have X ⊆ CFORM and A ∈ CFORM. Then we have ⌜¬A⌝ ∈ CFORM and it thus holds, with Theorem 4-16, that X ∪ {⌜¬A⌝} ⊢ A, and, with Theorem 4-15, it holds that X ∪ {⌜¬A⌝} ⊢ ⌜¬A⌝. According to Definition 3-24, we then have that X ∪ {⌜¬A⌝} is inconsistent.

(*R-L*): Now, suppose X ∪ {⌜¬A⌝} is inconsistent. According to Definition 3-24, we then have X ∪ {⌜¬A⌝} ⊆ CFORM and that there is a Γ ∈ CFORM such that X ∪ {⌜¬A⌝} ⊢ Γ and X ∪ {⌜¬A⌝} ⊢ ⌜¬Γ⌝. With Theorem 4-18-(xi), it then holds that X\{⌜¬A⌝} ⊢ ⌜¬¬A⌝ and thus, with Theorem 4-16, that X ⊢ ⌜¬¬A⌝. From this we get, with Theorem 4-18-(xii), that X ⊢ A. ■

Theorem 4-23. *A set of propositions is inconsistent if and only if all propositions can be derived from it*
X is inconsistent iff for all Γ ∈ CFORM: X ⊢ Γ.

Proof: (*L-R*): First, suppose X is inconsistent. According to Definition 3-24, we then have X ⊆ CFORM and that there is an A ∈ CFORM such that X ⊢ A and X ⊢ ⌜¬A⌝. Now, suppose Γ ∈ CFORM. Then we have ⌜¬Γ⌝ ∈ CFORM. With Theorem 4-16, it then holds that X ∪ {⌜¬Γ⌝} ⊢ A and X ∪ {⌜¬Γ⌝} ⊢ ⌜¬A⌝. Thus we have that X ∪ {⌜¬Γ⌝} is inconsistent. According to Theorem 4-22, we then have X ⊢ Γ.

(*R-L*): Now, suppose for all Γ ∈ CFORM it holds that X ⊢ Γ. There is a Δ ∈ CFORM. With Δ ∈ CFORM, we also have ⌜¬Δ⌝ ∈ CFORM. Then we have X ⊢ Δ and X ⊢ ⌜¬Δ⌝. With Definition 3-21, we then have X ⊆ CFORM. According to Definition 3-24, we hence have that X is inconsistent. ■

Theorem 4-24. *Generalisation theorem*
If $\xi \in$ VAR, $\Delta \in$ FORM, where FV(Δ) $\subseteq \{\xi\}$, $\alpha \in$ CONST and $X \vdash [\alpha, \xi, \Delta]$, where $\alpha \notin$ STSF($X \cup \{\Delta\}$), then $X \vdash \ulcorner\bigwedge\xi\Delta\urcorner$

Proof: Suppose $\xi \in$ VAR, $\Delta \in$ FORM, where FV(Δ) $\subseteq \{\xi\}$, $\alpha \in$ CONST and $X \vdash [\alpha, \xi, \Delta]$, where $\alpha \notin$ STSF($X \cup \{\Delta\}$). According to Theorem 3-12, there is then an $\mathfrak{H} \in$ RCS\{∅} such that AVAP($\mathfrak{H}$) $\subseteq X$ and C($\mathfrak{H}$) = $[\alpha, \xi, \Delta]$. There is a $\beta \in$ PAR\STSEQ($\mathfrak{H}$). With Theorem 4-9, there is then an $\mathfrak{H}^* \in$ RCS\{∅} such that:

a) $\alpha \notin$ STSEQ($\mathfrak{H}^*$),
b) AVAP($\mathfrak{H}$) = $\{[\alpha, \beta, B] \mid B \in$ AVAP($\mathfrak{H}^*$)$\}$, and
c) C($\mathfrak{H}$) = $[\alpha, \beta,$ C($\mathfrak{H}^*$)$]$.

Since it holds for all $\Gamma \in$ AVAP($\mathfrak{H}$) that $\alpha \notin$ ST(Γ), it holds with b) for all B $\in$ AVAP($\mathfrak{H}^*$) that $\beta \notin$ ST(B) and thus that $\beta \notin$ STSF(AVAP($\mathfrak{H}^*$)). For if $\beta \in$ ST(Γ) for a $\Gamma \in$ AVAP($\mathfrak{H}^*$), then we would have $\alpha \in$ ST($[\alpha, \beta, \Gamma]$) and, with b), we would have $[\alpha, \beta, \Gamma] \in$ AVAP($\mathfrak{H}$) $\subseteq X$. Thus we would have that $\alpha \in$ STSF(X), which contradicts the hypothesis. With b), we thus have AVAP($\mathfrak{H}$) = $\{[\alpha, \beta, B] \mid B \in$ AVAP($\mathfrak{H}^*$)$\}$ = $\{B \mid B \in$ AVAP($\mathfrak{H}^*$)$\}$ = AVAP($\mathfrak{H}^*$).

With c), it holds that $[\alpha, \xi, \Delta]$ = C($\mathfrak{H}$) = $[\alpha, \beta,$ C($\mathfrak{H}^*$)$]$. According to the initial assumption and with a), we have $\alpha \notin$ ST(Δ) $\cup$ ST(C($\mathfrak{H}^*$)). With Theorem 1-23, we thus have C($\mathfrak{H}^*$) = $[\beta, \xi, \Delta]$. Then we have $\beta \notin$ ST(Δ), because otherwise we would have, with $[\alpha, \xi, \Delta]$ = C($\mathfrak{H}$), that $\beta \in$ ST(C($\mathfrak{H}$)) $\subseteq$ STSEQ($\mathfrak{H}$), which contradicts the choice of β. With Definition 3-12, we thus have $\mathfrak{H}^* \cup \{($Dom($\mathfrak{H}^*$), ⌜Therefore $\bigwedge\xi\Delta$⌝$)\} \in$ UIF($\mathfrak{H}^*$) $\subseteq$ RCS\{∅}. With Theorem 3-26-(v), it then holds that AVAP($\mathfrak{H}^* \cup \{($Dom($\mathfrak{H}^*$), ⌜Therefore $\bigwedge\xi\Delta$⌝$)\}$) $\subseteq$ AVAP($\mathfrak{H}^*$) = AVAP($\mathfrak{H}$) $\subseteq X$. With Theorem 3-12, we hence have $X \vdash \ulcorner\bigwedge\xi\Delta\urcorner$. ■

Theorem 4-25. *Multiple IE*
If $k \in \mathbb{N}\setminus\{0\}$, $\{\theta_0, \ldots, \theta_{k-1}\}, \{\theta'_0, \ldots, \theta'_{k-1}\} \subseteq$ CTERM, $\{\xi_0, \ldots, \xi_{k-1}\} \subseteq$ VAR, where for all $i, j \in k$ with $i \neq j$ also $\xi_i \neq \xi_j$, $\Delta \in$ FORM, where FV(Δ) $\subseteq \{\xi_0, \ldots, \xi_{k-1}\}$, and $X \vdash [\langle\theta_0, \ldots, \theta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta]$ and for all $i < k$: $X \vdash \ulcorner\theta_i = \theta'_i\urcorner$, then $X \vdash [\langle\theta'_0, \ldots, \theta'_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta]$.

Proof: By induction on k. For k = 1, the statement follows with Theorem 4-18-(xviii). Now, suppose the statement holds for k and suppose $\{\theta_0, \ldots, \theta_k\}, \{\theta'_0, \ldots, \theta'_k\} \subseteq$

CTERM, $\{\xi_0, \ldots, \xi_k\} \subseteq$ VAR, where for all $i, j < k+1$ with $i \neq j$ also $\xi_i \neq \xi_j$, $\Delta \in$ FORM, where FV(Δ) $\subseteq \{\xi_0, \ldots, \xi_k\}$, and $X \vdash [\langle\theta_0, \ldots, \theta_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \Delta]$ and for all $i < k+1$: $X \vdash$ $\ulcorner\theta_i = \theta'_i\urcorner$.

With Theorem 1-28-(ii), we then have that $[\langle\theta_0, \ldots, \theta_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \Delta] = [\theta_k, \xi_k, [\langle\theta_1, \ldots, \theta_{k-1}\rangle, \langle\xi_1, \ldots, \xi_{k-1}\rangle, \Delta]]$ and thus that $X \vdash [\theta_k, \xi_k, [\langle\theta_1, \ldots, \theta_{k-1}\rangle, \langle\xi_1, \ldots, \xi_{k-1}\rangle, \Delta]]$, where, with FV($\Delta$) $\subseteq \{\xi_0, \ldots, \xi_k\}$, it holds that FV($[\langle\theta_1, \ldots, \theta_{k-1}\rangle, \langle\xi_1, \ldots, \xi_{k-1}\rangle, \Delta]$) $\subseteq \{\xi_k\}$. With $X \vdash \ulcorner\theta_k = \theta'_k\urcorner$ and Theorem 4-18-(xviii), we then have $X \vdash [\theta'_k, \xi_k [\langle\theta_0, \ldots, \theta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, \Delta]]$ and thus, again with Theorem 1-28-(ii), that $X \vdash [\langle\theta_0, \ldots, \theta_{k-1}, \theta'_k\rangle, \langle\xi_0, \ldots, \xi_{k-1}, \xi_k\rangle, \Delta]$. With Theorem 1-29-(ii), we have $[\langle\theta_0, \ldots, \theta_{k-1}, \theta'_k\rangle, \langle\xi_0, \ldots, \xi_{k-1}, \xi_k\rangle, \Delta] = [\langle\theta_0, \ldots, \theta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, [\theta'_k, \xi_k, \Delta]]$ and thus $X \vdash [\langle\theta_0, \ldots, \theta_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, [\theta'_k, \xi_k, \Delta]]$, where, with FV($\Delta$) $\subseteq \{\xi_0, \ldots, \xi_k\}$, it holds that FV($[\theta'_k, \xi_k, \Delta]$) $\subseteq \{\xi_0, \ldots, \xi_{k-1}\}$. According to the I.H., it then holds that $X \vdash [\langle\theta'_0, \ldots, \theta'_{k-1}\rangle, \langle\xi_0, \ldots, \xi_{k-1}\rangle, [\theta'_k, \xi_k, \Delta]]$ and thus, again with Theorem 1-29-(ii), that $X \vdash [\langle\theta'_0, \ldots, \theta'_k\rangle, \langle\xi_0, \ldots, \xi_k\rangle, \Delta]$. ■

5 Model-theory

In this chapter we will develop a classical model-theoretic consequence concept for the language L. First, we will define the concepts we need, in particular model-theoretic satisfaction and based on it the model-theoretic consequence relation, and prove some basic theorems about them (5.1). Subsequently, we will prove some theorems on the closure of the model-theoretic consequence relation (5.2). Consequently, in ch. 6, we can then prove the correctness and completeness of the Speech Act Calculus relative to the model-theoretic consequence concept developed in ch. 5.1.

5.1 Satisfaction Relation and Model-theoretic Consequence

The development of the model-theoretic consequence concept proceeds in the standard way.[14] First, we will define interpretation functions, models and parameter assignments. This suffices to assign each closed term a denotation (Definition 5-6), where the usual definition is mirrored in Theorem 5-2. Subsequently, we can determine under which conditions a model and a parameter assignment satisfy a formula (Definition 5-8). The usual definition is here mirrored by Theorem 5-4. Then, we will prove a coincidence and a substitution lemma (Theorem 5-5 and Theorem 5-6) as well as some other theorems that are needed for the further account. Finally, we will introduce further usual concepts, among them the model-theoretic consequence (Definition 5-10), which is used in the formulation of correctness and completeness.

Definition 5-1. *Interpretation function*

I <u>is an interpretation function for</u> D

iff

D is a set and I is a function with Dom(I) = CONST ∪ FUNC ∪ PRED and

(i) For all $\alpha \in$ CONST: $I(\alpha) \in D$,

(ii) For all $\varphi \in$ FUNC: If φ is r-ary, then $I(\varphi)$ is an r-ary function over D,

(iii) For all $\Phi \in$ PRED: If Φ is r-ary, then $I(\Phi) \subseteq {}^{r}D$, and

(iv) $I(\ulcorner = \urcorner) = \{\langle a, a\rangle \mid a \in D\}$.

[14] See, for example, EBBINGHAUS, H.-D.; FLUM, J.; THOMAS, W.: *Mathematische Logik*, p. 29–62, GRÄDEL, E.: *Mathematische Logik*, p. 49–53, and WAGNER, H.: *Logische Systeme*, p. 47–54.

Definition 5-2. *Model*
M is a model
iff
There is D, I such that I is an interpretation function for D and $M = (D, I)$.

Note: The non-emptiness of D is ensured by CONST $\neq \emptyset$ and clause (i) of Definition 5-1.

In contrast to the usual procedure, we will not use variable assignments, but parameter assignments. So, parameters, in keeping with their role in the calculus, fulfill tasks in the model-theory that are often given to free variables. Accordingly, quantificational formulas (e.g. $\ulcorner\bigwedge\xi\Delta\urcorner$) are not evaluated for Δ, but for a suitable parameter instantiation (e.g. $[\beta, \xi, \Delta]$) (cf. Definition 5-7 and Theorem 5-4).

Definition 5-3. *Parameter assignment*
b is a parameter assignment for D
iff
b is a function with $\mathrm{Dom}(b)$ = PAR and $\mathrm{Ran}(b) \subseteq D$.

Definition 5-4. *Assignment variant*
b' is in β an assignment variant of b for D
iff
b' and b are parameter assignments for D and $\beta \in$ PAR and $b' \setminus \{(\beta, b'(\beta))\} \subseteq b$.

Definition 5-5. *Term denotation functions for models and parameter assignments*
F is a term denotation function for D, I, b
iff
(D, I) is a model and b is a parameter assignment for D and F is a function on CTERM and:

(i) If $\alpha \in$ CONST, then $F(\alpha) = I(\alpha)$,

(ii) If $\beta \in$ PAR, then $F(\beta) = b(\beta)$, and

(iii) If $\varphi \in$ FUNC, φ r-ary, and $\theta_0, \ldots, \theta_{r-1} \in$ CTERM, then $F(\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner) = I(\varphi)(\langle F(\theta_0), \ldots, F(\theta_{r-1})\rangle)$.

Theorem 5-1. *For every model* (D, I) *and parameter assignment* b *for* D *there is exactly one term denotation function*
If (D, I) is a model and b is a parameter assignment for D, then there is exactly one F such that F is a term denotation function for D, I, b.

Proof: Suppose (D, I) is a model and b is a parameter assignment for D. With the theorems on unique readability (Theorem 1-10 and Theorem 1-11) there is then exactly one function F on CTERM such that clauses (i) to (iii) of Definition 5-5 are satisfied for F and thus, according to Definition 5-5, exactly one term denotation function for D, I, b. ■

Definition 5-6. *Term denotation operation (TD)*
$\text{TD}(\theta, D, I, b) = a$
iff
(i) There is a term denotation function F for D, I, b and $\theta \in$ CTERM and $a = F(\theta)$
or
(ii) There is no term denotation function for D, I, b or $\theta \notin$ CTERM and $a = \emptyset$.

The following theorem mirrors the usual definition of term denotations for models and parameter assignments:

Theorem 5-2. *Term denotations for models and parameter assignments*
If (D, I) is a model and b is a parameter assignment for D, then:
(i) If $\alpha \in$ CONST, then $\text{TD}(\alpha, D, I, b) = I(\alpha)$,
(ii) If $\beta \in$ PAR, then $\text{TD}(\beta, D, I, b) = b(\beta)$, and
(iii) If $\varphi \in$ FUNC, where φ r-ary ist, and $\theta_0, \ldots, \theta_{r-1} \in$ CTERM, then $\text{TD}(\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner, D, I, b) = I(\varphi)(\langle\text{TD}(\theta_0, D, I, b), \ldots, \text{TD}(\theta_{r-1}, D, I, b)\rangle)$.

Proof: Suppose (D, I) is a model and b is a parameter assignment for D. With Theorem 5-1, there is then exactly one term denotation function F for D, I, b. According to Definition 5-6, we then have for all $\theta \in$ CTERM: $\text{TD}(\theta, D, I, b) = F(\theta)$. From this, the statement then follows with Definition 5-5. ■

Definition 5-7. *Satisfaction functions for models and parameter assignments*

F is a satisfaction function for D, I

iff

(D, I) is a model, F is a function on CFORM × $\{b \mid b$ is a parameter assignment for $D\}$, Ran(F) = {0, 1} and for all parameter assignments b for D:

(i) If Φ ∈ PRED, Φ r-ary, and $\theta_0, \ldots, \theta_{r-1}$ ∈ CTERM then:
$F(\ulcorner\Phi(\theta_0, \ldots, \theta_{r-1})\urcorner, b) = 1$ iff $\langle TD(\theta_0, D, I, b), \ldots, TD(\theta_{r-1}, D, I, b)\rangle \in I(\Phi)$,

(ii) If A ∈ CFORM, then: $F(\ulcorner\neg A\urcorner, b) = 1$ iff $F(A, b) = 0$,

(iii) If A, B ∈ CFORM, then $F(\ulcorner A \wedge B\urcorner, b) = 1$ iff $F(A, b) = 1$ and $F(B, b) = 1$,

(iv) If A, B ∈ CFORM, then $F(\ulcorner A \vee B\urcorner, b) = 1$ iff $F(A, b) = 1$ or $F(B, b) = 1$,

(v) If A, B ∈ CFORM, then $F(\ulcorner A \rightarrow B\urcorner, b) = 1$ iff $F(A, b) = 0$ or $F(B, b) = 1$,

(vi) If A, B ∈ CFORM, then $F(\ulcorner A \leftrightarrow B\urcorner, b) = 1$ iff $F(A, b) = F(B, b)$,

(vii) If ξ ∈ VAR, Δ ∈ FORM and FV(Δ) ⊆ {ξ}, then
$F(\ulcorner\bigwedge\xi\Delta\urcorner, b) = 1$
iff
there is β ∈ PAR\ST(Δ) such that for all b' that are in β assignment variants of b for D: F([β, ξ, Δ], b') = 1, and

(viii) If ξ ∈ VAR, Δ ∈ FORM and FV(Δ) ⊆ {ξ}, then
$F(\ulcorner\bigvee\xi\Delta\urcorner, b) = 1$
iff
there is β ∈ PAR\ST(Δ) and b' that is in β an assignment variant of b for D such that $F([\beta, \xi, \Delta], b') = 1$.

Theorem 5-3. *For every model (D, I) there is exactly one satisfaction function*

If (D, I) is a model, then there is exactly one satisfaction function for D, I.

Proof: Suppose (D, I) is a model. With the theorems on unique readability (Theorem 1-10 and Theorem 1-11), there is then exactly one function F on CFORM × $\{b \mid b$ is a parameter assignment for $D\}$ such that clauses (i) to (viii) of Definition 5-7 are satisfied for F. Hence there is exactly one satisfaction function for D, I. ■

Definition 5-8. *4-ary model-theoretic satisfaction predicate ('.., .., .., ⊨ ..')*

$D, I, b \models \Gamma$

iff

Γ ∈ CFORM, b is a parameter assignment for D and there is a satisfaction function F for D, I such that $F(\Gamma, b) = 1$.

The following theorem mirors the usual definition of model-theoretic consequence in the grammatical framework chosen here. In this, we use the contradictory predicate for '.., .., .. ⊨ ..', i.e. '.., .., .. ⊭ ..', in the usual way.

Theorem 5-4. *Usual satisfaction concept*

If (D, I) is a model, b is a parameter assignment for D, A, B ∈ CFORM, ξ ∈ VAR, Φ ∈ PRED, Φ r-ary, $\theta_0, \ldots, \theta_{r-1}$ ∈ CTERM, Δ ∈ FORM , where FV(Δ) ⊆ {ξ}, then:

(i) $D, I, b \vDash \ulcorner\Phi(\theta_0, \ldots, \theta_{r-1})\urcorner$ iff $\langle \mathrm{TD}(\theta_0, D, I, b), \ldots, \mathrm{TD}(\theta_{r-1}, D, I, b)\rangle \in I(\Phi)$,

(ii) $D, I, b \vDash \ulcorner\neg A\urcorner$ iff $D, I, b \nvDash A$,

(iii) $D, I, b \vDash \ulcorner A \wedge B\urcorner$ iff $D, I, b \vDash A$ and $D, I, b \vDash B$,

(iv) $D, I, b \vDash \ulcorner A \vee B\urcorner$ iff $D, I, b \vDash A$ or $D, I, b \vDash B$,

(v) $D, I, b \vDash \ulcorner A \rightarrow B\urcorner$ iff $D, I, b \nvDash A$ or $D, I, b \vDash B$,

(vi) $D, I, b \vDash \ulcorner A \leftrightarrow B\urcorner$ iff
$D, I, b \vDash A$ and $D, I, b \vDash B$ or $D, I, b \nvDash A$ and $D, I, b \nvDash B$,

(vii) $D, I, b \vDash \ulcorner\bigwedge\xi\Delta\urcorner$ iff
there is a β ∈ PAR\ST(Δ) such that for all b' that are in β assignment variants of b for D: $D, I, b' \vDash [\beta, \xi, \Delta]$, and

(viii) $D, I, b \vDash \ulcorner\bigvee\xi\Delta\urcorner$ iff
there is a β ∈ PAR\ST(Δ) and a b' that is in β an assignment variant of b for D such that $D, I, b' \vDash [\beta, \xi, \Delta]$.

Proof: Let (D, I) be a model, b a parameter assignment for D, A, B ∈ CFORM, ξ ∈ VAR, Φ ∈ PRED, Φ r-ary, $\theta_0, \ldots, \theta_{r-1}$ ∈ CTERM, Δ ∈ FORM, where FV(Δ) ⊆ {ξ}. With Theorem 5-3, there is then exactly one satisfaction function F for D, I. With Definition 5-8, it then follows that for all Γ ∈ CFORM: $D, I, b \vDash \Gamma$ iff $F(\Gamma, b) = 1$ and $D, I, b \nvDash \Gamma$ iff $F(\Gamma, b) = 0$. From this, the statement then follows with Definition 5-7. ■

Theorem 5-5. *Coincidence lemma*

If (D, I) and (D, I') are models and b, b' are parameter assignments for D, then:

(i) For all θ ∈ CTERM: If $I{\restriction}\mathrm{SE}(\theta) = I'{\restriction}\mathrm{SE}(\theta)$ and $b{\restriction}\mathrm{ST}(\theta) = b'{\restriction}\mathrm{ST}(\theta)$, then $\mathrm{TD}(\theta, D, I, b) = \mathrm{TD}(\theta, D, I', b')$, and

(ii) For all Γ ∈ CFORM: If $I{\restriction}\mathrm{SE}(\Gamma) = I'{\restriction}\mathrm{SE}(\Gamma)$ and $b{\restriction}\mathrm{ST}(\Gamma) = b'{\restriction}\mathrm{ST}(\Gamma)$, then $D, I, b \vDash \Gamma$ iff $D, I', b' \vDash \Gamma$.

Proof: *Ad (i)*: Let (D, I) and (D, I') be models and b, b' parameter assignments for D. The proof is carried out by induction on the complexity of θ ∈ TERM. First, suppose θ ∈ ATERM ∩ CTERM and suppose $I{\restriction}\mathrm{SE}(\theta) = I'{\restriction}\mathrm{SE}(\theta)$ and $b{\restriction}\mathrm{ST}(\theta) = b'{\restriction}\mathrm{ST}(\theta)$. Then we have $\theta \in$ CONST $\cup$ PAR. Now, suppose $\theta \in$ CONST. Then it holds with $\{\theta\}$ = SE(θ) $\cap$ CONST and $I{\restriction}$SE(θ) = $I'{\restriction}$SE(θ) and Theorem 5-2-(i) that TD(θ, D, I, b) = $I(\theta)$ = $I'(\theta)$ = TD(θ, D, I', b'). Now, suppose $\theta \in$ PAR. Then it holds with $\{\theta\}$ = ST(θ) $\cap$ PAR and $b{\restriction}$ST(θ) = $b'{\restriction}$ST(θ) and Theorem 5-2-(ii) that TD(θ, D, I, b) = $b(\theta)$ = $b'(\theta)$ = TD(θ, D, I', b').

Now, suppose the statement holds for $\theta_0, \ldots, \theta_{r-1} \in$ TERM and suppose $\varphi \in$ FUNC, φ r-ary, and suppose $\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner \in$ FTERM $\cap$ CTERM and suppose $I{\restriction}$SE($\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner$) = $I'{\restriction}$SE($\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner$) and $b{\restriction}$ST($\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner$) = $b'{\restriction}$ST($\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner$). With FV($\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner$) = $\bigcup\{$FV(θ_i) $\mid i < r\}$, it then holds for all θ_i with $i < r$ that $\theta_i \in$ CTERM. We also have, with $\bigcup\{$SE(θ_i) $\mid i < r\} \subseteq$ SE($\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner$) and $\bigcup\{$ST(θ_i) $\mid i < r\} \subseteq$ ST($\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner$), for all $i < r$: $I{\restriction}$SE(θ_i) = $I'{\restriction}$SE(θ_i) and $b{\restriction}$ST(θ_i) = $b'{\restriction}$ST(θ_i). With the I.H., itthus holds for all $i < r$ that TD(θ_i, D, I, b) = TD(θ_i, D, I', b'). With $\varphi \in$ SE($\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner$) $\cap$ FUNC, we have by hypothesis that $I(\varphi)$ = $I'(\varphi)$. Thus it holds that

TD($\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner$, D, I, b)
=
$I(\varphi)(\langle$TD(θ_0, D, I, b), …, TD(θ_{r-1}, D, I, b)$\rangle$)
=
$I'(\varphi)(\langle$TD(θ_0, D, I', b'), …, TD(θ_{r-1}, D, I', b')$\rangle$)
=
TD($\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner$, D, I', b').

