Axiomatic methods: Lawvere's mathematical interpretation of Hegel's logic

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Abstract

Hilbert's axiomatic thinking was an influential philosophical model that motivated movements such as positivism in the early twentieth century in various areas within and outside philosophy, such as epistemology and meta-mathematics. Axiomatic formalism provides through the use of first order logic an important foundation for formal logic models, which for Hilbert would represent a universal model of empirical research, not only for mathematics, but for all natural sciences, and by the positivist view, also philosophy. However, in the more specific case of mathematics, there is a certain lack of communication between the foundations of mathematics and its practice, where informal methods still promote elegant tools for mathematicians in various areas, including when certain paradigms try to be broken. Exactly this asynchronicity between the foundationsof mathematics and its practice that we will investigate in this study. Lawvere, dissatisfied with the "unfounded foundation" of the axiomatic method proposed by Hilbert, and inspired by Hegelian dialectics sought to revise the foundations of mathematics by categorical logic and Category Theory. We see in this study how Lawvere's interpretations of Hegel's logic, such as equivalence, unity of opposites and "*aufheben*", allow a new mathematical approach with a philosophical positioning that seeks, in a way, to transcend the dichotomy between analytical and continental schools. Lawvere treats Hegel's objective logic as a possible strategy to solve the problem of logical grounding in metaphysics. Finally, we see how Lawvere's contributions to the axiomization of categorical logic have had innovative impacts on meta-mathematics, especially in the development of Vladimir Voevodsky's univalent foundations.

Keywords: Axiomatic methods, Hegelianism, Lawvere, Categories, Homotopy.

Axiomatic Method in Hilbert's Vision

The axiomatic method, also known as "axial thinking" was a philosophical and mathematical model prominently advocated by David Hilbert (1996), where the concept of axiomization in epistemological terms would logically and incontestably be the only way to "think with conscience", that is, to rationalize. In Hilber's words:

> If we consider more closely a particular theory we always see that some basic proposals underlie the construction of the concept framework, and these proposals are then processed by themselves for the construction, according to logical principles of the whole framework/theory. These fundamental propositions can be considered as the axioms of a theory of knowledge: the progressive development of each proposition of knowledge then resides only in the logical construction of the framework of already assumed axioms. This view is especially prevalent in pure mathematics. Anything that can be the object of scientific thought becomes dependent upon the axiomatic method, and thus indirectly upon mathematics (HILBERT, 1996, p. 1108).

Hilber's idea of logic can be interpreted as the concatenation of propositions, each one derived from the previous ones down to the axiomatic basis of the system, its prioris, in accordance with

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the rules of a logical system. Thus, axiomatic thinking is a form of vehicle that allows the thinker to externalize into a proper language, and supposedly "free" of ambiguity, the implications that such premises cause rational thinking to arrive (BOUBRBAKI, 1950). In this way axiomatic formalism provides what we could not achieve by using logic alone, giving importance to axiomatic foundations.

The informal idea of axiomatic thinking is that: axioms are important, without them we do not know about "what" we are talking about. Axioms allow us to restrict the space of possibility, for example: through the axioms of Peano Arithmetic we can define what natural numbers are: defining 0 as a natural number, being the only one that does not have a successor (S), and the succession of "entities" after 0, informally as follows: the number 1 can be defined as S(0), 2 as S(S(0)) (which is also S(1)) and, in general, any natural number n as $S^n(0)$. in other words, $\{0, S(0), S(S(0)), ...\} \subseteq \mathbb{N}$, where all natural numbers are contained in $\mathbb{N}$ (PEIRCE, 1881).

Hilbert's enthusiasm for the Axiomatic Method, though not universally accepted, is still the pattern shaping the modern notion of axiomatic theory, and his *Fundamentals of Geometry* is still considered one of the paradigms for the fundamentals of mathematics and philosophy (HILBERT, 1899). In comparison with other proposed models, Hilbert's mathematical philosophy, called Formalism, differs from, for example, Brouwer's intuitionism, and Russell's rationalism. We may ask ourselves: how well has the axiomatic method impacted and contributed to practical mathematics? On the one hand, Hilbert's formalism remains the standard method of building physical and mathematical theories, yet in both mathematics and the natural sciences, advances in these areas remain being made by "informal" methods, that is, they somehow escape (intelligently) from formal axiomatic logic methods (RODIN, 2012).

Nowadays there is a somewhat paradoxical situation: Hilbert in his Foundations of Mathematics (1967) suggested that his Formal Axiomatic Method be used as the basic instrument of all scientific, and even philosophical, research, and when we speak of contemporary axiomatic methods we refer to Axiomatic Set Theory. However, this is not what has historically happened. The consensus about the usefulness of the axiomatic method among several areas of natural sciences is: the Axiomatic Method matters only in the fundamentals, that is, in the meta-theory, while conventional science cares very little about its own fundamentals. Thus the question of the foundations of mathematics, for example, is left to logicioans and philosophers, while the realisation of mathematics itself is left to mathematicians.

This way of interpreting the problem of an apparent asynchronicity between the foundations of mathematics, and the practice of mathematics, may sound unsatisfactory to the reader, and indeed to many other theorists, mathematicians, and philosophers. A more satisfactory notion of foundation is described by Lawvere and Rosebrugh (2003) as follows:

> A foundation explains the essential general characteristics, ingredients and operations of a science, as well as its origins and general laws of development. The purpose of making them explicit is to provide a guide to the learning, use and future development of science. A "pure" foundation that forgets this purpose and pursues a "speculative" foundation by itself is clearly an unsupported foundation (LAWVERE, ROSEBRUGH, 2003, p. 235).

