Fragments of Science: Festschrift for Mendel Sachs
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INFORMATION-THEORETIC LOGIC AND TRANSFORMATION-THEORETIC LOGIC

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There is a symbiotic relationship between science and logic. The rational activities of scientists provide the content of logic. Logic, after assimilating and codifying that content, provides science with an account of its logical foundations. But when logic turns its foundational probings back on itself, it discovers problems it can not now solve. The opposition between information-theoretic and transformation-theoretic approaches to foundations of logic raises profound ontic and epistemic issues concerning the grounding of the two most fundamental logical activities: that of determining that a given conclusion is a consequence of, or is implied by, given premises; and that of determining that a given conclusion is independent of, or is not implied by, given premises.

1 Introduction

Greetings to our guest of honor, Prof. Mendel Sachs, to our honored guests, to our civic leaders, to our academic leaders, and to all in attendance. Thank you for being here for my presentation.

My field is philosophy and my specialties are in logic: history of logic, philosophy of logic and mathematical logic. Today I would like to discuss with you the role of logic in science and the role of science in logic. Toward the end of my presentation I hope to clarify for you some of the deepest problems in philosophy of logic. I will introduce a constellation of problems that has at its core a single ontological problem and a pair of epistemological problems.

As you know, ontology seeks to determine the nature of things as they are in themselves without regard to how or even whether they are known, whereas epistemology seeks to determine how our knowledge comes about. For example, we can ask the ontic (or ontological) question what is matter or, say, what is the siphon. And we can ask the epistemic (or epistemological) question, how do we know that matter exists (and that our belief in matter is not an illusion) or we can ask how do we know that the phenomenon explained in terms of the siphon really comes about through the siphon (and not by some other mechanism).

2 The unity of science and logic

In a celebration of science, it is fitting that logic be mentioned in a substantive way because without logic there could be no science—logic is used in comprehending, testing, confirming, and refuting scientific hypotheses, and logic is used in the construction of mathematical proof without which the other sciences would be held back.

Moreover, without science there would be no field of logic. If it had not been for the rational activities of scientists and mathematicians, it is unlikely that logic would have been discovered. The development of mathematics, especially number theory, and the development of physics, especially geometry, have provided the practical content of logic—because logic aims to reduce to theory the rational practice found in science. By making known the theoretical ideals implicit in the practice of science, logic serves as a critic of science. When logic criticizes science, logic says to science: you can do better, you have done better, live up to the potential you have already demonstrated. Thus, logic is the patient and objective observer of science, the student of science, who learns science's lessons well; so well that logic is justified in becoming science's teacher.

Almost every great logician was also either a scientist, a mathematician, or a person steeped in scientific and mathematical knowledge. The fact that Aristotle, the founder of logic, was investigating the underlying logic of science and mathematics is suggested by the fact that in the *Prior Analytics* he mentions the proof of the incommensurability of the diagonal with the side no less than eight times. As other examples of scientifically or mathematically informed logicians I can mention Galen, Ockham, Boole, Peano, Russell, Hilbert, Gödel, Tarski and, of course, Alonzo Church, who received an honorary doctorate from our University in 1990 in connection with a symposium much like this one.

The field of science, and this includes what we now call mathematics, and the field of philosophy, which has included logic since ancient times, are both manifestations of human striving. As Aristotle first observed, every human by nature desires to know. The class of scientists and the class of philosophers are not mutually exclusive and neither is exclusionary. On the contrary, each of us is a scientist to some extent though few of us concentrate on science enough to warrant the honorific title of scientist. Likewise, each of us is a philosopher to some extent though few of us aspire to contribute to the field of philosophy. This makes Mendel Sachs a triply rare human being, at once rare for being a scientist, again rare for being a philosopher, and most rare of all for combining two demanding callings. Mendel Sachs exemplifies the theme which was touched on above and which pervades my presentation: the theme of the unity of science and philosophy in the human spirit.