Ad (ii): The proof is carried out by induction on the degree of a formula. For this, suppose the theorem holds for all A $\in$ FORM with FDEG(A) < k. Now, let (D, I), (D, I') be models, b, b' parameter assignments for D and suppose $\Gamma \in$ CFORM and suppose $I{\restriction}$SE(Γ) = $I'{\restriction}$SE(Γ) and $b{\restriction}$ST(Γ) = $b'{\restriction}$ST(Γ) and suppose FDEG(Γ) = k.

Suppose FDEG(Γ) = 0. Then we have $\Gamma \in$ AFORM. Then there are $\theta_0, \ldots, \theta_{r-1} \in$ TERM and $\Phi \in$ PRED, Φ r-ary, such that Γ = $\ulcorner\Phi(\theta_0, \ldots, \theta_{r-1})\urcorner$. Then it holds, with FV($\ulcorner\Phi(\theta_0, \ldots, \theta_{r-1})\urcorner$) = $\bigcup\{$FV(θ_i) $\mid i < r\}$, $\bigcup\{$SE(θ_i) $\mid i < r\} \subseteq$ SE($\ulcorner\Phi(\theta_0, \ldots, \theta_{r-1})\urcorner$) and $\bigcup\{$ST(θ_i) $\mid i < r\} \subseteq$ ST($\ulcorner\Phi(\theta_0, \ldots, \theta_{r-1})\urcorner$) and with $\Gamma \in$ CFORM, for all $i < r$ that $\theta_i \in$ CTERM, $I{\restriction}$SE(θ_i) = $I'{\restriction}$SE(θ_i) and $b{\restriction}$ST(θ_i) = $b'{\restriction}$ST(θ_i). With (i), we thus have for all $i < r$: TD(θ_i, D, I, b) = TD(θ_i, D, I', b'). With $\Phi \in$ SE($\ulcorner\Phi(\theta_0, \ldots, \theta_{r-1})\urcorner$) $\cap$ PRED, we have by hypothesis $I(\Phi)$ = $I'(\Phi)$. With Theorem 5-4-(i), it thus holds that

$D, I, b \vDash \Gamma$
iff
$D, I, b \vDash \ulcorner\Phi(\theta_0, \ldots, \theta_{r-1})\urcorner$
iff
$\langle TD(\theta_0, D, I, b), \ldots, TD(\theta_{r-1}, D, I, b)\rangle \in I(\Phi)$
iff
$\langle TD(\theta_0, D, I', b'), \ldots, TD(\theta_{r-1}, D, I', b')\rangle \in I'(\Phi)$
iff
$D, I', b' \vDash \ulcorner\Phi(\theta_0, \ldots, \theta_{r-1})\urcorner$
iff
$D, I', b' \vDash \Gamma$.

Now, suppose FDEG(Γ) ≠ 0. Then we have Γ ∈ CONFORM ∪ QFORM. We can distinguish *seven* cases. *First*: Suppose Γ = ⌜¬A⌝. Then we have FDEG(A) < FDEG(Γ). According to the assumption for Γ, we then have that A ∈ CFORM, $I\upharpoonright$SE(A) = $I'\upharpoonright$SE(A) and $b\upharpoonright$ST(A) = $b'\upharpoonright$ST(A). With Theorem 5-4-(ii) and the I.H., we thus have

$D, I, b \vDash \Gamma$
iff
$D, I, b \vDash \ulcorner\neg A\urcorner$
iff
$D, I, b \nvDash A$
iff
$D, I', b' \nvDash A$
iff
$D, I', b' \vDash \ulcorner\neg A\urcorner$
iff
$D, I', b' \vDash \Gamma$.

Second: Suppose Γ = ⌜A ∧ B⌝. Then we have FDEG(A) < FDEG(Γ) and FDEG(B) < FDEG(Γ). According to assumption for Γ, we then have A, B ∈ CTERM, $I\upharpoonright$(SE(A) ∪ SE(B)) = $I'\upharpoonright$(SE(A) ∪ SE(B)) and $b\upharpoonright$(ST(A) ∪ ST(B)) = $b'\upharpoonright$(ST(A) ∪ ST(B)). With Theorem 5-4-(iii) and the I.H., it then holds that

$D, I, b \vDash \Gamma$
iff
$D, I, b \vDash \ulcorner A \wedge B\urcorner$
iff
$D, I, b \vDash A$ and $D, I, b \vDash B$
iff
$D, I', b' \vDash A$ and $D, I', b' \vDash B$

iff
$D, I', b' \vDash \ulcorner A \wedge B \urcorner$
iff
$D, I', b' \vDash \Gamma$.

The *third* to *fifth* cases are treated analogously.

Sixth: Suppose $\Gamma = \ulcorner \bigwedge\zeta\Delta \urcorner$. According to the assumption for Γ, we then have $FV(\Delta) \subseteq \{\zeta\}$, $I{\restriction}SE(\Delta) = I'{\restriction}SE(\Delta)$ and $b{\restriction}ST(\Delta) = b'{\restriction}ST(\Delta)$. Now, suppose $D, I, b \vDash \ulcorner \bigwedge\zeta\Delta \urcorner$. With Theorem 5-4-(vii), there is then a $\beta \in PAR \backslash ST(\Delta)$ such that for all b^+ that are in β assignment variants of b for D it holds that $D, I, b^+ \vDash [\beta, \zeta, \Delta]$. Now, suppose b'_1 is in β an assignment variant of b' for D. Now, let $b_1 = (b \backslash \{(\beta, b(\beta))\}) \cup \{(\beta, b'_1(\beta))\}$. Then b_1 is in β an assignment variant of b for D and thus it holds that $D, I, b_1 \vDash [\beta, \zeta, \Delta]$. Since $\beta \notin ST(\Delta)$ and $b{\restriction}ST(\Delta) = b'{\restriction}ST(\Delta)$, we have for all $\beta' \in ST(\Delta) \cap PAR$ that $b_1(\beta') = b(\beta') = b'(\beta') = b'_1(\beta')$. Since also $b_1(\beta) = b'_1(\beta)$ and $ST([\beta, \zeta, \Delta]) \subseteq ST(\Delta) \cup \{\beta\}$, we thus have that $b_1{\restriction}ST([\beta, \zeta, \Delta]) = b'_1{\restriction}ST([\beta, \zeta, \Delta])$. Also, we have $I{\restriction}SE([\beta, \zeta, \Delta]) = I{\restriction}(SE([\beta, \zeta, \Delta]) \cap (CONST \cup FUNC \cup PRED)) = I{\restriction}(SE(\Delta) \cap (CONST \cup FUNC \cup PRED)) = I{\restriction}SE(\Delta) = I'{\restriction}SE(\Delta) = I'{\restriction}(SE(\Delta) \cap (CONST \cup FUNC \cup PRED)) = I'{\restriction}(SE([\beta, \zeta, \Delta]) \cap (CONST \cup FUNC \cup PRED)) = I'{\restriction}(SE([\beta, \zeta, \Delta])$ and thus that $I{\restriction}SE([\beta, \zeta, \Delta]) = I'{\restriction}SE([\beta, \zeta, \Delta])$. Moreover, we have $[\beta, \zeta, \Delta] \in CFORM$ and, with Theorem 1-13, we have $FDEG([\beta, \zeta, \Delta]) = FDEG(\Delta) < FDEG(\Gamma)$. According to the I.H., we thus have that with $D, I, b_1 \vDash [\beta, \zeta, \Delta]$ it also holds that $D, I', b'_1 \vDash [\beta, \zeta, \Delta]$. Therefore we have for all b'^+ that are in β assignment variants of b' for D: $D, I', b'^+ \vDash [\beta, \zeta, \Delta]$ and hence, according to Theorem 5-4-(vii), $D, I', b' \vDash \ulcorner \bigwedge\zeta\Delta \urcorner$. The right-left-direction is shown analogously.

Seventh: Suppose $\Gamma = \ulcorner \bigvee\zeta\Delta \urcorner$. According to the assumption for Γ, we then have $FV(\Delta) \subseteq \{\zeta\}$, $I{\restriction}SE(\Delta) = I'{\restriction}SE(\Delta)$ and $b{\restriction}ST(\Delta) = b'{\restriction}ST(\Delta)$. Now, suppose $D, I, b \vDash \ulcorner \bigvee\zeta\Delta \urcorner$. With Theorem 5-4-(viii), there is then $\beta \in PAR \backslash ST(\Delta)$ and b_1 that is in β assignment variant of b for D such that $D, I, b_1 \vDash [\beta, \zeta, \Delta]$. Now, let $b'_1 = (b' \backslash \{(\beta, b'(\beta))\}) \cup \{(\beta, b_1(\beta))\}$. Then b'_1 is in β an assignment variant of b' for D. Since $\beta \notin ST(\Delta)$ and $b{\restriction}ST(\Delta) = b'{\restriction}ST(\Delta)$, it then holds for all $\beta' \in ST(\Delta) \cap PAR$ that $b_1(\beta') = b(\beta') = b'(\beta') = b'_1(\beta')$. Since also $b_1(\beta) = b'_1(\beta)$ and $ST([\beta, \zeta, \Delta]) \subseteq ST(\Delta) \cup \{\beta\}$, we thus have that $b_1{\restriction}ST([\beta, \zeta,$

Δ]) = b'_1↾ST([β, ζ, Δ]). Also, we have I↾SE([β, ζ, Δ]) = I↾(SE([β, ζ, Δ]) ∩ (CONST ∪ FUNC ∪ PRED)) = I↾(SE(Δ) ∩ (CONST ∪ FUNC ∪ PRED)) = I↾SE(Δ) = I'↾SE(Δ) = I'↾(SE(Δ) ∩ (CONST ∪ FUNC ∪ PRED)) = I'↾(SE([β, ζ, Δ]) ∩ (CONST ∪ FUNC ∪ PRED)) = I'↾(SE([β, ζ, Δ]) and hence I↾SE([β, ζ, Δ]) = I'↾SE([β, ζ, Δ]). Moreover, we have [β, ζ, Δ] ∈ CFORM and, with Theorem 1-13, FDEG([β, ζ, Δ]) = FDEG(Δ) < FDEG(Γ). According to the I.H., we thus have, with D, I, b_1 ⊨ [β, ζ, Δ], also D, I', b'_1 ⊨ [β, ζ, Δ] and hence, according to Theorem 5-4-(viii), D, I', b' ⊨ ⌜⋁ζΔ⌝. The right-left-direction is shown analogously. ■

Using the coincidence lemma, we can now prove the substitution lemma:

Theorem 5-6. *Substitution lemma*
If (D, I), (D, I') are models, b, b' are parameter assignments for D, ξ ∈ VAR, θ, θ' ∈ CTERM and TD(θ, D, I, b) = TD(θ', D, I', b') then:

(i) For all θ^+ ∈ TERM with FV(θ^+) ⊆ {ξ}, I↾SE(θ^+) = I'↾SE(θ^+) and b↾ST(θ^+) = b'↾ST(θ) it holds that TD([θ, ξ, θ^+], D, I, b) = TD([θ', ξ, θ^+], D, I', b'), and

(ii) For all Δ ∈ FORM with FV(Δ) ⊆ {ξ}, I↾SE(Δ) = I'↾SE(Δ) and b↾ST(Δ) = b'↾ST(Δ) it holds that D, I, b ⊨ [θ, ξ, Δ] iff D, I', b' ⊨ [θ', ξ, Δ].

Proof: *Ad (i)*: Let (D, I), (D, I') be models, b, b' parameter assignments for D, ξ ∈ VAR, θ, θ' ∈ CTERM and TD(θ, D, I, b) = TD(θ', D, I', b'). The proof is carried out by induction on the complexity of θ^+ ∈ TERM. First, suppose θ^+ ∈ ATERM, where FV(θ^+) ⊆ {ξ}, I↾SE(θ^+) = I'↾SE(θ^+) and b↾ST(θ^+) = b'↾ST(θ^+). Then we have θ^+ ∈ CONST ∪ PAR ∪ VAR. Now, suppose θ^+ ∈ CONST. Then we have [θ, ξ, θ^+] = θ^+ = [θ', ξ, θ^+] and thus it holds, with SE(θ^+) = {θ^+}, I↾SE(θ^+) = I'↾SE(θ^+) and Theorem 5-2-(i), that TD([θ, ξ, θ^+], D, I, b) = TD(θ^+, D, I, b) = I(θ^+) = I'(θ^+) = TD(θ^+, D, I', b') = TD([θ', ξ, θ^+], D, I', b'). Now, suppose θ^+ ∈ PAR. Then we have [θ, ξ, θ^+] = θ^+ = [θ', ξ, θ^+] and thus it holds, with ST(θ^+) = {θ^+}, b↾ST(θ^+) = b'↾ST(θ^+) and Theorem 5-2-(ii), that TD([θ, ξ, θ^+], D, I, b) = TD(θ^+, D, I, b) = b(θ^+) = b'(θ^+) = TD(θ^+, D, I', b') = TD([θ', ξ, θ^+], D, I', b'). Now, suppose θ^+ ∈ VAR. Then we have θ^+ = ξ. Then we have [θ, ξ, θ^+] = θ and [θ', ξ, θ^+] = θ'. By hypothesis, we thus have TD([θ, ξ, θ^+], D, I, b) = TD(θ, D, I, b) = TD(θ', D, I', b') = TD([θ', ξ, θ^+], D, I', b').

Now, suppose the statement holds for $\theta^+_0, \ldots, \theta^+_{r\text{-}1} \in$ TERM and suppose $\varphi \in$ FUNC, φ r-ary, and suppose $\theta^+ = \ulcorner\varphi(\theta^+_0, \ldots, \theta^+_{r\text{-}1})\urcorner \in$ FTERM, where FV$(\ulcorner\varphi(\theta^+_0, \ldots, \theta^+_{r\text{-}1})\urcorner) \subseteq \{\xi\}$, $I\upharpoonright$SE$(\ulcorner\varphi(\theta^+_0, \ldots, \theta^+_{r\text{-}1})\urcorner) = I'\upharpoonright$SE$(\ulcorner\varphi(\theta^+_0, \ldots, \theta^+_{r\text{-}1})\urcorner)$ and $b\upharpoonright$ST$(\ulcorner\varphi(\theta^+_0, \ldots, \theta^+_{r\text{-}1})\urcorner) = b'\upharpoonright$ST$(\ulcorner\varphi(\theta^+_0, \ldots, \theta^+_{r\text{-}1})\urcorner)$. Then it holds, with FV$(\ulcorner\varphi(\theta^+_0, \ldots, \theta^+_{r\text{-}1})\urcorner) = \bigcup\{FV(\theta^+_i) \mid i < r\}$, $\bigcup\{$SE$(\theta^+_i) \mid i < r\} \subseteq$ SE$(\ulcorner\varphi(\theta^+_0, \ldots, \theta^+_{r\text{-}1})\urcorner)$ and $\bigcup\{$ST$(\theta^+_i) \mid i < r\} \subseteq$ ST$(\ulcorner\varphi(\theta^+_0, \ldots, \theta^+_{r\text{-}1})\urcorner)$, for all $i < r$ that FV$(\theta^+_i) \subseteq \{\xi\}$, $I\upharpoonright$SE$(\theta^+_i) = I'\upharpoonright$SE$(\theta^+_i)$ and $b\upharpoonright$ST$(\theta^+_i) = b'\upharpoonright$ST$(\theta^+_i)$. With the I.H., it thus holds for all $i < r$ that TD$([\theta, \xi, \theta^+_i], D, I, b) =$ TD$([\theta', \xi, \theta^+_i], D, I', b')$. With $\varphi \in$ SE$(\ulcorner\varphi(\theta^+_0, \ldots, \theta^+_{r\text{-}1})\urcorner) \cap$ FUNC, we have, by hypothesis, also $I(\varphi) = I'(\varphi)$. With Theorem 5-2-(iii), we hence have

$$
\begin{aligned}
&\text{TD}([\theta, \xi, \ulcorner\varphi(\theta^+_0, \ldots, \theta^+_{r\text{-}1})\urcorner], D, I, b)\\
&=\\
&\text{TD}(\ulcorner\varphi([\theta, \xi, \theta^+_0], \ldots, [\theta, \xi, \theta^+_{r\text{-}1}])\urcorner, D, I, b)\\
&=\\
&I(\varphi)(\langle\text{TD}([\theta, \xi, \theta^+_0], D, I, b), \ldots, \text{TD}([\theta, \xi, \theta^+_{r\text{-}1}], D, I, b)\rangle)\\
&=\\
&I'(\varphi)(\langle\text{TD}([\theta', \xi, \theta^+_0], D, I', b'), \ldots, \text{TD}([\theta', \xi, \theta^+_{r\text{-}1}], D, I', b')\rangle)\\
&=\\
&\text{TD}(\ulcorner\varphi([\theta', \xi, \theta^+_0], \ldots, [\theta', \xi, \theta^+_{r\text{-}1}])\urcorner, D, I', b')\\
&=\\
&\text{TD}([\theta', \xi, \ulcorner\varphi(\theta^+_0, \ldots, \theta^+_{r\text{-}1})\urcorner], D, I', b').
\end{aligned}
$$

Ad (ii): The proof is carried out by induction on the degree of a formula. For this, suppose the theorem holds for all A $\in$ FORM with FDEG(A) $< k$. Let now (D, I), (D, I') be models, b, b' parameter assignments for D, $\xi \in$ VAR, $\theta, \theta' \in$ CTERM and TD$(\theta, D, I, b) =$ TD(θ', D, I', b') and suppose $\Delta \in$ FORM, where FV$(\Delta) \subseteq \{\xi\}$, $I\upharpoonright$SE$(\Delta) = I'\upharpoonright$SE$(\Delta)$ and $b\upharpoonright$ST$(\Delta) = b'\upharpoonright$ST$(\Delta)$, and suppose FDEG$(\Delta) = k$. Suppose FDEG$(\Delta) = 0$. Then we have $\Delta \in$ AFORM. Then there are $\theta^+_0, \ldots, \theta^+_{r\text{-}1} \in$ TERM and $\Phi \in$ PRED, where Φ is r-ary, such that $\Delta = \ulcorner\Phi(\theta^+_0, \ldots, \theta^+_{r\text{-}1})\urcorner$. With FV$(\ulcorner\Phi(\theta^+_0, \ldots, \theta^+_{r\text{-}1})\urcorner) = \bigcup\{FV(\theta^+_i) \mid i < r\}$, $\bigcup\{$SE$(\theta^+_i) \mid i < r\} \subseteq$ SE$(\ulcorner\Phi(\theta^+_0, \ldots, \theta^+_{r\text{-}1})\urcorner)$ and $\bigcup\{$ST$(\theta^+_i) \mid i < r\} =$ ST$(\ulcorner\Phi(\theta^+_0, \ldots, \theta^+_{r\text{-}1})\urcorner)$ and the assumption for Δ, it then holds for all $i < r$ that FV$(\theta^+_i) \subseteq \{\xi\}$, $I\upharpoonright$SE$(\theta^+_i) = I'\upharpoonright$SE$(\theta^+_i)$ and $b\upharpoonright$ST$(\theta^+_i) = b'\upharpoonright$ST$(\theta^+_i)$. With (i), we thus have for all $i < r$ that TD$([\theta, \xi, \theta^+_i], D, I, b) =$ TD$([\theta', \xi, \theta^+_i], D, I', b')$. With $\Phi \in$ SE$(\ulcorner\Phi(\theta^+_0, \ldots, \theta^+_{r\text{-}1})\urcorner) \cap$ PRED, we have, by hypothesis, that $I(\Phi) = I'(\Phi)$. With Theorem 5-4-(i), we hence have

$D, I, b \vDash [\theta, \xi, \Delta]$
iff
$D, I, b \vDash [\theta, \xi, \ulcorner\Phi(\theta^{+}_{0}, \ldots, \theta^{+}_{r\text{-}1})\urcorner]$
iff
$D, I, b \vDash \ulcorner\Phi([\theta, \xi, \theta^{+}_{0}], \ldots, [\theta, \xi, \theta^{+}_{r\text{-}1}])\urcorner$
iff
$\langle \mathrm{TD}([\theta, \xi, \theta^{+}_{0}], D, I, b), \ldots, \mathrm{TD}([\theta, \xi, \theta^{+}_{r\text{-}1}], D, I, b)\rangle \in I(\Phi)$
iff
$\langle \mathrm{TD}([\theta', \xi, \theta^{+}_{0}], D, I', b'), \ldots, \mathrm{TD}([\theta', \xi, \theta^{+}_{r\text{-}1}], D, I', b')\rangle \in I'(\Phi)$
iff
$D, I', b' \vDash \ulcorner\Phi([\theta', \xi, \theta^{+}_{0}], \ldots, [\theta', \xi, \theta^{+}_{r\text{-}1}])\urcorner$
iff
$D, I', b' \vDash [\theta', \xi, \ulcorner\Phi(\theta^{+}_{0}, \ldots, \theta^{+}_{r\text{-}1})\urcorner]$
iff
$D, I', b' \vDash [\theta', \xi, \Delta]$.

Now, suppose $\mathrm{FDEG}(\Delta) \neq 0$. Then we have $\Delta \in \mathrm{CONFORM} \cup \mathrm{QFORM}$. We can distinguish *seven* cases. *First*: Suppose $\Delta = \ulcorner\neg\mathrm{A}\urcorner$. Then we have $\mathrm{FDEG(A)} < \mathrm{FDEG}(\Delta)$. According to the assumption for Δ, we also have $\mathrm{FV(A)} \subseteq \{\xi\}$, $I\upharpoonright\mathrm{SE(A)} = I'\upharpoonright\mathrm{SE(A)}$ and $b\upharpoonright\mathrm{ST(A)} = b'\upharpoonright\mathrm{ST(A)}$. With the I.H. and Theorem 5-4-(ii), it then follows that

$D, I, b \vDash [\theta, \xi, \Delta]$
iff
$D, I, b \vDash [\theta, \xi, \ulcorner\neg\mathrm{A}\urcorner]$
iff
$D, I, b \vDash \ulcorner\neg[\theta, \xi, \mathrm{A}]\urcorner$
iff
$D, I, b \nvDash [\theta, \xi, \mathrm{A}]$
iff
$D, I', b' \nvDash [\theta', \xi, \mathrm{A}]$
iff
$D, I', b' \vDash \ulcorner\neg[\theta', \xi, \mathrm{A}]\urcorner$
iff
$D, I', b' \vDash [\theta', \xi, \ulcorner\neg\mathrm{A}\urcorner]$
iff
$D, I', b' \vDash [\theta', \xi, \Delta]$.

Second: Suppose $\Delta = \ulcorner\mathrm{A} \wedge \mathrm{B}\urcorner$. Therefore $\mathrm{FDEG(A)} < \mathrm{FDEG}(\Delta)$ and $\mathrm{FDEG(B)} < \mathrm{FDEG}(\Delta)$. According to the assumption for Δ, we also have $\mathrm{FV(A)} \cup \mathrm{FV(B)} \subseteq \{\xi\}$, $I\upharpoonright(\mathrm{SE(A)} \cup \mathrm{SE(B)}) = I'\upharpoonright(\mathrm{SE(A)} \cup \mathrm{SE(B)})$ and $b\upharpoonright(\mathrm{ST(A)} \cup \mathrm{ST(B)}) = b'\upharpoonright(\mathrm{ST(A)} \cup \mathrm{ST(B)})$. With the I.H. and Theorem 5-4-(iii), it then follows that

$D, I, b \vDash [\theta, \xi, \Delta]$
iff
$D, I, b \vDash [\theta, \xi, \ulcorner A \wedge B \urcorner]$
iff
$D, I, b \vDash \ulcorner[\theta, \xi, A] \wedge [\theta, \xi, B]\urcorner$
iff
$D, I, b \vDash [\theta, \xi, A]$ and $D, I, b \vDash [\theta, \xi, B]$
iff
$D, I', b' \vDash [\theta', \xi, A]$ and $D, I', b' \vDash [\theta', \xi, B]$
iff
$D, I', b' \vDash \ulcorner[\theta', \xi, A] \wedge [\theta', \xi, B]\urcorner$
iff
$D, I', b' \vDash [\theta', \xi, \ulcorner A \wedge B \urcorner]$
iff
$D, I', b' \vDash [\theta', \xi, \Delta]$.

The *third* to *fifth* cases are treated analogously.