To clarify the limitations of an, in Lawvere's words, "unfounded foundation", it is interesting to analyze situations where the axiomatic model fails to provide a satisfactory answer. Since Zermelo's revolutionary work (1903) which in the future gave rise to the Axiomatic Set Theory (Zermelo-Fraenkel Set Theory), modern mathematics has used its formalism and language as the basis for its identity, which makes set theory an example of a theory developed entirely within a formal axiomatic setting. A famous example of a problem that eludes Zermelo-Fraenkel's formalism is one of the 23 problems listed by Hilbert (1902), and considered by him to be the most important, the Continuum Hypothesis (CH) (1892).

Continuum Hypothesis

CH is a mathematical problem where until today there is a certain ambiguity about whether the problem was "really" solved or not. This situation occurs, besides the considerable difficulty of the problem itself, perhaps to the fact that the Zermelo-Fraenkel Set Theory is based on a formal axiomatic environment. The Continuum Hypothesis concerns the cardinality of the number of elements within a set, and if there is a bijection, a one-to-one correspondence between the elements of two different sets (COHEN, 2008).

Intuitively, two sets X e Y has the same cardinality if it is possible to "pair" the elements of X with elements of Y so that each element of X corresponds with exactly one element of Y and vice versa. The set {apple, watermelon, orange} has the same cardinality as {square, triangle, circle}. CH tries to answer the question if it is always possible to find a bijection between sets with infinite elements. For example, the set of Natural numbers, has the same cardinality as the Integers, and the same corresponds to the Rationals, and there is a one-to-one correspondence between the sets mentioned, all having the same cardinality, all being countable, for example:

$$N = 1, 2, 3, 4, 5, 6, 7, 8, 9, \dots \to \infty$$

$$I = 1, -1, 2, -2, 3, -3, 4, -4, 5, \dots \to \infty$$

Both sets, $\mathbb{N}$ (Naturals) and I (Integers) can be listed and paired by a bijection, therefore, both infinite sets have the same cardinality, both have the same "type" of infinity.

However, is there a bijection between the Natural and the Real numbers? This is the question that inspired Cantor to conjecture his Continuous Hypothesis. Through his diagonal argument

Cantor proved that the cardinality of the set of Natural numbers is strictly smaller than that of the set of Real numbers. However, Cantor's proof doesn't indicates the extent to which the cardinality of Natural numbers is smaller than that of Real numbers. Thus, Cantor proposed the Continuous Hypothesis as a possible solution to this issue (CANTOR, 1892), stating:

There is no set whose cardinality is strictly between that of Natural and Real numbers.

Cantor's Diagonal argument

The Diagonal argument method is an elegant and powerful tool that has influenced several other important proofs in areas such as logic and meta-mathematics, sucha as Gödel's Incompleteness Theorems and the incomputability of the Halting Problem. We will show below directly the method, which through its understanding, allows the extension to the other cited problems and paradoxes.

First, we will assume that the Real numbers can be paired in a one-to-one correspondence with the Natural numbers, which will lead us to a contradiction. Suppose we can list all the numbers between 0 and 1 that can be represented by decimals (the method works for any numerical basis, like binary), like 0.25, or decimals that repeat infinitely, like 0.123123123... For convenience we will use infinite representations in all cases, so that we attach an infinite number of zeros at the end of each outgoing Real number. Now suppose we list all the Real numbers, all the infinite decimal representations:

$$\begin{array}{l}
0.\mathbf{0}000000000000\ldots \\
0.1\mathbf{0}00000000000\ldots \\
0.50\mathbf{0}0000000000\ldots \\
0.333\mathbf{3}333333333\ldots \\
0.6666\mathbf{6}66666666\ldots \\
0.25000\mathbf{0}0000000\ldots \\
0.750000\mathbf{0}000000\ldots \\
0.2000000\mathbf{0}00000\ldots \\
0.40000000\mathbf{0}0000\ldots \\
0.600000000\mathbf{0}000\ldots \\
0.8000000000\mathbf{0}00\ldots \\
0.16666666666\mathbf{6}6\ldots \\
0.833333333333\mathbf{3}\ldots
\end{array}$$

Imagine that this list is infinitely long, and contains all possible decimal representations of Real numbers. Now we will manipulate the red diagonal highlighted in this list. The diagonalization technique shows us how to build a number out of this list, digit by digit, a representation that is not in the list, even though it is supposed to be the list of "all possible Real numbers". There are several procedures to achieve the same result, however, we will use the following algorithm: find the i-th Real number and its i-th decimal representation, then add "1" to this number, if it is a "9", turn the number into a "0". This procedure would produce the Real number:

0.1114711111174. . .

The number produced cannot be found at any position in the list. By definition the number created differs from the first Real number on the first place after the decimal point, differs from the second Real number on the second place after the decimal point, and so on. However, the list should contain all the Real numbers, we could simply add the number created in the list. But at the same time, the diagonalization technique can simply be repeated again, creating numbers that, at first, are not listed in the list of all Real numbers, what leads us to a contradiction. Our mistake was in assuming that the Real numbers could be listed, and they can't, therefore they are countless, innumerable.

Cantor (1892) used this method to prove the result that the Real numbers are not countable, and that no infinite list of decimals can contain all their elements. Thus, any such list must be incomplete, there being no bijection between the Real numbers and the Natural numbers, which implies the same difference between the Real numbers the Integers and the Rational numbers too, making these sets cardinally different. Formally, the cardinality of the Real numbers is much, much greater than that of the Natural numbers:

$$\mathbb{R} \gg \mathbb{N}$$

This result caused a split in Zermelo-Fraenkel's axiomatic system, because in 1938 Gödel (1938) proved that Zermelo-Fraenkel's axiomatic system is consistent with CH. However, in contrast, in 1963 Cohen (1963) proved that the Zermelo-Fraenkel set theory is also consistent with the negation of CH. Therefore, what is modernly accepted is that the continuous hypothesis, or its denial, can’t be derived from Zermelo-Fraenkel's axioms, being a matter of total ambiguity, depending on which Zermelo-Fraenkel model you want to adopt (KUNEN, 1980). This controversial result shows the independence of the CH from the very foundations of mathematics, which makes such foundations questionable. Perhaps new axioms, or some totally new system of axioms for the established theory may eventually help to establish the truth or denial of CH, however, this still remains an open question.

