CONCEPTUAL STRUCTURE OF CLASSICAL LOGIC

Among other things logic is concerned with the correctness of arguments. An examination of the kinds of things pronounced correct and incorrect in logic reveals two distinct kinds of arguments, each with its own characteristic kind of correctness. Indeed, each kind of correctness is so distinctively related to its corresponding kind of argument that nonsense generally results from predicating one kind of correctness (and/or incorrectness) of the other kind of argument. Indeed, it appears that the set of arguments of one sort *is* the range of applicability of the corresponding kind of correctness. Similarly for the other.

The distinction between the two kinds of arguments is closely related to Aristotle's distinction between syllogisms and perfect syllogisms [*Prior Analytics*, 25a22].[1] Moreover, there is an important parallel between this distinction and the distinction implicit in the English verbs "to imply" and "to infer." "Imply" can take a non-human subject whereas "infer" always takes a human subject: a sentence (or set of sentences) *implies* another sentence, but a person *infers* one sentence from another (or from a set of sentences) [Cohen and Nagel, pp. 7, 175]. Finally, within the framework of mathematical logic, this distinction between the two kinds of correctness is reflected in the difference between semantic (or model-theoretic) consequence [Tarski, p. 409ff.] and "natural" derivability consequence [Kalish and Montague, p. ix and p. 24].

One purpose of this paper is to discuss the two kinds of argument[2] so that it will be clear how consideration of the kinds of "correctness" appropriate to each leads to distinct logical problems and to distinct philosophical problems. The problem, in logic, of determining the conditions under which the truth of a set of sentences "guarantees"

[1] In order to avoid excessive footnotes bracketed expressions are used to refer by author (or by title) and location to items in the list of references at the end of this article. Dates of publication of cited editions are used to distinguish different works by the same author.

[2] This phrase is not intended to suggest one overarching genus "argument" subsuming two species.

the truth of another sentence contrasts with the problem of determining "norms of correct reasoning." In philosophy, questions concerning the nature of the relationship which obtains between a set of sentences and each of its consequences are to be contrasted with questions concerning the nature of deductive reasoning.

I. Premise-Conclusion Arguments

In the following discussion, I want to consider a particular way that a declarative sentence may or may not be related to a set of declarative sentences. For this purpose let us agree that the letter P will always indicate a set of sentences and that the letter c will always indicate a single sentence. Moreover, we will never consider the possibility of P being related or not related to c unless c and all sentences in P are in the same language. It is best to choose the language in such a way that all its sentences are on the same subject [cf. Mates, 1965, p. 173]. For example, P may be the axioms of Euclid's geometry and c might be the denial of the parallel postulate. Again, P may be all of the declarative sentences of the Declaration of Independence and c might be a sentence of English saying that the government has the duty to usurp the rights of individuals for the good of the state. If P happens to be the empty set then c can be any sentence whatever. In each context let us think of ourselves as having first chosen a language with a fixed, coherent, narrowly circumscribed subject matter and *then* having chosen P and c from within that language.

Admittedly the idea of "being in the same language" is vague. I doubt that anything that I have to say below depends very heavily on how this idea is spelled out — however, if one takes as a language something as broad as all of English then some remarks lose a certain amount of plausibility and a few concepts become useless

Usually, the actual truth-values of the members of P are irrelevant to logic. Often some sentences in P are true and some false — but this is no concern to logic. In logic one asks

If all sentences in P were true would c necessarily also be true?

or

If c were false, would some of the sentences in P necessarily also be false?

What these questions ask is, of course, whether c is a logical consequence of P, i.e., whether the truth of the sentences in P logically guarantees the truth of c. Several other locutions have been used to say the same thing. Some people prefer to ask whether P implies c.

Others prefer to ask whether c follows from P. The terminology is a secondary concern. The primary concern is the relation of logical consequence.

One reason that the logical consequence relation is so important is because, in a sense, if we do not understand the consequences of what we say we do not fully understand what we are saying. When we make a set P of assertions, we are implicitly asserting all of the logical consequences of P because we are just as much in error when a consequence is false as when one of our actual assertions is false. Indeed, if one of the consequences is false, then one of the actual assertions must also be false. To articulate some of the consequences of our assertions is to be more explicit without adding any "new information."

The logical consequence relation is a far-reaching interconnection of sets of sentences to individual sentences: it connects each set of sentences to each of an infinity of consequences.

In order to focus on the possibility of the relation of logical consequence holding between a set P of sentences and single sentence c in abstraction from other possibly irrelevant considerations, logicians have used the word *argument* to indicate *a set P of sentences together with a single sentence c.* Any set of sentences together with any single sentence forms an argument. The set of sentences of an argument is conventionally called *the premises* and the single sentence is conventionally *the conclusion* [Mates, 1953, p. 58 and Mates, 1965, p. 3].

To be more explicit, one could say that an argument is an ordered pair (P, c) where P is a set of sentences arbitrarily designated as the premises and c is a single sentence arbitrarily designated as the conclusion. To the extent that this terminology suggests either that someone has accepted the premises as "his premises" or that someone has concluded the conclusion it is misleading. Nothing of the sort is intended. Moreover, in calling the pair (P, c) an argument, there is no intention to suggest that the conclusion actually follows from the premises. If the conclusion does follow then the argument is said to be *valid* and if it does not follow, the argument is called *invalid.*

As was said, the interest in defining the argument is to be found in the desire to focus on the possibility of a logical consequence connection while ignoring other possibly irrelevant factors. In logic, one wants to consider arbitrary sets P of sentences and arbitrary single sentences c, that is, in logic one wants to consider the possibility of logical consequence generally without being limited to cases more or less accidentally singled out. Prominent among the factors ignored by the definition are the following: (1) the truth-values of the sentences involved, (2) whether the sentences were ever asserted, (3) whether anyone has ever inferred the conclusion from the premises, and (4) whether the conclusion follows. All these things are irrelevant to the definitional fact that any set of sentences together with any single sentence forms an argument.