Sixth: Suppose $\Delta = \ulcorner\bigwedge\zeta A\urcorner$. According to the assumption for Δ, we then have $FV(A) \subseteq \{\xi, \zeta\}$, $I{\upharpoonright}SE(A) = I'{\upharpoonright}SE(A)$ and $b{\upharpoonright}ST(A) = b'{\upharpoonright}ST(A)$. Suppose $\zeta = \xi$. Then we have $[\theta, \xi, \Delta] = [\theta, \zeta, \ulcorner\bigwedge\zeta A\urcorner] = \ulcorner\bigwedge\zeta A\urcorner = [\theta', \zeta, \ulcorner\bigwedge\zeta A\urcorner] = [\theta', \xi, \Delta]$ and hence $[\theta, \xi, \Delta] = \Delta = [\theta', \xi, \Delta]$. Also, we have $FV(\Delta) = \emptyset$ and hence $\Delta \in CFORM$. Since, by hypothesis, $I{\upharpoonright}SE(\Delta) = I'{\upharpoonright}SE(\Delta)$ and $b{\upharpoonright}ST(\Delta) = b'{\upharpoonright}ST(\Delta)$ we thus have, with Theorem 5-5-(ii), that $D, I, b \vDash [\theta, \xi, \Delta]$ iff $D, I, b \vDash \Delta$ iff $D, I', b' \vDash \Delta$ iff $D, I', b' \vDash [\theta', \xi, \Delta]$. Now, suppose $\zeta \neq \xi$. Then we have $[\theta, \xi, \Delta] = \ulcorner\bigwedge\zeta[\theta, \xi, A]\urcorner$ and $[\theta', \xi, \Delta] = \ulcorner\bigwedge\zeta[\theta', \xi, A]\urcorner$. With $\zeta \neq \xi$ and $\zeta, \xi \notin ST(\theta^{\#})$ for all $\theta^{\#} \in CTERM$ and Theorem 1-25-(ii), we also have for all $\beta^{+} \in PAR$: $[\beta^{+}, \zeta, [\theta, \xi, A]] = [\theta, \xi, [\beta^{+}, \zeta, A]]$ and $[\beta^{+}, \zeta, [\theta', \xi, A]] = [\theta', \xi, [\beta^{+}, \zeta, A]]$.

Now, suppose $D, I, b \vDash \ulcorner\bigwedge\zeta[\theta, \xi, A]\urcorner$. With Theorem 5-4-(vii), there is then a $\beta^{+} \in PAR\setminus ST([\theta, \xi, A])$ such that for all b^{+} that are in β^{+} assignment variants of b for D it holds that $D, I, b^{+} \vDash [\beta^{+}, \zeta, [\theta, \xi, A]]$. Now, let $\beta^{\#} \in PAR\setminus(ST([\theta, \xi, A]) \cup ST(\theta) \cup ST(\theta'))$. Now, suppose b'_1 is in $\beta^{\#}$ an assignment variant of b' for D. Now, let $b_1 = (b\setminus\{(\beta^{\#}, b(\beta^{\#}))\}) \cup \{(\beta^{\#}, b'_1(\beta^{\#}))\}$. Then b_1 is in $\beta^{\#}$ an assignment variant of b for D and $b_1(\beta^{\#}) = b'_1(\beta^{\#})$. Now, let $b_2 = (b\setminus\{(\beta^{+}, b(\beta^{+}))\}) \cup \{(\beta^{+}, b'_1(\beta^{\#}))\}$. Then b_2 is in β^{+} an assignment variant of b for D and thus we have $D, I, b_2 \vDash [\beta^{+}, \zeta, [\theta, \xi, A]]$. Also, we have $TD(\beta^{+}, D, I, b_2) = b_2(\beta^{+}) = b'_1(\beta^{\#}) = b_1(\beta^{\#}) = TD(\beta^{\#}, D, I, b_1)$. Also, we have, according to the assumption for β^{+} and $\beta^{\#}$, that $\beta^{+}, \beta^{\#} \notin ST([\theta, \xi, A])$ and thus $b_2{\upharpoonright}ST([\theta, \xi, A]) =$

b↾ST([θ, ξ, A]) = b_1↾ST([θ, ξ, A]). Also, we trivially have that *I*↾SE([θ, ξ, A]) = *I*↾SE([θ, ξ, A]). Further, we have FV([θ, ξ, A]) ⊆ {ζ} and, with Theorem 1-13, we have FDEG([θ, ξ, A]) = FDEG(A) < FDEG(Δ). By the I.H., we thus have, because of *D*, *I*, b_2 ⊨ [β^+, ζ, [θ, ξ, A]], that also *D*, *I*, b_1 ⊨ [$\beta^\#$, ζ, [θ, ξ, A]] = [θ, ξ, [$\beta^\#$, ζ, A]].

With $\beta^\#$ ∉ ST(θ), we have that b_1↾ST(θ) = *b*↾ST(θ) and, with $\beta^\#$ ∉ ST(θ'), we have that b'_1↾ST(θ') = *b*'↾ST(θ'), and, because we trivially have *I*↾SE(θ) = *I*↾SE(θ) and *I*'↾SE(θ') = *I*'↾SE(θ'), we thus have, according to Theorem 5-5-(i), that TD(θ, *D*, *I*, b_1) = TD(θ, *D*, *I*, *b*) and TD(θ', *D*, *I*', b'_1) = TD(θ', *D*, *I*', *b*'). By our intial hypothesis, we thus have TD(θ, *D*, *I*, b_1) = TD(θ', *D*, *I*', b'_1). With *b*↾ST(A) = *b*'↾ST(A), $b_1(\beta^\#)$ = $b'_1(\beta^\#)$ and ST([$\beta^\#$, ζ, A]) ⊆ ST(A) ∪ {$\beta^\#$}, we also have b_1↾ST([$\beta^\#$, ζ, A]) = b'_1↾ST([$\beta^\#$, ζ, A]). We also have: *I*↾SE([$\beta^\#$, ζ, A]) = *I*↾(SE([$\beta^\#$, ζ, A]) ∩ (CONST ∪ FUNC ∪ PRED)) = *I*↾(SE(A) ∩ (CONST ∪ FUNC ∪ PRED)) = *I*↾SE(A) = *I*'↾SE(A) = *I*'↾(SE(A) ∩ (CONST ∪ FUNC ∪ PRED)) = *I*'↾(SE([$\beta^\#$, ζ, A]) ∩ (CONST ∪ FUNC ∪ PRED)) = *I*'↾(SE([$\beta^\#$, ζ, A]) and hence *I*↾SE([$\beta^\#$, ζ, A]) = *I*'↾SE([$\beta^\#$, ζ, A]). Further, we have FV([$\beta^\#$, ζ, A]) ⊆ {ξ} and, with Theorem 1-13, we have FDEG([$\beta^\#$, ζ, A]) < FDEG(Δ). By the I.H. it thus holds, because of *D*, *I*, b_1 ⊨ [θ, ξ, [$\beta^\#$, ζ, A]], that also *D*, *I*', b'_1 ⊨ [θ', ξ, [$\beta^\#$, ζ, A]] = [$\beta^\#$, ζ, [θ', ξ, A]]. Therefore we have for all b'^{+} that are in $\beta^\#$ assignment variants of *b*' for *D* that *D*, *I*', b'^{+} ⊨ [$\beta^\#$, ζ, [θ', ξ, A]] and hence we have, according to Theorem 5-4-(vii), that *D*, *I*', *b*' ⊨ ⌜⋀ζ[θ', ξ, A]⌝. The right-left-direction is shown analogously.

Seventh: Suppose Δ = ⌜⋁ζA⌝. According to the assumption for Δ, we then have FV(A) ⊆ {ξ, ζ}, *I*↾SE(A) = *I*'↾SE(A) and *b*↾ST(A) = *b*'↾ST(A). Suppose ζ = ξ. Then we have [θ, ξ, Δ] = [θ, ζ, ⌜⋁ζA⌝] = ⌜⋁ζA⌝ = [θ', ζ, ⌜⋁ζA⌝] = [θ', ξ, Δ] and hence [θ, ξ, Δ] = Δ = [θ', ξ, Δ]. Also, we have FV(Δ) = ∅ and hence Δ ∈ CFORM. Since by hypothesis *I*↾SE(Δ) = *I*'↾SE(Δ) and *b*↾ST(Δ) = *b*'↾ST(Δ), we thus have, with Theorem 5-5-(ii) that *D*, *I*, *b* ⊨ [θ, ξ, Δ] iff *D*, *I*, *b* ⊨ Δ iff *D*, *I*', *b*' ⊨ Δ iff *D*, *I*', *b*' ⊨ [θ', ξ, Δ]. Now, suppose ζ ≠ ξ. Then we have [θ, ξ, Δ] = ⌜⋁ζ[θ, ξ, A]⌝ and [θ', ξ, Δ] = ⌜⋁ζ[θ', ξ, A]⌝. With ζ ≠ ξ and ζ, ξ ∉ ST($\theta^\#$) for all $\theta^\#$ ∈ CTERM and Theorem 1-25-(ii), it holds for all β^+ ∈ PAR that [β^+, ζ, [θ, ξ, A]] = [θ, ξ, [β^+, ζ, A]] and [β^+, ζ, [θ', ξ, A]] = [θ', ξ, [β^+, ζ, A]].

Now, suppose $D, I, b \vDash \ulcorner\bigvee\zeta[\theta, \xi, A]\urcorner$. With Theorem 5-4-(viii), there is then $\beta^+ \in PAR\backslash ST([\theta, \xi, A])$ and b_1, that is in β^+ an assignment variant of b for D such that $D, I, b_1 \vDash [\beta^+, \zeta, [\theta, \xi, A]]$. Now, let $\beta^\# \in PAR\backslash(ST([\theta, \xi, A]) \cup ST(\theta) \cup ST(\theta'))$. Now, let $b'_1 = (b'\backslash\{(\beta^\#, b'(\beta^\#))\}) \cup \{(\beta^\#, b_1(\beta^+))\}$. Then b'_1 is in $\beta^\#$ an assignment variant of b' for D and $b'_1(\beta^\#) = b_1(\beta^+)$. Now, let $b_2 = (b\backslash\{(\beta^\#, b(\beta^\#))\}) \cup \{(\beta^\#, b'_1(\beta^\#))\}$. Then b_2 is in $\beta^\#$ an assignment variant of b for D and $TD(\beta^\#, D, I, b_2) = b_2(\beta^\#) = b'_1(\beta^\#) = b_1(\beta^+) = TD(\beta^+, D, I, b_1)$. According to the assumption for β^+ and $\beta^\#$, we also have that $\beta^+, \beta^\# \notin ST([\theta, \xi, A])$ and thus that $b_2{\upharpoonright}ST([\theta, \xi, A]) = b{\upharpoonright}ST([\theta, \xi, A]) = b_1{\upharpoonright}ST([\theta, \xi, A])$. We trivially have $I{\upharpoonright}SE([\theta, \xi, A]) = I{\upharpoonright}SE([\theta, \xi, A])$. Also, we have $FV([\theta, \xi, A]) \subseteq \{\zeta\}$ and, with Theorem 1-13, we have $FDEG([\theta, \xi, A]) = FDEG(A) < FDEG(\Delta)$. By the I.H., it thus holds, because of $D, I, b_1 \vDash [\beta^+, \zeta, [\theta, \xi, A]]$, that $D, I, b_2 \vDash [\beta^\#, \zeta, [\theta, \xi, A]] = [\theta, \xi, [\beta^\#, \zeta, A]]$.

With $\beta^\# \notin ST(\theta)$ and $\beta^\# \notin ST(\theta')$, we have $b_2{\upharpoonright}ST(\theta) = b{\upharpoonright}ST(\theta)$ and $b'_1{\upharpoonright}ST(\theta') = b'{\upharpoonright}ST(\theta')$ and hence, according to Theorem 5-5-(i), we have $TD(\theta, D, I, b_2) = TD(\theta, D, I, b)$ and $TD(\theta', D, I', b'_1) = TD(\theta', D, I', b')$. By our initial hypothesis, we thus have $TD(\theta, D, I, b_2) = TD(\theta', D, I', b'_1)$. With $b{\upharpoonright}ST(A) = b'{\upharpoonright}ST(A)$, $b_2(\beta^\#) = b'_1(\beta^\#)$ and $ST([\beta^\#, \zeta, A]) \subseteq ST(A) \cup \{\beta^\#\}$, we also have $b_2{\upharpoonright}ST([\beta^\#, \zeta, A]) = b'_1{\upharpoonright}ST([\beta^\#, \zeta, A])$ and it holds that $I{\upharpoonright}SE([\beta^\#, \zeta, A]) = I{\upharpoonright}(SE([\beta^\#, \zeta, A]) \cap (CONST \cup FUNC \cup PRED)) = I{\upharpoonright}(SE(A) \cap (CONST \cup FUNC \cup PRED)) = I{\upharpoonright}SE(A) = I'{\upharpoonright}SE(A) = I'{\upharpoonright}(SE(A) \cap (CONST \cup FUNC \cup PRED)) = I'{\upharpoonright}(SE([\beta^\#, \zeta, A]) \cap (CONST \cup FUNC \cup PRED)) = I'{\upharpoonright}(SE([\beta^\#, \zeta, A])$ and hence it holds that $I{\upharpoonright}SE([\beta^\#, \zeta, A]) = I'{\upharpoonright}SE([\beta^\#, \zeta, A])$. Further we have $FV([\beta^\#, \zeta, A]) \subseteq \{\xi\}$ and, with Theorem 1-13, we have $FDEG([\beta^\#, \zeta, A]) < FDEG(\Delta)$. By the I.H., it thus holds, because of $D, I, b_2 \vDash [\theta, \xi, [\beta^\#, \zeta, A]]$, that $D, I', b'_1 \vDash [\theta', \xi, [\beta^\#, \zeta, A]] = [\beta^\#, \zeta, [\theta', \xi, A]]$ and hence, according to Theorem 5-4-(viii), that $D, I', b' \vDash \ulcorner\bigvee\zeta[\theta', \xi, A]\urcorner$. The right-left-direction is shown analogously. ■

Now we will proof some consequences of the substitution lemma in order to facilitate some later proofs.

Theorem 5-7. *Coreferentiality*

If (D, I) is a model, b is a parameter assignment for D, $\xi \in$ VAR, θ, $\theta' \in$ CTERM and TD(θ, D, I, b) = TD(θ', D, I, b), then:

(i) For all $\theta^+ \in$ TERM with FV(θ^+) $\subseteq \{\xi\}$ it holds that TD([θ, ξ, θ^+], D, I, b) = TD([θ', ξ, θ^+], D, I, b), and

(ii) For all $\Delta \in$ FORM with FV(Δ) $\subseteq \{\xi\}$ it holds that D, I, $b \vDash$ [θ, ξ, Δ] iff D, I, $b \vDash$ [θ', ξ, Δ].

Proof: Suppose (D, I) is a model, b is a parameter assignment for D, $\xi \in$ VAR, θ, $\theta' \in$ CTERM and TD(θ, D, I, b) = TD(θ', D, I, b). Then we trivially have for all $\mu \in$ TERM $\cup$ FORM: $I\restriction$SE(μ) = $I\restriction$SE(μ) and $b\restriction$ST(μ) = $b\restriction$ST(μ) and thus the statement follows with Theorem 5-6. ■

Theorem 5-8. *Invariance of the satisfaction of quantificational formulas with respect to the choice of parameters*

If (D, I) is a model, b is a parameter assignment for D, $\xi \in$ VAR, $\Delta \in$ FORM, with FV(Δ) $\subseteq$ $\{\xi\}$ and $\beta \in$ PAR\ST(Δ), then:

(i) D, I, $b \vDash$ ⌜$\bigwedge\xi\Delta$⌝ iff for all b' that are in β assignment variants of b for D it holds that D, I, $b' \vDash$ [β, ξ, Δ], and

(ii) D, I, $b \vDash$ ⌜$\bigvee\xi\Delta$⌝ iff there is a b' that is in β assignment variant of b for D such that D, I, $b' \vDash$ [β, ξ, Δ].

Proof: Suppose (D, I) is a model, b is a parameter assignment for D, $\xi \in$ VAR, $\Delta \in$ FORM with FV(Δ) $\subseteq$ $\{\xi\}$ and $\beta \in$ PAR\ST(Δ). *Ad (i)*: The right-left-direction follows directly with Theorem 5-4-(vii). Now, for the left-right-direction, suppose D, I, $b \vDash$ ⌜$\bigwedge\xi\Delta$⌝. Then there is a $\beta^* \in$ PAR\ST(Δ) such that for all b^* that are in β^* assignment variants of b for D it holds that D, I, $b^* \vDash$ [β^*, ξ, Δ]. Now, suppose b' is in β an assignment variant of b for D. Now, let b^* = (b\{(β^*, b(β^*))}) $\cup$ {(β^*, b'(β))}. Then b^* is in β^* an assignment variant of b for D and hence we have D, I, $b^* \vDash$ [β^*, ξ, Δ]. We also have TD(β^*, D, I, b^*) = b^*(β^*) = b'(β) = TD(β, D, I, b'). With β, $\beta^* \notin$ ST(Δ), we further have $b^*\restriction$ST(Δ) = $b\restriction$ST(Δ) = $b'\restriction$ST(Δ). With Theorem 5-6-(ii), we hence have D, I, $b' \vDash$ [β, ξ, Δ].

Ad (ii): The right-left-direction follows directly with Theorem 5-4-(viii). Now, for the left-right-direction, suppose D, I, $b \vDash$ ⌜$\bigvee\xi\Delta$⌝. Then there is $\beta^* \in$ PAR\ST(Δ) and b^* that is in β^* an assignment variant of b for D such that $D, I, b^* \vDash [\beta^*, \xi, \Delta]$. Now, let $b' = (b \setminus \{(\beta, b(\beta))\}) \cup \{(\beta, b^*(\beta^*))\}$. Then b' is in β an assignment variant of b for D and we have $\mathrm{TD}(\beta^*, D, I, b^*) = b^*(\beta^*) = b'(\beta) = \mathrm{TD}(\beta, D, I, b')$. With $\beta, \beta^* \notin \mathrm{ST}(\Delta)$ we have again $b^*{\upharpoonright}\mathrm{ST}(\Delta) = b'{\upharpoonright}\mathrm{ST}(\Delta)$. With Theorem 5-6-(ii), we hence have $D, I, b' \vDash [\beta, \xi, \Delta]$. ■

Theorem 5-9. *Simple substitution lemma for parameter assignments*
If (D, I) is a model, b is a parameter assignment for D, $\xi \in \mathrm{VAR}$, $\beta \in \mathrm{PAR}$ and $\theta \in \mathrm{CTERM}$, then:

(i) If b' is in β an assignment variant of b for D and $b'(\beta) = \mathrm{TD}(\theta, D, I, b)$, then for all $\theta^+ \in \mathrm{TERM}$ with $\mathrm{FV}(\theta^+) \subseteq \{\xi\}$ and $\beta \notin \mathrm{ST}(\theta^+)$: $\mathrm{TD}([\theta, \xi, \theta^+], D, I, b) = \mathrm{TD}([\beta, \xi, \theta^+], D, I, b')$, and

(ii) If b' is in β an assignment variant of b for D and $b'(\beta) = \mathrm{TD}(\theta, D, I, b)$, then for all $\Delta \in \mathrm{FORM}$ with $\mathrm{FV}(\Delta) \subseteq \{\xi\}$ and $\beta \notin \mathrm{ST}(\Delta)$: $D, I, b \vDash [\theta, \xi, \Delta]$ iff $D, I, b' \vDash [\beta, \xi, \Delta]$.

Proof: Suppose (D, I) is a model, b is a parameter assignment for D, $\xi \in \mathrm{VAR}$, $\beta \in \mathrm{PAR}$ and $\theta \in \mathrm{CTERM}$. Now, suppose b' is in β an assignment variant of b for D, where $b'(\beta) = \mathrm{TD}(\theta, D, I, b)$. Now, suppose $\mu \in \mathrm{TERM} \cup \mathrm{FORM}$ with $\mathrm{FV}(\mu) \subseteq \{\xi\}$ and $\beta \notin \mathrm{ST}(\mu)$. Then we trivially have $I{\upharpoonright}\mathrm{SE}(\mu) = I{\upharpoonright}\mathrm{SE}(\mu)$. With $\beta \notin \mathrm{ST}(\mu)$, we also have $b{\upharpoonright}\mathrm{ST}(\mu) = b'{\upharpoonright}\mathrm{ST}(\mu)$. By hypothesis, we also have $\mathrm{TD}(\beta, D, I, b') = b'(\beta) = \mathrm{TD}(\theta, D, I, b)$.

According to Theorem 5-6-(i), we then have for all $\theta^+ \in \mathrm{TERM}$ with $\mathrm{FV}(\theta^+) \subseteq \{\xi\}$ and $\beta \notin \mathrm{ST}(\theta^+)$: $\mathrm{TD}([\theta, \xi, \theta^+], D, I, b) = \mathrm{TD}([\beta, \xi, \theta^+], D, I, b')$, and, with Theorem 5-6-(ii), we have for all $\Delta \in \mathrm{FORM}$, where $\mathrm{FV}(\Delta) \subseteq \{\xi\}$ and $\beta \notin \mathrm{ST}(\Delta)$: $D, I, b \vDash [\theta, \xi, \Delta]$ iff $D, I, b' \vDash [\beta, \xi, \Delta]$. ■

Definition 5-9. *4-ary model-theoretic satisfaction for sets*
$D, I, b \;{}_{\mathrm{M}}{\vDash}\; X$
iff
(D, I) is a model, b is a parameter assignment for D, $X \subseteq \mathrm{CFORM}$ and for all $\Delta \in X$: $D, I, b \vDash \Delta$.

Definition 5-10. *Model-theoretic consequence*
$X \vDash \Gamma$
iff
$X \cup \{\Gamma\} \subseteq$ CFORM and for all D, I, b: If $D, I, b\ {}_{M}\!\vDash X$, then $D, I, b \vDash \Gamma$.

Definition 5-11. *Validity*
$\vDash \Gamma$ iff $\emptyset \vDash \Gamma$.

Definition 5-12. *Satisfiability*
Γ <u>is satisfiable</u>
iff
$\Gamma \in$ CFORM and there is D, I, b such that $D, I, b \vDash \Gamma$.

In Definition 5-8 to Definition 5-12 we introduced some of the usual model-theoretic concepts. With the next Definition, we will now add a 3-ary satisfaction concept for propositions that aims especially at parameter-free propositions. Subsequently, we will introduce concepts for sets of propositions that are analogous to the concepts we introduced for closed formulas in Definition 5-10 to Definition 5-13, in the same way as we did with Definition 5-9 for the satisfaction concept for closed formulas defined in Definition 5-8.

Definition 5-13. *3-ary model-theoretic satisfaction*
$D, I \vDash \Gamma$
iff
(D, I) is a model and for all b that are parameter assignments for D it holds that $D, I, b \vDash \Gamma$.

Definition 5-14. *3-ary model-theoretic satisfaction for sets*
$D, I\ {}_{M}\!\vDash X$
iff
(D, I) is a model, $X \subseteq$ CFORM and for all $\Delta \in X$ it holds that $D, I \vDash \Delta$.

Definition 5-15. *Model-theoretic consequence for sets*
$X\ {}_{M}\!\vDash Y$
iff
$X \cup Y \subseteq$ CFORM and for all $\Delta \in Y$ it holds that $X \vDash \Delta$.

Definition 5-16. *Validity for sets*
$_{M}\vDash X$
iff
$X \subseteq$ CFORM and for all $\Delta \in X$ it holds that $\vDash \Delta$.

Definition 5-17. *Satisfiability for sets*
X is satisfiable$_{M}$
iff
$X \subseteq$ CFORM and there is D, I, b such that $D, I, b \vDash X$.

In the following the context will always indicate if we deal with propositions or with sets of propositions. Therefore, we will supress the index 'M' when using concepts defined in Definition 5-9 and Definition 5-14 to Definition 5-17. Now, we will define the closure of a set of propositions under the model-theoretic consequence relation. The remaining part of this section contains only some simple supporting theorems.

Definition 5-18. *The closure of a set of propositions under model-theoretic consequence*
$X^{\vDash} = \{\Delta \mid \Delta \in$ CFORM and $X \vDash \Delta\}$.

Theorem 5-10. *Satisfaction carries over to subsets*
If $D, I, b \vDash X$, then it holds for all $Y \subseteq X$ that $D, I, b \vDash Y$.

Proof: Follows directly from Definition 5-9. ■

Theorem 5-11. *Satisfiability carries over to subsets*
If X is satisfiable, then it holds for all $Y \subseteq X$ that Y is satisfiable.

Proof: Follows directly from Definition 5-17 and Theorem 5-10. ■

Theorem 5-12. *Consequence relation and satisfiability*
If $X \cup \{\Gamma\} \subseteq$ CFORM, then: $X \vDash \Gamma$ iff $X \cup \{\ulcorner\neg\Gamma\urcorner\}$ is not satisfiable.

Proof: Suppose $X \cup \{\Gamma\} \subseteq$ CFORM. Suppose $X \vDash \Gamma$. Then we have for all D, I, b: If $D, I, b \vDash X$, then $D, I, b \vDash \Gamma$. Suppose for contradiction that $X \cup \{\ulcorner\neg\Gamma\urcorner\}$ is satisfiable. Then there would be D, I, b such that $D, I, b \vDash X \cup \{\ulcorner\neg\Gamma\urcorner\}$. With Definition 5-9 and Theorem 5-4-(ii), it then follows that $D, I, b \nvDash \Gamma$. On the other hand, we would have, with Theorem 5-10, that $D, I, b \vDash X$ and thus, by hypothesis, that $D, I, b \vDash \Gamma$. Contradiction!

