*ment of that form with true premisses and false conclusion.* But if the claim that a particular argument is valid is to be spelled out by appeal to other arguments of that form, it is hopeless to try to justify that form of argument by appeal to the validity of its instances. (Indeed, it is not a simple matter to specify of what schema a particular argument is an instance. Our decision about what the logical form of an argument is may depend upon our view about whether the argument is valid.) Third, since a valid schema has infinitely many instances, if the validity of the schema were to be proven on the basis of the validity of its instances, the justification of the schema would have to be inductive, and would in consequence inevitably fail to establish a result of the desired strength. (Cf. Section 1.)

In rejecting this suggestion I do not, of course, deny the genetic point, that the *codification* of valid forms of inference, the *construction* of a formal system, may proceed in part via generalisation over cases—though in part, I think, the procedure may also go in the opposite direction. (This genetic point is, I think, related to the one Carnap (1968) is making when he observes that we could not convince a man who is 'deductively blind' of the validity of MPP.) But I do claim that the *justification* of a form of inference cannot derive from intuition of the validity of its instances.

6. What I have said in this paper should, perhaps, be already familiar—it is foreshadowed in Carroll (1895), and more or less explicit in Quine (1936) and Carnap (1968) ('. . . the epistemological situation in inductive logic . . . is not worse than that in deductive logic, but quite analogous to it', p. 266). But the point does not seem to have been taken.

The moral of the paper might be put, pessimistically, as that deduction is no less in need of justification than induction; or, optimistically, as that induction is in no more need of justification than deduction. But however we put it, the presumption, that induction is shaky but deduction is firm, is impugned. And this presumption is quite crucial, e.g. to Popper's proposal (1959) to replace inductivism by deductivism. Those of us who are sceptical about the analytic/synthetic distinction will, no doubt, find these consequences less unpalatable than will those who accept it. And those of us who take a tolerant attitude to nonstandard logics—who regard logic as a theory, revisable, like other theories, in the light of experience—may even find these consequences welcome.[1]

1. I have profited from comments made when an earlier version of this paper was read to the Research Students' Seminar in Cambridge, May 1972.



# 7

# Meanings of Implication

*John Corcoran*

In philosophical and mathematical discourse as well as in ordinary scholarly contexts the term 'implies' is used in several clear senses, many of which have already been noticed and explicated. The first five sections of this article codify and interrelate the most widely recognized meanings. Section 6 discusses a further significant and common use. Section 7 discusses and interrelates Tarski's notion of logical consequence, the "model-theoretic" notion of logical consequence, and Bolzano's two grounding relations. The eighth section employs the use-mention distinction to separate the three common grammatical categories of 'implies'. Section 8 also shows that criteria based on use-mention are not reliable indications of intended usage of 'implies'. The ninth and last section relates the above to the counterfactual and gives reasons for not expecting to find 'implies' used to express counterfactuals. A summary is provided.

1. It is already a widely recognized (and widely lamented) fact that mathematicians, needing a short single word to replace 'if . . . then' in its truth-functional sense, have adopted the term 'implies' for this purpose. In this sense "A implies B" means simply that A is false or B is true.[1] Let us use 'implies$_1$' to distinguish this sense from others to be noted below. Incidentally, as will become even more obvious below, it is only rarely, if at all, that 'implies' is used in this sense in current English. Some authors express 'implies$_1$' by the phrase 'materially implies'.

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1. In order to avoid unnecessary intricacy, notation for the use-mention distinction is not strictly observed in the first seven sections. To some extent section 8 offers further justification for somewhat neglecting this otherwise important distinction.

2. "A implies B" is also used to mean that B is already logically implicit in A, i.e., that one would be redundant if he were to assert A and then also assert B in that asserting B would be making another statement without adding any new information (not already conveyed by A). For example, using 'implies' in this sense we would say that "The area of a triangle is one-half the base times the height" *implies* "The area of an isosceles triangle is one-half the base times the height." It is perhaps more usual to say "B is a logical consequence of A" or "A logically implies B" to mean that A implies B in this sense. We use 'implies$_2$' to distinguish this usage.

Clearly if A implies$_2$ B then A implies$_1$ B, but not necessarily conversely. For example, "Cats bark" *implies*$_1$ "Dogs bark," but "Dogs bark" is certainly not a logical consequence of "Cats bark." Moreover, in the case of sentences which can have different truth-values at different times, a sentence which is true at a certain time has different implications$_1$ (at that time) than it has at a time when it is false. A false sentence implies$_1$ all sentences whereas a true sentence implies$_1$ only true sentences (Lewis and Langford 1959, p. 261). On the other hand, implication$_2$ is completely independent of the actual truth-value of A. A implies$_2$ the same sentences when true as when false. Implication$_2$ is a *logical* relation between sentences not a so-called material relation.