Given an argument, the only thing that is relevant to its being valid or invalid is whether the conclusion is implied by the premises. Consider for a moment the class of all arguments in a given language. Some are valid and the rest invalid, or so it would seem. Of the valid arguments some have been thought about but most have not. Of the valid arguments that have been thought about, some have been "recognized" as possibly valid, others not. Finally, among the valid arguments that have been studied and recognized to be possibly valid, some have been shown to be valid while the rest have been left as open questions. The point is that being valid, on one hand, and having been shown to be valid, on the other, are very different properties. Presumably there are infinitely many valid arguments, only a small portion of which have been shown to be valid.

Presumably, in order to know that an argument is valid one must understand why the conclusion must be true if the premises were true, i.e., how the truth of the premises guarantees the truth of the conclusion. On the other hand, in order to know that an argument is invalid, it is sufficient to notice that the premises are true and the conclusion false [cf. Ockham p. 93]. But of course, not all invalid arguments have true premises and false conclusion and in such cases some less trivial method is needed (see Section III below). In any case, as indicated above, there are many valid arguments not known to be valid, and, also, many invalid arguments not known to be invalid. Indeed there are many famous arguments which fall into one or the other of these two classes (but, of course, we do not know which class). For example, if we take P to be the axioms of arithmetic, then we can make several "unknown" arguments by taking c to be various of the famous open questions of arithmetic — e.g., Goldbach's Conjecture, the Twin Prime Conjecture, Fermat's Conjecture [Davenport, pp. 37-39, 160].

This is a good place to point out what is by now obvious, viz., that there is no argumentation, no deduction, no inference, no reasoning involved in arguments as the term is being used here — not even in valid arguments. Of course, in order to know that a valid argument is valid, one must certainly do *some* reasoning — but the reasoning is not *part* of the argument. The argument consists in premises and conclusion — period. The valid arguments do not demonstrate anything — they are par excellence the things that require demonstration.

If P is the axioms of arithmetic and c is a sentence which asserts that the square root of two is not a fraction then (P, c) is a valid argument. Clearly the argument shows nothing — it is the very kind of thing for which we require a demonstration. These considerations should suggest that the present sense of 'argument' is not the more usual one. It is unfortunate that this technical sense of the term has become so entrenched. The only term that is sometimes used in a sense synonymous with this sense of 'argument' is 'syllogism" and this term admits of an even less acceptable range of interpretations.

Incidentally, in order to gain some additional familiarity with logical consequence, it is worthwhile to consider the possible combinations of validity and invalidity of argument pairs (P, c) and (P, not-c). In the first place, notice that (P, c) and (P, not-c) may both be valid, i.e., that a particular c could follow from a particular P and the negation of c could also follow from P. In this case, P implies a contradiction and is called a *contradictory* set of sentences [Tarski, p. 414]. It can also happen that both (P, c) and (P, not-c) are invalid; c does not follow from P, nor does not-c. Here c is said to be *independent* of P. (Some authors [Tuller, p. 28] say that c is "independent" of P to mean merely that (P, c) is invalid. This has awkward consequences, e.g. that '$1 \neq 1$' is "independent" of the axioms of arithmetic.) A well-known example of independence, in our sense, arises when P is taken to be the axioms of Euclidean geometry *less* the parallel postulate, and c is taken as the parallel postulate. Other interesting examples have recently been discovered in set theory [Cohen and Hersh]. In fact, this is a common situation; i.e., if one takes an arbitrary set P of sentences then the chances are very good that there exists a sentence c in the same language which is such that neither (P, c) nor (P, not-c) is valid. However, there are several famous cases of sets P of sentences such that for any given sentence c in the same language, either (P, c) is valid or (P, not-c) is valid. Such sets of sentences are said to be *complete (with respect to consequences)*. As an example, one may take any one of several modern sets of axioms for Euclidean geometry [Forder, p. 296]. Other examples of axiom sets which are complete with respect to consequences are the following: the axioms for arithmetic (second-order) [Montague, p. 135], the axioms for the real numbers (second-order) [Montague, p. 135], the axioms for the theory of strings of symbols [Tarski, p. 173], the axioms for the rational numbers [Huntington, p. 35]. Each of these axiom sets was developed in order to organize the laws of some branch of science. The significance of completeness is that the process of axiomatization is finished, completed — if an axiom set is complete then (assuming all of its axioms are true of the subject-matter) every other sentence in the language which is also true follows from it. Thus a complete set of (true) axioms on a given subject implicitly contains all laws of that subject. A final "thus" — in the (perhaps Pickwickian) sense of "understand" used above — one could say that knowledge of all axioms in a complete set on a given subject would constitute complete understanding of that subject [cf. Boole, 1952, p. 143]. The reason that this is Pickwickian is that one might know the axioms of arithmetic mentioned above but still not have the vaguest hint of an idea whether, for example, the Twin Prime Conjecture is true.

Incidentally, Euclid's "axioms and postulates" do not form a complete set [Heath, Vol 1, pp. 232-240] and, of course, this is regarded as a defect of his system. On the other hand, few of the "abstract" axiomatizations studied in modern algebra are complete (e.g., "commutativity" is independent of the group axioms) but these systems are not regarded as deficient on that account because they were not intended to characterize any "particular" subject matter. In a sense, each of them is merely a set of sentences whose consequences are of permanent interest and usefulness to mathematicians and scientists [Mates, 1965, p. 176].