In the axiomatic model proposed by Hilbert, like Zermelo-Fraenkel, the only thing mathematics can do according to this point of view is: provide true proposals of the kind if - then, if the deductible proposals are true given the axioms of the system in question. The absence of dialectics makes this model incompatible not only with common mathematical practice, but with the practice of meta-mathematics, as in the example of CH, since its proposition is not expressed in the "if-and-then" form, but as an "absolute" truth, which does not refer to any particular formal structure. Thus, Hilbert's axiomatic method is not able to differentiate between an axiomatic theory that contains valuable knowledge of cyclical theories without any epistemic value, since it only presents a logical form of how to proceed given certain axioms. Surely Hilbert's progress has enabled great progress in mathematical theories constructed with these formal logical systems, but he does not serve, as Hilbert suggested, as a foundation on which to build all scientific theories.

In this study we will review the work of the mathematician William Lawvere, who, inspired by Hegel's logic and dialectics sought to explored new forms of foundations in mathematics with the use of categorical logic and Category Theory.

New Forms of Logic

With the rise of analytical philosophy in the 20th century, and the adoption and development of the new logic forged by Frege and Peano, came a radical rejection of the existing philosophical paradigm, the idealism of Hegel. Hegel's understanding of the term "*Logic*" was much broader than is usually the case, and his "objective logic" proposed on his “Science of Logic” was considered a kind of metaphysics by the positivist movement. An embodiment of the positivist criticism of metaphysics was made by the movement known as the Circle of Vienna, where the aim was to reconceptualize empiricism from the new scientific discoveries and demonstrate the falsehoods of Metaphysics. The logical positivism, later called logical empiricism, and also known as neo-positivism, was a philosophical movement whose central thesis was the principle of verification, where it was stated that only facts that were verifiable through direct observation or logical proof are significant, discarding metaphysics as meaningless (CARNAP, 1932).

Bertrand Russell was categorical in stating that the new logic would break the philosophical paradigm established by the logical idealism of, for example, Kant and Hegel:

> The old logic puts thought into shackles, while the new logic gives it wings. It has, in my opinion, introduced the same kind of advance in philosophy that Galileo introduced in physics, making it finally possible to see what kinds of problems may be capable of solution, and what kinds are beyond human powers. And where a solution seems possible, the new logic provides a method that allows us to obtain results that not only incorporate personal idiosyncrasies, but should command the consent of all who are competent to form an opinion (SULLIVAN, 2003, p. 277).

The positivist doctrine somehow summarized the vision of analytical philosophy about Hegel's, and perhaps other continental thinkers, work for several decades. Reichenbach wrote:

> Hegel was called the successor to Kant; this is a serious misunderstanding of Kant and an unwarranted elevation of Hegel. The system of Kant, though proved unsustainable by later developments, was the attempt of a great mind to establish rationalism on a scientific basis. Hegel's system is the poor construction of a fanatic who saw an empirical truth and tried to make it a logical law, within the most unscientific of all logics. While Kant's system

> marks the peak of the historical line of rationalism, Hegel's system belongs to the decadence of speculative philosophy that characterizes the nineteenth century (REICHENBACH, 1951, p. 72).

Although the analytical philosophy has rejected Hegel's metaphysics and logic in favor of the analysis of mathematical logic, in particular propositional logic, recent developments in the foundations of mathematics through Category Theory, use Hegelian logic to suggest a new form of mathematical thinking, not by propositional logic, but by modal logic.

Modal logic refers to an enrichment of propositional logic where standard operations (&, ¬, →, ≡, etc) are accompanied by certain extra operations called modal operators, denoted by "□" or similar symbols. Thus, a proposition "□P" is a new proposition whose interpretation depends on the type of modal operator, for example: "P is possibly true", or "P will eventually become true", or "P could be true" (GARSON, 2018). There is no established axiom what a modal operator must satisfy in propositions, allowing great flexibility and applicability to modal logic techniques.

In the 20th century the rise of modal logic allowed for a resurgence of analytical metaphysics, for example: In Lewis' theory of counterfactuality, also known as Lewis-Stalnaker's model of possible worlds (LEWIS, 1973), a logical proposition consists in its realization in all possible neighboring (similar/nearer) worlds, and through a modal operator, ">", where X > Y means, "If it were X, then it would be Y", which allows the possibility of bringing meaning to these possible worlds. Currently topics of analytical metaphysics include causality, necessity, space and time, identity, and such concepts are only expressed in a formal way, in philosophy, through modal logic and not propositional logic.

Given the limitations of analytical philosophy, much due to its foundation in first order propositional logic, and despite the long initial rejection of Hegelian ideas, Hegel's writings have been placed under a new perspective, as, for example, by mathematician Francis William Lawvere, who is an American mathematician known for his work on category theory, topos and philosophy of mathematics. Inspired by the concept of "Unity of Opposites" present in Hegel's logic, Lawvere sought to formalize categorical logic by the dialectics of Hegel's logic, a link that led modal logic and category theory to give rise to the Homotopy Type Theory, a theory that promises to modify the foundations of mathematics (LAWVERE, 1991)..