Now, suppose $X \cup \{\ulcorner\neg\Gamma\urcorner\}$ is not satisfiable. Then there is no D, I, b such that $D, I, b \vDash X \cup \{\ulcorner\neg\Gamma\urcorner\}$. With Definition 5-9 there is then no D, I, b such that $D, I, b \vDash X$ and $D, I, b \vDash \ulcorner\neg\Gamma\urcorner$. Now, suppose $D, I, b \vDash X$. Then (D, I) is a model and b is a parameter assignment for D and $D, I, b \nvDash \ulcorner\neg\Gamma\urcorner$. According to Theorem 5-4-(ii), we then have $D, I, b \vDash \Gamma$. Therefore we have for all D, I, b: If $D, I, b \vDash X$, then $D, I, b \vDash \Gamma$. Hence we have $X \vDash \Gamma$. ■

5.2 Closure of the Model-theoretic Consequence Relation

The following section leads to correctness. For each rule of the Speech Act Calculus (cf. ch. 3.1) (or for each extension operation (cf. ch. 3.2)), we will therefore prove a model-theoretic theorem that corresponds to the respective closure clause in ch. 4.2, i.e. to Theorem 4-15 (AR) or to one of the clauses of Theorem 4-18. First, however, we will prove the monotony of the model-theoretic consequence relation (cf. Theorem 4-16).

Theorem 5-13. *Model-theoretic monotony*
If $X' \subseteq X \subseteq$ CFORM and $X' \vDash \Gamma$, then $X \vDash \Gamma$.

Proof: Suppose $X' \subseteq X \subseteq$ CFORM and $X' \vDash \Gamma$. Then we have for all D, I, b: If $D, I, b \vDash X'$, then $D, I, b \vDash \Gamma$. Now, suppose $D, I, b \vDash X$. Then it holds, with $X' \subseteq X$ and Theorem 5-10, that $D, I, b \vDash X'$. By hypothesis, it thus holds that $D, I, b \vDash \Gamma$. Therefore we have for all D, I, b: If $D, I, b \vDash X$, then $D, I, b \vDash \Gamma$. Therefore $X \vDash \Gamma$. ■

Theorem 5-14. *Model-theoretic counterpart of AR*
If $X \subseteq$ CFORM and A $\in X$, then $X \vDash$ A.

Proof: Suppose $X \subseteq$ CFORM and A $\in X$. According to Definition 5-9, we then have for all D, I, b: If $D, I, b \vDash X$, then $D, I, b \vDash$ A and thus we have $X \vDash$ A. ■

Theorem 5-15. *Model-theoretic counterpart of CdI*
If $X \vDash$ B and A $\in X$, then $X \setminus \{A\} \vDash \ulcorner A \rightarrow B \urcorner$.

Proof: Suppose $X \vDash$ B and A $\in X$. Now, suppose $D, I, b \vDash X \setminus \{A\}$. Then (D, I) is a model and b is a parameter assignment for D and for all $\Delta \in X \setminus \{A\}$ it holds that $D, I, b \vDash \Delta$. Then we have either $D, I, b \vDash$ A or $D, I, b \nvDash$ A. In the first case, it holds that $D, I, b \vDash \Delta$ for all $\Delta \in X$, and hence we have $D, I, b \vDash X$. By hypothesis, it then follows that also $D, I, b \vDash$ B. With Theorem 5-4-(v), it then follows that $D, I, b \vDash \ulcorner A \rightarrow B \urcorner$. The same holds if $D, I, b \nvDash$ A. Therefore we have for all D, I, b that if $D, I, b \vDash X \setminus \{A\}$, then $D, I, b \vDash \ulcorner A \rightarrow B \urcorner$. Therefore $X \setminus \{A\} \vDash \ulcorner A \rightarrow B \urcorner$. ■

Theorem 5-16. *Model-theoretic counterpart of CdE*
If $X \vDash \ulcorner A \rightarrow B \urcorner$ and $Y \vDash A$, then $X \cup Y \vDash B$.

Proof: Suppose $X \vDash \ulcorner A \rightarrow B \urcorner$ and $Y \vDash A$. Suppose $D, I, b \vDash X \cup Y$. Then (D, I) is a model and b is a parameter assignment for D and, with Theorem 5-10, we have $D, I, b \vDash X$ and $D, I, b \vDash Y$. By hypothesis, it then follows that $D, I, b \vDash A$ and $D, I, b \vDash \ulcorner A \rightarrow B \urcorner$. With $D, I, b \vDash \ulcorner A \rightarrow B \urcorner$ and Theorem 5-4-(v), we then have $D, I, b \nvDash A$ or $D, I, b \vDash B$. With $D, I, b \vDash A$, we thus have $D, I, b \vDash B$. Therefore we have for all D, I, b, that if $D, I, b \vDash X \cup Y$, then also $D, I, b \vDash B$. Therefore $X \cup Y \vDash B$. ■

Theorem 5-17. *Model-theoretic counterpart of CI*
If $X \vDash A$ and $Y \vDash B$, then $X \cup Y \vDash \ulcorner A \wedge B \urcorner$.

Proof: Suppose $X \vDash A$ and $Y \vDash B$. Suppose $D, I, b \vDash X \cup Y$. Then (D, I) is a model and b is a parameter assignment for D and, with Theorem 5-10, we have $D, I, b \vDash X$ and $D, I, b \vDash Y$. By hypothesis, it then follows that also $D, I, b \vDash A$ and $D, I, b \vDash B$. With Theorem 5-4-(iii), it then follows that $D, I, b \vDash \ulcorner A \wedge B \urcorner$. Therefore we have for all D, I, b that if $D, I, b \vDash X \cup Y$, then also $D, I, b \vDash \ulcorner A \wedge B \urcorner$. Therefore $X \cup Y \vDash \ulcorner A \wedge B \urcorner$. ■

Theorem 5-18. *Model-theoretic counterpart of CE*
If $X \vDash \ulcorner A \wedge B \urcorner$, then $X \vDash A$ and $X \vDash B$.

Proof: Suppose $X \vDash \ulcorner A \wedge B \urcorner$. Suppose $D, I, b \vDash X$. Then (D, I) is a model and b is a parameter assignment for D and by hypothesis we have $D, I, b \vDash \ulcorner A \wedge B \urcorner$. With Theorem 5-4-(iii), it then follows that $D, I, b \vDash A$ and $D, I, b \vDash B$. Therefore we have for all D, I, b that if $D, I, b \vDash X$, then also $D, I, b \vDash A$ and $D, I, b \vDash B$. Therefore $X \vDash A$ and $X \vDash B$. ■

Theorem 5-19. *Model-theoretic counterpart of BI*
If $X \vDash \ulcorner A \rightarrow B \urcorner$ and $Y \vDash \ulcorner B \rightarrow A \urcorner$, then $X \cup Y \vDash \ulcorner A \leftrightarrow B \urcorner$.

Proof: Suppose $X \vDash \ulcorner A \rightarrow B \urcorner$ and $Y \vDash \ulcorner B \rightarrow A \urcorner$. Suppose $D, I, b \vDash X \cup Y$. Then (D, I) is a model and b is a parameter assignment for D and, with Theorem 5-10, we have $D, I, b \vDash X$ and $D, I, b \vDash Y$. By hypothesis, it then follows that $D, I, b \vDash \ulcorner A \rightarrow B \urcorner$ and $D, I, b \vDash \ulcorner B \rightarrow A \urcorner$. With Theorem 5-4-(v), it then follows that (i) $D, I, b \nvDash$ A or $D, I, b \vDash$ B and (ii) that $D, I, b \nvDash$ B or $D, I, b \vDash$ A. Suppose (the first case of (i)) $D, I, b \nvDash$ A. With (ii), it then holds that $D, I, b \nvDash$ B. Suppose (the second case of (i)) $D, I, b \vDash$ B. With (ii), it then holds that $D, I, b \vDash$ A. Therefore we have $D, I, b \vDash$ A and $D, I, b \vDash$ B or $D, I, b \nvDash$ A and $D, I, b \nvDash$ B. With Theorem 5-4-(vi), it then follows that $D, I, b \vDash \ulcorner A \leftrightarrow B \urcorner$. Therefore we have for all D, I, b that if $D, I, b \vDash X \cup Y$, then also $D, I, b \vDash \ulcorner A \leftrightarrow B \urcorner$. Therefore $X \cup Y \vDash \ulcorner A \leftrightarrow B \urcorner$. ■

We include a variant of Theorem 5-19 as a corollary. Here it is not required that some conditionals have to be model-theoretic consequences of some sets of propositions.

Theorem 5-20. *Model-theoretic counterpart of BI**
If $X \vDash$ B and A $\in X$ and $Y \vDash$ A and B $\in Y$, then $(X \backslash \{A\}) \cup (Y \backslash \{B\}) \vDash \ulcorner A \leftrightarrow B \urcorner$.

Proof: Suppose $X \vDash$ B and A $\in X$ and $Y \vDash$ A and B $\in Y$. According to Theorem 5-15, we then have $X \backslash \{A\} \vDash \ulcorner A \rightarrow B \urcorner$ and $Y \backslash \{B\} \vDash \ulcorner B \rightarrow A \urcorner$. With Theorem 5-19, it then follows that $(X \backslash \{A\}) \cup (Y \backslash \{B\}) \vDash \ulcorner A \leftrightarrow B \urcorner$. ■

Theorem 5-21. *Model-theoretic counterpart of BE*
If $X \vDash \ulcorner A \leftrightarrow B \urcorner$ or $X \vDash \ulcorner B \leftrightarrow A \urcorner$ and $Y \vDash$ A, then $X \cup Y \vDash$ B.

Proof: Suppose $X \vDash \ulcorner A \leftrightarrow B \urcorner$ or $X \vDash \ulcorner B \leftrightarrow A \urcorner$ and $Y \vDash$ A. Now, suppose $D, I, b \vDash X \cup Y$. Then (D, I) is a model and b is a parameter assignment for D and, with Theorem 5-10, we have $D, I, b \vDash X$ and $D, I, b \vDash Y$. By hypothesis, it then follows that $D, I, b \vDash$ A. Now, suppose $X \vDash \ulcorner A \leftrightarrow B \urcorner$. Then we have $D, I, b \vDash \ulcorner A \leftrightarrow B \urcorner$. With Theorem 5-4-(vi), it then follows that $D, I, b \vDash$ A and $D, I, b \vDash$ B or $D, I, b \nvDash$ A and $D, I, b \nvDash$

B. Now, suppose $X \vDash$ ⌜ B ↔ A⌝. Then we have $D, I, b \vDash$ ⌜ B ↔ A⌝. With Theorem 5-4-(vi), it then follows again that $D, I, b \vDash$ A and $D, I, b \vDash$ B or $D, I, b \nvDash$ A and $D, I, b \nvDash$ B. However, since $D, I, b \vDash$ A, it cannot be the case that $D, I, b \nvDash$ A and $D, I, b \nvDash$ B. Thus we have $D, I, b \vDash$ A and $D, I, b \vDash$ B. Therefore we have for all D, I, b that if $D, I, b \vDash X \cup Y$, then also $D, I, b \vDash$ B. Therefore $X \cup Y \vDash$ B. ■

Theorem 5-22. *Model-theoretic counterpart of DI*
If $X \vDash$ A or $X \vDash$ B, then $X \vDash$ ⌜A ∨ B⌝.

Proof: Suppose $X \vDash$ A or $X \vDash$ B. Suppose $D, I, b \vDash X$. Then (D, I) is a model and b is a parameter assignment for D. By hypothesis, we also have $D, I, b \vDash$ A or $D, I, b \vDash$ B. With Theorem 5-4-(iv), we have in both cases $D, I, b \vDash$ ⌜A ∨ B⌝. Therefore we have for all D, I, b that if $D, I, b \vDash X$, then also $D, I, b \vDash$ ⌜A ∨ B⌝. Therefore $X \vDash$ ⌜A ∨ B⌝. ■

Theorem 5-23. *Model-theoretic counterpart of DE*
If $X \vDash$ ⌜A ∨ B⌝ and $Y \vDash$ ⌜A → Γ⌝ and $Z \vDash$ ⌜B → Γ⌝, then $X \cup Y \cup Z \vDash$ Γ.

Proof: Suppose $X \vDash$ ⌜A ∨ B⌝ and $Y \vDash$ ⌜A → Γ⌝ and $Z \vDash$ ⌜ B → Γ⌝. Suppose $D, I, b \vDash X \cup Y \cup Z$. Then (D, I) is a model and b is a parameter assignment for D and, with Theorem 5-10, we have $D, I, b \vDash X$ and $D, I, b \vDash Y$ and $D, I, b \vDash Z$. By hypothesis, it then follows that $D, I, b \vDash$ ⌜A ∨ B⌝ and $D, I, b \vDash$ ⌜A → Γ⌝ and $D, I, b \vDash$ ⌜ B → Γ⌝. With Theorem 5-4-(iv) and -(v), we then have: (i) $D, I, b \vDash$ A or $D, I, b \vDash$ B and (ii) $D, I, b \nvDash$ A or $D, I, b \vDash$ Γ and (iii) $D, I, b \nvDash$ B or $D, I, b \vDash$ Γ. Suppose (the first case of (i)) $D, I, b \vDash$ A. With (ii), we then have $D, I, b \vDash$ Γ. Suppose (the second case of (i)) $D, I, b \vDash$ B. With (iii), we then have $D, I, b \vDash$ Γ. Thus we have in both cases $D, I, b \vDash$ Γ. Therefore we have for all D, I, b that if $D, I, b \vDash X \cup Y \cup Z$, then also $D, I, b \vDash$ Γ. Therefore $X \cup Y \cup Z \vDash$ Γ. ■

We include a variant of Theorem 5-23 as a corollary. Here it is not required that some conditionals have to be model-theoretic consequences of some sets of propositions.

Theorem 5-24. *Model-theoretic counterpart of DE**
If $X \vDash \ulcorner A \vee B \urcorner$ and $Y \vDash \Gamma$ and $A \in Y$ and $Z \vDash \Gamma$ and $B \in Z$, then $X \cup (Y \backslash \{A\}) \cup (Z \backslash \{B\}) \vDash \Gamma$.

Proof: Suppose $X \vDash \ulcorner A \vee B \urcorner$ and $Y \vDash \Gamma$ and $A \in Y$ and $Z \vDash \Gamma$ and $B \in Z$. According to Theorem 5-15, we then have $Y \backslash \{A\} \vDash \ulcorner A \rightarrow \Gamma \urcorner$ and $Z \backslash \{B\} \vDash \ulcorner B \rightarrow \Gamma \urcorner$. With Theorem 5-23, it then follows that $X \cup (Y \backslash \{A\}) \cup (Z \backslash \{B\}) \vDash \Gamma$. ■

Theorem 5-25. *Model-theoretic counterpart of NI*
If $X \vDash B$ and $Y \vDash \ulcorner \neg B \urcorner$ and $A \in X \cup Y$, then $(X \cup Y) \backslash \{A\} \vDash \ulcorner \neg A \urcorner$.

Proof: Suppose $X \vDash B$ and $Y \vDash \ulcorner \neg B \urcorner$ and $A \in X \cup Y$. Suppose $D, I, b \vDash (X \cup Y) \backslash \{A\}$. Then (D, I) is a model and b is a parameter assignment for D such that for all $\Delta \in (X \cup Y) \backslash \{A\}$ it holds that $D, I, b \vDash \Delta$. Suppose for contradiction that $D, I, b \vDash A$. Then we would have for all $\Delta \in X$ and for all $\Delta \in Y$: $D, I, b \vDash \Delta$ and thus $D, I, b \vDash X$ and $D, I, b \vDash Y$. By hypothesis, it would then follows that $D, I, b \vDash B$ and $D, I, b \vDash \ulcorner \neg B \urcorner$. With Theorem 5-4-(ii), it would then follow that $D, I, b \vDash B$ and $D, I, b \nvDash B$. Sed certe hoc esse non potest. Therefore $D, I, b \nvDash A$ and thus $D, I, b \vDash \ulcorner \neg A \urcorner$. Therefore we have for all D, I, b that if $D, I, b \vDash (X \cup Y) \backslash \{A\}$, then also $D, I, b \vDash \ulcorner \neg A \urcorner$. Therefore $(X \cup Y) \backslash \{A\} \vDash \ulcorner \neg A \urcorner$. ■

Theorem 5-26. *Model-theoretic counterpart of NE*
If $X \vDash \ulcorner \neg\neg A \urcorner$, then $X \vDash A$.

Proof: Suppose $X \vDash \ulcorner \neg\neg A \urcorner$. Suppose $D, I, b \vDash X$. Then (D, I) is a model and b is a parameter assignment for D and, by hypothesis, we also have $D, I, b \vDash \ulcorner \neg\neg A \urcorner$. With Theorem 5-4-(ii), it then follows that $D, I, b \nvDash \ulcorner \neg A \urcorner$. Applying Theorem 5-4-(ii) again yields $D, I, b \vDash A$. Therefore we have for all D, I, b: If $D, I, b \vDash X$, then $D, I, b \vDash A$. Therefore $X \vDash A$. ■

Theorem 5-27. *Model-theoretic counterpart of UI*
If β ∈ PAR, ξ ∈ VAR, A ∈ FORM, where FV(A) ⊆ {ξ}, and *X* ⊨ [β, ξ, A] and β ∉ STSF(*X* ∪ {A}), then *X* ⊨ ⌜⋀ξA⌝.

Proof: Suppose β ∈ PAR, ξ ∈ VAR, A ∈ FORM, where FV(A) ⊆ {ξ}, *X* ⊨ [β, ξ, A] and β ∉ STSF(*X* ∪ {A}). Suppose *D*, *I*, *b* ⊨ *X*. Then (*D*, *I*) is a model and *b* is a parameter assignment for *D*. Suppose *b*' in β an assignment variant of *b* for *D*. Suppose Δ ∈ *X*. Therefore *D*, *I*, *b* ⊨ Δ. We have, by hypothesis, β ∉ ST(Δ). Therefore we have *b*↾ST(Δ) = *b*'↾ST(Δ). According to Theorem 5-5-(ii) it then follows that also *D*, *I*, *b*' ⊨ Δ. Therefore *D*, *I*, *b*' ⊨ Δ for all Δ ∈ *X* and hence *D*, *I*, *b*' ⊨ *X*. With *X* ⊨ [β, ξ, A], we then have also *D*, *I*, *b*' ⊨ [β, ξ, A]. Therefore we have for all *b*' that are in β an assignment variant of *b* for *D*: *D*, *I*, *b*' ⊨ [β, ξ, A]. With Theorem 5-4-(vii) follows *D*, *I*, *b* ⊨ ⌜⋀ξA⌝. Therefore we have for all *D*, *I*, *b*: If *D*, *I*, *b* ⊨ *X*, then also *D*, *I*, *b* ⊨ ⌜⋀ξA⌝. Therefore *X* ⊨ ⌜⋀ξA⌝. ■

Theorem 5-28. *Model-theoretic counterpart of UE*
If θ ∈ CTERM, ξ ∈ VAR, A ∈ FORM, where FV(A) ⊆ {ξ}, and *X* ⊨ ⌜⋀ξA⌝, then *X* ⊨ [θ, ξ, A].

Proof: Suppose θ ∈ CTERM, ξ ∈ VAR, A ∈ FORM, where FV(A) ⊆ {ξ}, and *X* ⊨ ⌜⋀ξA⌝. Suppose *D*, *I*, *b* ⊨ *X*. Then (*D*, *I*) is a model and *b* is a parameter assignment for *D* and, by hypothesis, *D*, *I*, *b* ⊨ ⌜⋀ξA⌝. According to Theorem 5-4-(vii) there is then a β ∈ PAR\ST(A) such that for all *b*' that are in β an assignment variant of *b* for *D* it holds that *D*, *I*, *b*' ⊨ [β, ξ, A]. Suppose *b** = (*b*\{(β, *b*(β))}) ∪ {(β, TD(θ, *D*, *I*, *b*))}. Obviously *b** is in β an assignment variant of *b* for *D*. Therefore *D*, *I*, *b** ⊨ [β, ξ, A]. With *b**(β) = TD(θ, *D*, *I*, *b*) and β ∉ ST(A) it follows then with Theorem 5-9-(ii) that *D*, *I*, *b* ⊨ [θ, ξ, A]. Therefore we have for all *D*, *I*, *b*: If *D*, *I*, *b* ⊨ *X*, then *D*, *I*, *b* ⊨ [θ, ξ, A]. Therefore *X* ⊨ [θ, ξ, A]. ■

Theorem 5-29. *Model-theoretic counterpart of PI*
If θ ∈ CTERM, ξ ∈ VAR, A ∈ FORM, where FV(A) ⊆ {ξ}, and X ⊨ [θ, ξ, A], then X ⊨ ⌜⋁ξA⌝.

Proof: Suppose θ ∈ CTERM, ξ ∈ VAR, A ∈ FORM, where FV(A) ⊆ {ξ}, and X ⊨ [θ, ξ, A]. Suppose D, I, b ⊨ X. Then (D, I) is a model and b is a parameter assignment for D and, by hypothesis, we have D, I, b ⊨ [θ, ξ, A]. Now, let β ∈ PAR\ST(A) and let b^* = (b\{(β, b(β))}) ∪ {(β, TD(θ, D, I, b))}. Then b^* is in β an assignment variant of b for D. With b^*(β) = TD(θ, D, I, b), β ∉ ST(A) and Theorem 5-9-(ii), it then follows that D, I, b^* ⊨ [β, ξ, A]. With Theorem 5-4-(viii), it then follows that D, I, b ⊨ ⌜⋁ξA⌝. Therefore we have for all D, I, b: If D, I, b ⊨ X, then D, I, b ⊨ ⌜⋁ξA⌝. Therefore X ⊨ ⌜⋁ξA⌝. ■

Theorem 5-30. *Model-theoretic counterpart of PE*
If β ∈ PAR, ξ ∈ VAR, A ∈ FORM, where FV(A) ⊆ {ξ}, and X ⊨ ⌜⋁ξA⌝ and Y ⊨ B and {[β, ξ, A]} ∈ Y and β ∉ STSF((Y\{[β, ξ, A]}) ∪ {A, B}), then X ∪ (Y\{[β, ξ, A]}) ⊨ B.

Proof: Suppose β ∈ PAR, ξ ∈ VAR, A ∈ FORM, where FV(A) ⊆ {ξ}, X ⊨ ⌜⋁ξA⌝, Y ⊨ B, {[β, ξ, A]} ∈ Y and β ∉ STSF((Y\{[β, ξ, A]}) ∪ {A, B}). Suppose D, I, b ⊨ X ∪ (Y\{[β, ξ, A]}). Then (D, I) is a model and b is a parameter assignment for D and, with Theorem 5-10, we have D, I, b ⊨ X and D, I, b ⊨ Y\{[β, ξ, A]}. By hypothesis, it then follows that D, I, b ⊨ ⌜⋁ξA⌝. Since β ∉ ST(A), there is then, according to Theorem 5-8-(ii), a b' that is in β an assignment variant of b for D such that D, I, b' ⊨ [β, ξ, A]. Now, suppose Δ' ∈ Y. Then we have Δ' ∈ Y\{[β, ξ, A]} or Δ' = [β, ξ, A]. In the first case, we have D, I, b ⊨ Δ'. Since β ∉ ST(Δ'), we have b↾ST(Δ') = b'↾ST(Δ'). By Theorem 5-5-(ii), it then follows that D, I, b' ⊨ Δ'. For the second case, we already have D, I, b' ⊨ [β, ξ, A]. Therefore D, I, b' ⊨ Δ' for all Δ' ∈ Y and hence D, I, b' ⊨ Y. By hypothesis, it then follows that D, I, b' ⊨ B. Since β ∉ ST(B), we have b↾ST(B) = b'↾ST(B). With Theorem 5-5-(ii), it then follows that D, I, b ⊨ B. Therefore we have for all D, I, b: If D, I, b ⊨ X ∪ (Y\{[β, ξ, A]}), then D, I, b ⊨ B. Therefore X ∪ (Y\{[β, ξ, A]}) ⊨ B. ■

Theorem 5-31. *Model-theoretic counterpart of II*
For all $X \subseteq$ CFORM and $\theta \in$ CTERM: $X \vDash \ulcorner\theta = \theta\urcorner$.

Proof: Suppose $X \subseteq$ CFORM and $\theta \in$ CTERM. Suppose $D, I, b \vDash X$. Then (D, I) is a model and b is a parameter assignment for D. With $\langle$TD(θ, D, I, b), TD$(\theta, D, I, b)\rangle \in \{\langle a, a\rangle \mid a \in D\}$, we have $\langle$TD(θ, D, I, b), TD$(\theta, D, I, b)\rangle \in I(\ulcorner=\urcorner)$. According to Theorem 5-4-(i), it then follows that $D, I, b \vDash \ulcorner\theta = \theta\urcorner$. Therefore we have for all D, I, b: If $D, I, b \vDash X$, then $D, I, b \vDash \ulcorner\theta = \theta\urcorner$. Therefore $X \vDash \ulcorner\theta = \theta\urcorner$. ■

Theorem 5-32. *Model-theoretic counterpart of IE*
If $\theta_0, \theta_1 \in$ CTERM, $\xi \in$ VAR, $\Delta \in$ FORM, where FV$(\Delta) \subseteq \{\xi\}$, and $X \vDash \ulcorner\theta_0 = \theta_1\urcorner$ and $Y \vDash [\theta_0, \xi, \Delta]$, then $X \cup Y \vDash [\theta_1, \xi, \Delta]$.