Another way that 'implies$_1$' and 'implies$_2$' may be contrasted is this: "A implies$_1$ B" amounts to "it is *not true* that A is true and B is false" whereas "A implies$_2$ B" amounts to "It is *logically impossible* that A is true and B is false" (Cf. Lewis and Langford 1959, pp. 243–44)

It is worth explicitly noting that logical implication is intimately related to the traditional notion of validity of a premise-conclusion argument. To say that A logically implies B is to say neither more nor less than that the argument (A,B) [premise A, conclusion B] is valid. And, as has often been noted, to say that (A,B) is valid is to say neither more nor less than that B simply "restates" part (or all) of what is said in A.

3. It also happens, both in mathematical contexts and in common parlance, but perhaps not as frequently, that "A implies B" is used to mean that B can be deduced (or derived or inferred) by logical reasoning from A. The reader should note that one logically deduces B from A for the sole purpose of establishing that B is already logically implicit in A, i.e., that A implies$_2$ B. It is usually taken for granted that A implies$_2$ B when B is correctly inferable from A (otherwise one could not rely on logical reasoning). "A implies$_3$ B" is used to indicate "A implies B" in the sense of "B is logically derivable from A."



Suppose for the moment that B actually is logically deducible from A. Now suppose that a student, Mr. S., goes through the process of step-by-step deducing B from A. Does this mean that Mr. S. did the deduction correctly? Of course not. As we all know by experience in elementary geometry, often B is logically deducible from A even though the students do it incorrectly. Thus, to say that B is logically deducible from A is not to say that anyone who deduces B from A is doing it correctly. To say that B is logically deducible from A is to say that it is theoretically possible to carry out step-by-step, in a logically correct way, a process of deduction leading from A to B. There is no sufficient reason to think that B is logically deducible from A whenever A logically implies B. It is obvious that there could be cases where, although the deduction is theoretically possible, it is practically impossible for reasons of time. The deduction may require thousands of years to complete, for example. But, in a certain sense, the situation is worse than this. As a result of Gödel's work in the thirties, many mathematicians and philosophers believe that in some cases where A implies$_2$ B, A does not imply$_3$ B. These would be cases wherein a certain sentence B actually is a logical consequence of A (say the axioms of some branch of mathematics stated as a single sentence) but where it is impossible to deduce B from A.

Although Tarski has stated (1946, p. 410) that some logicians believed logical consequence (implies$_2$) to be sufficient as well as necessary for logical deducibility (implies$_3$), apart from some obvious confusions (e.g. Lewis and Langford 1959, p. 337; Wolf 1938, p. 40), none seem to have done so. In any case I have not been able to find any reference to corroborate Tarski's statements (cf. Corcoran 1972).

4. To compare the above three senses of 'implies' let us consider a particular example. Let A be one sentence which states Peano's postulates for arithmetic (Montague 1965, p. 135) *and* the definitions of addition, multiplication and prime number. Let B be Goldbach's conjecture "Every even number greater than two is the sum of two primes" (whose truth or falsity is yet unknown, see Forder 1958, p. 6). Now consider the following sentence.

(I) A implies B or A implies not-B

*In the first sense of 'implies'*, statement I is a completely trivial remark whose truth is deducible by the law of excluded middle: If B is true then A $implies_1$ B and if not-B is true then A $implies_1$ not-B. But one or the other *is* true, hence statement I.

*In the second sense of 'implies'*, statement I says, in effect, that Goldbach's conjecture is not logically independent of A.[2] Literally, statement I says: *either* Goldbach's conjecture is already logically implicit in A (so it would be redundant to add it as a new axiom) *or* the negation of Goldbach's conjecture is already logically implicit in A (so it would be redundant to add *it* as a new axiom). Under this reading I is not trivial. It is actually a rather deep statement involving the logical properties of the usual axiomatization of arithmetic. It so happens that statement I (in this sense) is known to be true.[3]

*In the third sense of 'implies'*, statement I says, in effect, that *either* it is possible to deduce Goldbach's conjecture from the axioms and definitions of arithmetic *or* it is possible to deduce the negation of Goldbach's conjecture from the axioms and definitions of arithmetic. I think that it is safe to say that no one has any reason either to think that statement I, in this sense, is true *or* to think that it is false.

In summary, in the first sense statement I is trivially true, one need know essentially nothing to determine its truth; in the second sense it is true, but it is a fairly deep truth, knowledge of which involves a fairly extensive background in mathematics (say that of a college senior); in the third sense, it is a very deep statement whose truth (or falsity) is as yet not known. In fact, one mathematician (see Forder 1958, p. 6) writing in the 1920's seems to suggest that previous to his remarks on the subject *questions* of that sort had not even been discussed. As far as I know he is perfectly correct (cf. Corcoran 1972).