It is now important to recall that whenever we refer to an argument (P, c) we are presupposing that a language containing c and all sentences in P has been specified in advance. Generally speaking, it is only relative to a language that a set P can be said to be complete. When we say that "the" modern axioms of geometry form a complete set of sentences, we mean only that given any sentence c *in the language of geometry* either it follows from them or its negation follows from them. Certainly, for example, the law of natural selection is "independent" of the axioms of geometry. If one always insists on considering each set of sentences as a subclass of some vast language system such as all of English, then, of course, no small set of sentences about a limited subject matter will be complete — unless it happens to be contradictory.

Given that this point is made I hope that I will be permitted to express my opinion that one should only consider arguments whose sentences are all about some one coherent and limited subject matter. For example, I feel that to *ask* whether the law of natural selection follows from the axioms of geometry is to commit a category mistake, to ask a pseudoquestion, to somehow be facetious or silly or something of the sort. Such a question is on a par with questions of the number of angels that will fill a telephone booth and whether twenty-seven is heavier than causation and how furiously green ideas sleep. If this opinion could be substantiated it would then be possible to answer *some* of the current objections to the classical concept of logical consequence.

Before closing the discussion of this type of argument, I would like to make a few obvious observations concerning linguistic usage. On the face of it each argument is either valid or invalid — given the premises, either the conclusion follows or it does not. Thus one can *meaningfully* say of any argument that it is valid and that it is invalid. Similarly, one can meaningfully say of any declarative sentence that it is true, and that it is false. But one cannot say of an argument that it is true or false. Arguments are not the sorts of things to which truth or falsity apply, i.e., arguments are not in the range of applicability of truth or falsity [cf. Martin, pp. 288, 292]. To say that an argument is true would be to make a "category mistake" or a "confusion of spheres" [Carnap, pp. 51-56]. For exactly the same reasons it is improper to speak of "asserting an argument." One can, of course, make assertions about arguments.

There are several accepted formulas for asserting that an argument is valid. All of the following are acceptable: (1) the argument is valid, (2) the conclusion follows from the premises, (3) the premises imply the conclusion, (4) if the premises were true, the conclusion would necessarily also be true, (5) the conclusion is a logical consequence of the premises. There are others as well.

Sometimes one encounters a series of assertions followed by the word 'therefore' and a final assertion. That is, one encounters a complex assertion of the form: premises, therefore, conclusion. A person making such an assertion is generally understood to have done two things: (1) asserted that each of the premises is true and (2) asserted that the conclusion follows. The point is that the traditional way of representing arguments (premises, therefore, conclusion) is misleading because the form used already represents something else — viz., a complex assertion of the truth of the premises and the validity of the argument.

Finally on the subject of usage, it seems to me that the word 'argument' is rarely used in the above sense outside of elementary logic. This will become clear as we proceed to consider a more common sense of the term below.

Let me end the discussion of this type of argument by mentioning some things that have not been treated. In the first place it should be clear that very little has been said concerning criteria or tests of validity. Nothing has been said about logical form, formal validity or about related ideas involving them. Moreover, no definition of logical consequence has been given. Since validity of arguments is defined in terms of logical consequence, the vagueness in the concept of logical consequence is inherited by the concept of validity. The above discussion was intended to focus attention on the concept of logical consequence and to clarify how the technical concept of argument facilitates consideration of logical consequence in itself without the encumbrance of other possible irrelevant factors.

II. Demonstrative Arguments

It is characteristic of a rational person to be interested in justifying his own beliefs and in learning how others justify theirs. A discourse, written or spoken, which is offered in justification of a belief is frequently referred to as an argument. In some cases an argument is offered as establishing the truth of a statement, but in most cases, perhaps, the argument is not claimed to establish the truth of a statement but only to support it. The words "demonstration" and "proof" are also used in this connection.

Below when I use the term "argument" in connection with demonstration or justification, I will say "demonstrative argument." Where I intend merely premises and conclusion I will say "premise-conclusion argument." Often the term will be used without a modifier where the sense is obvious from the context.

Perhaps the two most famous arguments in philosophy are Anselm's argument for the existence of God and Russell's argument establishing the falsity of the principle of abstraction. Another famous argument, which rivals the above two in elegance, but which is more intricate, is the Pythagorean argument which "proves" that the square root of two is not a fraction. Zeno's arguments are also well-known. In intellectual pursuits arguments are encountered more than frequently. Indeed, arguments play such a prominent role in the formation of rational beliefs that one is probably justified in refusing a belief in the honorific title of rational unless it is supported by argument.

In any of the above-mentioned examples it can be debated whether the argument establishes its conclusion. Generally speaking it is always appropriate to ask of a given demonstrative argument whether it actually establishes its conclusion. Even though people may disagree concerning whether a given argument establishes its conclusion, there seems to be general agreement that *an argument which proceeds by "correct" reasoning from established premises does establish its conclusion.* In other words, in order for an argument to establish its conclusion it is both necessary and sufficient that the argument begin with established premises and that it show by "correct" logical reasoning that the conclusion must be true given the truth of the premises. This makes explicit the obvious fact that in practical situations a demonstrative argument may be criticized either by questioning the truth of the premises or by questioning the correctness of the reasoning (or both).