It is a happy coincidence that Mendel Sachs retires in 1997, on the 150^{th} anniversary of the publication of the seminal work of another scientist-philosopher. I refer of course to George Boole (1815-1864) and to his 1847 work *Mathematical Analysis of Logic* which was to revolutionize our thinking about logic. Boole's work revolutionized logic not by destroying or rejecting the Aristotelian legacy but rather by helping us to understand better what Aristotle was trying to do and what Aristotle was trying to tell us, and by helping to advance our progress toward the goals that Aristotle set for us.

3 The axiomatization of number theory

Information-theoretic logic, or information logic, for short, contrasts with transformation-theoretic logic, or transformation, logic for short. These are not two *systems* of logic, but rather two conceptions of modern classical logic, two competing ways of understanding the system of logic that is the currently accepted version of the logic that is incipient in the seminal work of Aristotle, the logic toward which Aristotle aimed.

Instead of speaking of information logic or of transformation logic we should speak more fully of the information-theoretic philosophy of logic and of the transformation-theoretic philosophy of logic. To grasp the opposition between these two philosophies we need to sketch what they agree on. Every disagreement presupposes agreement. Without at least a core framework of mutually accepted concepts, principles and methods, there would only be failure to communicate. The broader and more comprehensive the agreement, the richer and more significant the disagreement.

To exemplify what information logic and transformation logic agree on we can recall the axiomatization of number theory due originally to Giuseppe Peano (1858-1932) in the 1880s and the refinement of it due to Kurt Gödel (1906-1978) in the 1930s. As you know number theory, or higher arithmetic, is the branch of mathematics that takes as its domain the class of natural numbers. Peano took unity, the number one, to be the first natural number; but Gödel took zero. The difference is insignificant. I follow Gödel's convention whereby the natural numbers are zero, one, two, and so on; zero and the positive integers.

Number theory traces its origins to antiquity. In the classical periods in Europe, especially Greece, and in Asia, especially China, results were established that are still admired, not as signs of the incipient intelligence of primitive peoples but as respectable scientific advances. This is not the place to give examples of the most intricate of the ancient theorems but a simple example might give a hint. It was known over two thousand years ago that the sum of any given number of consecutive odd numbers beginning with one is the square of that given number. Take the first two odd numbers; one and three. One plus three is the square of two. Take the first three: one, three, and five. One plus three plus five is the square of three.

The field of number theory has flourished and in the last few years number theorists have solved problems that had been open for centuries. Indeed the sophistication of the ancient number theorists can be gauged by the fact that they were able to formulate hypotheses that are still hypotheses to this day; they worked to settle questions that have not been answered to this day. The proposition known as the perfect even hypothesis (to the effect that every perfect number is even) is still a hypothesis, not known to be true and not known to be false.

By the time Peano came on the scene number theory was already a vast science with dozens of concepts: number, zero, unity; the successor function, the squaring function, the Fibonacci function, and so on; the properties of being even, being odd, being square, being prime, being composite, being perfect, and so on; the operations of addition, multiplication, and so on; the relations: exceeds, precedes, divides, is a multiple of, is a power of, and so on. Besides its concepts it had hundreds of theorems and thousands of proofs, not to mention the growing treasure of open problems including hypotheses. As above, I use the word 'hypothesis' to mean "proposition not known to be true and not known to be false".

The thousands of thinkers who developed this science knew intuitively that a proof is a discourse which establishes the truth of its conclusion; they knew that a proof establishes that its conclusion follows logically from propositions already known to be true. It was also tacitly known at first, and later made explicit by Aristotle, that a proof is constructed by chaining together obvious inferences; the non-obvious, complex inference is made obvious by reducing it to a chain of obvious, simple inferences. To establish that a given conclusion follows from a given premise set it is sufficient to chain together obvious cases of conclusions following from premise sets. One point that is of crucial importance here is that throughout the course of the development of number theory mathematicians repeatedly made affirmative and negative judgements of logical implication. And these judgements constituted a practice quite separate from any deliberately constructed logical theory. In logic as elsewhere, practice comes before theory.