It is already obvious that no two of the above three notions of implication are intensionally the same, i.e. each has a distinct meaning. It can happen that distinct notions are nevertheless extensionally equivalent, i.e. that they apply truly to exactly the same things (or pairs of things in the case of relations). However, as we have just seen, no two of these are extensionally equivalent. The extension of $implies_3$ is

2. Since current usage of the term "independent" (and its variants) is not uniform, the following conventions of this essay should be noted. "B is logically independent of A" means that neither B nor its negation is a logical consequence of A. To say that a set of sentences is *independent* is to say that no consequence of A. To say that a set of sentences is *independent* is to say that no one of them is a logical consequence of the rest. Saying that two sentences are independent is to say that the pair is independent.

3. See Forder (1958, *loc. cit.*). The basic fact needed to discover its truth can be gotten by combining the discussion of Peano's postulates in Birkhoff and Mac Lane (1953, pp. 54–56) with the general discussion of axiom systems in Forder (1958, pp. 1–12). Also cf. Montague (1965, p. 136).



properly included in that of $implies_2$ which itself is properly included in the extension of $implies_1$.

Russell was just one in a long line of otherwise sensitive philosophers who confused $implies_1$ (material implication) with $implies_2$ (logical implication). This is clear from many passages in Russell's writing where he uses "follows from" as indicating the converse of "implies." Remarkably, he is quite clear in his statement of the following three principles (Russell 1937, p. 15).

(1) A false proposition $implies_1$ every proposition.
(2) A true proposition is $implied_1$ by every proposition.
(3) Of any two propositions, one $implies_1$ the other.

Moreover, Russell would accept principle 4 below without hesitation; it is simply I (above) in the first sense discussed.

(4) For any two propositions, one $implies_1$ the other or else it $implies_1$ the negation of the other.

No one versed in elementary logic could accept Russell's identification of these two notions because, using the usual interconnection between logical consequence and validity of premise-conclusion arguments, principles 1 and 2 would imply that any argument with a false premise or a true conclusion is valid. No one versed in the history of mathematics could accept it because it would reduce the historically difficult, logical question of the independence of the parallel postulate to a triviality: principle 4 would say that no postulate is independent. Russell could save himself from this mistake by urging, with good reason, that the question of the independence of the fifth postulate is really about "$implies_3$" i.e., whether the fifth postulate and/or its negation is logically derivable from the conjunction of the other four. But what could he answer to the question of why anyone should care whether the fifth is independent (with respect to derivability) of the other four? If he were to answer that the real point is whether the fifth postulate and/or its negation is materially implied by the others, then he is caught in his triviality again.

My view is that no sense can be made of the importance of the problem of the independence of the parallel postulate without presupposing the notion of $implies_2$ (logical implication). My conclusion here is that Russell, for all his greatness, was insensitive to history and this insensitivity not only made possible his conflation of material and logical implication but it also effectively blocked his discovery of his error.

Needless to say, the four principles all become blatant falsehoods if '$implies_2$' or '$implies_3$' replaces '$implies_1$'.

At the cost of seeming to flog a dead horse, I would like to discuss what I think are causes for confusing material implication and logical implication with (logical) derivability or deducibility.

There is a subtle fallacy involved in confusing material implication with derivability. Suppose we want to "show" that A materially implies B if and only if B is derivable from A. In the first place it is obvious enough and true besides that if B is derivable from A then A materially implies B. The fallacy comes in doing the converse. Suppose that A materially implies B. Then, if we also assume A we can derive B by modus ponens. This would seem to show that then B is derivable from A—but it doesn't! What it shows is that B is derivable from 'A materially implies B' *and* A (taken together), something that we did not need to be shown. (Cf. Russell 1937, p. 33 and Bolzano 1972, p. 209.)

Remember 'A materially implies B' means simply that A is false or B is true while 'B is derivable from A' means that it is theoretically possible to write out a deduction, possibly a very long one, which would show that B must be true were A true.

One confusion between logical implication and derivability seems to turn on a systematic ambiguity in English use of the suffix 'able' related to the ambiguity of the 'incorrect' use of 'can'. In the first place, deriving (inferring, deducing) B from A is not simply accomplished by pronouncing a performative, e.g. "I hereby infer B from A." Something must be done and it is usually something very complicated.

