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Hodeson, Kate (4-BRST-Q)

Mathematical representation: playing a role. (English summary)

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The article under review—hereafter MRPR—concerns recent aspects of a two-century-old development. In the last half of the twentieth century this development spawned several philosophies known as mathematical structuralisms; in philosophy-of-mathematics contexts, they are often known simply as structuralisms. Unfortunately, the term “structuralism” carries different senses in other contexts, notably in philosophy of science as well as in other fields such as linguistics, anthropology, and psychology.

Even though there are disagreements among mathematical structuralisms, the group of philosophies is sometimes regarded as one movement—in which case the singular of the common noun is used as a proper name of a single field.

Mathematical structuralism has its origins in now-familiar logico-mathematical insights dating to the 1800s, if not earlier. One of these insights was prominent in algebra even before Boole, but it became more prominent in Boole’s 1847 booklet [G. Boole, *Mathematical analysis of logic*, Macmillan, Cambridge, 1847] and 1854 book [*The laws of thought*, reprint of the 1854 original, Gt. Books Philos., Prometheus, Amherst, NY, 2003; [MR1994936 \(2004i:03001\)](#)]: Every mathematical language can be reinterpreted in a new “universe of discourse” so that sentences expressing truths in the old “intended interpretation” remain true in the new reinterpretation—although they convey completely different propositions about completely different objects. The word “equivalent”, sometimes with modifiers such as “first-order” or “truth-value”, is used with such reinterpretations: Two interpretations of the same language are equivalent if each sentence of the language has the same truth-value in one interpretation as in the other.

The fact that every science admits multiple equivalent reinterpretations is referred to repeatedly in structuralism. MRPR refers to it at least 17 times using the misleading expression “multiple realizability of mathematical objects”—often shortened to “multiple realizability”—that occurs nowhere else in the literature of structuralism. Equivalent reinterpretability does not concern the nature of mathematical objects as much as it concerns the nature of language. Moreover, it does not concern mathematical objects any more than physical, anthropological, psychological, or any other kind of objects. The two facts premising these objections to MRPR’s “multiple realizability of mathematical objects” can be found in MRPR on pages 775 and 781—after the expression has been used for 6 pages. The last sentence of MRPR is “I attribute [...] multiple realizability, not to the nature of mathematical objects, but instead to the activity of mathematics”.

Related to equivalent reinterpretability is the fact that sentences can change truth-values under reinterpretation, a fact used in independence proofs [J. Corcoran, in *History and philosophy of logic, Vol. 1*, 187–207, Abacus, Tunbridge Wells, 1980; [MR0626358 \(82j:03034\)](#)]. Incidentally, Boole never noticed that non-equivalent reinterpretability could be used in independence proofs even though a form of such independence proofs had been used by Aristotle [J. Corcoran and S. Wood, *Notre Dame J. Formal Logic* **21** (1980), no. 4, 609–638; [MR0592521 \(81j:03003\)](#)], whose logic Boole admired and built on.

The expression “universe of discourse”, once pervasive in structuralism, was coined by Boole in 1854 [see *Cambridge dictionary of philosophy*, second edition, Cambridge

University Press, Cambridge, 1999 (“Universe of discourse”, p. 546)] although reinterpretation had already been used in his 1847 booklet. Roughly, in 1847 Boole used one and the same language to express a calculus of classes, a calculus of propositions, and a two-valued algebra—each with its own universe of discourse [J. Corcoran and S. Wood, *op. cit.*; [MR0592521 \(81j:03003\)](#)].

The word “ontology” was traditionally a proper name denoting the branch of philosophy that studies “being as such”. The proper name “ontology” came from the Greek: *ontos* (about being) + *logos* (discourse) yielding “discourse about being”. Recently, following Quine’s lead, it has come to be used as a common noun applicable to universes of discourse among other things. In MRPR, as in other current structuralist literature, the universe of discourse of a science is or is included in what is called its ontology.

Another insight, of post-Boole vintage, leading to structuralism is that axioms determining a mathematical science about one intended interpretation can be construed—not as *stating* truths and not as defining *the* intended interpretation—but as *defining* the class of all reinterpretations that satisfy the axioms, including the intended interpretation. From here it is an easy step to consider the axioms to define—not a class of interpretations—but a postulated abstraction that all such concrete reinterpretations have in common. The postulated abstractions—variously called structures, patterns, forms, arrangements, morphs, shapes, and so on—are among the things known as structures in structuralism.

Another idea creating the philosophies known as mathematical structuralisms involves taking mathematics—in whole or in part—as a science of these abstract structures, in addition to or instead of more traditional entities [M. D. Resnik, *Mathematics as a science of patterns*, Oxford Univ. Press, New York, 1997; [MR1474238 \(98j:00008\)](#)]. According to some mathematical structuralisms, mathematics is not about or is not just about the entities—numbers, geometrical figures, strings of symbols, sets, equations, algorithms, and the like—populating the universes of discourse of the intended interpretations of the languages of mathematics. Rather, mathematics is about abstract structures—whether in addition to or instead of the traditional entities.