Now comes Giuseppe Peano. This amazing Italian humanitarian, logician, geometer, analyst, reduced all of the known propositions of number theory to five axioms and a series of definitions. Peano showed that the vast information content of number theory could be concentrated in a small kernel of propositions. In the process he showed how to reduce all of the concepts of number theory to three: the concept of number, the concept of zero, the concept of the successor. (The successor of zero is one, the successor of one is two, the successor of a given number is the number immediately following the given number.) It is easy to define successor in terms of number, addition, and one; Peano defined addition in terms of number, successor, and one. Among all of Peano's stunning successes his reduction of number theory is just one, but it is the one that creates a permanent place in history for him-regardless of what judgements future logicians may come to make about the correctness of Peano's details.

Ironically, Peano contributed to the development of modern logic whose progress and objectivity require it to make somewhat negative judgements about Peano's own work. The students can be taught so well that they are able to find flaws in the teachings. This is what every teacher hopes for: to be instructed by the student. This is what science gets from logic.

In the first place, there is an awkwardness, perhaps a real mistake, in Peano's system due to the fact that Peano did not know about universes of discourse. Gödel's correction of this flaw has as a by-product the elimination of two of Peano's axioms, or rather the ability to get along with only three axioms. In the second place, even apart from ramifications of the awkwardness, modern logic finds an inescapable mistake in Peano's system. In fact, number theory is *not* reducible to Peano's actual system but to what Peano's system becomes once the mistake is corrected. The mistake is that the simple concept "successor" is treated as a complex, "one plus," and consequently the remaining "additions" ("two plus," "three plus," and so on) are not treated at all, leaving a gap in Peano's system. Whether *you* will regard this mistake as major or minor is partly a matter of taste and judgement.

Let us turn to Gödel's three axioms: the zero axiom, ZA; the successor axiom, SA; the induction axiom, IA. The zero axiom, ZA, is to the effect that the successor function does not carry any number to zero. We take it to be the following number-universal proposition: given any number, its successor is not zero. The successor axiom, SA, is to the effect that the successor function is one-to-one. We take it to be another number-universal proposition: given any two distinct numbers, the successor of one is distinct from the successor of the other. The induction axiom, IA, is to the effect that every number other than zero can be obtained by applying the successor function finitely many times to zero. In accord with Gödel we take it to be the following property-universal proposition: every property belonging to zero and to the successor of any number to which it belongs also belongs without exception to every number.

Of course each of these axioms has its own set of consequences, or implications; each of the three pairs has its own set of implications; and there are propositions that follow from all three together but not from any one alone or from any pair.

For example, the induction axiom by itself implies the proposition that every number is either zero or a successor. This proposition, that every number is either zero or a successor, which has come to be known as the Robinson axiom, RA, of course, does not imply the induction axiom. IA implies RA, but not conversely.

For a proposition that follows from the three but not from any two we can take the so-called "no-fixed-point principle," NFP, to the effect that the successor function never carries a number back to itself, explicitly the proposition that no number is its own successor. Again this example is like the previous one: the Gödel Axiom Set implies NFP, but NFP does not imply any one of the Gödel axioms.

Like any other axiom set, the Gödel Axiom Set has obvious consequences readily determinable by logical intuition. It also has hidden consequences that nevertheless have been determined by chaining logical intuitions. And it has hidden consequences that have yet to be determined. For example, using Peano's definitions it is possible to reduce the perfect even hypothesis to a proposition involving only the concepts "number," "zero" and "successor" that may well be a hidden consequence of the Gödel Axiom Set; no one knows. In fact, one of the discoveries that came out of the 1931 Gödel paper is that there is no algorithmic method to determine of an arbitrary number-theoretic proposition whether it is a consequence of the Gödel Axiom Set.

4 Substitutional transformations

In one sense the subject matter of the Gödel Axiom Set can be taken to be its triple of concepts, "number-zero-successor." By substituting "new" subject-matter, propositions are transformed into different propositions having the same [logical] form and, as a rule but not in every case, having different information. For example, when the subject matter is changed to "integer-two-square," the zero axiom transforms to a true proposition to the effect that two is not a square, but the successor axiom transforms to a false proposition to the effect that distinct integers have distinct squares. The distinct integers, minus one and plus one, have the same square, of course.

The exceptions to the rule that changing subject matter changes information content are the degenerate cases; tautologies, which contain no information, and contradictions, which contain "all" information. Each tautology, as you know, is logically implied by each and every proposition without exception. Each contradiction, as you know, logically implies each and every proposition. This is classical logic, of course.