From the Dialectic Triad in Kant to the Aufhebung in Hegel

The "Science of Logic" (1st ed. 1812-1816, HEGEL, 2010) can be considered as one of the main texts of Hegel's philosophy, as the "Phenomenology of the Spirit". The Science of Logic has to be seen in the context of the philosophy of the early 19th century, being a form of response to Kant's “refoundation” of metaphysics. The idealistic system proposed by Kant changed much of the metaphysical and epistemological visions of the time, but it left traditional logic untouched, and precisely at this point that Hegel proposes to extend the critical examination of the foundations of knowledge initiated by Kant to logic itself.

The "*rediscovery of the dialectic triad*" that Hegel attributes to Kant (1763), provides a preliminary concept of the distinction between contradictions and real oppositions, something that anticipates Hegel's analytic-synthetic division, the dialectic triad: divided into itself (subject), for itself (object), and into itself (object as experienced by the subject, and subject reflected to itself when experiencing the object) (HEGEL, 2010). The dialectic triad here originates the term "*Aufhebung*", the synthesis of oppositions, a Hegelian concept expressed in the use of the terms "*thesis*", "*antithesis*" and "*synthesis*”.

Kant had organized the synthesis of terms into positive and negative, something that for him would cause “reason” to become necessarily entangled in the contradictions of transcendental dialectics by its very nature, but for Hegel, this inconclusive interpretation would in fact be an indicator of the positive role of contradictions as an essential aspect of thought. Here dialectics and *Aufhebung* come into play, for Hegel's criticism of Kant would be that, in fact, categories are not given a priori, but rather as a result of continuous transformation, of *becoming*, thus, logic ceases to be an inventory of categories, but becomes a system of transformations of categories. Curiously, this Hegelian terminology would be appropriated by mathematics in the formal study of categories in the 20th century (EILENBERG; MACLANE, 1945). Hegel explains the concept of *Aufheben* (HEGEL, 2010, p.113) as one of the most important in whole of philosophy, repeating itself constantly everywhere, as, for example: in Spinoza's concept of "*omnis determinatio est negatio*". For Hegel, Aufhebung is the mode of this negation-affirmation coexistence.

Aufhebung is an extremely general concept which is one of the central buildings of the Hegelian thesis, and for Lawvere, this concept proposes interesting introspections for mathematics.

Lawvere's interpretation of Hegel's objective logic

Lawvere (2000) suggests a particularly simple example where the idea of equivalence is given a dialectical interpretation, allowing a new meaning for "*equivalence*" within categorical theory.

Let N be the natural numbers *{0,1,...}* seen as a category through their usual ordering. That $E, D: N \to N$ be two functors, a functor being a mapping between categories, “even” and “odd” defined by $E(n) \coloneqq 2n$ e $D(n) \coloneqq 2n + 1$.

Both categories correspond to two sub-categories of inclusions, *N-even* and *N-odd*. In this situation both subcategories "oppose" each other $N - even \neq N - odd$, imparesHowever, they are "*identical*" because there is a bijection, a one-to-one mapping from $N_{even} \to\simeq N_{odd}$. In addition, both are encompassed as part of a whole, the Natural Numbers, whose general structure can be represented by both parallel functors:

$$N_{even} \to\simeq \mathbb{N} \simeq\leftarrow N_{odd}$$

Normally, and perhaps analytically, opposing concepts are not considered equivalent, but in this interpretation of equivalence as a mapping between sets (bijection), or in Hegelian terms, a transformation, will always be a pair consisting of two subcategories, one reflexive and one co-reflexive. Lawvere (2000) suggests that if we add a third functor, that we call T, we can encapsulate the relationships between E and D forming a triple adjunct $E \dashv T \dashv D$, where this triple expresses the unity of the (co)reflexive subcategories:

- $E \circ T \dashv D \circ T$ representing the opposition between E and D at the same time;
- $T \circ E \simeq T \circ D$ representing the identity between E and D for the equivalence involved.

Informally, T unites, opposes and identifies E e D at the same time.

The above concept of adjunct expresses a duality, specifically, a duality between opposites. The concept of correspondence and equivalence in terms of triple adjuncts was suggested by Lawvere (1991, p. 7; 1994, p. 11) in order to formalize the concept of "*Unity and Identity of Opposites*" as they appear informally in Hegel's Science of Logic (HEGEL, 2010), capturing the notion of Hegelian dialectics.

The "unity of opposites": *aufhebung*, and Lawvere's categorical logic

To understand Lawvere's formalization of the concept of units of opposites, it is important to review Hegel's thesis on the Science of Logic, where Hegel provides an account of his objective logic as follows:

> What we are dealing with logically is not a thought about something that exists independently as a basis for our thinking and apart from it, nor forms that supposedly provide mere signs or distinctive marks of truth; on the contrary, the necessary forms and self-determinations of thinking are the content and ultimate truth itself (HEGEL, 1990, p.50).

One possible interpretation is that Hegel seeks a logic that will reason about the things with which it reasons. And this concept is in the essence of categorical logic, a type of formal modal logic in mathematics that allows the development of various concepts and applications for Category Theory and Type Theories, because in these disciplines there is no distinction between the objects of theory (elements, groups, categories, types, morphisms, functions) and the logic (propositions and proofs) used to reason on the objects of theory, something known as "*propositions as types*".

Lawvere (1991) makes this comparison of Hegel's objective logic with categorical mathematical logic, where logic takes the form of tools, which can also be considered objective elements/categories, a form of object that has functionality. In this interpretation, truth and falsehood are not considered opposite and disconnected states, but the ends of a transformation, truth being represented by a terminal object, and falsehood by an initial object. These principles can be formalized in terms of transformations (adjunctions), as in the example explained above where opposite pairs are adjuncts on the left (E) and on the right (D) of a common functor (T): falsehood and truth are opposite sides (E and D) of a terminal functor (T). We return to Hegel because the fundamental guiding principle of objective logic is the "*unity of opposites*", Aufhebung summing up to "*sublimating*" an opposition.