Frequently critical examination of a demonstrative argument involves much more than appraisal of the truth of the premises and the soundness of the reasoning [cf. Boole, 1958, pp. 185-187]. More frequently than not we find arguments beginning with a claim to the effect that the author will demonstate such-and-such a conclusion where no mention of premises is made. Reading on through such an argument, we find premises interwined with the reasoning, the premises being introduced as needed to make the reasoning proceed. Anselm's argument [Anselm, pp. 7, 8] is of this sort, as are many versions of the Pythagorean argument for the irrationality of the square root of two [e. g. Hardy, pp. 34-36]. Relatively rarely do we find an author stating his premises, avowing his intention to demonstrate from them a certain conclusion, and then proceeding to give his reasoning. Lack of organization in the presentation of arguments is often justified when it is obvious what the premises are. When they are cited as premises in the reasoning it would be redundant to have them listed at the beginning. However, as we all quickly discover, it is more usually the case that lack of organization makes it difficult to determine from a presentation of an argument exactly what the premises are and what the reasoning is. Frequently, before we can critically appraise an argument we must try to determine the basic premises and sometimes, we even discover that some of the statements obviously intended as premises do not even occur in the presentation [cf. Boole, 1958, pp. 185-218 for some revealing examples from philosophy].

In this connection, sometimes people are surprised to learn that Russell's argument for the falsity of the principle of abstraction has *no premises at all.* Presentations begin with a statement of the principle and an avowal that it will be shown to be false. Then, from the assumption of the principle, a contradiction is deduced, thereby showing that it could not possibly be true [cf. Suppes, p. 6]. Incidentally, this is also true of presentations of arguments demonstrating "the barber paradox" (there is no barber who shaves *exactly* those barbers who do not shave themselves).

Needless to say, it is of utmost importance to notice that demonstrative arguments generally consist of much more than merely premises and conclusion. Russell's argument, for example, could require five or six sentences, only one of which is the conclusion, none of which are premises. The additional material expresses the reasoning from the set of premises (in Russell's case, the null set) to the conclusion. In fact, all of the above-mentioned *demonstrative arguments consist in a premise-conclusion argument together with additional discourse intended to* show *that the conclusion follows from the premises.*

By considering a few examples, one can easily become convinced of the fact that the additional material, the discourse expressing the reasoning, cannot be taken as a set of sentences — the order in which the sentences occur is of crucial importance. Generally speaking, there are many other considerations besides order that are important. Some of these will be mentioned below. In any case, since we can describe a demonstrative argument as consisting of premises, "reasoning," and conclusion, in order to study them abstractly we can define a *demonstrative argument* as an ordered triple (P, R, c) where P is a set of sentences, R is a discourse, and c is a single sentence. Now we could wonder which demonstrative arguments establish their conclusions — but this question goes beyond logic because it involves the question of whether the premises have been established. However, the question of the correctness of the reasoning is a concern of logic and in order to discuss this we can agree to call a demonstrative argument (P, R, c) *sound* whenever the discourse R expresses correct reasoning from P to c. In the absence of all norms of correct reasoning, this definition is not very useful in practical cases. All that the definition specifies is what we shall mean by the word *sound* — it does *not* supply us with criteria for its application.

There are several simple things about demonstrative arguments which should be pointed out. In the first place, it is not always problematic to determine that a demonstrative argument is unsound. If the premises were true and the conclusion false, then the premise-conclusion argument contained in the demonstrative argument is invalid. Thus, in this case, the reasoning must be incorrect — one cannot show by correct reasoning *that* a conclusion follows from premises unless the conclusion actually follows from them. Moreover, even if the premises happen to be false (or the conclusion true), we may still be able to determine that the premise-conclusion argument is invalid and therefore that the reasoning is incorrect. However, and this important point is sometimes overlooked [Beth, p. 58 and Copi, p. 2], it can even happen that the premise-conclusion is valid but that the reasoning is incorrect. For example, a student might offer a new demonstrative argument for the irrationality of the square root of two from the axioms of arithmetic — and have the reasoning done incorrectly. Actually, this sort of situation is surprisingly common in mathematics — mathematicians have been known to publish a demonstrative argument whose included premise-conclusion argument is valid and then later discover its unsoundness [Bell, p. 301, Davenport, p. 37, Kershner and Wilcox, p. 77, Reid, p. 200].

Examples of this sort of thing can be easily constructed: take as P and c the premises and conclusions of any valid argument which is sufficiently complicated as to require reasoning to see that it is valid, then take R to be any inappropriate discourse, say, the Pledge of Allegiance (if the sentences of the argument are in the same language).

Another thing to be noticed is that the above definition of demonstrative argument abstracts from considerations which would be irrelevant to logic. For example, the truth-values of the premises are irrelevant, whether the argument was ever offered is irrelevant, whether anyone ever thought of the argument is irrelevant, etc. Moreover, it is even left open whether the conclusion follows from the premises and whether the reasoning is correct. This definition includes anything that could possibly be considered as a demonstrative argument (informal sense), so that logic can focus on the peculiarly logical question of determining the conditions under which they are sound.

I have defined a demonstrative argument (P, R, c) to be sound if the discourse R expresses correct reasoning from P to c, i.e., if R shows that c must be true when all sentences in P are true. By a discourse, I mean any sort of text consisting of possibly several sentences. A discourse is something that could be written down. I am explicitly assuming that such discourse is finite [cf. Braithwaite, p. 22 and Tarski, pp. 294-295]. Of all discourses, some express correct reasoning and some do not. The discourses which express correct reasoning I am calling *proof-discourses* to emphasize (1) that they are discourses and (2) that they prove that something follows from something.