Not only is every tautology transformed into a tautology by every substitutional transformation, every contradiction is transformed into a contradiction by every such transformation. But the most important point is that these transformations preserve all logical relationships: if one given proposition implies or contradicts a second then, given any transformation, the same relationship holds between the transform of the first and the transform of the second. This gives rise to the far-reaching *economy of thought* permitting knowledge of logical relationships among one set of propositions to be transferred to propositions having an entirely or partly different subject matter. In particular, these transformations can be used to establish logical independence, i.e., to establish that one given proposition is not a consequence of a second. Since no false proposition is a consequence of a true proposition, in order to show that one given proposition is independent of a second it is sufficient to find a transformation carrying the first to a falsehood and the second to a truth.

Whenever it is intuitively obvious in a given case that one proposition is logically independent of a second, it is easy to find a substitutional transformation that confirms our logical "intuitions." For example, the induction axiom, to the effect that every number is either zero or is generated from zero by applying successor, is obviously independent of the Robinson axiom, to the effect that every number is either zero or the successor of a number. Substitution of "integer" for "number" carries the induction axiom to a falsehood (negative one is neither zero nor obtained from zero by applying successor). But this transformation carries the Robinson axiom to a truth. By the way, what was described above as substitution of "integer" for "number" is more properly described as substitution of the triple "integer-zero-successor" for the triple "number-zero-successor" because the concept of "successor on the integers" is not the same as the concept of "successor on the natural numbers," quite apart from the vexed question of whether the integer zero is different from the natural number zero.

This method of proving independence, the method of countertransformations, which is found already in the writings of Aristotle, has been used in modern times to prove the independence of the parallel postulate in geometry and the independence of the continuum hypothesis in set theory. It is known by various names according to peculiarities of the various forms it takes: the method of countermodels, the method of counterinterpretations, the method of counterarguments. It is so widely used that many logicians and mathematicians have stopped relying on their logical intuitions where judgements of independence or non-implication are called for. There is a subtle irony here because it is unlikely that the method would ever have been accepted if it had not been found to agree with logical intuitions in all cases where logical intuition was conclusive.

In fact, some recent logical writings have denied any scientific or cognitive basis for logical intuitions not grounded in transformations. These transformation-theoretic writings assert that any intuitive judgement of independence, which is a genuine cognition and not just a guess, is really an application of the method of countertransformations. Interestingly, even though this transformation-theoretic conception of independence judgements is recent, when we go back to Aristotle's writings we find nothing that contradicts it, as might have been expected given that Aristotle repeatedly uses a form of the method of countertransformations.

5 The ground of logical intuitions: information contents or transformations?

Information logic and transformation logic agree on absolutely every question of implication or independence among the propositions of number theory and, in particular, among all propositions involving only the three concepts "number-zero-successor." Whether information logic says that a certain conclusion follows from a certain premise set or that a certain conclusion is independent of a certain premise set, transformation logic agrees. They also agree on the unknown cases, of which there are infinitely many, some having been with us for centuries.

Without trying to sound paradoxical we can say the following: although information logic and transformation logic agree on every question of implication or independence, they disagree both on *what* it is they are agreeing on and on *how* they came to have the agreed on knowledge. To be more explicit, they disagree on the ontic question of the nature of implication and they disagree on the two epistemic questions of how knowledge of implication is arrived at and of how knowledge of independence (or non-implication) is arrived at.

Information logic appeals to the information contained in each proposition. It says that implication is entirely a matter of information containment: in order for a given premise set to imply a given conclusion it is necessary and sufficient for the information of the premise set to contain all of the information of the conclusion. When a premise set implies a conclusion, once the premise set has been asserted no information is added by asserting the conclusion. But when the premise set does not imply the conclusion, there is information added by asserting the conclusion in the presence of prior assertion of the premise set.

For example, once the induction axiom has been asserted no new information is added by asserting the Robinson axiom. But the Robinson axiom contains only a small part of the information contained in the induction axiom.