In a certain sense, logical implication is a warrant for derivation (inference, deduction). But even the presence of the warrant is no guarantee that the action can be carried out—either theoretically or actually. Of course, it is a tautology that if the warrant is present then the warrant is present. Interestingly enough, it is possible to use a word of the form *X-able* to indicate not the theoretical or actual possibility of doing X but merely that a warrant for doing X exists. For example, in a state park the mountain faces which have been approved for climbing could be called 'climbable' even though some of the so-called 'climbable faces' are not even theoretically possible to climb. Thus, if 'logically implies' is used as Russell (1937, p. 33) and others did use it, as indicating the existence of a warrant for logical deducibility, and if 'logically deducible' is also used to indicate the existence of the warrant—then the confusion results from an equivocation on the tautology: "A logically implies B if and only if B is logically deducible from A."

It is also relevant to note here that some writers seem to think that to deduce B from A is simply to form a belief that A logically implies B (where A actually does logically imply B). This use of "to deduce" would support the view that logical implication of B by A is a warrant for deducing B from A (cf. Lewis and Langford 1959, p. 337). This in turn would be consonant with using "deducible from" as a synonym for "logically implied by." However, it should be noted that analysis of philosophic, scientific and mathematical practice does not support the above use of "to deduce." Indeed, to deduce B from A *is* to form a belief that A logically implies B—but not simply that. In order to deduce B from A one must form the belief in a logically correct way which, in non-trivial cases, involves substantial logical discovery, discovery of a proof, a chain of logical reasoning from A to B. For example, Fermat *claimed* to have deduced his last "theorem," but to this day no one knows whether he did and no one has been able to do it (again?).* In any case, those of us who happen to believe that Fermat's last "theorem" does follow logically from the axioms and definitions of arithmetic do not say of others of similar belief that they have deduced the "theorem." Moreover, there are many people who believe true logical implications without having deduced them.



5. It has also already been noticed by others that "A implies B" is also sometimes used to mean that "A-and-C implies$_3$ B" where C is some "obvious" statement tacitly taken by the speaker to be presumed by anyone following the conversation. For example, one could say that "Marion is a football player" implies "Marion is a male" under the presumption that all football players are male. As another example, it is often noted in set-theory courses that the axiom of choice implies the well-ordering principle. Here the sentence C being presumed must express at least the definition of well-ordering and usually some of the more elementary axioms of set theory as well. To indicate this sense of 'implies' we could write "C-implies$_3$" where C indicates that a presumption is involved. Naturally, one would expect that "A implies B" is also used in the sense of "A-and-C implies$_2$ B" and in the sense of "A-and-C implies$_1$ B" where C indicates a presumption as above. We use "C-implies$_2$" and "C-implies$_1$" to indicate the last two senses.

It may well be the case that the three last-mentioned meanings of 'implies' account for the majority of actual usages. We call the last three usages *elliptical* or *enthymematic*. Enthymematic usage of 'implies' is particularly handy when it suits one's purpose to be vague while still conveying the idea of some sort of connection between two sentences.

* [Editor's note] A proof of the Fermat "theorem" by the British mathematician Andrew Wiles of Princeton University was announced in June, 1993 (*Nature*, vol. 364, 1 July 1993, pp. 13–14).

6. An additional class of uses I wish to discuss will at first seem very strange and perverse to those who carefully use 'implies' in one or more of the above senses. In one of the new senses, "A implies B" is used to mean that B can be logically concluded *as a fact* on the strength of A. In other words, "A implies B" means that A *is* sufficient evidence for B. As Frege insisted, nothing can be concluded as a fact on the strength of a false statement (cf. Jourdain 1912, p. 240) and, for that matter, a false statement cannot *be* evidence for anything although, of course, false statements are often (erroneously) accepted as evidence. In any case, if one knows that A is false (or at least does not know that A is true) then even if one knows that B follows logically from A one cannot conclude B as a fact on the strength of A. The point is that "A implies B" in this sense amounts to "A is true and A implies$_2$ B."

Another more general way of putting this involves the linguistic observation that when we say "The fact that A . . ." we intend to convey that A is true (plus whatever else is said). For example, "The fact that Samuel Clemens is alive implies that certain newspaper accounts are incorrect" means both that Samuel Clemens actually is alive and that a certain implication holds. Given this observation we can explain that "A implies B," in the senses of this section, means "The-fact-that-A implies B" which in turn is paraphrased "A is true and A implies B," where 'implies' is here taken to indicate ambiguously any one of the other senses (usually 'implies$_2$' or "C-implies$_2$"). This yields six new senses of 'implies'—one for each of the previous senses.

Each of these six present senses presupposes the truth of the antecedent sentence and it is only in these senses that "A implies B" presupposes the truth of A. In all other senses here considered, only a relation between A and B is asserted and no indication of the truth of A is suggested. Indeed, in the other senses the proposition "A implies A" is trivially true regardless of the truth-value of A whereas in the present senses "A implies A" logically implies A and so is false whenever A is false.