For Hegel the primary idea of opposition comes from the unity between emptiness, applying to nothing, and tautology, applying to anything, the unity between *Nothing* and *Being*. Thus, in the Hegelian vision, for the subject to entertain any idea it is necessary to be able to entertain its opposite, otherwise its idea is empty in the sense that it could apply to anything. Hegel explains it in the following way: *to speak even of Nothingness is to consider it as a thing, it is to make it to be, even being something without characteristics, a pure Being, the lack of content implies the absence of characteristics, therefore, the pure Being is Nothingness* (HEGEL, 2010, p. 59-60).

Lawvere's (1991) interpretation of this opposition can be interpreted by representing Nothing as the initial null object "$\emptyset$" and Being as the terminal object "1". In a category (*topos*) space, "$\emptyset$" is an empty category and "1" is a single point/element of a category. These are opposite in the sense that they are distinct but unified in the sense that they are the left and right adjuncts E and D of the same functor T. In Category Theory the adjunct is a relationship that two functors can have, when in this relationship they are called adjoint functors (KAN, 1958). For example, an adjunct among the categories A and B is a pair of functors E and D:

$$E: B \rightarrow A \; e \; D: A \rightarrow B$$

and for all objects X in A and Y in B a bijection between the respective sets of morphism is:

$$hom_A(EY, X) \cong hom_B(Y, DX)$$

so that this family of bijections has a one-to-one relationship between the elements X and Y. The functor E is called the left adjunct functor, or simply left adjunct to D, while D is called the right adjunct functor to E.

This logical formalism is essential for the definition of concepts such as "equivalence" in this area of mathematics, and Lawvere's idea of unity between oppositions is deeply inspired by the Hegelian opposition of Nothingness and Being. Just as Hegel says that we cannot think of something without its opposite, here we recover the opposite points of a dialectic transformation through the singularity of adjunctions.

Categorical logic and Hegelian dialectics

Lawvere's criticism of the Axiomatic Method proposed by Hilbert is supported by Hegel's philosophy, which, in addition to the *Aufhebung* concept cited above, uses the Hegelian distinction between objective and subjective logic. Lawvere's Hegelian perspective on mathematics, specifically categorical theory and logic, is crucial for a philosophical understanding of his work. Lawvere states that the appreciation of the importance of Hegel's dialectics in categorical logic does not imply the need to be a "*Hegelian*", but that the importance of the concept of dialectics as proposed by Hegel would be something of intrinsic value enough to transcend the "*endless battle*" between analytical and continental traditions. In Lawvere's words:

> It is my conviction that in the next decade and the next century, the technique forged by categorical theory will be of value to dialectical philosophy, lending precious formalisms with mathematical models to aid controversial ancient philosophical distinctions, such as: general vs. particular, objective vs. subjective, being vs. becoming, space vs. quantity, equality vs. difference, quantitative vs. qualitative, etc. The explicit attention of mathematicians to such philosophical questions is necessary to achieve the goal of making mathematics (therefore, other sciences) more widely learned and usable. Of course, this will require philosophers to learn mathematics and mathematicians to learn philosophy (LAWEVERE, 1992, p. 16.

Lawvere is primarily a mathematician that shares with other philosophers, including analysts such as Russell, the idea that mathematics allows philosophy to operate more clearly, a view that is sometimes denied by continental tradition. Thus his influence to philosophy by coming form another area, mathematics, seems unaffected by the analytical/continental dichotomy, which we believe is a sign of "philosophical maturity", that is, *the understanding that both sides possibly tell different versions of the same story*.

Hegel's distinction between objective and subjective logic is exposed in the following passage:

> What is to be considered is the whole *Concept*, firstly as the *Concept* in the form of being, secondly, the *Concept*; in the first case, the Concept is only in itself, the *Concept* of reality or of being; in the second case, it is the *Concept* as such, the *Concept* existing for itself. Thus, the logic should be divided primarily into the logic of the *Concept* as being and the *Concept* as *Concept*, or, using the usual terms in "*objective*" and "*subjective*" logic (HEGEL, 2010, p. 79).

The Concept of Hegel can be understood as a category that comprises both:

(i) reality (being);
(ii) thinking about reality (being thinking).

This notion forms Hegel's objective idealism, dividing logic into (i) the logic of Being, which can be compared to a kind of ontology, and (ii) the logic of thinking, which is commonly called logic. Hegel's objective logic, or Hegelian ontology, is not only limited to a notion of metaphysics, like Kant's transcendental logic, which for Hegel gave little importance to the objective part and focused only on the subjective portion of logic. As Hegel puts it:

> Recently, Kant opposed what is usually called another logic, that is, a transcendental logic. What has been called objective logic here would correspond in part to what is transcendental logic. Kant distinguishes it from what he calls general logic in this way, which deals with the notions that refer a priori to objects, and consequently does not abstract from the whole content of objective cognition, or, in other words, contains the rules of pure thought of an object, and at the same time deals with the origin of our cognition [...]. It is to this second aspect that Kant's philosophical interest is directed exclusively (HEGEL, 2010, p. 81).

Given the above passage we can make the following comparison: Hegel's objective logic "*corresponds in part*" to Kant's transcendental logic, or, Kant's transcendental logic is an "*approximation*" to Hegel's objective logic. For Hegel, Kant differentiates transcendental logic from "*general logic*" because transcendental logic is not totally unambiguous to the topic, which for Kant results in the implication that general logic is applicable in mathematics and empirical sciences, but not to the transcendental, so the interpretation of the transcendental would be a departure from logic to metaphysics.