There is a lot to be learned about proof-discourses but a few simple things seem clear already. In the first place each proof-discourse has a final conclusion. Secondly, even though in some proof-discourses some of the assumptions are made "for purpose of argument only" and are not used as premises of the final conclusions [e.g. Kleene, p. 27], in many there are assumptions from which the final conclusion is deduced. These latter we will call *the premises of the proof-discourse*. In Russell's proof-discourse mentioned above, the principle of abstraction is assumed for purposes of argument only and from it a contradiction is deduced. The final conclusion is the denial of the principle of abstraction and there are no assumptions used as premises. There is a third point which is almost too obvious to mention, viz., if R is a proof-discourse then R expresses correct reasoning from P to c if and only if all the premises of the proof-discourse R are among the premises P of the argument and c is the final conclusion. In other words, in proving c from P: (1) your proof discourse must have c as its final conclusion, (2) you cannot smuggle in premises not in P but (3) you do not have to use all of the premises in P.

Already Russell's argument [Suppes, *loc. cit.*] serves as a counterexample to the ancient view [Mates, 1961, p. 78] that a proof-discourse of c from P is simply a list of sentences beginning with premises in P and such that such subsequent sentence is an immediate inference obtained by "applying a simple argument pattern" to some combination of previously derived conclusions and initial premises. Later logicians apparently also allowed "Laws of thought" (e.g. the "laws" of identity, excluded middle and non-contradiction) to appear and Boole reduced the allowed argument patterns to instances of "substitution of identities" [cf. Boole, 1952, pp. 142ff.].

Before proceeding to consider further interrelations among premise-conclusion arguments and demonstrative arguments let us introduce one more important concept. Consider all sound demonstrative arguments (made up of sentences from one language). Also consider an arbitrary premise-conclusion argument (P, c). Either there exists among the sound demonstrative arguments one of the form (P, R, c) or there does not. In case there is such a sound demonstrative argument showing that c follows from P, we will say that (P, c) is *demonstrable* — otherwise, *indemonstrable.* It may very well be the case that a given argument (P, c) is demonstrable but that no one has ever thought of a demonstration of it (i.e., a sound demonstrative argument (P, R, c)). Moreover, *there does not seem to be any good reason for thinking that all valid arguments are demonstrable:* there may very well be valid arguments which are so complex as to be "unknowable"— but more on this subject below. The possible combinations of demonstrability and indemonstrability of arguments (P, c) and (P not-c) are the same as the combinations of validity and invalidity — but different terms are used because, as we just pointed out, validity and demonstrability might be extensionally different [cf. Tarski, p. 295]. If (P, c) and (P, not-c) are both demonstrable, then P is called *inconsistent.* If neither (P, c) nor (P, not-c) is demonstrable, then c is *deductively independent* of P. If no sentence c is deductively independent of P, then P is *deductively complete.* Incidentally, since the terms 'contradictory' and 'inconsistent' received parallel definitions in terms of validity and demonstrability, respectively, one might expect that if there are valid arguments which are not demonstrable, then there should be contradictory sets of sentences which are not inconsistent [cf. Beth, pp. 70-71].

One final note on usage: the word argument used in connection with justification is more vague than I have indicated. Sometimes it is used in the sense of a discourse offered as a proof-discourse. Sometimes it is used to indicate, not the discourse, but "what the discourse says." For example, one often hears it said that one and the same argument has been expressed several different ways. The term "demonstrative argument" is intended to reflect the ideal situation in which a man states the premise-conclusion argument that he wants to demonstrate before he expresses his reasoning from the premises to the conclusion.

III. Two Logical Problems and Their Interrelations

At the beginning of the paper I expressed the idea that logic is concerned with the correctness of arguments. We have seen that there are two distinct types of arguments and corresponding distinct types of correctness. Correctness in premise-conclusion arguments is validity, i.e., a premise-conclusion argument is "correct" if its conclusion follows from its premises. On the other hand, correctness in demonstrative arguments is soundness, i.e., a demonstrative argument is correct if it shows by correct reasoning that its conclusion follows from its premises. In addition, I indicated that certain other possible kinds of correctness are irrelevant to logic.

Thus, we have two central concepts of logic: validity and soundness. Given that these are the appropriate types of correctness and given that logic is concerned with correctness of arguments, we are now ready to raise two central problems in logic. In the first place, there is the *problem of validity:* what is a necessary and sufficient condition for premise-conclusion arguments to be valid? In other words: which premise-conclusion arguments are valid? We could also call this the problem of implication (or the problem of logical consequence) and state it as follows: what is a necessary and sufficient condition for P to imply c (or for c to be a logical consequence of P)? Secondly, we have the problem: what is a necessary and sufficient condition for demonstrative arguments to be sound? Although it would be natural *here* to refer to this problem as "the problem of soundness" we would be inviting confusion in a wider context. Leaving the statement as above we will call it the problem of demonstration because we could equivalently ask for a necessary and sufficient condition that R express a correct demonstration of c from P.

In the two thousand years since Aristotle started investigations into logical questions many partial solutions to the above problems have been offered. Let us consider what partial solutions to the problem of validity would be like. One may approach the problem head-on by aiming at a necessary and sufficient condition for an arbitrary argument to be valid. On the other hand, one may first delimit a manageable subclass of premise-conclusion arguments and then aim at a necessary and sufficient condition for a member of the subclass to be valid. Aristotle can be seen as taking the latter approach, perhaps unwittingly. Apparently, (at one point) he restricted his attention to what are *now* sometimes called categorical syllogisms. Then by listing *all* of the valid "forms" he gave his necessary and sufficient condition for validity of *categorical syllogisms.* Since there are valid arguments which are not categorical syllogisms. Aristotle gave only a sufficient condition and hence only a partial solution to the problem of validity. From the point of view of the "head-on" approach a partial solution would consist in either a (merely) necessary condition or a (merely) sufficient condition. For another example, since truth-functional validity is a sufficient condition for validity it may be offered as a partial solution to the problem.