When I first became convinced that some students were actually using the term in one (or more) of the present senses I was at a loss to figure out exactly what, in their linguistic activity, had "induced" me to notice it. Then I made the following observations: (1) they were uncomfortable when I would say "A implies B" when A was obviously false, (2) one student actually said that a false sentence does not imply anything and (3) when A was obviously false they were reluctant to say "A implies B" but they often said "A *would* imply B" meaning, I suppose, that A would imply B if A were true.



The above is not conclusive evidence for my claim that 'implies' is actually used in the senses of this section. To make the claim more plausible—or at least more understandable—I will list a few other common ways of saying "A is true and A implies B" (usually "implies$_2$" or "C-implies$_2$").

(1) A; therefore, B.
(2) A; hence, B.
(3) A; consequently, B.
(4) A; thus B.
(5) A; so B.
(6) Since A, it follows that B.
(7) Since A; B.
(8) That A implies that B.

What this list is designed to show is that "A is true and A implies B" expresses a rather widely used idea.[4] This in turn makes it more plausible to think that 'implies' is sometimes used in some of these senses which, I repeat, are the only ones which presuppose the truth of the antecedent sentence.[5] Reflection on English usage will settle the matter.

"A implies B" in the sense of "A is true and A implies$_2$ B" is especially important in interpretation of Frege's views on logic. It may very well be the case that Frege developed only a logistic system (for proving logical truths) and did not go on to develop a system for deducing conclusions from (non-logical) premises because he was taking 'implies' in the latter sense. Going beyond a logistic system would have involved him in determination of truth-values of logically con-

4. There is nothing novel about this list. For example, Russell (1937, p. 14) discussed the first item and the rest are obvious once the relevant facts about the first are noticed.

5. It is already clear that I am using the term "presuppose" in one of its ordinary senses and *not* in the technical sense of Keenan (1973), Strawson (1952), and the modern linguistic semanticists according to which a sentence S presupposes P if and only if P must be true in order for S to have *either* truth-value. According to this usage, a sentence and its negation have the same presuppositions. For example, 'Fred was surprised that Mary won' and 'Fred was not surprised that Mary won' both presuppose 'Mary won.' I am by no means asserting that there is no use of "implies" in which "A implies B" presupposes A in Keenan's sense. On the contrary, such usage does exist. It is worth noting, however, that such use is not synonymous with the "genuine" conditional, "if A then B," which, even though it has a truth-value only when A is true, still does not presuppose the truth of A. In fact, the whole point of the genuine conditional is to avoid implying and/or presupposing the antecedent. Cf. Quine (1959, p. 12).

tingent sentences—thus exceeding the bounds of pure logic (cf. Jourdain 1912, esp. p. 240 and Russell 1937, p. 16). [No special notation will be used for the senses of this section.]

7. There is another class of meanings which might be attached to the term "implies." It is certain that some of them have been so attached and, if Bolzano's work ever gains the attention it deserves, several of the others will be also.

The easiest way of getting into this class of meanings is through some of Russell's remarks in *Principles of Mathematics* (1937). Russell considers the following sentence:

(1) Socrates is a man implies Socrates is mortal.

This appears to be a case of enthymematic implication where the presumption is that all men are mortal. But Russell says (1937, p. 14).

> . . . it appears at once that we may substitute not only another man, but any other entity whatever, in the place of Socrates. Thus although what is explicitly stated, in such a case, is a material implication ["$\text{implies}_1$" above], what is meant is a formal implication; and some effort is needed to confine our imagination to material implication.

By a formal implication Russell means a proposition of the following kind.

(2) For all values of $x$, A($x$) $\text{implies}_1$ B($x$).

In other words Russell is claiming that sentence 1 above would normally be understood, not as an enthymematic implication, but rather as equivalent in meaning to sentences 3 and 4 below.

(3) Everything which is a man is mortal.

(4) For every $x$, if $x$ is a man then $x$ is mortal.

At this point Russell is clear about formal implication, although he loses his clarity later on. Formal implication is a relation between propositional functions (or, in the terminology of this essay, between sentential expressions involving free variables) which holds when the universal closure of the appropriate conditional is true. In the above-quoted passage Russell says that it is natural to understand 'implies' between two sentences as indicating that formal implication holds between two sentential expressions gotten from the sentences by putting variables for terms (also cf. Bolzano 1972, p. 252). But Russell never bothered to say exactly which terms should be replaced by variables. There seem to be three obvious possibilities in explicating Russell. First, that there is no rule for determining which terms should be varied even in a given sentence. If the hearer is uncertain about what is being said in a given case the speaker must say which terms he wants to 'vary'. For example, sentence 5 below could be used



to say that whoever eats fish likes fish, or that whatever Socrates eats he likes, or even that whatever anyone eats he likes.

(5) Socrates eats fish implies Socrates likes fish.