One of the central ideas of the Hegelian thesis is to continue doing something logical, instead of moving on to speculative physics (metaphysics), by making logic *objective*. Hegel compares his objective logic with the traditional metaphysics in the following passage:

> Objective logic takes the place of ancient metaphysics, which was intended to be the scientific construction of the world only in terms of thoughts. If we take into account the form of this science, then it is the first and immediate ontology whose place is taken by objective logic [...] furthermore, objective logic also understands the rest of metaphysics, in the sense that it is an

> attempt to understand how pure thought forms, particular substrata taken mainly from the conception of the soul [...] however, it considers these forms free from these substrata, from the subjects of conception, considering them, their nature and their value, in their own character. [...]. The objective logic is therefore a genuine criticism of them [Kant]-a criticism of those who do not consider them as contrasted under the abstract forms of the a priori and the a posteriori [...](HEGEL, 2010, p. 85).

In this way, Hegel gets rid of the a priori versus a posteriori distinction of Kant, which somewhat diminishes the role of empirical data in the sciences, and suggests that the construction of theories about the world can be done only in terms of "*thoughts*", in a new dialectical form. The price of using this method, which compromises the empirical character of science, gives reason to criticism to the Hegelian thesis of objective logic.

The subjective logic, for Hegel, is the second stage of a single process of dialectical reasoning, which is made up of three parts: the logic of Being, the logic of the Essence and the logic of the Concept, the last part being called "*objectification of logic*". For Hegel the distinction between objective and subjective logic must be thought of as disputable, because in his thesis what is really aimed at is not the demarcation that divides both, but the transformations that interact between them resulting in the logic of the Concept (HEGEL, 2010, p. 86). This form of Hegelian philosophy is reflected in Lawvere's objectives, which are (i) to reformulate Hegel's dialectical logic in mathematical terms, by categorical logic, and (ii) to use Hegel's dialectical logic as a guide to mathematical research. However, Lawvere's ultimate goal is to unify (i) and (ii) into a single mathematical-philosophical project where the distinction between stages, (i) and (ii), is totally unnecessary, analogous to the objectification of logic, Hegel's logic of Concept (LAWVERE, 1992).

Lawvere (1993) distinguishes between objective and subjective logic in the context of categorical logic in the following way: objective logic - a guide to complex but non-arbitrary constructions, concepts and their interactions that grow from the needs of geometry and the study of *space* and *quantity*. We can replace the concept of "*space and quantity*" with "*any serious object of study*", which, like Hegel, defines for Lawvere the negation of objective logic from Kant's perspective, since the concept of space and quantity penetrate any type of scientific study, empirical or not. Subjective logic on the other hand deals with the inference between the propositions, being of interest only those that concretize the concepts, that is, objectify them.

The denial of Lawvere's objective logic, even though not fully representing it, uses some key ideas coming from Hegel: while Lawvere relates his objective logic to categories of space and quantity, Hegel develops all categories, after dialectic development, from the categories of *Being*, *Nothingness* and *Becoming*. Lawvere defines, from the concepts of "*space*" and "*quantity*", a "*theoretical category guide for conceptual constructions that grow out of any* *serious field of study*", in the philosophical sense the concepts of space and quantity, for Hegel, Being and Nothingness (the opposites), permeate any serious field of study, which we can interpret as empirical, so that both space and quantity, Nothingness and Being, are in fact equivalent, making the logic of Concept not restricted to generalities.

From a point of view shared by both Lawvere and Hegel, the concept of Kantian magnitude (KANT, 1763[1992]), which allows a clear distinction between useful mathematical constructions and "metaphysical", or "speculative" mathematics, is a dubious proposal. Wigner (1960) in his seminal article "*The unreasonable effectiveness of mathematics in the natural sciences*" expresses this point in the sense that given the immense reach of mathematics and its ability to deal with totally abstract concepts, its efficiency and functionality in explaining natural phenomena is fantastic. It is important to point out that many of the contributions that pure mathematics has had are inspired by reasons, perhaps for Kant, of total metaphysical interest, thus considering that only "applied mathematics" is relevant to the natural sciences, is a contradiction, because the source of all applicable theory is its pure and abstract essence.

Therefore, Hegel's dialectics allows us a more flexible way than Kant's of thinking about concepts, theories, empirical and metaphysical applicability. Lawvere takes this starting point proposed by Hegel and uses it in the formalization of categorical logic, based on the concepts of quantity and space, in the application of empirical studies, and in particular, in Measurement Theory. For Lawvere, for objective logic to be qualified as a dialectic process, it must be a progress in the general theory of space and quantity, as a necessary condition for progress in any fundamental empirical research (LAWVERE, 2005).

Lawvere promotes an internalization of logic with respect to categories, in the sense that logical concepts, such as propositions, logical operators, quantifiers, truth values, etc, are considered only as a form of categories. Lawvere describes it in Hegelian terms as an "*objectification*" of logic, where subjective logic, the one concerning inference between statements should not be thought of as a system of laws and rules which provide the basic foundations of mathematics and natural sciences, but rather as something emerging from conceptual constructions, from the topos, in question, and how it relates in terms of space and quantity to other topoi (plural) (LAWVERE, 2005). And at this point Lawvere's categorical logic promotes a new proposal for the axiomatic foundations of mathematics through Hegelian philosophy.

The concept of topos in Category Theory can be understood as follows: during our mathematical trainning we learn various concepts and methods such as: arithmetic, algebra, geometry, calculus, analysis, number theory, topology, etc. What do all these mathematical theories have in common? They can all be described within the formalism of Set Theory. This axiomization has proved extremely useful because it allows a "language" to speak about all known mathematical structures. Lawvere's proposal is that "groups" may not be the only language to reason and talk about mathematics, because there are other "places" to do mathematics, where each theory may have a different interpretation, perhaps better equipped to meet the needs of a given problem.