Proof: Suppose $\theta_0, \theta_1 \in$ CTERM, $\xi \in$ VAR, $\Delta \in$ FORM, where FV$(\Delta) \subseteq \{\xi\}$, and $X \vDash \ulcorner\theta_0 = \theta_1\urcorner$ and $Y \vDash [\theta_0, \xi, \Delta]$. Now, suppose $D, I, b \vDash X \cup Y$. Then (D, I) is a model and b is a parameter assignment for D and, with Theorem 5-10, we have $D, I, b \vDash X$ and $D, I, b \vDash Y$. By hypothesis, it then follows that $D, I, b \vDash \ulcorner\theta_0 = \theta_1\urcorner$ and $D, I, b \vDash [\theta_0, \xi, \Delta]$. By Theorem 5-4-(i), we then have that $\langle$TD(θ_0, D, I, b), TD$(\theta_1, D, I, b)\rangle \in I(\ulcorner=\urcorner) = \{\langle a, a\rangle \mid a \in D\}$. Thus we have TD$(\theta_0, D, I, b)$ = TD(θ_1, D, I, b). According to Theorem 5-7-(ii), it then follows, with $D, I, b \vDash [\theta_0, \xi, \Delta]$, that also $D, I, b \vDash [\theta_1, \xi, \Delta]$. Therefore we have for all D, I, b: If $D, I, b \vDash X \cup Y$, then $D, I, b \vDash [\theta_1, \xi, \Delta]$. Therefore $X \cup Y \vDash [\theta_1, \xi, \Delta]$. ■

6 Correctness and Completeness of the Speech Act Calculus

After having established the Speech Act Calculus and a model-theory, we now have to show that the respective consequence relations are equivalent. As usual, this adequacy proof contains two parts: *First* the proof of the correctness of the Speech Act Calculus relative to the model-theory. Informally: Everthing that is derivable also follows model-theoretically (6.1). *Second* the proof of the completeness of the Speech Act Calculus relative to the model-theory. Informally: Everthing that follows model-theoretically is also derivable (6.2).

Note that our talk of the *correctness and completeness of the Speech Act Calculus* follows the usual custom. On the other hand, one could also read the two results obversely, i.e. so that we show in ch. 6.1 that the model-theoretic consequence relation is complete relative to the calculus. In ch. 6.2 we would then accordingly show that the model-theoretic consequence relation is correct relative to the calculus. We do not follow this alternative way of interpreting the results in order to avoid confusion. However, even if we speak of correctness and completeness in the usual way, we do not want to insinuate that the model-theoretic consequence relation is in some way superior to the deductive consequence relation established by the calculus or that calculi have to be justified by reference to model-theoretic concepts of consequence and not the other way round. The adequacy result just says that Speech Act Calculus and classical first-order model-theory are associated with equivalent consequence relations.

6.1 Correctness of the Speech Act Calculus

The following section consists mainly of one single proof, namely the proof of Theorem 6-1, which says that in each derivation $\mathfrak{H}$ the conclusion is a model-theoretic consequence of AVAP($\mathfrak{H}$). The proof is carried out by induction on the length of a derivation. Using the I.H., we will show that for all 17 possible extensions of $\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1 to $\mathfrak{H}$ it holds that AVAP($\mathfrak{H}$) $\vDash$ C($\mathfrak{H}$). In doing this, we will first deal with the more ›interesting‹ cases, i.e. those cases in which the set of available assumptions is reduced or augmented by the extension of $\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1 to $\mathfrak{H}$. These four cases are AF, CdIF, NIF and PEF (or AR, CdI, NI and PE). For the remaining 13 cases, we can then exlcude that the the last step in the derivation under consideration belongs to one of the first four cases. The correctness of the Speech Act Calculus relative to the model-theory is then established at the end of the section in Theorem 6-2.

Theorem 6-1. *Main correctness proof*
If $\mathfrak{H} \in \mathrm{RCS}\setminus\{\emptyset\}$, then $\mathrm{AVAP}(\mathfrak{H}) \vDash \mathrm{C}(\mathfrak{H})$.

Proof: Proof by induction on $|\mathfrak{H}|$. For this, suppose the theorem holds for all $l < |\mathfrak{H}|$ and suppose $\mathfrak{H} \in \mathrm{RCS}\setminus\{\emptyset\}$. According to Definition 3-19, we then have $\mathfrak{H} \in \mathrm{SEQ}$ and for all $j < \mathrm{Dom}(\mathfrak{H})$: $\mathfrak{H}\upharpoonright j{+}1 \in \mathrm{RCE}(\mathfrak{H}\upharpoonright j)$. Also, with Theorem 3-8, it holds for all $j \in \mathrm{Dom}(\mathfrak{H})$ that $\mathfrak{H}\upharpoonright j{+}1 \in \mathrm{RCS}\setminus\{\emptyset\}$. With this and the I.H., we have for all $0 < j < \mathrm{Dom}(\mathfrak{H})$: $\mathrm{AVAP}(\mathfrak{H}\upharpoonright j) \vDash \mathrm{C}(\mathfrak{H}\upharpoonright j)$. According to Theorem 3-6 and Definition 3-18, we also have $\mathfrak{H} \in \mathrm{AF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \mathrm{CdIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \mathrm{CdEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \mathrm{CIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \mathrm{CEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \mathrm{BIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \mathrm{BEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \mathrm{DIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \mathrm{DEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \mathrm{NIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \mathrm{NEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \mathrm{UIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \mathrm{UEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \mathrm{PIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \mathrm{PEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \mathrm{IIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \in \mathrm{IEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$.

We further have that $\mathfrak{H} \in \mathrm{AF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) \cup \mathrm{CdIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) \cup \mathrm{NIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) \cup \mathrm{PEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\mathfrak{H} \notin \mathrm{AF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) \cup \mathrm{CdIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) \cup \mathrm{NIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) \cup \mathrm{PEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. Thus we can distinguish *two* major cases. Now, for the *first case*, suppose $\mathfrak{H} \in \mathrm{AF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) \cup \mathrm{CdIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) \cup \mathrm{NIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) \cup \mathrm{PEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. Then we can distinguish four subcases, where, with Definition 3-2, Definition 3-10 and Definition 3-16, we have for the three latter ones: $\mathrm{Dom}(\mathfrak{H})\text{-}1 \neq 0$ and thus $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1 \in \mathrm{RCS}\setminus\{\emptyset\}$ and $\mathrm{AVAP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) \vDash \mathrm{C}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$.

(*AF*): Suppose $\mathfrak{H} \in \mathrm{AF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. According to Theorem 3-15-(viii), we then have $\mathrm{C}(\mathfrak{H}) \in \mathrm{AVAP}(\mathfrak{H})$. Theorem 5-14 then yields $\mathrm{AVAP}(\mathfrak{H}) \vDash \mathrm{C}(\mathfrak{H})$.

(*CdIF*): Suppose $\mathfrak{H} \in \mathrm{CdIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. According to Theorem 3-19-(x), we then have $\mathrm{C}(\mathfrak{H}) = \ulcorner\mathrm{P}(\mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)))}) \rightarrow \mathrm{C}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)\urcorner$. We have $\mathrm{AVAP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) \vDash \mathrm{C}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. With Theorem 3-19-(ix), we have $\mathrm{AVAP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) = \mathrm{AVAP}(\mathfrak{H}) \cup \{\mathrm{P}(\mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)))})\}$ and thus we have $\mathrm{AVAP}(\mathfrak{H}) \cup \{\mathrm{P}(\mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)))})\} \vDash \mathrm{C}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. With Theorem 5-15, it then follows that $\mathrm{AVAP}(\mathfrak{H})\setminus\{\mathrm{P}(\mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)))})\} \vDash \ulcorner\mathrm{P}(\mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)))})$

$\rightarrow$ C($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1)$\urcorner$. Theorem 5-13 then yields AVAP($\mathfrak{H}$) $\vDash$ $\ulcorner$P($\mathfrak{H}_{\max(\text{Dom}(\text{AVAS}(\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-}1)))}$) $\rightarrow$ C($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1)$\urcorner$ and thus AVAP($\mathfrak{H}$) $\vDash$ C($\mathfrak{H}$).

(*NIF*): Suppose $\mathfrak{H}$ $\in$ NIF($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1). According to Theorem 3-20-(x), we then have C($\mathfrak{H}$) = $\ulcorner\neg$P($\mathfrak{H}_{\max(\text{Dom}(\text{AVAS}(\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-}1)))}$)$\urcorner$. With Theorem 3-20-(i) and Theorem 2-92, there is Γ $\in$ CFORM and j $\in$ Dom($\mathfrak{H}$)-1 such that max(Dom(AVAS($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1))) $\leq j$ and either P($\mathfrak{H}_j$) = Γ and P($\mathfrak{H}_{\text{Dom}(\mathfrak{H})\text{-}2}$) = $\ulcorner\neg\Gamma\urcorner$ or P($\mathfrak{H}_j$) = $\ulcorner\neg\Gamma\urcorner$ and P($\mathfrak{H}_{\text{Dom}(\mathfrak{H})\text{-}2}$) = Γ and (j, $\mathfrak{H}_j$) $\in$ AVS($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1). Thus we have either Γ = C($\mathfrak{H}\upharpoonright j$+1) and $\ulcorner\neg\Gamma\urcorner$ = C($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1) or $\ulcorner\neg\Gamma\urcorner$ = C($\mathfrak{H}\upharpoonright j$+1) and Γ = C($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1). First suppose Γ = C($\mathfrak{H}\upharpoonright j$+1) and $\ulcorner\neg\Gamma\urcorner$ = C($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1). Then we have AVAP($\mathfrak{H}\upharpoonright j$+1) $\vDash$ Γ and AVAP($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1) $\vDash$ $\ulcorner\neg\Gamma\urcorner$. Also, we have that Γ is available in $\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1 at j and thus, according to Theorem 3-29-(iv), AVAP($\mathfrak{H}\upharpoonright j$+1) $\subseteq$ AVAP($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1). With Theorem 5-13, we thus also have AVAP($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1) $\vDash$ Γ. Second suppose $\ulcorner\neg\Gamma\urcorner$ = C($\mathfrak{H}\upharpoonright j$+1) and Γ = C($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1). Then we have AVAP($\mathfrak{H}\upharpoonright j$+1) $\vDash$ $\ulcorner\neg\Gamma\urcorner$ and AVAP($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1) $\vDash$ Γ. Also, $\ulcorner\neg\Gamma\urcorner$ is then available in $\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1 at j and hence we have, again with Theorem 3-29-(iv), that AVAP($\mathfrak{H}\upharpoonright j$+1) $\subseteq$ AVAP($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1) and thus, with Theorem 5-13, that AVAP($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1) $\vDash$ $\ulcorner\neg\Gamma\urcorner$. Thus we have in both cases that AVAP($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1) $\vDash$ Γ and AVAP($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1) $\vDash$ $\ulcorner\neg\Gamma\urcorner$. With Theorem 3-20-(ix), we have AVAP($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1) = AVAP($\mathfrak{H}$) $\cup$ {P($\mathfrak{H}_{\max(\text{Dom}(\text{AVAS}(\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-}1)))}$)}. Thus we have AVAP($\mathfrak{H}$) $\cup$ {P($\mathfrak{H}_{\max(\text{Dom}(\text{AVAS}(\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-}1)))}$)} $\vDash$ Γ and AVAP($\mathfrak{H}$) $\cup$ {P($\mathfrak{H}_{\max(\text{Dom}(\text{AVAS}(\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-}1)))}$)} $\vDash$ $\ulcorner\neg\Gamma\urcorner$. With Theorem 5-25 (where X as well as Y are instantiated by AVAP($\mathfrak{H}$) $\cup$ {P($\mathfrak{H}_{\max(\text{Dom}(\text{AVAS}(\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-}1)))}$)}) and Theorem 5-13, it then follows that AVAP($\mathfrak{H}$) $\vDash$ $\ulcorner\neg$P($\mathfrak{H}_{\max(\text{Dom}(\text{AVAS}(\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-}1)))}$)$\urcorner$ and thus that AVAP($\mathfrak{H}$) $\vDash$ C($\mathfrak{H}$).

(*PEF*): Suppose $\mathfrak{H}$ $\in$ PEF($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1). According to Theorem 3-21-(x), we then have C($\mathfrak{H}$) = C($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1). According to Theorem 3-21-(i) and Theorem 2-93, there are β $\in$ PAR, ξ $\in$ VAR, Δ $\in$ FORM with FV(Δ) $\subseteq$ {ξ}, and Γ $\in$ CFORM such that P($\mathfrak{H}_{\max(\text{Dom}(\text{AVAS}(\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-}1)))\text{-}1}$) = $\ulcorner\bigvee\xi\Delta\urcorner$ and (max(Dom(AVAS($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1)))-1, $\mathfrak{H}_{\max(\text{Dom}(\text{AVAS}(\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-}1)))\text{-}1}$) $\in$ AVS($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1) and P($\mathfrak{H}_{\max(\text{Dom}(\text{AVAS}(\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-}1)))}$) = [$\beta$, ξ, Δ] and β $\notin$ STSF({Δ, C($\mathfrak{H}$)}) and there is no $j \leq$ max(Dom(AVAS($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1)))-1 such that β $\in$ ST($\mathfrak{H}_j$). Then we have AVAP($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1) $\vDash$ C($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1) = C($\mathfrak{H}$). With Theorem 3-21-(ix), we have AVAP($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1) = AVAP($\mathfrak{H}$) $\cup$ {P($\mathfrak{H}_{\max(\text{Dom}(\text{AVAS}(\mathfrak{H}\upharpoonright\text{Dom}(\mathfrak{H})\text{-}1)))}$)} = AVAP($\mathfrak{H}$) $\cup$ {[β, ξ, Δ]} and thus AVAP($\mathfrak{H}$) $\cup$ {[β, ξ, Δ]} $\vDash$ C($\mathfrak{H}$). Also, we have AVAP($\mathfrak{H}\upharpoonright$max(Dom(AVAS($\mathfrak{H}\upharpoonright$Dom($\mathfrak{H}$)-1)))) $\vDash$ $\ulcorner\bigvee\xi\Delta\urcorner$.

It holds that $\mathrm{AVAP}(\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)))) \subseteq \mathrm{AVAP}(\mathfrak{H})$. According to Theorem 3-21-(iii), we first have $(\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)))\text{-}1, \ulcorner\bigvee\xi\Delta\urcorner) \in \mathrm{AVS}(\mathfrak{H})$ because $(\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)))\text{-}1, \mathfrak{H}_{\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)))\text{-}1}) \in \mathrm{AVS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ and $\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)))\text{-}1 < \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)))$. Therefore $\ulcorner\bigvee\xi\Delta\urcorner$ is available in $\mathfrak{H}$ at $\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)))\text{-}1$. With Theorem 3-29-(ii), it then follows that $\mathrm{AVAP}(\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)))) \subseteq \mathrm{AVAP}(\mathfrak{H})$. With Theorem 5-13, we then have $\mathrm{AVAP}(\mathfrak{H}) \vDash \ulcorner\bigvee\xi\Delta\urcorner$.

We already have $\beta \notin \mathrm{STSF}(\{\Delta, \mathrm{C}(\mathfrak{H})\})$. Since there is no $j \leq \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)))\text{-}1$ such that $\beta \in \mathrm{ST}(\mathfrak{H}_j)$, there is no $j \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright\max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)))))$ such that $\beta \in \mathrm{ST}(\mathfrak{H}_j) = \mathrm{ST}(\mathrm{P}(\mathfrak{H}_j))$ and $j \neq \max(\mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)))$. With Theorem 3-21-(iv) und -(v), we therefore have that there is no $j \in \mathrm{Dom}(\mathrm{AVAS}(\mathfrak{H}))$ such that $\beta \in \mathrm{ST}(\mathrm{P}(\mathfrak{H}_j))$. Thus we have $\beta \notin \mathrm{STSF}(\mathrm{AVAP}(\mathfrak{H}))$ and thus $\beta \notin \mathrm{STSF}(\mathrm{AVAP}(\mathfrak{H}) \cup \{\Delta, \mathrm{C}(\mathfrak{H})\})$ and finally $\beta \notin \mathrm{STSF}((\mathrm{AVAP}(\mathfrak{H})\setminus\{[\beta, \xi, \Delta]\}) \cup \{\Delta, \mathrm{C}(\mathfrak{H})\})$. According to Theorem 5-30 (where X is instantiated by $\mathrm{AVAP}(\mathfrak{H})$ and Y is instantiated by $\mathrm{AVAP}(\mathfrak{H}) \cup \{[\beta, \xi, \Delta]\}$), we hence have $\mathrm{AVAP}(\mathfrak{H}) \vDash \mathrm{C}(\mathfrak{H})$.

Second case: Now, suppose $\mathfrak{H} \notin \mathrm{AF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) \cup \mathrm{CdIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) \cup \mathrm{NIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) \cup \mathrm{PEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. According to Theorem 3-28, we then have $\mathrm{AVAP}(\mathfrak{H}) = \mathrm{AVAP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. We can distinguish 13 subcases.

(*CdEF*, *CIF*, *BIF*, *BEF*, *IEF*): Suppose $\mathfrak{H} \in \mathrm{CdEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. According to Definition 3-3, there is then $\Delta \in \mathrm{CFORM}$ such that $\Delta, \ulcorner\Delta \rightarrow \mathrm{C}(\mathfrak{H})\urcorner \in \mathrm{AVP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. Because of $\Delta, \ulcorner\Delta \rightarrow \mathrm{C}(\mathfrak{H})\urcorner \in \mathrm{AVP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ there are $j, l \in \mathrm{Dom}(\mathfrak{H})\text{-}1$ such that Δ is available in $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1$ at j and $\ulcorner\Delta \rightarrow \mathrm{C}(\mathfrak{H})\urcorner$ is available in $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1$ at l. Then we have $\mathrm{C}(\mathfrak{H}\upharpoonright j\text{+}1) = \Delta$ and $\mathrm{C}(\mathfrak{H}\upharpoonright l\text{+}1) = \ulcorner\Delta \rightarrow \mathrm{C}(\mathfrak{H})\urcorner$. Then we have $\mathrm{AVAP}(\mathfrak{H}\upharpoonright j\text{+}1) \vDash \Delta$ and $\mathrm{AVAP}(\mathfrak{H}\upharpoonright l\text{+}1) \vDash \ulcorner\Delta \rightarrow \mathrm{C}(\mathfrak{H})\urcorner$. With Theorem 3-29-(iv), it then follows that $\mathrm{AVAP}(\mathfrak{H}\upharpoonright j\text{+}1) \subseteq \mathrm{AVAP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ and $\mathrm{AVAP}(\mathfrak{H}\upharpoonright l\text{+}1) \subseteq \mathrm{AVAP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. Since $\mathrm{AVAP}(\mathfrak{H}) = \mathrm{AVAP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$, we thus have $\mathrm{AVAP}(\mathfrak{H}\upharpoonright j\text{+}1) \subseteq \mathrm{AVAP}(\mathfrak{H})$ and $\mathrm{AVAP}(\mathfrak{H}\upharpoonright l\text{+}1) \subseteq \mathrm{AVAP}(\mathfrak{H})$ and thus, with Theorem 5-13, also $\mathrm{AVAP}(\mathfrak{H}) \vDash \Delta$ and $\mathrm{AVAP}(\mathfrak{H}) \vDash \ulcorner\Delta \rightarrow \mathrm{C}(\mathfrak{H})\urcorner$. Theorem 5-16 then yields $\mathrm{AVAP}(\mathfrak{H}) \vDash \mathrm{C}(\mathfrak{H})$. Similarly one shows for CIF with Theorem 5-17, for BIF with Theorem 5-19, for BEF with Theorem 5-21 and for IEF with Theorem 5-32 that $\mathrm{AVAP}(\mathfrak{H}) \vDash \mathrm{C}(\mathfrak{H})$.

(*CEF*, *DIF*): Suppose $\mathfrak{H} \in \mathrm{CEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. According to Definition 3-5, there is then $\Delta \in \mathrm{CFORM}$ such that $\ulcorner\Delta \wedge \mathrm{C}(\mathfrak{H})\urcorner \in \mathrm{AVP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\ulcorner\mathrm{C}(\mathfrak{H}) \wedge \Delta\urcorner \in \mathrm{AVP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. Because of $\ulcorner\Delta \wedge \mathrm{C}(\mathfrak{H})\urcorner \in \mathrm{AVP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ or $\ulcorner\mathrm{C}(\mathfrak{H}) \wedge \Delta\urcorner \in \mathrm{AVP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ there is $j \in \mathrm{Dom}(\mathfrak{H})\text{-}1$ such that $\ulcorner\Delta \wedge \mathrm{C}(\mathfrak{H})\urcorner$ or $\ulcorner\mathrm{C}(\mathfrak{H}) \wedge \Delta\urcorner$ is available in $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1$ at j. Then we have $\mathrm{C}(\mathfrak{H}\upharpoonright j\text{+}1) = \ulcorner\Delta \wedge \mathrm{C}(\mathfrak{H})\urcorner$ or $\mathrm{C}(\mathfrak{H}\upharpoonright j\text{+}1) = \ulcorner\mathrm{C}(\mathfrak{H}) \wedge \Delta\urcorner$. Then we have $\mathrm{AVAP}(\mathfrak{H}\upharpoonright j\text{+}1) \vDash \ulcorner\Delta \wedge \mathrm{C}(\mathfrak{H})\urcorner$ or $\mathrm{AVAP}(\mathfrak{H}\upharpoonright j\text{+}1) \vDash \ulcorner\mathrm{C}(\mathfrak{H}) \wedge \Delta\urcorner$. With Theorem 3-29-(iv), it follows that $\mathrm{AVAP}(\mathfrak{H}\upharpoonright j\text{+}1) \subseteq \mathrm{AVAP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) = \mathrm{AVAP}(\mathfrak{H})$. With Theorem 5-13, we thus have $\mathrm{AVAP}(\mathfrak{H}) \vDash \ulcorner\Delta \wedge \mathrm{C}(\mathfrak{H})\urcorner$ or $\mathrm{AVAP}(\mathfrak{H}) \vDash \ulcorner\mathrm{C}(\mathfrak{H}) \wedge \Delta\urcorner$. Theorem 5-18 yields in both cases $\mathrm{AVAP}(\mathfrak{H}) \vDash \mathrm{C}(\mathfrak{H})$. For DIF one shows similarly, with Theorem 5-22, that $\mathrm{AVAP}(\mathfrak{H}) \vDash \mathrm{C}(\mathfrak{H})$.

(*DEF*): Suppose $\mathfrak{H} \in \mathrm{DEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. According to Definition 3-9, there are then $\mathrm{B}, \Delta \in \mathrm{CFORM}$ such that $\ulcorner\mathrm{B} \vee \Delta\urcorner$, $\ulcorner\mathrm{B} \rightarrow \mathrm{C}(\mathfrak{H})\urcorner$, $\ulcorner\Delta \rightarrow \mathrm{C}(\mathfrak{H})\urcorner \in \mathrm{AVP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. Then there are $j, k, l \in \mathrm{Dom}(\mathfrak{H})\text{-}1$ such that $\ulcorner\mathrm{B} \vee \Delta\urcorner$ is available in $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1$ at j and $\ulcorner\mathrm{B} \rightarrow \mathrm{C}(\mathfrak{H})\urcorner$ is available in $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1$ at k and $\ulcorner\Delta \rightarrow \mathrm{C}(\mathfrak{H})\urcorner$ is available in $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1$ at l. Then we have $\mathrm{C}(\mathfrak{H}\upharpoonright j\text{+}1) = \ulcorner\mathrm{B} \vee \Delta\urcorner$ and $\mathrm{C}(\mathfrak{H}\upharpoonright k\text{+}1) = \ulcorner\mathrm{B} \rightarrow \mathrm{C}(\mathfrak{H})\urcorner$ and $\mathrm{C}(\mathfrak{H}\upharpoonright l\text{+}1) = \ulcorner\Delta \rightarrow \mathrm{C}(\mathfrak{H})\urcorner$. Then it holds that $\mathrm{AVAP}(\mathfrak{H}\upharpoonright j\text{+}1) \vDash \ulcorner\mathrm{B} \vee \Delta\urcorner$ and $\mathrm{AVAP}(\mathfrak{H}\upharpoonright k\text{+}1) \vDash \ulcorner\mathrm{B} \rightarrow \mathrm{C}(\mathfrak{H})\urcorner$ and $\mathrm{AVAP}(\mathfrak{H}\upharpoonright l\text{+}1) \vDash \ulcorner\Delta \rightarrow \mathrm{C}(\mathfrak{H})\urcorner$. With Theorem 3-29-(iv), it then follows that $\mathrm{AVAP}(\mathfrak{H}\upharpoonright j\text{+}1) \subseteq \mathrm{AVAP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ and $\mathrm{AVAP}(\mathfrak{H}\upharpoonright k\text{+}1) \subseteq \mathrm{AVAP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ and $\mathrm{AVAP}(\mathfrak{H}\upharpoonright l\text{+}1) \subseteq \mathrm{AVAP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ and thus $\mathrm{AVAP}(\mathfrak{H}\upharpoonright j\text{+}1) \subseteq \mathrm{AVAP}(\mathfrak{H})$ and $\mathrm{AVAP}(\mathfrak{H}\upharpoonright k\text{+}1) \subseteq \mathrm{AVAP}(\mathfrak{H})$ and $\mathrm{AVAP}(\mathfrak{H}\upharpoonright l\text{+}1) \subseteq \mathrm{AVAP}(\mathfrak{H})$. With Theorem 5-13, we thus have $\mathrm{AVAP}(\mathfrak{H}) \vDash \ulcorner\mathrm{B} \vee \Delta\urcorner$ and $\mathrm{AVAP}(\mathfrak{H}) \vDash \ulcorner\mathrm{B} \rightarrow \mathrm{C}(\mathfrak{H})\urcorner$ and $\mathrm{AVAP}(\mathfrak{H}) \vDash \ulcorner\Delta \rightarrow \mathrm{C}(\mathfrak{H})\urcorner$. Theorem 5-23 then yields $\mathrm{AVAP}(\mathfrak{H}) \vDash \mathrm{C}(\mathfrak{H})$.