Given the character of *Principles of Mathematics* I think that this is the answer, i.e., that Russell was commenting on an ambiguous usage of 'implies'. The ambiguous usage tends to move the possibilities for 'implies' closer to the sense of logical implication by allowing some implications which would be false as logical implications to be counted as false. For example, sentence 6 is false when 'implies' means logical implication and it is false when taken in the sense of formal implication with 'meow' replaced by a variable; but, of course, it is a true material implication.

(6) Dogs meow implies cats meow.

A second way of understanding Russell is to let all shared terms vary. Here sentence 5 would mean that whatever anyone eats he likes. This has the advantage of being unambiguous. It also moves closer to logical implication but it still holds between sentences which are not related by logical implication. In this sense of 'implies', all generalizable material implications would hold as implications.

A third way of understanding Russell is to let all non-logical terms vary. Under this interpretation of 'implies' many of the generalizable material implications would fail and this would bring us very close to logical implication. Let us use '$\text{implies}_4$' to indicate this sense of implies.

In fact, the last move brings us to a sense of implication which is consonant with Aristotelian logic to the extent that an Aristotelian argument is valid if and only if the conclusion is $\text{implied}_4$ by the conjunction of the premises. Moreover, Lewis and Langford (1959, pp. 342–46) offer something like $\text{implication}_4$ as an explication of the normal mathematical usage, and Tarski's explication (1956, pp. 410ff) differs from $\text{implication}_4$ only incidentally for the purposes of this article.[6]

Russell, Lewis-Langford and Tarski were all working in a framework of an interpreted language having a fixed universe of discourse. In addition, they all distinguished logical and non-logical terms. Finally, they all considered relations between A and B which hold when

6. From a broader perspective there are two highly significant further refinements in Tarski's work. In the first place Tarski recognized the possibility of a metalinguistic notion of implication (i.e. one not necessarily expressible in the object language in question) whereas Lewis and Langford followed Russell in trying to consider implication as an object language concept. Secondly, Tarski recognized the fact that in many scientific contexts implication relates sets (possibly infinite) of sentences to individual sentences.

universal closure of "$A^*$ $\text{implies}_1$ $B^*$" is true (where $A^*$ and $B^*$ are obtained by appropriately substituting variables for non-logical terms of A and B).

From the present point of view, the most significant refinement found in Tarski's explication is that all non-logical terms are to be varied. That Lewis and Langford did not explicitly lay down this requirement may be more of an oversight in exposition than an oversight in research—but this is unlikely given their comments (*op. cit.* on p. 340).

The explication of logical consequence which is most widely accepted today diverges from the above-mentioned Tarskian notion only by allowing universes of discourse "to vary." Using '$\text{implies}_5$' for this notion restricted to sentences we would have that A $\text{implies}_5$ B if and only if the universal closure of $A^*$ $\text{implies}_1$ $B^*$ is true in every universe of discourse.[7] The reason for preferring "$\text{implies}_5$" to "$\text{implies}_4$" as an explication of logical consequence turns on an insight which was developed in the course of criticism of the axiom of infinity in type theory—viz. that the number of objects in the universe should not be a logical presupposition. A related reason for preferring "$\text{implies}_5$" is thought by some to be a reason for rejecting it—viz. that use of $\text{implication}_5$ makes clear that logic presupposes "logically possible worlds." This brings us to fringes of philosophy of logic which are beyond the compass of an essay designed to clarify the interrelations among the multitude of meanings of implication.

In connection with formal implication and Tarskian implication it would be unfair not to at least mention Bolzano's *Theory of Science* (1972), first published in 1837. Bolzano defined a notion which we may call relative implication. Let A and B be sentences and let S be a set of symbols, logical and/or non-logical. Bolzano's idea is to say that *A implies B relative to S* if and only if every uniform substitution for occurrences of members of S in A and B making A true makes B true (cf. Bolzano 1972, p. 209).

If S is taken to be empty then implication relative to S is material implication. If S is an appropriate set of non-logical terms shared by A and B then implication relative to S can be made to coincide with one reading of the ambiguous use of 'implies' which Russell thought he had noticed. If S is the set of all non-logical terms then implication relative to S is $\text{implication}_4$ or Tarskian implication. Bolzano let S be arbitrary and, consequently, he seems to have defined a notion which

7. This wording is adequate only for quantificationally closed languages which, like the language of type theory, contain universal generalizations of each sentence containing one or more nonlogical constants. For other languages the wording must be changed. (See, e.g. Quine 1959, p. 147).



was never studied before or since and which is much broader than any of the senses of implication mentioned above.

Bolzano did not believe that his notion of relative implication coincided with logical implication. He has devoted a section of his book to discussing the relation between relative implication and logical implication (1972, section 223). There he considers two examples of relative implication. He observes that they amount to generalized conditionals and he observes that knowledge of those implications is outside of the province of logic. One example is sentence 7 below as an implication relative to 'Caius'.