Each place can be thought of as a universe with its own laws governing the mathematical objects that inhabit them, and in the same way that it is possible to compare objects from the same universe, it is sometimes possible to compare objects from different universes. A mathematical universe is called a topos (RODIN, 2012). An analogy can be drawn this way: consider the commutative property of multiplication:

$$a\ x\ b = b\ x\ a$$

Each top has its interpretation of this rule, or theory, where the interpretation given above is the one we commonly find in an arithmetic or algebra class. Group theory does not allow a universal construction of commutative laws, but if we allow constructions that allow us to "get out" of a certain topos, there is a universal construction of commutative properties, it just does not inhabit the specific topos of group theory.

What we see is the idea that the laws of logic are not self-sustaining, but only a possible form of metaphysics. What the positivist movement at the beginning of the 20th century tried to do was to revive a metaphysics based only on first order (propositional) and second order logic (Group Theory) (RUSSEL, 1918), and inferring that it was the "real" metaphysics, or, the nonspeculative one. However, the problem of logical grounding remains open, the objectification of Hegel's logic, formalized in part by Lawvere, is a possible strategy to solve the problem of logical grounding in metaphysics.

For Lawvere (2005), "*logic is a special case of geometry*", this thought goes in total opposite to the background of Hilbert's Axiomatic Method, where logic provides a basis for mathematical theories. Lawvere's axiomatic dialectics suggests to us that geometry, not logic, is the main aspect, and that logic is a special kind of geometry. Lawvere refers to the internal logic of a given topos, and every topos also has a geometric content, which allows a dialectic interaction between geometry and logic. Something important in this new notion of logic is the concept of context and locality, because the internal logic on a topos usually does not fully coincide with its external logic (meta-logic), something that can be interpreted geometrically where a space S can be interpreted as embedded in some other space T, or, intrinsically as the incorporation of a space P (MCLARTY, 1992). For example: an "L" drawn on a sheet of paper, if the paper is crumpledor folded, extrinsically L is no longer straight, but intrinsically nothing has changed.

Thus, the extrinsic and intrinsic views, the geometries and logics in question about the situation are no longer the same.

In the Hegelian sense, the geometrization of Lawvere's topos logic can be philosophically compared to a form of objectification of logic, resulting in the logic of Concept, but of course, only as an approximation of the Hegelian thesis. This objectification of logic, in the mathematical sense proposed by Lawvere, requires that a topos and topoi are not abstract entities, but something that provides a connection between pure mathematics and the empirical sciences, that is, geometric intuition and the world of experience.

For Hegel (2010) subjective logic emerges from the objective, in Lawvere's interpretation objective logic is primarily constructed by empirically significant objective categories, i.e., dealing with notions of space and quantity. On this objective basis is superimposed a system of subjective logic, another type of category that encompasses: truth values, connectives, quantifiers and all formal and modal logic machinery. This subjective logic would not be reactive to the general characteristics of the objective categories, and to the objective topos in question, that is, the internal logic A of a given topos T is determined by the geometric nature of that topos, which is an empirically meaningful objective conceptual construction (LAWVERE, 2005).

However, we can still ask ourselves which logical model should be the "truest", or which metaphysics best represents real physics? Possibly, such questions are more directed to philosophy and metamathematics, and not to mathematics itself. What mathematics can do is choose your point of view, your logical system, and pragmatically do your work based on it, but what Lawvere suggests to us and what you are asking is actually totally mathematical. By using categorical logic, Lawvere objectifies logic as Hegel, and, in a certain way makes it manipulable, being able to be analyzed intrinsically, within a certain topos (mathematical universe), or extrinsically as the connection between different topoi (mathematical multi-verse).

Lawvere's contribution to the axiomatization of logic and categorical mathematics inspired by Hegelian philosophy, mainly the idea of geometric interpretation of logical quantifiers, which, within the formal axiomatic context proposed by Hilbert would be impossible, had transforming impacts on 20th and 21st century mathematical theories. Lawvere's idea of internalization and objectification of categorical logic reappears in a geometric interpretation in areas of mathematics that had their origin, in a way, in categorical mathematics, specifically in topology, in Type Theory, and finally in Homotopy Type Theory, which has been proposed as the new language for the foundations of mathematics.

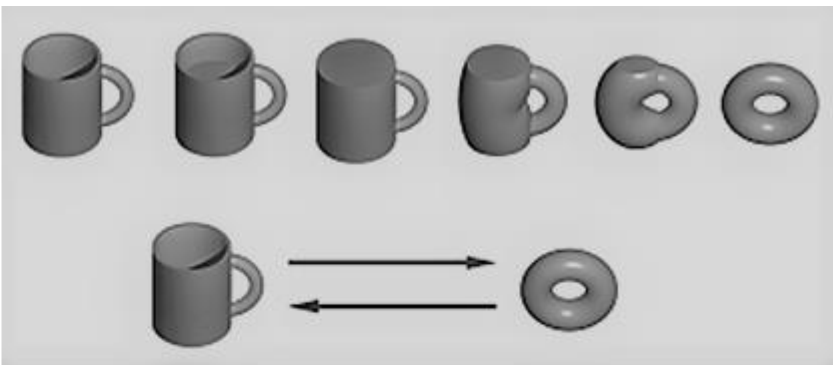
Defining "homotopy" in topological terms

In topology we find a similar definition to Lawvere and Hegel's proposal (unity of opposites) of the concept of equality as a form of transformation. Two continuous functions from one topological space to another are called homotopic, or equivalent, if one can be "continuously deformed" in the other, such deformation being called a homotopy between the two functions (HATCHER, 2002).

Formally, a homotopy between two continuous functions f and g from a topological space B to a topological space C is defined as a continuous function $H: B \times [0,1] \rightarrow C$ as the product of space B with the unit range $[0,1]$ to C in such a way that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$ (HATCHER, 2002). If we think about the second parameter H as time then H describes a continuous deformation of f to g, in time 0 we have the function f and in time 1 we have the function g. We can also think of the second parameter as a "*slider control*" that allows us to smoothly pass from f to g as the slider moves from 0 to 1, and vice versa, along the line of the Real numbers.