(*NEF*, *UEF*, *PIF*): Suppose $\mathfrak{H} \in \mathrm{NEF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. According to Definition 3-11, we then have $\ulcorner\neg\neg\mathrm{C}(\mathfrak{H})\urcorner \in \mathrm{AVP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. Then there is $j \in \mathrm{Dom}(\mathfrak{H})\text{-}1$ such that $\ulcorner\neg\neg\mathrm{C}(\mathfrak{H})\urcorner$ is available in $\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1$ at j. Then we have $\mathrm{C}(\mathfrak{H}\upharpoonright j\text{+}1) = \ulcorner\neg\neg\mathrm{C}(\mathfrak{H})\urcorner$. Then we have $\mathrm{AVAP}(\mathfrak{H}\upharpoonright j\text{+}1) \vDash \ulcorner\neg\neg\mathrm{C}(\mathfrak{H})\urcorner$. With Theorem 3-29-(iv), it follows that $\mathrm{AVAP}(\mathfrak{H}\upharpoonright j\text{+}1) \subseteq \mathrm{AVAP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1) = \mathrm{AVAP}(\mathfrak{H})$. With Theorem 5-13, we thus have $\mathrm{AVAP}(\mathfrak{H}) \vDash \ulcorner\neg\neg\mathrm{C}(\mathfrak{H})\urcorner$. Theorem 5-26 then yields $\mathrm{AVAP}(\mathfrak{H}) \vDash \mathrm{C}(\mathfrak{H})$. Similarly, one shows for UEF with Theorem 5-28 and for PIF with Theorem 5-29 that in both cases $\mathrm{AVAP}(\mathfrak{H}) \vDash \mathrm{C}(\mathfrak{H})$.

(*UIF*): Suppose $\mathfrak{H} \in \mathrm{UIF}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$. According to Definition 3-12 there is then $\beta \in \mathrm{PAR}$, $\xi \in \mathrm{VAR}$ and $\Delta \in \mathrm{FORM}$, where $\mathrm{FV}(\Delta) \subseteq \{\xi\}$, such that $[\beta, \xi, \Delta] \in \mathrm{AVP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1)$ and $\beta \notin \mathrm{STSF}(\{\Delta\} \cup \mathrm{AVAP}(\mathfrak{H}\upharpoonright\mathrm{Dom}(\mathfrak{H})\text{-}1))$ and $\mathrm{C}(\mathfrak{H}) = \ulcorner\bigwedge\xi\Delta\urcorner$.

Then there is $j \in$ Dom(𝔖)-1 such that [β, ξ, Δ] is available in 𝔖↾Dom(𝔖)-1 at j. Then we have C(𝔖↾j+1) = [β, ξ, Δ]. Then it holds that AVAP(𝔖↾j+1) ⊨ [β, ξ, Δ]. With Theorem 3-29-(iv), it follows that AVAP(𝔖↾j+1) ⊆ AVAP(𝔖↾Dom(𝔖)-1) = AVAP(𝔖). With Theorem 5-13, we thus have AVAP(𝔖) ⊨ [β, ξ, Δ]. With AVAP(𝔖↾Dom(𝔖)-1) = AVAP(𝔖), it follows from β ∉ STSF({Δ} ∪ AVAP(𝔖↾Dom(𝔖)-1)) that β ∉ STSF({Δ} ∪ AVAP(𝔖)). Theorem 5-27 then yields AVAP(𝔖) ⊨ C(𝔖).

(*IIF*): Suppose 𝔖 ∈ IIF(𝔖↾Dom(𝔖)-1). According to Definition 3-16 there is then θ ∈ CTERM such that C(𝔖) = ⌜θ = θ⌝. Theorem 5-31 yields AVAP(𝔖) ⊨ C(𝔖). ■

Theorem 6-2. *Correctness of the Speech Act Calculus relative to the model-theory*
For all X, Γ: If X ⊢ Γ, then X ⊨ Γ.

Proof: Suppose X ⊢ Γ. According to Theorem 3-12, we then have that X ⊆ CFORM and that there is 𝔖 ∈ RCS\{∅} such that Γ = C(𝔖) and AVAP(𝔖) ⊆ X. Theorem 6-1 then yields AVAP(𝔖) ⊨ Γ. With Theorem 5-13 and AVAP(𝔖) ⊆ X, it follows that X ⊨ Γ. ■

6.2 Completeness of the Speech Act Calculus

In the following we will prove the completeness of the Speech Act Calculus relative to the model-theoretic consequence relation for L defined in Definition 5-10. To do this, we will show that consistent sets are satisfiable. Since CFORM, the set of closed L-formulas, is denumerably infinite, it suffices to show this for denumerably infinite sets. For this, we choose the method of constructing Hintikka sets and showing that Hintikka sets are satisfied by the respective canonical term structure.[15] For this purpose, L has to be expanded to the language L_H, which results from L by adding denumerably infinitely many new individual constants to the vocabulary of L:

Definition 6-1. *The vocabulary of* L_H *(CONSTEXP, PAR, VAR, FUNC, PRED, CON, QUANT, PERF, AUX)*

The vocabulary of L_H contains the following pairwise disjunct sets: the denumerably infinite set CONSTEXP = CONST $\cup$ CONSTNEW, where CONSTNEW = $\{c^*_i \mid i \in \mathbb{N}\}$ (and for all i, $j \in \mathbb{N}$ with $i \neq j$: $c^*_i \neq c^*_j$ and $c^*_i \in \{c^*_i\}$ and CONST $\cap$ CONSTNEW = $\emptyset$), and PAR, VAR, FUNC, PRED, CON, QUANT, PERF, AUX.

Note: In the remainder of this section we adopt the following notation: For all expressions P that are defined by definition D let P_H be the expression defined for L_H instead of L and let D_H be the corresponding definition and for all theorems T let T_H be the corresponding theorem for L_H. As for the relationship of P and P_H, it holds that suitable restrictions of P_H and $P_H(a)$ to L lead back to P and $P(a)$, respectively. For example, we have: (i) PEXP = $\mathrm{PEXP}_H \cap$ PEXP, TERM = $\mathrm{TERM}_H \cap$ PEXP, FORM = $\mathrm{FORM}_H \cap$ PEXP, SENT = $\mathrm{SENT}_H \cap$ PEXP, SEQ = $\mathrm{SEQ}_H \cap$ SEQ, RCS = $\mathrm{RCS}_H \cap$ SEQ. (ii) ST = $\mathrm{ST}_H\restriction$PEXP, STSEQ = $\mathrm{STSEQ}_H\restriction$SEQ, STSF = $\mathrm{STSF}_H\restriction$Pot(FORM), P = $\mathrm{P}_H\restriction$SENT, C = $\mathrm{C}_H\restriction$SEQ, AVAP = $\mathrm{AVAP}_H\restriction$SEQ. (iii) If $\mathfrak{H} \in$ SEQ, then RCE($\mathfrak{H}$) = $\mathrm{RCE}_H(\mathfrak{H}) \cap$ SEQ. Many of these relationships can be shown without much technical difficulties but require quite some tedious writing. Therefore, we will not reproduce the proofs here. Where the relationships are not immediately obvious or where there are particular complications in a proof, we will execute the proofs. For example, we will show that RCS $\subseteq$ RCS_H in

[15] See, for example, GRÄDEL, E.: *Mathematische Logik*, p. 109–119, WAGNER, H.: *Logische Systeme*, p. 97–101, and KLEINKNECHT, R.: *Grundlagen der modernen Definitionstheorie*, p. 154–157.

Theorem 6-6. In Theorem 6-3-(i), we will show that models_H can be transformed into models by restricting the respective interpretation function_H on PEXP (or, more precisely: CONST ∪ FUNC ∪ PRED). For the substitution operation, the equivalence for L-arguments is trivial. To avoid a clutter of indices behind square brackets (cf. the proof of Theorem 6-10), we will therefore suppress the H-index for the substitution operator.

The following theorems first secure the connection between satisfiability in L and L_H (Theorem 6-3 to Theorem 6-5) and between consistency in L and L_H (Theorem 6-6 to Theorem 6-8). Then we will define Hintikka sets (Definition 6-2). Subsequently, we will show that all consistent sets of L-propositions have a Hintikka superset (Theorem 6-9) and that all Hintikka sets are satisfiable_H (Theorem 6-10). From this, we will then derive the completeness of the Speech Act Calculus (Theorem 6-11).

Theorem 6-3. *Restrictions of L_H-models on L are L-models*

(i) If (D, I) is a model_H, then $(D, I\restriction(\text{CONST} \cup \text{FUNC} \cup \text{PRED}))$ is a model,

(ii) b is a parameter assignment_H for D iff b is a parameter assignment for D, and

(iii) b' is in β an assignment variant_H of b for D iff b' is in β an assignment variant of b for D.

Proof: *Ad (i)*: Suppose (D, I) is a model_H. According to Definition 5-2_H, I is then an interpretation function_H for D. According to Definition 5-1_H, we then have $\text{Dom}(I)$ = CONSTEXP ∪ FUNC ∪ PRED. With CONST ⊆ CONSTEXP, we then have $\text{Dom}(I\restriction(\text{CONST} \cup \text{FUNC} \cup \text{PRED})) = \text{CONST} \cup \text{FUNC} \cup \text{PRED}$ and for all $\mu \in \text{CONST} \cup \text{FUNC} \cup \text{PRED}$ it holds that $I\restriction(\text{CONST} \cup \text{FUNC} \cup \text{PRED})(\mu) = I(\mu)$. Thus it follows, with Definition 5-1_H and Definition 5-1, that $I\restriction(\text{CONST} \cup \text{FUNC} \cup \text{PRED})$ is an interpretation function for D and thus that $(D, I\restriction(\text{CONST} \cup \text{FUNC} \cup \text{PRED}))$ is a model.

Ad (ii): With Definition 5-3_H and Definition 5-3 it holds that

b is a parameter assignment_H for D
iff
b is a function with $\text{Dom}(b)$ = PAR such that for all $\beta \in \text{PAR}$: $b(\beta) \in D$
iff
b is a parameter assignment for D.

Ad (iii): With Definition 5-4_H, (ii) and Definition 5-4 it holds that

b' is in β an assignment variant$_{H}$ of b for D
iff
b' and b are parameter assignments$_{H}$ for D and $\beta \in$ PAR and $b' \setminus \{(\beta, b'(\beta))\} \subseteq b$
iff
b' and b are parameter assignments for D and $\beta \in$ PAR and $b' \setminus \{(\beta, b'(\beta))\} \subseteq b$
iff
b' is in β an assignment variant of b for D.

■

Theorem 6-4. *L_H-models and their L-restrictions behave in the same way with regard to L-entities*

If (D, I) is a model$_{H}$ and b is a parameter assignment$_{H}$ for D, then for all $\theta \in$ CTERM, $\Gamma \in$ CFORM and $X \subseteq$ CFORM:

(i) TD$_{H}$(θ, D, I, b) = TD(θ, $D, I\upharpoonright$(CONST ∪ FUNC ∪ PRED), b),

(ii) $D, I, b \vDash_{H} \Gamma$ iff $D, I\upharpoonright$(CONST ∪ FUNC ∪ PRED), $b \vDash \Gamma$, and

(iii) $D, I, b \vDash_{H} X$ iff $D, I\upharpoonright$(CONST ∪ FUNC ∪ PRED), $b \vDash X$.

Proof: The proof for (i) and (ii) is analogous to the proof of the coincidence lemma (Theorem 5-5) by induction on the complexity of terms and formulas. Additionally, one has to use Theorem 6-3. (iii) then follows from (ii) and Definition 5-9$_{H}$ and Definition 5-9. ■

Theorem 6-5. *A set of L-propositions is L_H-satisfiable if and only if it is L-satisfiable*

If $X \subseteq$ CFORM, then: X is satisfiable$_{H}$ iff X is satisfiable.

Proof: Suppose $X \subseteq$ CFORM. Now, suppose X is satisfiable$_{H}$. According to Definition 5-17$_{H}$, there are then D, I, b such that $D, I, b \vDash_{H} X$. With Theorem 6-4, it then follows that $D, I\upharpoonright$(CONST ∪ FUNC ∪ PRED), $b \vDash X$ and thus we have that X is satisfiable. Now, suppose X is satisfiable. Then there is D^-, I^-, b^- such that $D^-, I^-, b^- \vDash X$. We have that there is an $a \in D$. Now, let $I^+ = I^- \cup$ (CONSTNEW × $\{a\}$). Then (D, I^+) is a model$_{H}$ and b^- is a parameter assignment$_{H}$ and $I^+\upharpoonright$(CONST ∪ FUNC ∪ PRED) = I^-. With Theorem 6-4, it then follows that $D^-, I^+, b^- \vDash_{H} X$ and hence that X is satisfiable$_{H}$. ■

Theorem 6-6. *L-sequences are RCS$_H$-elements if and only if they are RCS-elements*
If $\mathfrak{H} \in$ SEQ, then: $\mathfrak{H} \in \mathrm{RCS_H}$ iff $\mathfrak{H} \in$ RCS.

Proof: The proof is to be carried out by induction on Dom($\mathfrak{H}$). The induction basis is given with $\emptyset \in \mathrm{RCS_H} \cap \mathrm{RCS}$ and one easily shows for $\mathfrak{H} \in$ SEQ with $0 <$ Dom($\mathfrak{H}$) that if the statement holds for $\mathfrak{H}\restriction$Dom($\mathfrak{H}$)-1, it also holds for $\mathfrak{H}$. ■

Theorem 6-7. *An L-proposition is L$_H$-derivable from a set of L-propositions if and only if it is L-derivable from that set*
If $X \cup \{\Gamma\} \subseteq$ CFORM, then: $X \vdash_{\mathrm{H}} \Gamma$ iff $X \vdash \Gamma$.

Proof: Suppose $X \cup \{\Gamma\} \subseteq$ CFORM. Then the right-left-direction follows directly with Theorem 3-12, Theorem 6-6 and Theorem 3-12$_{\mathrm{H}}$. Now, for the left-right-direction, suppose $X \vdash_{\mathrm{H}} \Gamma$. According to Theorem 3-12$_{\mathrm{H}}$, there is then an $\mathfrak{H} \in \mathrm{RCS_H}\setminus\{\emptyset\}$ such that $\mathrm{AVAP_H}(\mathfrak{H}) \subseteq X$ and $\mathrm{K_H}(\mathfrak{H}) = \Gamma$. Now we can show by induction on $|\mathrm{CONSTNEW} \cap \mathrm{STSEQ_H}(\mathfrak{H})| \in \mathbb{N}$ that there is an $\mathfrak{H}^* \in \mathrm{SEQ} \cap (\mathrm{RCS_H}\setminus\{\emptyset\})$ with $\mathrm{AVAP_H}(\mathfrak{H}^*) = \mathrm{AVAP_H}(\mathfrak{H})$ and $\mathrm{C_H}(\mathfrak{H}^*) = \mathrm{C_H}(\mathfrak{H})$. With Theorem 6-6, we then have for such $\mathfrak{H}^*$ that $\mathfrak{H}^* \in \mathrm{RCS}\setminus\{\emptyset\}$, $\mathrm{AVAP}(\mathfrak{H}^*) = \mathrm{AVAP_H}(\mathfrak{H}^*) = \mathrm{AVAP_H}(\mathfrak{H}) \subseteq X$ and $\mathrm{C}(\mathfrak{H}^*) = \mathrm{C_H}(\mathfrak{H}^*) = \mathrm{C_H}(\mathfrak{H}) = \Gamma$. From this, we then get $X \vdash \Gamma$.

Suppose $|\mathrm{CONSTNEW} \cap \mathrm{STSEQ_H}(\mathfrak{H})| = k$ and suppose the statement holds for all $\mathfrak{H}^*$ with $|\mathrm{CONSTNEW} \cap \mathrm{STSEQ_H}(\mathfrak{H}^*)| < k$. Suppose $k = 0$. Then $\mathfrak{H}$ itself is the desired $\mathfrak{H}^* \in \mathrm{SEQ} \cap (\mathrm{RCS_H}\setminus\{\emptyset\})$ with $\mathrm{AVAP_H}(\mathfrak{H}^*) = \mathrm{AVAP_H}(\mathfrak{H})$ and $\mathrm{C_H}(\mathfrak{H}^*) = \mathrm{C_H}(\mathfrak{H})$. Now, suppose $0 < k$. Let α be the individual constant with the greatest index in $\mathrm{CONSTNEW} \cap \mathrm{STSEQ_H}(\mathfrak{H})$. There is a $\beta \in \mathrm{PAR}\setminus\mathrm{STSEQ_H}(\mathfrak{H})$. According to Theorem 4-9$_{\mathrm{H}}$, there is then an $\mathfrak{H}^* \in \mathrm{RCS_H}\setminus\{\emptyset\}$ with $\alpha \notin \mathrm{STSEQ_H}(\mathfrak{H}^*)$, $\mathrm{STSEQ_H}(\mathfrak{H}^*)\setminus\{\beta\} \subseteq \mathrm{STSEQ_H}(\mathfrak{H})$, $\mathrm{AVAP_H}(\mathfrak{H}) = \{[\alpha, \beta, \mathrm{B}] \mid \mathrm{B} \in \mathrm{AVAP_H}(\mathfrak{H}^*)\}$ and $\mathrm{K_H}(\mathfrak{H}) = [\alpha, \beta, \mathrm{K_H}(\mathfrak{H}^*)]$. Since $\mathrm{AVAP_H}(\mathfrak{H}) \subseteq X$, it holds that $\alpha \notin \mathrm{STSF_H}(\mathrm{AVAP_H}(\mathfrak{H}))$. Therefore we have $\beta \notin \mathrm{STSF_H}(\mathrm{AVAP_H}(\mathfrak{H}^*))$ and thus $[\alpha, \beta, \mathrm{B}] = \mathrm{B}$ for all $\mathrm{B} \in \mathrm{AVAP_H}(\mathfrak{H}^*)$. Therefore we have $\mathrm{AVAP_H}(\mathfrak{H}) = \mathrm{AVAP_H}(\mathfrak{H}^*)$. Since $\mathrm{C_H}(\mathfrak{H}) = \Gamma \in \mathrm{CFORM}$, we also have $\alpha \notin \mathrm{ST_H}(\mathrm{C_H}(\mathfrak{H}))$. Therefore we have $\beta \notin \mathrm{ST_H}(\mathrm{C_H}(\mathfrak{H}^*))$ and thus $\mathrm{C_H}(\mathfrak{H}) = [\alpha, \beta, \mathrm{C_H}(\mathfrak{H}^*)] = \mathrm{C_H}(\mathfrak{H}^*)$. Therefore we have $\mathrm{C_H}(\mathfrak{H}) = \mathrm{C_H}(\mathfrak{H}^*)$. From $\alpha \notin \mathrm{STSEQ_H}(\mathfrak{H}^*)$ and $\mathrm{STSEQ_H}(\mathfrak{H}^*)\setminus\{\beta\} \subseteq \mathrm{STSEQ_H}(\mathfrak{H})$, it follows that $|\mathrm{CONSTNEW} \cap \mathrm{STSEQ_H}(\mathfrak{H}^*)| < |\mathrm{CONSTNEW} \cap$

$STSEQ_H(\mathfrak{H}^*)|$. According to the I.H., there is then an $\mathfrak{H}'$ such that $AVAP_H(\mathfrak{H}') = AVAP_H(\mathfrak{H}^*) = AVAP_H(\mathfrak{H})$ and $C_H(\mathfrak{H}') = C_H(\mathfrak{H}^*) = C_H(\mathfrak{H})$ and $\mathfrak{H}' \in SEQ \cap RCS_H \backslash \{\emptyset\}$. ■

Theorem 6-8. *A set of L-propositions is L_H-consistent if and only if it is L-consistent*
If $X \subseteq$ CFORM, then: X is consistent$_H$ iff X is consistent.

Proof: Suppose $X \subseteq$ CFORM and suppose X is not consistent$_H$. With Theorem 4-23$_H$, it then holds for all $\Delta \in CFORM_H$ that $X \vdash_H \Delta$. Then we have $X \vdash_H \ulcorner c_0 = c_0 \urcorner$ and $X \vdash_H \ulcorner \neg(c_0 = c_0)\urcorner$. It holds that $\ulcorner c_0 = c_0 \urcorner, \ulcorner \neg(c_0 = c_0)\urcorner \in$ CFORM and thus it follows with Theorem 6-7 that $X \vdash \ulcorner c_0 = c_0 \urcorner$ and $X \vdash \ulcorner \neg(c_0 = c_0)\urcorner$. Hence X is not consistent. Now, suppose X is not consistent. Then there is $A \in CFORM \subseteq CFORM_H$ such that $X \vdash A$ and $X \vdash \ulcorner \neg A \urcorner$. With Theorem 6-7 we then also have $X \vdash_H A$ and $X \vdash_H \ulcorner \neg A \urcorner$ and thus that X is not consistent$_H$. ■

Definition 6-2. *Hintikka set*
X is a Hintikka set
iff
$X \subseteq CFORM_H$ and:

(i) If $A \in AFORM_H \cap X$, then $\ulcorner \neg A \urcorner \notin X$,
(ii) If $A \in CFORM_H$ and $\ulcorner \neg\neg A \urcorner \in X$, then $A \in X$,
(iii) If $A, B \in CFORM_H$ and $\ulcorner A \wedge B \urcorner \in X$, then $\{A, B\} \subseteq X$,
(iv) If $A, B \in CFORM_H$ and $\ulcorner \neg(A \wedge B)\urcorner \in X$, then $\{\ulcorner \neg A \urcorner, \ulcorner \neg B \urcorner\} \cap X \neq \emptyset$,
(v) If $A, B \in CFORM_H$ and $\ulcorner A \vee B \urcorner \in X$, then $\{A, B\} \cap X \neq \emptyset$,
(vi) If $A, B \in CFORM_H$ and $\ulcorner \neg(A \vee B)\urcorner \in X$, then $\{\ulcorner \neg A \urcorner, \ulcorner \neg B \urcorner\} \subseteq X$,
(vii) If $A, B \in CFORM_H$ and $\ulcorner A \rightarrow B \urcorner \in X$, then $\{\ulcorner \neg A \urcorner, B\} \cap X \neq \emptyset$,
(viii) If $A, B \in CFORM_H$ and $\ulcorner \neg(A \rightarrow B)\urcorner \in X$, then $\{A, \ulcorner \neg B \urcorner\} \subseteq X$,
(ix) If $A, B \in CFORM_H$ and $\ulcorner A \leftrightarrow B \urcorner \in X$, then $\{A, B\} \subseteq X$ or $\{\ulcorner \neg A \urcorner, \ulcorner \neg B \urcorner\} \subseteq X$,
(x) If $A, B \in CFORM_H$ and $\ulcorner \neg(A \leftrightarrow B)\urcorner \in X$, then $\{A, \ulcorner \neg B \urcorner\} \subseteq X$ or $\{\ulcorner \neg A \urcorner, B\} \subseteq X$,
(xi) If $\xi \in VAR$, $\Delta \in FORM_H$, where $FV_H(\Delta) \subseteq \{\xi\}$, and $\ulcorner \bigwedge \xi \Delta \urcorner \in X$, then it holds for all $\theta \in CTERM_H$ that $[\theta, \xi, \Delta] \in X$,
(xii) If $\xi \in VAR$, $\Delta \in FORM_H$, where $FV_H(\Delta) \subseteq \{\xi\}$, and $\ulcorner \neg \bigwedge \xi \Delta \urcorner \in X$, then there is a $\theta \in CTERM_H$ such that $\ulcorner \neg[\theta, \xi, \Delta]\urcorner \in X$.
(xiii) If $\xi \in VAR$, $\Delta \in FORM_H$, where $FV_H(\Delta) \subseteq \{\xi\}$, and $\ulcorner \bigvee \xi \Delta \urcorner \in X$, then there is a $\theta \in CTERM_H$ such that $[\theta, \xi, \Delta] \in X$,
(xiv) If $\xi \in VAR$, $\Delta \in FORM_H$, where $FV_H(\Delta) \subseteq \{\xi\}$, and $\ulcorner \neg \bigvee \xi \Delta \urcorner \in X$, then it holds for all $\theta \in CTERM_H$ that $\ulcorner \neg[\theta, \xi, \Delta]\urcorner \in X$,

(xv) If $\theta \in CTERM_H$, then $\ulcorner\theta = \theta\urcorner \in X$,

(xvi) If $\theta_0, \ldots, \theta_{r-1} \in CTERM_H$, $\theta'_0, \ldots, \theta'_{r-1} \in CTERM_H$, for all $i < r$: $\ulcorner\theta_i = \theta'_i\urcorner \in X$ and $\varphi \in$ FUNC, φ r-ary, then $\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1}) = \varphi(\theta'_0, \ldots, \theta'_{r-1})\urcorner \in X$, and

(xvii) If $\theta_0, \ldots, \theta_{r-1} \in CTERM_H$, $\theta'_0, \ldots, \theta'_{r-1} \in CTERM_H$, for all $i < r$: $\ulcorner\theta_i = \theta'_i\urcorner \in X$ and $\Phi \in$ PRED, Φ r-ary, and $\ulcorner\Phi(\theta_0, \ldots, \theta_{r-1})\urcorner \in X$, then $\ulcorner\Phi(\theta'_0, \ldots, \theta'_{r-1})\urcorner \in X$.