Information logic says that implication is an informational relationship depending exclusively on information contents of propositions. This is the ontological point. Epistemologically, it postulates a human faculty of comparing information contents and of judging containment and non-containment.

When we focus solely on the deductive process of establishing implicational relationships, on showing that given premise sets imply given conclusions, no transformations are in evidence and information logic seems to carry the day by explaining deduction as information processing.

Transformation logic, to the contrary, says that implication is a transformational relationship: in order for a given premise set to imply a given conclusion it is necessary and sufficient for every transformation carrying the premises into truths to carry the conclusion into a truth. Put negatively, this is the famous no-countertransformation-conception of implication: the existence of a implicational relation from a premise set to a conclusion is the absence of a countertransformation, carrying the premises to truths and the conclusion to a falsehood. Transformation logic reduces implication to a relationship connecting the premise set and conclusion pair, on one hand, to the class of all transformations, on the other. This is the ontological point. Epistemologically, transformation logic postulates a human faculty capable of surveying all transformations and judging truth-values of the transforms. For the transformation theorist, the assertion that the induction axiom implies the Robinson axiom is an assertion about all transformations, that every transformation carrying the induction axiom to a truth automatically carries the Robinson axiom to a truth.

When we focus solely on independence results establishing the absence of implication relations, establishing that a given conclusion is not implied by a given premise set, no appeal to information seems relevant and transformation logic seems to carry the day. The flood of independence results unleashed in Europe by the Italian School and on this continent by the American Postulate Theorists seems to create a strong presumption in favor of transformation logic.

The opposition between information logic and transformation logic could hardly be more diametrical. Their ontological positions and their ontological presuppositions seem irreconcilable. And when we come to their epistemological positions the situation becomes more extreme. Information logic seems strongest when dealing with establishing implications, i.e., when dealing with deductive reasoning, and at its weakest when dealing with establishing non-implication, i.e. independence. On the other hand, transformation logic seems strongest when dealing with independence and weakest when dealing with deduction.

6 Conclusion

This opposition between the information-theoretic conception of classical logic and the transformation-theoretic conception has thus raised one ontic question and two epistemic questions. The ontic question is this: what is implication? What is the ontic nature of the relationship most fundamental to scientific thought, without which scientific thought would be groundless? The epistemic questions are these: first, how does a human being come to know that a given conclusion is implied by a given premise set that implies it? second, how does a human being come to know that a given conclusion is independent of a given premise set that does not imply it? In other words, the epistemic questions are: how is it possible to validate an argument? how is it possible to invalidate an argument?

Where can we go for help to decide whether the answers are information-theoretic or transformation-theoretic, or neither? Aristotle gives us no help and the great modern logicians did not really address these questions either, although their writings clearly and unequivocally favor the transformation-theoretic viewpoint. There are two exceptions. The American Postulate Theorist, C.I. Lewis, whose co-worker C.H. Langford favored the transformation theoretic approach, was himself unable to enthusiastically adopt it. But the one great modern logician to explicitly oppose the transformational approach was Tarski's teacher, J. Lukasiewicz, who went so far as to criticize what he took to be Aristotle's tacit adoption of it.

Perhaps before these questions can be addressed we will need to deal with ontic and epistemic questions concerning the nature of propositions, and the source of our acquaintance with them and our judgements about them. Perhaps we will need to determine the nature of information and the relation of information to the propositions containing it. Perhaps we will need to determine the ontic status of transformations and the epistemic status of the ability to deal with them in the way so crucial to the method of countertransformations.

We are on our own here, as is usual with important philosophical issues. By grasping a philosophical question, we do not gain new knowledge in the sense of science but we come to understand much better what human capabilities we have and the depth of our own ignorance about the nature and origin of these capabilities.

7 Acknowledgments

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Crystallography
— Herbert Hauptman
Herbert Hauptman
Source of X-Rays
Crystal
B10H14
Boron
Hydrogen
=0 if r ≠ rj
The phase is lost in the diffraction
John Corcoran
John Corcoran
John Corcoran
F(EH)=F(|EH|, ΦH)=0
G(ΦH/|EH|)=0
Without mathematics, the Science cannot be understood,
nor made clear, nor taught, nor learned