The figure below provides an example of a homotopy between two points, B and C where a toroid B at R_3 (Euclidean 3D space), f being some continuous function of the toroid by R_3 which takes the toroid to the embedded surface in the form of a "mug" C, g being some continuous function that takes the mug to the embedded surface of the toroid, as shown in Figure 1. It is a notion of similarity as a transformation between opposite states, where the transformation itself encapsulates the whole " toroid + mug" in a topological interpretation of the concept of equivalence proposed by Lawvere and inspired by Hegel's concept of *Aufhebung*. Developments in the area of topology and algebraic topology led to the development of theories such as the Homotopy Type Theory.

Figura 1 - Transformação topológica de $\boldsymbol{X} \rightarrow \boldsymbol{Y}$ em $\boldsymbol{R_3}$.



Homotopy type theory

Homotopy Type Theory can be used as a fundamental language for mathematics, that is, an alternative to the Zermelo-Fraenkel Set Theory, and models both have several important distinctions. Set theory has two distinct "*layers*":

- the deductive system, which involves first (propositional) logic;
- the second layer are the formulations created within this system, the axioms of Zermelo-Fraenkel being the formal standard language of mathematics.

Thus, the Zermelo-Fraenkel Sets Theory is about the interaction between the objects of the second layer (sets) and the objects of the first layer (propositions). Meanwhile, Homotopy Type Theory is its own deductive system, not needing to be formulated within any sub-structure as first or second order logic, and, unlike the basic distinction of propositions and sets from set theory, Homotopiy Type Theory has only one basic notion: types (ACZEL et, al. 2013, p. 17).

All these approaches can be seen as different developments of Lawvere's 1970 axiomatic proposals on the internalization of categorical logic and its geometric nature. In 2006 these proposals culminated with Voevodsky (2011), when he published a manuscript in which he proposed a major research program on type systems by homotopic methods.

Univalent Foundations of Mathematics

The ambition of the research program proposed by Voevodsky would be to build new foundations for mathematics, which he calls Univalent Foundations. In Voevodsky's words:

> The broad motivation behind univalent foundations is the desire to have a system where mathematics can be formalized as naturally as possible. Although it is possible to codify all mathematics in Zermelo-Fraenkel's theory, the way it is done is often ugly; worse still, when you do that, there are still many statements in Zermelo-Fraenkel that are mathematically meaningless. This problem becomes particularly present in the attempt to formalize mathematics by computers; in the standard essentials, writing down even the most basic deductions, such as isomorphism between sets, or the group structure on a set requires many pages of symbols. Univalent foundations seek to improve this situation through a system based on homotopy type theory [...](VOEVODSKY, 2011, p. 7).

Besides its theoretical aspect this initiative has an extremely practical objective: to allow a systematic use of computer assistants like Coq in everyday mathematical practice. Coq is an interactive computer program that allows the expression of mathematical statements as well as the verification and production of mathematical proofs and theorems (GONTHIER, 2008). The univalent foundations aim to provide contemporary mathematics with a universal language that has a formal, intuitive, and programmable aspect.

When we compare the formalization proposed by Voevodsky with Hilbert's formal axiomatic method we see beyond some key differences. Voevodsky formalism is similar to what Lawvere cited as "*a foundation of mathematics connected with its practice*”. In both cases we have a symbolic syntax and an interpretation of this syntax, but the relationships between the syntax and its interpretation are not the same. In the case of Hilbert's interpretation the dualistic model works by separating logical concepts (subjectives) and groups (objectives), while Voevodsky's approach does not involve the same difference between logical and non-logical categories. Voevodsky's intention is to merge the two categories, objective logic and subjective logic in Hegelian terms, as close as possible where both logical symbols and objects are interpreted both in logical and geometric terms. The dialectics is established in a totally internalized way, whose final result can be analogous to the objectification, here used the term internalization, of logic.

These features make Voevodsky's axiomatic method significantly different from Hilbert's axiomatic method, and promote a new way of interpreting mathematics. The design of the univalent foundations remains open, being an area of intense research and development in mathematics, with the promise of making the foundations of this discipline simpler, more accessible, dialectic, and intuitive, something that with the help of mathematical augmentation tools such as Coq among other similar programs promises to propel mathematics into a new age.

Conclusion

Categorical logic suggests a very different strategy of unification from the axiomization proposed by Hilbert. While the formal axiomatic method, i.e. Zermelo-Fraenkel Set Theory, uses a central invariant structure supposedly shared by all geometric spaces, organizing its universe in terms of maps/objects, logic and sets. In the case of the Univalent Foundations, the universal deontological rule of having first order logic as an invariant structure is abandoned and converted into a form of logical pluralism, where the Axiomatic Formal Method is only a special case of its generality.

Lawvere's influence can be traced in mathematics and categorical logic from the mid-20th century to the 21st century, until Voevodsky and the Univalent Foundations of Mathematics based on homotopy type theory. The source of such shared ideas concerns the relationship between geometry and logic, Lawvere in his terms describes these relationships in Hegelian formalisms as a dialectical contradiction. Ironically Hegel's logical idealism criticized by the positivist’s movement was something extremely important to rejuvenate and improve the foundations of mathematics itself, which, as was reported above in this study, for much of its history had been completely disconnected from the very practice of its discipline.

Hilbert's approach and the formal axiomatic model itself assumes an asymmetrical relationship between geometry and logic, between the way of thinking and the object of interest, geometry being supposedly logical while logic is not geometric. Lawvere approaches inspired by Hegel dialectic concept of equivalence as a form of transformation, the unity of opposites, make this relationship symmetric, their geometry being logical and logic itself geometric.

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