(7) Caius is a man implies Caius has an immortal soul.

This, of course, amounts to the sentence 9 below.

(8) For every *x*, if *x* is a man then *x* has an immortal soul.

He went on to indicate that for logical implication all except the logical concepts would have to be varied. Bolzano was explicit in these passages and all of his examples of logical implications clearly fall under Tarskian implication. In my opinion Bolzano thought that logical implication is $\text{implication}_4$ above. If this is so then Bolzano truly deserves credit for explication of logical consequence, if Tarski does, because in my opinion Bolzano offered precisely the same idea.[8]

Interestingly enough, Bolzano mentions two other places where 'implies' might be used. One is where Bolzano's "ground-consequence" relation holds. He explains that A is the ground of B (and B the consequence of A) when A and B are both true and A is the "reason why" B is true. The ground-consequence relation is not the same as logical implication because, as Bolzano himself points out, logical implication can hold between false sentences. He also points out that ground-consequence is not simply logical implication between true sentences, although he conjectures that whenever ground-consequence holds logical implication also holds (1972, pp. 274–75). The other place Bolzano mentions is where the "ground-judgment" relation holds, though he does not use these terms. Here we could say that *A yields B* if knowledge of A would be evidence for concluding B. Bolzano speaks of A being "the cause of knowing" B. He claims, with good argument, that this relation often goes in the opposite direction from the ground-consequence relation, i.e., that A is sometimes the ground of B (the "reason why" of B) when in fact B is "the cause of our knowledge" of A. For example, we know that it is hot outside because we know that a certain thermometer reads high but the rea-

8. The reason that I did not quote Bolzano is that the section in question (223) is not self-contained and Bolzano is not concise. Other passages which support my interpretation are found in *op. cit.*, pp. 38, 198, and 199.

son why the thermometer reads high is because it is hot outside. This is close to Bolzano's example. He uses the terms "real ground" and "ground of knowledge."

8. To some readers my failure to strictly observe the use-mention notation will seem unfortunate. It seems to me, however, that rigid observance of the distinction would add nothing to the paper and would actually detract from its clarity by making it unnecessarily intricate. Of course, the use-mention distinction and its accompanying notation are essential for avoiding certain kinds of confusion. But, as we will argue presently, the notation is not normally or necessarily observed and thus cannot be used as a sign indicative of intended meaning.

'Implies' can be used in all of the above senses in any and all of the grammatical categories of 'implies' usually distinguished by means of use-mention. There are three candidates for "the" grammatical category of "implies'. First, it can be used as a (binary) sentential connective—roughly a word which, when placed between two *sentences* of a language, forms a third sentence of the same language. Second, it can be used as a factive (or propositional) verb in the object language. This means that when placed between two factive (or propositional) noun phrases (usually "that . . .") of the object language it forms an object language sentence. Third, it can be used as a meta-linguistic verb, i.e., when placed between names of two object language sentences it forms a sentence of the metalanguage. 'Implies' actually occurs as a word in each category and in each category it can have a meaning corresponding to many of the distinctions made above.

To exemplify the use of 'implies' in each of the three categories let P and Q be (object language) sentences and let $p$ and $q$ be (object language) names of P and Q, respectively. Let '$q$-implies-$p$' be a name of $\ulcorner q$ implies $p\urcorner$, also in the object language.

*Connective*

$\ulcorner$P implies Q$\urcorner$
$\ulcorner$P implies (Q implies P)$\urcorner$

*Object language factive verb*

$\ulcorner$That P implies that Q$\urcorner$
$\ulcorner$That P implies that that Q implies that P$\urcorner$

*Metalinguistic verb*

$\ulcorner p$ implies $q\urcorner$
$\ulcorner p$ implies $q$-implies-$p\urcorner$



I do not disagree with the logicians who believe that "implies$_1$" is best expressed with a connective and that "implies$_2$" and "implies$_3$" are best expressed by metalinguistic verbs. My point is that the preferred usage is conventional and that the convention has not been universally accepted. There is nothing to prevent "implies$_1$" from being expressed either as a factive verb or metalinguistically. More importantly, there is nothing to prevent "implies$_2$" from being expressed by a connective (necessarily non-truth-functional). Bolzano (1972, p. 44) seems to have thought that implication$_2$ was normally expressed by a connective. In *Prior Analytics*, especially in I.44, Aristotle seems to express a non-truth-functional implication by a connective. Even in current English we often express a non-truth-functional implication by "if A then necessarily B" and it may be possible to argue that 'if . . . then necessarily . . .' is a complex, discontinuous connective.