Some mathematical structuralists even attempt to use these new structures to explain the metaphysical nature and ontological status of the traditional entities: An integer is identified with a *place* in the structural domain of the structure associated with the traditional theory of integers; a geometric point is identified with a *place* in the structural domain of the structure associated with the traditional point-based geometric theory. Other ways of putting this deploy—instead of the spatial metaphor of place—the theatrical metaphor of *role* or the institutional metaphor of *office*. The word “position” can be used with the spatial metaphor or with the institutional metaphor. As MRPR says on page 770: “According to structuralism, it is the structure itself that is the subject-matter of arithmetic, and its positions, rather than the things filling them, that are the mathematical objects.”

Mathematical structuralism has no affiliation with mathematical formalism—the view that mathematics is the manipulation of meaningless symbolic forms according to arbitrary rules. Ironically, David Hilbert’s name has been associated with both philosophies. Moreover, in the structuralist sense, structures are not what Quine and others call logical structures, nor are they syntactic structures in senses associated with Tarski, Chomsky, and others. Further, a structuralist’s structure is not a “framework” as in expressions such as “the conceptual structure of logic”. It is also worth explicitly noting that the word “structure” is not being used in the sense sometimes found in the literature of model theory where, e.g., the intended interpretation of Peano Arithmetic is a structure that includes the set of natural numbers as its universe of discourse. In these model-theoretic uses, the word “structure” is often nearly synonymous with the

older term “system” [J. Corcoran, op. cit.; MR0626358 (82j:03034); M. R. Cohen and E. Nagel, *An introduction to logic*, second edition, Hackett, Indianapolis, IN, 1993 (pp. xli–xlii; pp. 129–150); MR1241880 (94h:03002)]. Combining the structuralist and the model-theorist usages leads to awkward expressions: An isomorphism is a structure-preserving mapping between two structures; two structures are isomorphic if they have the same structure; concrete model-theoretic structures have abstract structure or abstract structures.

To understand MRPR’s scope and style it is useful to study its entire four-sentence summary:

“The primary justification for mathematical structuralism is its capacity to explain two observations about mathematical objects, typically natural numbers. Non-eliminative structuralism attributes these features to the particular ontology of mathematics. I argue that attributing the features to an ontology of structural objects conflicts with claims often made by structuralists to the effect that their structuralist theses are versions of Quine’s ontological relativity or Putnam’s internal realism. I describe and argue for an alternative explanation for these features which instead explains the attributes them to the *mathematical practice* of representing numbers using more concrete tokens, such as sets, strokes and so on.” [sic]

The first sentence implies (1) that mathematical structuralism is justified, (2) that mathematical structuralism explains certain (unspecified) two observations about mathematical objects, and (3) that these two observations are established facts that can be verified by observing something: they are called phenomena on page 770. Notice that the abstract gives no clues as to what these observations are or who observed them. MRPR describes one on page 770, the other on page 771.

The paper never supports the summary’s first sentence. On page 773, the third implication is retracted if not contradicted: the two “observations” are demoted to assumptions. On page 770 the second implication is weakened by replacing “explains” with “seeks to explain”. On page 777, MRPR seems even to contradict the second implication by suggesting that mathematical structuralism doesn’t explain the two “observations”. On page 780, in its conclusion, MRPR opposes structuralism, thus at least suggesting that the first implication is doubtful. Consequently, the first sentence of the summary, introduced as an unsupported assertion, seems to be undermined by further statements in the text of the paper.

On a first reading of the summary’s second sentence, the reader encounters the word “features”, an expression that also occurs in its third and fourth sentences. Once the reader notes that the word “observations” doesn’t occur after the first sentence, the inference that “features” refers to observations seems warranted. But how can a feature be an observation?

The summary’s last sentence is especially puzzling. But in response to a request by the reviewer, the author sent a list of five corrections: one deletes “explains the” in the summary’s last sentence. The reviewer noticed several more corrections: some in the same sentence. For example, something needs to be done with the summary’s last phrase: “the *mathematical practice* of representing numbers using more concrete tokens, such as sets, strokes and so on”. In one reading this phrase seems to imply that numbers are tokens: tokens that are less concrete than strokes—presumably as /, //, ///, . . . in stroke notation represent 1, 2, 3, . . . It also seems to imply that sets are concrete tokens. In another reading, the four-word expression “using more concrete tokens” is elliptical for the ten words “using tokens that are more concrete than those usually used”. This reading also implies that sets are concrete tokens but adds that sets are more concrete than numerals.

MRPR is very difficult to read. It is peppered with the technical terminology of structuralism: non-eliminative structuralism, ontological relativity, internal realism, in rebus, ante rem, referential underdetermination, etc. MRPR is written for an audience of structuralists; it makes no effort to provide the introductory background needed to reach a mathematical audience.

MRPR's main thesis seems to be that "multiple realizability of objects" should be understood as reinterpretability of names and, more generally, as concerning the fact that whatever represents one thing can represent another.

MRPR does not reference its author's similarly titled Ph.D. thesis [*Playing a role: structuralism & the underdetermination of representation*, Univ. Bristol, 2013].

J. Corcoran

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