Theorem 6-9. *Hintikka-supersets for consistent sets of L-propositions*
If $X \subseteq$ CFORM and X is consistent, then there is a $Y \subseteq CFORM_H$ such that
(i) Y is a Hintikka set, and
(ii) $X \subseteq Y$.

Proof: Suppose $X \subseteq$ CFORM and X is consistent. Now, let g be a bijection between $\mathbb{N}$ and $CFORM_H$. Using g and the (inverse of) the CANTOR pairing function C, we will now define an enumeration of the $\Gamma \in CFORM_H$ in which each proposition occurs denumerably infinitely many times as value.[16] For this, let $F = \{(k, \Gamma) \mid$ There is $i, j \in \mathbb{N}$, $k = \frac{(i+j)\cdot(i+j+1)}{2}+j$ and $\Gamma = g(j)\}$. Then F is a function from $\mathbb{N}$ to $CFORM_H$. First, we have $Dom(F) \subseteq \mathbb{N}$. Now, suppose $k \in \mathbb{N}$. With the surjectivity of the CANTOR pairing function and $Dom(g) = \mathbb{N}$, it then holds that there are $i, j \in \mathbb{N}$ and $\Gamma \in CFORM_H$ such that $k = \frac{(i+j)\cdot(i+j+1)}{2}+j$ and $\Gamma = g(j)$. Therefore we have also $\mathbb{N} \subseteq Dom(F)$ and hence $Dom(F) = \mathbb{N}$. According to the definitions of F and g, we have $Ran(F) \subseteq CFORM_H$. Now, suppose $(k, \Gamma), (k, \Gamma^*) \in F$. Then there are i, j and i', j' so that $\frac{(i+j)\cdot(i+j+1)}{2}+j = k = \frac{(i'+j')\cdot(i'+j'+1)}{2}+j'$ and $\Gamma = g(j)$ and $\Gamma^* = g(j')$. Because of the injectivity of the CANTOR pairing function, we then have $i = i'$ and $j = j'$ and thus $\Gamma = g(j) = g(j') = \Gamma^*$. Also, we have for all $l \in \mathbb{N}$ and all $\Gamma \in CFORM_H$: There is a $k > l$ such that $F(k) = \Gamma$. To see this, suppose $l \in \mathbb{N}$ and $\Gamma \in CFORM_H$. Then there is an $s \in \mathbb{N}$ such that $\Gamma = g(s)$. Then we have $l \leq \frac{(l+s)\cdot(l+s+1)}{2}+s < \frac{(l+1+s)\cdot(l+1+s+1)}{2}+s$ and $F(\frac{(l+1+s)\cdot(l+1+s+1)}{2}+s) = g(s) = \Gamma$.

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[16] For the CANTOR pairing function $C: \mathbb{N} \times \mathbb{N} \overset{bij}{\rightarrow} \mathbb{N}$ with $C(i, j) = (i+j)\cdot(i+j+1)/2+j$ see, for example, DEISER, O.: *Mengenlehre*, p. 112–113.

Using F, we will now define a function G on $\mathbb{N}$, with which we will generate the desired Hintikka-superset for X. For this, let $G(0) = X$. For all $k \in \mathbb{N}$ let $G(k+1)$ be as follows: If $F(k) \in G(k)$, then:

(i*) If $F(k) = \ulcorner\Phi(\theta_0, \ldots, \theta_{r-1})\urcorner$, then $G(k+1) = G(k) \cup \{\ulcorner\Phi(\theta'_0, \ldots, \theta'_{r-1})\urcorner \mid$ For all $i < r$: $\ulcorner\theta_i = \theta'_i\urcorner \in G(k)\} \cup \{\ulcorner\varphi(\theta^*_0, \ldots, \theta^*_{s-1}) = \varphi(\theta^+_0, \ldots, \theta^+_{s-1})\urcorner \mid \ulcorner\varphi(\theta^*_0, \ldots, \theta^*_{s-1})\urcorner = \theta_0$ and for all $i < s$: $\ulcorner\theta^*_i = \theta^+_i\urcorner \in G(k)\}$,

(ii*) If $F(k) = \ulcorner\neg\Phi(\theta_0, \ldots, \theta_{r-1})\urcorner$, then $G(k+1) = G(k)$,

(iii*) If $F(k) = \ulcorner\neg\neg A\urcorner$, then $G(k+1) = G(k) \cup \{A\}$,

(iv*) If $F(k) = \ulcorner A \wedge B\urcorner$, then $G(k+1) = G(k) \cup \{A, B\}$,

(v*) If $F(k) = \ulcorner\neg(A \wedge B)\urcorner$, then $G(k+1) = G(k) \cup \{\ulcorner\neg A\urcorner\}$, if $G(k) \cup \{\ulcorner\neg A\urcorner\}$ is consistent$_H$, $G(k+1) = G(k) \cup \{\ulcorner\neg B\urcorner\}$ otherwise,

(vi*) If $F(k) = \ulcorner A \vee B\urcorner$, then $G(k+1) = G(k) \cup \{A\}$, if $G(k) \cup \{A\}$ is consistent$_H$, $G(k+1) = G(k) \cup \{B\}$ otherwise,

(vii*) If $F(k) = \ulcorner\neg(A \vee B)\urcorner$, then $G(k+1) = G(k) \cup \{\ulcorner\neg A\urcorner, \ulcorner\neg B\urcorner\}$,

(viii*) If $F(k) = \ulcorner A \rightarrow B\urcorner$, then $G(k+1) = G(k) \cup \{\ulcorner\neg A\urcorner\}$, if $G(k) \cup \{\ulcorner\neg A\urcorner\}$ is consistent$_H$, $G(k+1) = G(k) \cup \{B\}$ otherwise,

(ix*) If $F(k) = \ulcorner\neg(A \rightarrow B)\urcorner$, then $G(k+1) = G(k) \cup \{A, \ulcorner\neg B\urcorner\}$,

(x*) If $F(k) = \ulcorner A \leftrightarrow B\urcorner$, then $G(k+1) = G(k) \cup \{A, B\}$, if $G(k) \cup \{A, B\}$ is consistent$_H$, $G(k+1) = G(k) \cup \{\ulcorner\neg A\urcorner \; \ulcorner\neg B\urcorner\}$ otherwise,

(xi*) If $F(k) = \ulcorner\neg(A \leftrightarrow B)\urcorner$, then $G(k+1) = G(k) \cup \{A, \ulcorner\neg B\urcorner\}$, if $G(k) \cup \{A, \ulcorner\neg B\urcorner\}$ is consistent$_H$, $G(k+1) = G(k) \cup \{\ulcorner\neg A\urcorner, B\}$ otherwise,

(xii*) If $F(k) = \ulcorner\bigwedge\xi\Delta\urcorner$, then $G(k+1) = G(k) \cup \{[\theta, \xi, \Delta] \mid \theta \in STSF_H(G(k)) \cap CTERM_H\}$,

(xiii*) If $F(k) = \ulcorner\neg\bigwedge\xi\Delta\urcorner$, then $G(k+1) = G(k) \cup \{\ulcorner\neg[\alpha, \xi, \Delta]\urcorner\}$ for the $\alpha \in$ CONSTNEW with the smallest index for which it holds that $\alpha \notin STSF_H(G(k))$,

(xiv*) If $F(k) = \ulcorner\bigvee\xi\Delta\urcorner$, then $G(k+1) = G(k) \cup \{[\alpha, \xi, \Delta]\}$ for the $\alpha \in$ CONSTNEW with the smallest index for which it holds that $\alpha \notin STSF_H(G(k))$,

(xv*) If $F(k) = \ulcorner\neg\bigvee\xi\Delta\urcorner$, then $G(k+1) = G(k) \cup \{\ulcorner\neg[\theta, \xi, \Delta]\urcorner \mid \theta \in STSF_H(G(k)) \cap CTERM_H\}$.

If $F(k) \notin G(k)$, then: If $F(k) = \ulcorner\theta = \theta\urcorner$ for a $\theta \in$ CTERM$_H$, then $G(k+1) = G(k) \cup \{\ulcorner\theta = \theta\urcorner\}$, $G(k+1) = G(k)$ otherwise.

Note that G is well-defined, because no $\alpha \in$ CONSTNEW is a subterm of a $\Gamma \in X \subseteq$ CFORM and because for every $k \in \mathbb{N}$ at most one element of CONSTNEW can be added to the subterms of elements of $G(k)$ in the step from $G(k)$ to $G(k+1)$: For all $k \in \mathbb{N}$ it holds that CONSTNEW\STSF$_H(G(k))$ is denumerably infinite.

According to the construction of G it now holds that

a) $X = G(0) \subseteq \bigcup\text{Ran}(G)$,

b) For all $k \in \mathbb{N}$: $G(k)$ is consistent$_H$,

c) If $l \leq k$, then $G(l) \subseteq G(k)$,

d) If $Y \subseteq \bigcup\text{Ran}(G)$ and $|Y| \in \mathbb{N}$, then there is a $k \in \mathbb{N}$ such that $Y \subseteq G(k)$,

e) $\bigcup\text{Ran}(G)$ is consistent$_H$.

a) follows directly from the definition of G. *Now ad b)*: By hypothesis, $G(0) = X \subseteq$ CFORM is consistent and thus, with Theorem 6-8, also consistent$_H$. Now, suppose for k it holds that $G(k)$ is consistent$_H$. Suppose for contradiction that $G(k+1)$ is inconsistent$_H$. Then we have not for all $\Gamma \in G(k+1)$ that $G(k) \vdash \Gamma$, because otherwise, we would have, with Theorem 4-19$_H$ that $G(k)$ is also inconsistent$_H$. Thus it is not the case that $G(k+1) \subseteq G(k) \cup \{\ulcorner\theta = \theta\urcorner\}$ for a $\theta \in$ CTERM$_H$. Therefore we have $F(k) \in G(k)$. For this case, the cases (i*) to (iv*), (vii*), (ix*), (xii*) and (xv*) are exluded for the same reason (this is easily established with the L$_H$-versions of the theorems in ch. 4.2). Therefore we have $F(k) \in G(k)$ and $F(k) = \ulcorner\neg(A \wedge B)\urcorner$ or $F(k) = \ulcorner A \vee B\urcorner$ or $F(k) = \ulcorner A \rightarrow B\urcorner$ or $F(k) = \ulcorner A \leftrightarrow B\urcorner$ or $F(k) = \ulcorner\neg(A \leftrightarrow B)\urcorner$ or $F(k) = \ulcorner\neg\bigwedge\xi\Delta\urcorner$ or $F(k) = \ulcorner\bigvee\xi\Delta\urcorner$. Suppose $F(k) = \ulcorner\neg(A \wedge B)\urcorner$. According to (v*), we then have $G(k+1) = G(k) \cup \{\ulcorner\neg A\urcorner\}$, if $G(k) \cup \{\ulcorner\neg A\urcorner\}$ is consistent$_H$, $G(k+1) = G(k) \cup \{\ulcorner\neg B\urcorner\}$ otherwise. Then we have that $G(k) \cup \{\ulcorner\neg A\urcorner\}$ is inconsistent$_H$ and $G(k+1) = G(k) \cup \{\ulcorner\neg B\urcorner\}$ is inconsistent$_H$. With Theorem 4-22$_H$, it then holds that $G(k) \vdash_H$ A and $G(k) \vdash_H$ B and hence that $G(k) \vdash_H \ulcorner A \wedge B\urcorner$. Thus we would have that $G(k)$ is inconsistent$_H$. Contradiction! The other cases for connective formulas are shown analogously. Now, suppose $F(k) = \ulcorner\neg\bigwedge\xi\Delta\urcorner$. According to (xiii*), we then have $G(k+1) = G(k) \cup \{\ulcorner\neg[\alpha, \xi, \Delta]\urcorner\}$ for the $\alpha \in$ CONSTNEW with the smallest index for which it holds that $\alpha \notin \mathrm{STSF_H}(G(k))$. Then we would have that $G(k) \cup \{\ulcorner\neg[\alpha, \xi, \Delta]\urcorner\}$ is inconsistent$_H$. Then we would have $G(k) \vdash_H [\alpha, \xi, \Delta]$. But then we would have, because of $\alpha \notin \mathrm{STSF_H}(G(k))$ and $\ulcorner\neg\bigwedge\xi\Delta\urcorner \in G(k)$, that $\alpha \notin \mathrm{STSF_H}(G(k) \cup \{\Delta\})$ and thus, with Theorem 4-24$_H$, that $G(k) \vdash_H \ulcorner\bigwedge\xi\Delta\urcorner$. Then $G(k)$ would be inconsistent$_H$. Contradiction! The case $F(k) = \ulcorner\bigvee\xi\Delta\urcorner$ is treated analogously. Hence we have b).

By induction on k, one can easily show that c) holds by the definition of G. Thus we have also d). To see this, suppose $Y \subseteq \bigcup\mathrm{Ran}(G)$ and $|Y| \in \mathbb{N}$. Then we have for all $\Gamma \in Y$: There is an $l \in \mathbb{N}$ such that $\Gamma \in G(l)$. Now, let $k = \max(\{l \mid$ There is a $\Gamma \in Y$ such that $\Gamma \in G(l)\}$. Then it holds with c) for all $\Gamma \in Y$: $\Gamma \in G(k)$.

Thus we have also e). To see this, suppose for contradiction that $\bigcup\mathrm{Ran}(G)$ is inconsistent$_H$. Then there would be a finite inconsistent$_H$ subset Y of $\bigcup\mathrm{Ran}(G)$ and thus a $k \in \mathbb{N}$ such that $G(k)$ is inconsistent$_H$, which contradicts b).

Now, we can show that $\bigcup\mathrm{Ran}(G)$ is a Hintikka set. First we have, with e), that clause (i) of Definition 6-2 holds. Now, suppose $\ulcorner\neg\neg A\urcorner \in \bigcup\mathrm{Ran}(G)$. Then there is an $l \in \mathbb{N}$ such that $\ulcorner\neg\neg A\urcorner \in G(l)$. Then there is a $k > l$ such that $\ulcorner\neg\neg A\urcorner = F(k)$. With c), we then have $\ulcorner\neg\neg A\urcorner \in G(k)$. According to (iii*), we then have $A \in G(k+1)$ and thus $A \in \bigcup\mathrm{Ran}(G)$. Thus clause (ii) of Definition 6-2 holds. The other cases for connective formulas (clauses (iii) to (x) of Definition 6-2) and the two particular cases (clauses (xii) and (xiii) of Definition 6-2) are shown analogously.

Now, suppose $\theta \in \mathrm{CTERM_H}$. Then there is a $k \in \mathbb{N}$ such that $\ulcorner\theta = \theta\urcorner = F(k)$. Then it holds: If $\ulcorner\theta = \theta\urcorner \notin G(k)$, then $\ulcorner\theta = \theta\urcorner \in G(k+1)$ and hence in both cases: $\ulcorner\theta = \theta\urcorner \in \bigcup\mathrm{Ran}(G)$. Thus we have on the one hand, that clause (xv) of Definition 6-2 holds. On the other hand, we thus have that the two universal cases, clauses (xi) and (xiv) of Definition 6-2, hold. To see this, suppose $\ulcorner\bigwedge\xi\Delta\urcorner \in \bigcup\mathrm{Ran}(G)$. Now, suppose $\theta \in \mathrm{CTERM_H}$. Then we have (as we have just shown) $\ulcorner\theta = \theta\urcorner \in G(l)$ for an $l \in \mathbb{N}$ and we have $\ulcorner\bigwedge\xi\Delta\urcorner \in G(i)$ for an $i \in \mathbb{N}$. Then there is a $k > l, i$ such that $\ulcorner\bigwedge\xi\Delta\urcorner = F(k)$. With c), we then have $\ulcorner\bigwedge\xi\Delta\urcorner, \ulcorner\theta = \theta\urcorner \in G(k)$. According to (xii*), we then have $[\theta, \xi, \Delta] \in G(k+1)$ and thus $[\theta, \xi, \Delta] \in \bigcup\mathrm{Ran}(G)$. Thus clause (xi) of Definition 6-2 holds. Clause (xiv) is shown analogously.

Now, we still have to show the two IE-clauses, i.e. clauses (xvi) and (xvii), of Definition 6-2. *First ad (xvi)*: Suppose $\theta^*_0, \ldots, \theta^*_{s\text{-}1} \in \mathrm{CTERM_H}$, $\theta^+_0, \ldots, \theta^+_{s\text{-}1} \in \mathrm{CTERM_H}$, for all $i < s$: $\ulcorner\theta^*_i = \theta^+_i\urcorner \in \bigcup\mathrm{Ran}(G)$ and $\varphi \in \mathrm{FUNC}$, φ s-ary. As we have already shown, it holds that $\ulcorner\varphi(\theta^*_0, \ldots, \theta^*_{s\text{-}1}) = \varphi(\theta^*_0, \ldots, \theta^*_{s\text{-}1})\urcorner \in \bigcup\mathrm{Ran}(G)$. With d), there is thus an $l \in \mathbb{N}$ such that for all $i < s$: $\ulcorner\theta^*_i = \theta^+_i\urcorner \in G(l)$ and $\ulcorner\varphi(\theta^*_0, \ldots, \theta^*_{s\text{-}1}) = \varphi(\theta^*_0, \ldots, \theta^*_{s\text{-}1})\urcorner \in G(l)$. Then there is a $k > l$ such that the same holds for $G(k)$ and $F(k) = \ulcorner\varphi(\theta^*_0, \ldots, \theta^*_{s\text{-}1}) = \varphi(\theta^*_0, \ldots, \theta^*_{s\text{-}1})\urcorner$. With (i*), we then have $\ulcorner\varphi(\theta^*_0, \ldots, \theta^*_{s\text{-}1}) = \varphi(\theta^+_0, \ldots, \theta^+_{s\text{-}1})\urcorner \in G(k+1) \subseteq \bigcup\mathrm{Ran}(G)$.

Now ad (xvii): Suppose $\theta_0, \ldots, \theta_{r\text{-}1} \in \mathrm{CTERM_H}$, $\theta'_0, \ldots, \theta'_{r\text{-}1} \in \mathrm{CTERM_H}$, for all $i < r$: $\ulcorner\theta_i = \theta'_i\urcorner \in \bigcup\mathrm{Ran}(G)$ and $\Phi \in \mathrm{PRED}$, Φ r-ary, and $\ulcorner\Phi(\theta_0, \ldots, \theta_{r\text{-}1})\urcorner \in \bigcup\mathrm{Ran}(G)$. With d), there is then an $l \in \mathbb{N}$ such that for all $i < r$: $\ulcorner\theta_i = \theta'_i\urcorner \in G(l)$ and $\ulcorner\Phi(\theta_0, \ldots, \theta_{r\text{-}1})\urcorner \in G(l)$. Then there is a $k > l$ such that the same holds for $G(k)$ and $F(k) = \ulcorner\Phi(\theta_0, \ldots, \theta_{r\text{-}1})\urcorner$. With (i*), we then have $\ulcorner\Phi(\theta'_0, \ldots, \theta'_{r\text{-}1})\urcorner \in G(k+1) \subseteq \bigcup\mathrm{Ran}(G)$. ■

Theorem 6-10. *Every Hintikka set is L_H-satisfiable*
If X is a Hintikka set, then X is satisfiable$_\mathrm{H}$.

Proof: Suppose X is a Hintikka set. Now, let $A = \{(\theta, \theta') \mid (\theta, \theta') \in \mathrm{CTERM_H} \times \mathrm{CTERM_H}$ and $\ulcorner\theta = \theta'\urcorner \in X\}$.

Then it holds that A is an equivalence relation on $\mathrm{CTERM_H}$. *Concerning reflexivity*, we have, according to Definition 6-2-(xv), that $\ulcorner\theta = \theta\urcorner \in X$ and thus $(\theta, \theta) \in A$. Now *for symmetry*, suppose $(\theta, \theta') \in A$. Then we have $\ulcorner\theta = \theta'\urcorner \in X$ and, as we have just shown, $\ulcorner\theta = \theta\urcorner \in X$. Thus we have $\ulcorner\theta = \theta'\urcorner \in X$ and $\ulcorner\theta = \theta\urcorner \in X$ and thus (with θ for θ_0, θ_1, and θ'_1 and θ' for θ'_0 and $\ulcorner\theta = \theta\urcorner$ for $\ulcorner\Phi(\theta_0, \theta_1)\urcorner$ and $\ulcorner\theta' = \theta\urcorner$ for $\ulcorner\Phi(\theta'_0, \theta'_1)\urcorner$), according to Definition 6-2-(xvii), also $\ulcorner\theta' = \theta\urcorner \in X$. Therefore $(\theta, \theta') \in A$. Now *for transitivity*, suppose $(\theta, \theta') \in A$ and $(\theta', \theta^*) \in A$. Then it holds: $\ulcorner\theta = \theta'\urcorner \in X$ and $\ulcorner\theta' = \theta^*\urcorner \in X$. Also, as we have shown, it holds that $\ulcorner\theta = \theta\urcorner \in X$. Thus it holds (with θ for θ_0 and θ'_0 and θ' for θ_1 and θ^* for θ'_1 and $\ulcorner\theta = \theta'\urcorner$ for $\ulcorner\Phi(\theta_0, \theta_1)\urcorner$ and $\ulcorner\theta = \theta^*\urcorner$ for $\ulcorner\Phi(\theta'_0, \theta'_1)\urcorner$), according to Definition 6-2-(xvii), also that $\ulcorner\theta = \theta^*\urcorner \in X$ and thus that $(\theta, \theta^*) \in A$.

Now, for all $\theta \in \text{CTERM}_\text{H}$ let $[\theta]_A = \{\theta' \mid (\theta, \theta') \in A\}$. Since A is an equivalence relation on CTERM_H, it then follows that

a) For all $\theta \in \text{CTERM}_\text{H}$: $\theta \in [\theta]_A$.

b) For all $\theta, \theta' \in \text{CTERM}_\text{H}$: $[\theta]_A = [\theta']_A$ iff $(\theta, \theta') \in A$ iff $\ulcorner\theta = \theta'\urcorner \in X$.

c) For all $\theta, \theta' \in \text{CTERM}_\text{H}$: If $[\theta]_A \cap [\theta']_A \neq \emptyset$, then $[\theta]_A = [\theta']_A$.

The second equivalence in b) follows from the definition of A.

Now, let $D_X = \text{CTERM}_\text{H}/A = \{[\theta]_A \mid \theta \in \text{CTERM}_\text{H}\}$. In addition, let I_X be a function with $\text{Dom}(I_X) = \text{CONST} \cup \text{CONSTNEW} \cup \text{FUNC} \cup \text{PRED}$, where for all $\alpha \in \text{CONST} \cup \text{CONSTNEW}$: $I_X(\alpha) = [\alpha]_A$ and for all $\varphi \in \text{FUNC}$: If φ r-ary, then $I_X(\varphi) = \{(\langle[\theta_0]_A, \ldots, [\theta_{r-1}]_A\rangle, [\theta^*]_A) \mid (\langle\theta_0, \ldots, \theta_{r-1}\rangle, \theta^*) \in {}^r\text{CTERM}_\text{H} \times \text{CTERM}_\text{H}$ and $\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1}) = \theta^*\urcorner \in X\}$ and for all $\Phi \in \text{PRED}$: If Φ r-ary, then $I_X(\Phi) = \{\langle[\theta_0]_A, \ldots, [\theta_{r-1}]_A\rangle \mid \langle\theta_0, \ldots, \theta_{r-1}\rangle \in {}^r\text{CTERM}_\text{H}$ and $\ulcorner\Phi(\theta_0, \ldots, \theta_{r-1})\urcorner \in X\}$. Lastly, let b_X be a function with $\text{Dom}(b_X) = \text{PAR}$ and for all $\beta \in \text{PAR}$: $b_X(\beta) = [\beta]_A$.