9. The so-called counterfactual conditionals have been left out of the discussion because the word 'implies' is not normally involved in them. In the first place, the counterfactuals presuppose the *negation* of "the antecedent" whereas none of the uses of 'implies' just considered does this.[9] Indeed, use of "A implies B" in a sense that presupposes the negation of A seems so perverse as to be outside of the range of acceptable English. In the second place, a counterfactual cannot be constructed grammatically in any of the three ways for constructing implicational sentences. The only construction deserving of mention is the one which involves use of a connective between two sentences and this cannot be the counterfactual construction because the antecedent and consequent are commonly *not sentences.* This can be seen from the following example.

If I were Hughes then I would be rich.

The counterfactual is probably derived grammatically by applying a (nonparaphrastic) transformation to an ordinary conditional.[10] For the example above, the transformation would be applied to the following.

If I am Hughes then I am rich.

9. It is truly remarkable that treatments of the counterfactual which leave this out of account could be called "preferred analyses." Indeed, it has been suggested that the counterfactual of A and B be explicated as "A C-implies$_2$ B" (cf. Craig and Mates 1970, p. 303).

10. Although existence of nonparaphrastic (meaning-changing) transformations had been denied by most linguists, Harris has recognized them in his latest books (1968, pp. 60–63).

If this is so then the problem of counterfactuals does not involve merely analysis of "if . . . then" but rather also analysis of the semantic effect of the transformation.

SUMMARY AND CONCLUSION: In the first five sections we have distinguished twelve uses of the term 'implies'. At the outset we distinguished: $implies_1$ (truth-functional), $implies_2$ (logical consequence) and $implies_3$ (logical deducibility). Next we distinguished three elliptical or enthymematic varieties of implication: C-$implies_1$, C-$implies_2$ and C-$implies_3$. In none of these six senses did "A implies B" presuppose the truth of A. Then we discussed the cases wherein "A implies B" is used to mean "The-fact-that-A implies B," which *does* presuppose the truth of A. We paraphrased the latter as "A is true and A implies B" where 'implies' indicates any of the previous six senses of the term. Thus, at that point, twelve senses of implies were distinguished, six which do not presuppose the truth of the implying sentence and six which do. Of the six which do, three are enthymematic.

In addition, the three original senses were carefully distinguished and interrelated, and possible causes of confusion were identified.

Then, building on some off-hand observations of Russell, we related the truth-functional use of 'implies' to two further notions which have been used as explications of traditional logical consequence. We also brought in Bolzano's relative implication and his two grounding relations.

We argued briefly that counterfactuals are not normally expressed using 'implies' and that the distinction between use and mention cannot be used as a test for distinguishing different meanings of 'implies'.

Use of 'implies' as a transitive verb taking a human subject has been ignored.[11]

11. I am grateful to the following persons for criticisms and suggestions: William Frank (Oregon State University), John Herring and Charles Lambros (SUNY/Buffalo), Jack Meiland (University of Michigan), Marshall Spector (SUNY/Stony Brook), Frank Jackson (Latrobe University, Australia), William Wisdom (Temple University). The final version of this paper was read at the University of Puerto Rico in March 1973. Previous versions were read at the University of Pennsylvania and at SUNY/Buffalo. [This paper appears in Spanish translation by José M. Sagüillo as "Significados de la implicación," *Agora* 5 (1985) 279–94.]

# 8

# Truth and Proof

*Alfred Tarski*

The subject of this article is an old one. It has been frequently discussed in modern logical and philosophical literature, and it would not be easy to contribute anything original to the discussion. To many readers, I am afraid, none of the ideas put forward in the article will appear essentially novel; nonetheless, I hope they may find some interest in the way the material has been arranged and knitted together.

As the title indicates, I wish to discuss here two different though related notions: the notion of truth and the notion of proof. Actually the article is divided into three sections. The first section is concerned exclusively with the notion of truth, the second deals primarily with the notion of proof, and the third is a discussion of the relationship between these two notions.

## 1. THE NOTION OF TRUTH

The task of explaining the meaning of the term "true" will be interpreted here in a restricted way. The notion of truth occurs in many different contexts, and there are several distinct categories of objects to which the term "true" is applied. In a psychological discussion one might speak of true emotions as well as true beliefs; in a discourse from the domain of esthetics the inner truth of an object of art might be analyzed. In this article, however, we are interested only in what might be called the logical notion of truth. More specifically, we concern ourselves exclusively with the meaning of the term "true" when this term is used to refer to sentences. Presumably this was the original use of the term "true" in human language. Sentences are treated here as linguistic objects, as certain strings of sounds or written signs. (Of course, not every such string is a sentence.) Moreover, when speaking of sentences, we shall always have in mind what are called in grammar declarative sentences, and not interrogative or imperative sentences.

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