Apparently the only known necessary condition for validity involves (1) distinguishing content words and logical words and (2) defining two arguments (P, c) and (P', c') to be *in the same form* (as each other) if and only if one is obtained from the other by a one-one replacement of content words which "preserves logical categories" (proper names replace proper names, one-place predicates replace one-place predicates, etc.). Given this, a necessary condition for (P, c) to be valid is that no argument (P' c') in the same form has true premises and false conclusion. This principle, which we may call the *Principle of Form,* has been presupposed by logicians from Aristotle [*Prior Analytics,* 25a12, 25a25, 26a38, etc.] to Beth [*Mathematical Thought,* pp. 57-58].

There are also partial solutions to the problem of demonstration These take the form of sets of rules which will serve to construct *some* of the sound proof-discourses from sets of premises. The Stoics *apparently* discovered rules of this sort [Mates, 1953, pp. 77 ff.]. Modern interest in the problem of demonstration seems to originate in the 1920's with the ideas of Lukasiewicz and the work of his student Jaskowski [1934].

Once we are reasonably clear about the two problems we may begin to wonder whether a solution to one could serve as a solution for the other or whether they are essentially different problems. It is obvious that they are very closely related. To consider these questions let us first imagine that the problem of validity is solved and that we wish to develop a solution of the problem of demonstration. Thus we are assuming that we have a necessary and sufficient condition for validity of premise-conclusion arguments and we are seeking a necessary and *sufficient* condition for soundness of demonstrative arguments.

Already we have a necessary condition: in order for a demonstrative argument (P, R, c) to be sound it is necessary that the included premise-conclusion argument (P, c) be valid. But, of course, this is not a sufficient condition because incorrect reasoning may lead from premises to a conclusion which actually follows. In any case, it should be clear that in order to give sufficient conditions for soundness we must know something very specific about proof-discourses. In particular, we should have to know something about rules of inference and such knowledge does not seem to be implicit in conditions for validity. One may very well imagine that if, in addition to knowing the condition for validity, one were very clear about the nature of rules of inference then it would be possible to determine the sound rules and thus to solve the problem of demonstration. All that this would show is that a solution to the problem of validity *together with some other knowledge* would lead to a solution to the problem of soundness — not that a solution to validity would by itself constitute a solution to the demonstration problem.

Now let us consider the question of whether a solution to the problem of demonstration could serve as a solution to the problem of validity. In order to do this let us imagine that we have been given a necessary and sufficient condition for soundness of demonstrative arguments and we are seeking conditions for validity. Since we defined demonstrability in terms of soundness, we already have a sufficient condition for validity: in order for a premise-conclusion argument to be valid it is sufficient that it be demonstrable. Let us spell this out: Let (P, c) be any premise-conclusion argument. If there is a sound demonstrative argument (P, R, c) then (P, c) is valid. In other words, in order for (P, c) to be valid it is sufficient that there is a proof-discourse R showing that c follows from P. Now we are faced with the question of whether demonstrability is a necessary condition for validity. In other words, *are all valid arguments demonstrable?* This is the question I hinted at above. It is a very difficult question and all attempts to answer it — either way — necessarily must make some assumptions about the nature of proof-discourses. According to the assumptions presently accepted by many logicians the answer is no.

On first seeing this question the inclination seems to be to answer affirmatively, i.e., to suppose that all valid arguments are demonstrable and, indeed, it has been claimed [Tarski, p. 410] that some logicians held this view. However, I have not been able to find any references to support the claim. As far as I have been able to determine it seems that each logician either failed to raise the question, *or* raised it but refused to speculate concerning it, *or* expressed a negative opinion. For example, (1) Boole apparently never raised the question; (2) in the 1920's Forder [*op. cit.*, p. 6] raised the question and refused to speculate one way or the other noting merely that it is an interesting logical question which ". . . does not seem to have been discussed"; and (3) Tarski [*op. cit.*, p. 412] expressed a negative opinion.

There are two separate lines of thought based on different assumptions, either of which by itself leads to the conclusion that there are valid but indemonstrable arguments.

The first involves the existence of an infinite argument whose conclusion requires an infinite number of the premises. To see how this works, assume that we have a valid argument (P, c) where P contains an infinite number of premises. Let us also assume that c does not follow from any finite number of sentences in P. (P, c) cannot be demonstrable because of the finiteness of proof-discourses. Each proof-discourse has but a finite number of premises from P but since c does not follow from any finite number of premises in P, c cannot be the final conclusion of any correct proof-discourse having only premises in P.

The general principle to be gleaned from this discussion is that if (P, c) is demonstrable then c must follow from a finite number of sentences in P.

Thus in order to establish the existence of a valid but nondemonstrable argument (given the above assumptions) it is (more than) sufficient to exhibit a valid argument (P, c) where P is infinite and c does not follow from *any* proper subset of P. This can be done in many languages. For our example we take the language of geometry which contains a primitive predicate for points. This is the only predicate needed. Using the predicate for points, together with logical symbols, all of the following things can be said: there are at least two points; if there are at least two points then there are at least three points; if there are at least three points there are at least four points, and so on. All of these sentences we put in P. For c we take a sentence which "says" that there is an infinite number of points. (For example, one could use for c the result of writing "there is a relation F such that" in front of the first formula on page 117 of Hilbert and Ackerman [q.v.]).