According to Definition 5-1$_\text{H}$, I_X is then an interpretation function$_\text{H}$ for D_X. First, it holds for all $\alpha \in \text{CONST} \cup \text{CONSTNEW}$: $I_X(\alpha) = [\alpha]_A \in D_X$. Now, suppose $\varphi \in \text{FUNC}$, φ r-ary. Then we have $I_X(\varphi) = \{(\langle[\theta_0]_A, \ldots, [\theta_{r-1}]_A\rangle, [\theta^*]_A) \mid (\langle\theta_0, \ldots, \theta_{r-1}\rangle, \theta^*) \in {}^r\text{CTERM}_\text{H} \times \text{CTERM}_\text{H}$ and $\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1}) = \theta^*\urcorner \in X\}$. Thus we have $I_X(\varphi) \subseteq {}^rD_X \times D_X$. Now, suppose $\langle a_0, \ldots, a_{r-1}\rangle \in {}^rD_X$. Then there are $\theta_0, \ldots, \theta_{r-1} \in \text{CTERM}_\text{H}$ such that for all $i < r$: $a_i = [\theta_i]_A$. With Definition 6-2-(xv), we also have $\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1}) = \varphi(\theta_0, \ldots, \theta_{r-1})\urcorner \in X$ and thus $(\langle[\theta_0]_A, \ldots, [\theta_{r-1}]_A\rangle, [\varphi(\theta_0, \ldots, \theta_{r-1})]_A) \in I_X(\varphi)$ and therefore $\langle a_0, \ldots, a_{r-1}\rangle \in \text{Dom}(I_X(\varphi))$. Now, suppose $(\langle a_0, \ldots, a_{r-1}\rangle, a^*) \in I_X(\varphi)$ and $(\langle a_0, \ldots, a_{r-1}\rangle, a^+) \in I_X(\varphi)$. Then there are $\theta_0, \ldots, \theta_{r-1}$ and θ^* such that for all $i < r$: $a_i = [\theta_i]_A$ and $a^* = [\theta^*]_A$ and $(\langle\theta_0, \ldots, \theta_{r-1}\rangle, \theta^*) \in {}^r\text{CTERM}_\text{H} \times \text{CTERM}_\text{H}$ and $\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1}) = \theta^*\urcorner \in X$ and there are $\theta'_0, \ldots, \theta'_{r-1}$ and θ^+ such that for all $i < r$: $a_i = [\theta'_i]_A$ and $a^+ = [\theta^+]_A$ and $(\langle\theta'_0, \ldots, \theta'_{r-1}\rangle, \theta^+) \in {}^r\text{CTERM}_\text{H} \times \text{CTERM}_\text{H}$ and $\ulcorner\varphi(\theta'_0, \ldots, \theta'_{r-1}) = \theta^+\urcorner \in X$. Then we have for all $i < r$: $[\theta_i]_A = a_i = [\theta'_i]_A$. Thus it holds that for all $i < r$: $(\theta_i, \theta'_i) \in A$ and thus $\ulcorner\theta_i = \theta'_i\urcorner \in X$. According to Definition 6-2-(xvi), we then have that $\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1}) = \varphi(\theta'_0, \ldots, \theta'_{r-1})\urcorner \in X$ and thus, with b), that $[\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner]_A = [\ulcorner\varphi(\theta'_0, \ldots, \theta'_{r-1})\urcorner]_A$. With $\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1}) = \theta^*\urcorner \in X$ and $\ulcorner\varphi(\theta'_0, \ldots, \theta'_{r-1}) = \theta^+\urcorner \in X$ and b), we then also have $[\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner]_A = [\theta^*]_A$ and $[\ulcorner\varphi(\theta'_0, \ldots, \theta'_{r-1})\urcorner]_A = [\theta^+]_A$ and thus $a^* = [\theta^*]_A = [\theta^+]_A = a^+$. Altogether, we thus have that $I_X(\varphi)$ is an r-ary function over D_X. Furthermore, we have for all $\Phi \in$ PRED: If Φ is r-ary, then $I_X(\Phi) \subseteq {}^rD_X$. Lastly, we have $I_X(\ulcorner=\urcorner) = \{\langle a, a\rangle \mid a \in D_X\}$. To see this, suppose $\langle a, a'\rangle \in I_X(\ulcorner=\urcorner)$. Then there are $\theta, \theta' \in \text{CTERM}_\text{H}$ such that $a = [\theta]_A$ and $a' = [\theta']_A$ and $\ulcorner\theta = \theta'\urcorner \in X$. With b), we thus have $a = [\theta]_A = [\theta']_A = a'$. Now, suppose $a \in D_X$. Then there is a $\theta \in \text{CTERM}_\text{H}$ such that $a = [\theta]_A$. According to Definition 6-2-(xv), we have $\ulcorner\theta = \theta\urcorner \in X$ and thus $\langle a, a\rangle \in I_X(\ulcorner=\urcorner)$. According to Definition 5-2$_\text{H}$, (D_X, I_X) is hence a model$_\text{H}$. Also, we can easily convince ourselves that b_X is a parameter assignment$_\text{H}$ for D_X.

Morevover, it holds for all $\varphi \in$ FUNC that if φ is r-ary and $\theta_0, \ldots, \theta_{r-1} \in \text{CTERM}_\text{H}$, then $I_X(\varphi)(\langle[\theta_0]_A, \ldots, [\theta_{r-1}]_A\rangle) = [\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner]_A$. To see this, suppose $\varphi \in$ FUNC, φ is r-ary and $\theta_0, \ldots, \theta_{r-1} \in \text{CTERM}_\text{H}$. With Definition 6-2-(xv), we have $\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1}) = \varphi(\theta_0, \ldots, \theta_{r-1})\urcorner \in X$ and thus $(\langle[\theta_0]_A, \ldots, [\theta_{r-1}]_A\rangle, [\varphi(\theta_0, \ldots, \theta_{r-1})]_A) \in I_X(\varphi)$. Thus we have $I_X(\varphi)(\langle[\theta_0]_A, \ldots, [\theta_{r-1}]_A\rangle) = [\ulcorner\varphi(\theta_0, \ldots, \theta_{r-1})\urcorner]_A$.

Now we will show that for all $\Phi \in$ PRED: If Φ is r-ary and $\theta_0, \ldots, \theta_{r-1} \in \text{CTERM}_\text{H}$, then: $\langle[\theta_0]_A, \ldots, [\theta_{r-1}]_A\rangle \in I_X(\Phi)$ iff $\ulcorner\Phi(\theta_0, \ldots, \theta_{r-1})\urcorner \in X$. For this, suppose $\Phi \in$ PRED, Φ is r-ary and $\theta_0, \ldots, \theta_{r-1} \in \text{CTERM}_\text{H}$. First, suppose $\langle[\theta_0]_A, \ldots, [\theta_{r-1}]_A\rangle \in I_X(\Phi)$. Then there are $\theta'_0, \ldots, \theta'_{r-1}$ such that for all $i < r$: $[\theta_i]_A = [\theta'_i]_A$ and $\langle\theta'_0, \ldots, \theta'_{r-1}\rangle \in {}^r\text{CTERM}_\text{H}$ and $\ulcorner\Phi(\theta'_0, \ldots, \theta'_{r-1})\urcorner \in X$. With b), it then holds for all $i < r$: $\ulcorner\theta_i = \theta'_i\urcorner \in X$. With the symmetry shown above, it then follows that for all $i < r$: $\ulcorner\theta'_i = \theta_i\urcorner \in X$. Also, we have $\ulcorner\Phi(\theta'_0, \ldots, \theta'_{r-1})\urcorner \in X$ and thus, according to Definition 6-2-(xvii), also $\ulcorner\Phi(\theta_0, \ldots, \theta_{r-1})\urcorner \in X$. Now, suppose $\ulcorner\Phi(\theta_0, \ldots, \theta_{r-1})\urcorner \in X$. Then it follows easily that $\langle[\theta]_0, \ldots, [\theta]_{r-1}\rangle \in I_X(\Phi)$.

Moreover, it follows with Theorem 5-2$_\text{H}$ by induction on the complexity of θ that for all $\theta \in \text{CTERM}_\text{H}$: $\text{TD}(\theta, D_X, I_X, b_X) = [\theta]_A$. To see this, suppose $\alpha \in \text{CONST} \cup \text{CONSTNEW}$. Then we have $\text{TD}(\alpha, D_X, I_X, b_X) = I_X(\alpha) = [\alpha]_A$. Suppose $\beta \in$ PAR. Then we have $\text{TD}(\beta, D_X, I_X, b_X) = b_X(\beta) = [\beta]_A$. Now, suppose the statement holds for $\theta_0, \ldots, \theta_{r-1} \in$

$CTERM_H$ and suppose $\ulcorner\varphi(\theta_0, \ldots, \theta_{r\text{-}1})\urcorner \in FTERM_H$. Then we have $TD_H(\ulcorner\varphi(\theta_0, \ldots, \theta_{r\text{-}1})\urcorner, D_x, I_x, b_x) = I_x(\varphi)(\langle TD(\theta_0, D_x, I_x, b_x), \ldots, TD_H(\theta_{r\text{-}1}, D_x, I_x, b_x)\rangle)$ and thus, with the I.H., $TD_H(\ulcorner\varphi(\theta_0, \ldots, \theta_{r\text{-}1})\urcorner, D_x, I_x, b_x) = I_x(\varphi)(\langle[\theta_0]_A, \ldots, [\theta_{r\text{-}1}]_A\rangle) = [\ulcorner\varphi(\theta_0, \ldots, \theta_{r\text{-}1})\urcorner]_A$.

Furthermore, it follows that for all $A \in AFORM_H$: $D_x, I_x, b_x \vDash_H A$ iff $A \in X$. To see this, suppose $A \in AFORM_H$. Then there are $\Phi \in PRED$, Φ r-ary, and $\theta_0, \ldots, \theta_{r\text{-}1} \in CTERM_H$ such that $A = \ulcorner\Phi(\theta_0, \ldots, \theta_{r\text{-}1})\urcorner$. Then it holds that

$D_x, I_x, b_x \vDash_H A$

iff

$D_x, I_x, b_x \vDash_H \ulcorner\Phi(\theta_0, \ldots, \theta_{r\text{-}1})\urcorner$

iff

$\langle TD_H(\theta_0, D_x, I_x, b_x), \ldots, TD_H(\theta_{r\text{-}1}, D_x, I_x, b_x)\rangle \in I_x(\Phi)$

iff

$\langle[\theta]_0, \ldots, [\theta]_{r\text{-}1}\rangle \in I_x(\Phi)$

iff

$\ulcorner\Phi(\theta_0, \ldots, \theta_{r\text{-}1})\urcorner \in X$

iff

$A \in X$.

Now we will show by induction on $FDEG_H(\Gamma)$: If $\Gamma \in X$, then $D_x, I_x, b_x \vDash_H \Gamma$ and if $\ulcorner\neg\Gamma\urcorner \in X$, then $D_x, I_x, b_x \nvDash_H \Gamma$. From this follows immediately $D_x, I_x, b_x \vDash_H X$ and thus that X is satisfiable$_H$.

Suppose the statement holds for all $k < FDEG_H(\Gamma)$. Now, suppose $FDEG_H(\Gamma) = 0$. Then we have $\Gamma \in AFORM_H$. Now, suppose $\Gamma \in X$. Then it holds that $D_x, I_x, b_x \vDash_H \Gamma$. Now, suppose $\ulcorner\neg\Gamma\urcorner \in X$. With Definition 6-2-(i), we then have $\Gamma \notin X$ and thus $D_x, I_x, b_x \nvDash_H \Gamma$.

Now, suppose $FDEG_H(\Gamma) > 0$. Then we have $\Gamma \in CONFORM_H \cup QFORM_H$. First, we will now show: If $\Gamma \in X$, then $D_x, I_x, b_x \vDash_H \Gamma$. For this, suppose $\Gamma \in X$. We can distinguish *seven* cases. *First*: Suppose $\Gamma = \ulcorner\neg B\urcorner$. Then we have $FDEG_H(B) < FDEG_H(\Gamma)$ and thus, according to the I.H., $D_x, I_x, b_x \nvDash_H B$ and hence $D_x, I_x, b_x \vDash_H \ulcorner\neg B\urcorner = \Gamma$. *Second*: Suppose $\Gamma = \ulcorner A \wedge B\urcorner$. With Definition 6-2-(iii), it then holds that $A, B \in X$. Since $FDEG_H(A) < FDEG_H(\Gamma)$ and $FDEG_H(B) < FDEG_H(\Gamma)$, we thus have, according to the

I.H., that $D_x, I_x, b_x \vDash_H$ A and $D_x, I_x, b_x \vDash_H$ B and thus $D_x, I_x, b_x \vDash_H \ulcorner A \wedge B \urcorner = \Gamma$. The *third* to *fifth* case are treated analogously.

Sixth: Suppose $\Gamma = \ulcorner \bigwedge \xi \Delta \urcorner$. With Definition 6-2-(xi), it then holds that $[\theta, \xi, \Delta] \in X$ for all $\theta \in \mathrm{CTERM_H}$. Since, according to Theorem 1-13$_H$, it holds for all $\theta \in \mathrm{CTERM_H}$ that $\mathrm{FDEG_H}([\theta, \xi, \Delta]) < \mathrm{FDEG_H}(\Gamma)$, we thus have, according to the I.H., for all $\theta \in \mathrm{CTERM_H}$: $D_x, I_x, b_x \vDash_H [\theta, \xi, \Delta]$. Now, let $\beta \in \mathrm{PAR}\backslash\mathrm{ST_H}(\Delta)$ and let b' be in β an assignment variant$_H$ of b_x for D_x. Then we have $b'(\beta) \in D_x$ and hence there is a $\theta \in \mathrm{CTERM_H}$ such that $b'(\beta) = [\theta]_A$. Then we have $\mathrm{TD_H}(\theta, D_x, I_x, b_x) = [\theta]_A$ and hence $b'(\beta) = \mathrm{TD_H}(\theta, D_x, I_x, b_x)$. Because of $D_x, I_x, b_x \vDash_H [\theta, \xi, \Delta]$, it then follows, with Theorem 5-9$_H$-(ii), that D_x, I_x, b' $\vDash_H [\beta, \xi, \Delta]$. Therefore we have for all b' that are in β assignment variants$_H$ of b_x for D_x: $D_x, I_x, b' \vDash_H [\beta, \xi, \Delta]$. According to Theorem 5-8$_H$-(i), we hence have $D_x, I_x, b_x \vDash_H$ $\ulcorner \bigwedge \xi \Delta \urcorner = \Gamma$.

Seventh: Suppose $\Gamma = \ulcorner \bigvee \xi \Delta \urcorner$. With Definition 6-2-(xiii), there is then a $\theta \in \mathrm{CTERM_H}$ such that $[\theta, \xi, \Delta] \in X$. According to Theorem 1-13$_H$, we then have $\mathrm{FDEG_H}([\theta, \xi, \Delta]) <$ $\mathrm{FDEG_H}(\Gamma)$. According to the I.H., we thus have $D_x, I_x, b_x \vDash_H [\theta, \xi, \Delta]$. Now, let $\beta \notin$ $\mathrm{ST_H}(\Delta)$. Now, let $b' = (b_x\backslash\{(\beta, b_x(\beta))\} \cup \{(\beta, [\theta]_A)\}$. Then b' is in β an assignment variant$_H$ of b_x for D_x with $b'(\beta) = [\theta]_A$. Also, we have $\mathrm{TD_H}(\theta, D_x, I_x, b_x) = [\theta]_A$ and hence $b'(\beta) = \mathrm{TD_H}(\theta, D_x, I_x, b_x)$. Because of $D_x, I_x, b_x \vDash_H [\theta, \xi, \Delta]$, it then follows, with Theorem 5-9$_H$-(ii), that $D_x, I_x, b' \vDash_H [\beta, \xi, \Delta]$. Therefore there is a b' that is in β an assignment variant$_H$ of b_x for D_x such that $D_x, I_x, b' \vDash_H [\beta, \xi, \Delta]$. According to Theorem 5-8$_H$-(ii), we hence have $D_x, I_x, b_x \vDash_H \ulcorner \bigvee \xi \Delta \urcorner = \Gamma$.

Now, we will show that if $\ulcorner \neg\Gamma \urcorner \in X$, then $D_x, I_x, b_x \nvDash_H \Gamma$. Suppose $\ulcorner \neg\Gamma \urcorner \in X$. Remember that, by hypothesis, $0 < \mathrm{FDEG_H}(\Gamma)$. Thus we can distinguish *seven* cases. *First*: Suppose $\Gamma = \ulcorner \neg B \urcorner$. With Definition 6-2-(ii), we then have $B \in X$. Since $\mathrm{FDEG_H}(B) <$ $\mathrm{FDEG_H}(\Gamma)$, we then have, according to the I.H., that $D_x, I_x, b_x \vDash_H$ B. With Theorem 5-4$_H$-(ii), we then have $D_x, I_x, b_x \nvDash_H \ulcorner \neg B \urcorner = \Gamma$. *Second*: Suppose $\Gamma = \ulcorner A \wedge B \urcorner$. With Definition 6-2-(iv), we then have $\ulcorner \neg A \urcorner \in X$ or $\ulcorner \neg B \urcorner \in X$. Since $\mathrm{FDEG_H}(A) <$ $\mathrm{FDEG_H}(\Gamma)$ and $\mathrm{FDEG_H}(B) < \mathrm{FDEG_H}(\Gamma)$, we then have, according to the I.H., that $D_x, I_x,$

$b_x \nvDash_H$ A or D_x, I_x, $b_x \nvDash_H$ B. With Theorem 5-4$_H$-(iii), it follows that D_x, I_x, $b_x \nvDash_H \ulcorner A \wedge B\urcorner = \Gamma$. The *third* to *fifth* case are treated analogously.

Sixth: Suppose $\Gamma = \ulcorner\neg\bigwedge\xi\Delta\urcorner$. With Definition 6-2-(xii), there is then a $\theta \in$ CTERM$_H$ such that $\neg[\theta, \xi, \Delta] \in X$. According to Theorem 1-13$_H$, we have FDEG$_H$([θ, ξ, Δ]) < FDEG$_H$(Γ). According to the I.H., we thus have D_x, I_x, $b_x \nvDash_H [\theta, \xi, \Delta]$. Now, let $\beta \notin$ ST$_H$(Δ). Now, let b' be in β the assignment variant$_H$ of b_x for D_x with $b'(\beta) = [\theta]_A$. Then we have TD$_H$(θ, D_x, I_x, b_x) = $[\theta]_A$ and hence $b'(\beta)$ = TD$_H$(θ, D_x, I_x, b_x). Because of D_x, I_x, $b_x \nvDash_H [\theta, \xi, \Delta]$, it then follows, with Theorem 5-9$_H$-(ii), that D_x, I_x, $b' \nvDash_H [\beta, \xi, \Delta]$. Therefore there is a b' that is in β an assignment variant$_H$ of b_x for D_x such that D_x, I_x, $b' \nvDash_H [\beta, \xi, \Delta]$. With Theorem 5-8$_H$-(i), we hence have D_x, I_x, $b_x \nvDash_H \ulcorner\bigwedge\xi\Delta\urcorner = \Gamma$.

Seventh: Suppose $\Gamma = \ulcorner\neg\bigvee\xi\Delta\urcorner$. With Definition 6-2-(xiv), it then holds for all $\theta \in$ CTERM$_H$ that $\ulcorner\neg[\theta, \xi, \Delta]\urcorner \in X$. According to Theorem 1-13$_H$, it holds for all $\theta \in$ CTERM$_H$ that FDEG$_H$([θ, ξ, Δ]) < FDEG$_H$(Γ). According to the I.H., it thus holds for all $\theta \in$ CTERM$_H$ that D_x, I_x, $b_x \nvDash_H [\theta, \xi, \Delta]$. Now, let $\beta \notin$ ST$_H$(Δ) and suppose b' is in β an assignment variant$_H$ of b_x for D_x. Then we have $b'(\beta) \in D_x$ and hence there is a $\theta \in$ CTERM$_H$ such that $b'(\beta) = [\theta]_A$. Then we have TD$_H$(θ, D_x, I_x, b_x) = $[\theta]_A$ and hence $b'(\beta)$ = TD$_H$(θ, D_x, I_x, b_x). Because of D_x, I_x, $b_x \nvDash_H [\theta, \xi, \Delta]$, it then follows, with Theorem 5-9$_H$-(ii), that D_x, I_x, $b' \nvDash_H [\beta, \xi, \Delta]$. Therefore we have for all b' that are in β assignment variants$_H$ of b_x for D_x that D_x, I_x, $b' \nvDash_H [\beta, \xi, \Delta]$. With Theorem 5-8$_H$-(ii), we hence have D_x, I_x, $b_x \nvDash_H \ulcorner\bigvee\xi\Delta\urcorner$.

Thus we have shown: If $\Gamma \in X$, then D_x, I_x, $b_x \vDash_H \Gamma$ and if $\ulcorner\neg\Gamma\urcorner \in X$, then D_x, I_x, $b_x \nvDash_H \Gamma$. According to Definition 5-17$_H$ and Definition 5-9$_H$, it follows from the first part alone that X is satisfiable$_H$. ■

Theorem 6-11. *Model-theoretic consequence implies deductive consequence*
For all X, Γ: If $X \vDash \Gamma$, then $X \vdash \Gamma$.

Proof: Suppose $X \vDash \Gamma$. According to Definition 5-10, we then have $X \cup \{\Gamma\} \subseteq$ CFORM and thus also $X \cup \{\ulcorner\neg\Gamma\urcorner\} \subseteq$ CFORM. With Theorem 5-12, we have that $X \cup \{\ulcorner\neg\Gamma\urcorner\}$ is not satisfiable. Now, suppose for contradiction that $X \cup \{\ulcorner\neg\Gamma\urcorner\}$ is consistent. With

Theorem 6-9, there would then be a Hintikka set Z such that $X \cup \{\ulcorner\neg\Gamma\urcorner\} \subseteq Z$. With Theorem 6-10, Z would be satisfiable$_H$. With Theorem 5-11$_H$, we would then have that $X \cup \{\ulcorner\neg\Gamma\urcorner\}$ is satisfiable$_H$. But then we would have, with Theorem 6-5, that $X \cup \{\ulcorner\neg\Gamma\urcorner\}$ is satisfiable. Contradiction! Therefore $X \cup \{\ulcorner\neg\Gamma\urcorner\}$ is not consistent and thus inconsistent. With Theorem 4-22, it then follows that $X \vdash \Gamma$. ■

Theorem 6-12. *Compactness theorem*

(i) If $X \vDash \Gamma$, then there is a $Y \subseteq X$ such that $|Y| \in \mathbb{N}$ and $Y \vDash \Gamma$,

(ii) If $X \subseteq$ CFORM, then: X is satisfiable iff it holds for all $Y \subseteq X$ with $|Y| \in \mathbb{N}$ that Y is satisfiable.

Proof: *Ad (i)*: Suppose $X \vDash \Gamma$. With Theorem 6-11, it then follows that $X \vdash \Gamma$. According to Definition 3-21, there is therefore an $\mathfrak{H}$ such that $\mathfrak{H}$ is a derivation of Γ from AVAP($\mathfrak{H}$) and AVAP($\mathfrak{H}$) $\subseteq X$. According to Theorem 3-9, we then have |AVAP($\mathfrak{H}$)| $\in \mathbb{N}$. Accordingt to Definition 3-20, we also have $\mathfrak{H} \in$ RCS\{∅} and thus, with Theorem 6-1, also AVAP($\mathfrak{H}$) $\vDash \Gamma$. Hence we have (i).

Ad (ii): Suppose $X \subseteq$ CFORM. The left-right-direction follows directly from Theorem 5-11. Now, for the right-left-direction suppose all $Y \subseteq X$ with $|Y| \in \mathbb{N}$ are satisfiable. Suppose for contradiction that X is not satisfiable. With Definition 5-17, there would then be no D, I, b such that $D, I, b \vDash X$. According to Definition 5-10, we would then have $X \vDash \ulcorner(c_0 = c_0) \wedge \neg(c_0 = c_0)\urcorner$. With (i), there is then $Y \subseteq X$ such that $|Y| \in \mathbb{N}$ and $Y \vDash \ulcorner(c_0 = c_0) \wedge \neg(c_0 = c_0)\urcorner$. Suppose for contradiction that there are D, I, b such that $D, I, b \vDash Y$. According to Definition 5-9, (D, I) would then be a model and b would be a parameter assignment for D. According to Definition 5-10, we would also have $D, I, b \vDash \ulcorner(c_0 = c_0) \wedge \neg(c_0 = c_0)\urcorner$. With Theorem 5-4-(ii) and -(iii), it would then hold that $D, I, b \vDash \ulcorner c_0 = c_0\urcorner$ and $D, I, b \nvDash \ulcorner c_0 = c_0\urcorner$. Contradiction! Thus Y is not satisfiable though $|Y| \in \mathbb{N}$, which contradicts the assumption. Hence X is satisfiable. ■

7 Retrospects and Prospects

We have developed a pragmatised natural deduction calculus for which it holds that: (i) Every sentence sequence $\mathfrak{H}$ is not a derivation of a proposition from a set of propositions or there is exactly one proposition Γ and one set of propositions X such that $\mathfrak{H}$ is a derivation of Γ from X, where this can be determined for every sentence sequence without recourse to any meta-theoretical means of commentary. (ii) The classical first-order model-theoretic consequence relation is equivalent to the consequence relation for the calculus. We assumed a language L, where L is an arbitrary but fixed language with certain properties: The development of the calculus and its meta-theory can therefore be applied to all suitable languages.

We believe that this calculus is suited to support the claim that usual practices of inference can be established or modelled solely by setting up systems of rules, where the implementation of these practices does not require any meta-theoretical support practices (like, for example, an additional practice of commenting). Confessionally: Inferring in a language consists in the performance of (rule-respecting) speech acts in this language and not in the performance of speech acts in this language and concomitant meta-theoretical speech acts. For short: Inferring in a language is performing speech acts in *this* language. These theses have to be substantiated philosophically.

Also, some further meta-theoretical work seems in order, e.g. extending the completeness result to non-denumerably infinite languages and a precise investigation of the relationships between the individual rules of the calculus. So, one could investigate in which sense the logical operators are interdefinable. Also, it seems worthwhile to examine how the approach we have taken can be extended so as to include speech-act rules for the speech acts of positing-as-axiom, defining, stating and adducing-as-reason and for the use of modal and description operators etc. Further, it has to be examined how derivations in the calculus can be simplified by introducing admissible rules. *Last but not least*, a propaedeutic version of the calculus is to be established, where such a version should also demonstrate that in order to establish the availability concepts and the rules of the calculus solely for application purposes, one does not require genuinely set-theoretical vocabulary.

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