A moment's reflection should convince the reader that the above argument (P, c) is valid and that the deletion of any one or more premises would render it invalid. Thus, acording to what was said above, (P, c) is valid but not demonstrable. The air of paradox should vanish once one realizes that (1) in order to see that (P, c) is valid one must know that there are an infinite number of sentences in P but (2) *no* sentence or finite number of sentences in P says that. Thus in demonstrating that the argument is valid one must use information not among the premises. In fact, no proof *that* c follows from P can be a proof of c from P because any such proof must assume a description of P and no description of P is in P. (There are other ways of looking at this situation but the upshot is usually the same — viz., that (P, c) is valid but not demonstrable [cf. Beth, p. 71].)

The above argumentation assumes that although proof-discourses must be finite, infinite sets of premises are countenanced. But proof-discourses are finite . . . because discourses are all finite (Hilbert [p. 370] has ridiculed those who think that this point even deserves emphasis.) Moreover, since languages contain infinitely many sentences, there seems to be no objection to considering arguments involving an infinite number of premises. Of course, there are objections to speaking of infinite sets at all. I must admit to those who voice these objections that I have nothing to add to the literature on this subject.

As I said above, there are two lines of thought used to show that there are valid but nondemonstrable arguments. The second does not involve infinite sets of premises. The upshot of the second line of thought is that there are *finite* valid arguments which are not demonstrable. The details involved here go well beyond the scope of this paper by involving questions of undecidability — but the main ideas are simple:

Let P be any finite set of premises and let D be the set of all proof-discourses having premises only from P. Modern logicians generally accept the hypothesis that D must be becidable, i.e., that there is a mechanical method for deciding of any discourse whether or not it is in D. The motivation for accepting this hypothesis is patently a priori reasoning from the "essential character of a proof." Church [p. 53] puts it this way:

> It is essential to the idea of a proof that, to any one who admits the presuppositions on which it is based, a proof carries final conviction. The requirements of effectiveness (decidability) . . . may be thought of as intended just to preserve this essential characteristic of proof. (Parenthetical expression mine.)

From the hypothesis that D is decidable it follows that the set of theorems (demonstrable consequences) of P is enumerable, i.e., that there is a mechanical device (1) which will "print-out" in sequence only theorems of P and (2) which is such that any given theorem will be printed after a finite amount of time. One upshot of Godel's famous incompleteness theorem is that the set of consequences of Peano's (second-order) axioms is not enumerable. Thus, there must be a consequence which is not a demonstrable consequence.

Incidentally, it is true that all truth-functionally valid arguments are demonstrable. The same holds for all so-called first-order arguments. Once second-order arguments are considered, one finds both finite and infinite valid arguments that are not demonstrable. In particular, one finds arguments having no premises which are valid but not demonstrable [Henkin, p. 81].

Again incidentally, recall that I suggested above that if there are valid arguments which are not demonstrable then there should also be consistent sets of sentences which are contradictory. Those who appreciate infinite sets of sentences may take as an example the result of adjoining not-c to P where (P, c) is the infinite argument in the language of geometry which we considered above [cf. Beth, *loc. cit.*]

This whole discussion was focused on the question of whether a solution to the problem of validity could serve as a solution to the problem of demonstration or vice versa. I do not pretend to have settled either question.

Naturally, since I argued against positive answers to each, I very strongly suspect that both answers are negative. The reason that I do not think that knowledge of conditions for validity would solve the problem of demonstration is because I think that much more than validity is involved in norms of correct reasoning. The reason that I do not think that a solution to the problem of demonstration would solve the problem of validity is because I believe that the scope of validity exceeds human capabilities, i.e., that we can ask logical questions which we are in principle incapable of answering. I also think that the above discussion only shows, at best, how this *might be possible.*

Conclusions

I hope that I have made it clear that there are two fundamentally different senses of the word "argument," both of which are germane to logic. In addition, I tried to make clear how inquiry into the respective kinds of "correctness" of the two kinds of arguments leads, on one hand, to the problem of validity, and on the other, to the problem of demonstration. The problem of validity can be said to require formulation of the conditions under which a conclusion is a logical consequence of premises. The problem of demonstration is the problem of determing "the" norms of correct deductive reasoning.

In my opinion neither of the problems has been solved. Let us briefly consider the current state of affairs.

In 1936 Tarski [*loc. cit.*] exposed the fallacy (which may never have been committed) of trying to solve the problem of validity by identifying validity with demonstrability. This fallacy would have consisted in taking a merely sufficient condition (demonstrability) for a necessary and sufficient condition. In the same article Tarski suggests defining an argument (P, c) to be valid if and only if, in effect, there is no argument (P', c') in the same form having true premises and false conclusion. Instead of taking another argument (P', c') obtained by changing content words he actually leaves the content words in (P, c) alone and changes their meanings. There is room to doubt whether Tarski also permits changes in domain (universe of discourse). Both of these changes constitute major improvements on what was attributed "in effect" to Tarski above. In any case Tarski (and most modern logicians) would say that (P, c) is valid if there is no argument (P, c)* having true premises and false conclusion where (P, c)* is the result of assigning any nonempty domain and any appropriate (extensional) meanings to the content words of (P, c). I agree that this new condition (as well as the one it replaced) is a necessary condition for validity but I doubt that it is a sufficient condition. To my way of thinking Tarski made the same sort of mistake that he had just exposed — instead of taking a merely sufficient condition as necessary *and* sufficient, Tarski took what may well be a merely necessary condition as necessary *and* sufficient.

What I am suggesting is that there might well be an argument (P, c) which is invalid but for which there is no (re)interpretation making the premises true and the conclusion false. In any case according to the Tarskian definition of validity, the invalidity of an argument depends on *the existence* of a suitable domain and there might not be "enough" domains to provide "counter interpretations" for all invalid arguments. Moreover, even if there are enough domains (etc.) so that the Tarskian condition is extensionally equivalent to validity it seems incorrect to take this for granted without supporting reasons. In effect, taking the Tarskian condition as sufficient for validity is precisely analogous to taking recursiveness as a necessary condition for calculability (i.e., to accepting Church's Thesis [Kleene, p. 300]). In my view then, we should no longer refer to "the Tarskian definition of logical consequence" but rather to "Tarski's Thesis"— to indicate that some sort of justification is required [cf. Kleene, pp. 317-323]. As yet the question of whether the condition *is* sufficient does not seem to have been discussed.

Years before Tarski offered his definition of validity several logicians and mathematicians recognized the fact that the Tarskian condition was necessary for validity [e.g., Padoa, p. 122; Forder, pp. 4, 5; Lewis and Langford, pp. 342-346]. The question of whether it is also sufficient does not seem to have been raised except, perhaps, by Lewis and Langford [p. 346] who seem to speculate that it may well *not* be sufficient.

If the "no counterinterpretations" condition is extensionally equivalent to validity then the knowledge of that fact would be a solution to the problem of validity as raised above. Even assuming that this doubtful state of affairs comes about, it still seems incorrect to take the condition as the *defining* property of validity because that would amount to "identifying" a logical concept (involving possibilities) with a material concept (involving actualities).

The problem of demonstration is also largely unsolved although the number of known rules of inference is large and although there are known conditions that a rule must satisfy in order to be a correct rule of inference. The difficulty here is twofold: first, there are doubtless many correct rules of inference in current use which have not been discovered and there are probably even *kinds* of rules of inference which have not been recognized; second (as indicated in more detail elsewhere [Corcoran]) there is very little understanding of what conditions rules must satisfy in order to be rules of inference — over and above the now obvious conditions of being effective [cf. Church, *loc. cit.*] and being "truth-preserving" (in the wide sense). Moreover, it is possible to speculate that there may not be any objective solution to this problem. For example, some questions concerning whether a particular discourse shows that c follows from P may have to be relativized to individual thinkers or to small communities of thinkers. If this is so then some of my previous remarks (as well as some to come) will require similar relativization.

The problem of demonstration, as discussed above, is only approximately described as the problem of giving an exhaustive characterization of the system of correct deductive reasoning which is presupposed in the extant corpus of deductive texts (including prominently Euclid's *Elements*, Newton's *Principia*, etc. and excluding devised artifacts such as *Principia Mathematica*). Many early logicians eschewed this problem in favor of *devising* artificial systems of deduction. This is particularly striking in Aristotle where normal reasoning is used to prove the validity of the arguments which are then incorporated into his *devised* system. Lukasiewicz should probably be credited as being the first logician to give serious consideration to the problem of demonstration [Jaskowski, p. 1] although Boole apparently believed that his system was *the* "ideal standard" underlying "actual performance." (The quoted terms are Boole's [Boole, 1952, p. 246; also cf. Boole, 1958, p. 158].) However, instead of describing the actual system of proof-discourses Boole replaced them by a kind of computation which not only failed to express reasoning but which had literally uninterpretable parts — contrary to Boole's own admission that the "failure of correspondency between process and interpretation does not manifest itself in the *ordinary* applications of human reason" (emphasis Boole's) [1958, pp. 66, 67].

Closely related in spirit to Boole's work are modern attempts to provide quasicomputational methods (proof-procedures) for mechanically producing from a set of sentences its logical consequences [Gilmore, p. 201]. This sort of work is rarely (if ever) directed toward the problem of demonstration but rather it seems to aim at providing a syntactic criterion for validity. Some logicians speak of attempts to "capture by syntactic means the semantic concept of logical consequence" [cf. Tarski, p. 294; Copi, pp. 186-187]. From the point of view of intent, therefore, work on proof-procedures is naturally regarded as contributing toward solution of *the problem of validity* by restricted means rather than as contributing toward solution of the problem of demonstration.

I would like to end this article by relating some of the above ideas first to the teaching of logic and then to a small point in the history of logic.

To my mind it is unfortunate that many elementary logic texts concentrate on the correctness of arguments in only one sense, largely neglecting the other. To those who would focus on techniques for constructing demonstrative arguments, i.e., on the rules of correct reasoning, I suggest that equal emphasis on logical consequence may provide a valuable touchstone [Mates, 1964, pp. 134-136]. Failure to attend to logical consequence as a touchstone of correctness for rules has led to serious confusions and even outright mistakes [cf. Parry]. Those who focus on premise-conclusion arguments treat the techniques for constructing demonstrative arguments as "devices" for ascertaining (not understanding) the validity of premise-conclusion arguments or else eschew such techniques altogether in favor of proof-procedures. These writers generally give as an *excuse* for introducing a system of rules for first-order deduction the fact that no mechanical method exists for checking the validity of first-order premise-conclusion arguments. To these writers, I suggest that the study of deductive reasoning is important in itself.

Finally, at the cost of incurring the wrath of the antiquaries and for their benefit, I suggest that after the smell of the red herring of categorical statements has diffused, it will be clear that what Aristotle meant by *syllogism* includes both valid premise-conclusion arguments and sound demonstrative arguments and what he meant by *perfect syllogism* is sound demonstrative argument [Ross, p. 292].

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JOHN CORCORAN.

STATE UNIVERSITY OF NEW YORK AT BUFFALO.

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