JOHN CORCORAN

REMARKS ON STOIC DEDUCTION

The purpose of this note is to raise and clarify certain questions concerning deduction in Stoic logic. Despite the fact that the extant corpus of relevant texts is limited, it may nevertheless be possible to answer some of these questions with a considerable degree of certainty. Moreover, with the answers obtained one might be able to narrow the range of possible solutions to other problems concerning Stoic theories of meaning and inference.

The content of this note goes somewhat beyond the comments I made during the discussion of Professor Gould's paper [8], 'Deduction in Stoic Logic', in the symposium. I am grateful to Professors Gould and Kretzmann for pointing out the implications of those comments as well as for encouraging me to prepare them for this volume.

One of the obstacles to a careful discussion of Stoic logic is obscurity of terminology. Clarification of terminology may catalyze recognition of important historical facts. For example, in 1956 a modern logician suggested (incorrectly) in a historical note [4, fn. 529] that the distinction between implication and deduction *could* not have been made before the work of Tarski and Carnap. But once historians had clarified their own terminology it became obvious that this distinction played an important role in logic from the very beginning. Aristotle's distinction between imperfect and perfect syllogisms is a variant of the implication-deduction distinction and Gould [8] suggests the existence of a parallel distinction in Stoic logic.

1. Implication and Inference

Let us clarify our terminology. We use the two-placed verb 'to imply' (*P* implies *c*) to indicate the converse of the logical consequence relation. For us, its subject is always a set of sentences and its object is always a single sentence. For example, we might say that Euclid's Postulates *imply*

J. Corcoran (ed.), Ancient Logic and Its Modern Interpretations, 169–181.

Playfair's Postulate. As is common in ordinary English, we use the three-placed verb 'to infer' to indicate a certain rational action. Thus, we might say that Playfair *inferred* his postulate *from* Euclid's postulates. The subject is always human, the direct object is always a single sentence and the prepositional object is always a set of sentences (but it is sometimes omitted by ellipsis). 'To deduce' is a synonym for 'to infer'.

The more common English usage of 'implies' presupposes that the subject contains only truths. Occasionally a logician has adopted this convention, e.g., Frege [7; pp. 82, 105, 107] and Łukasiewicz [10, p. 55]. When it is not known whether the presupposition obtains, the common usage requires the verb to be put in the subjunctive in order to 'cancel' the presupposition. Thus Frege might say something like the following: the axiom of choice, if true, *would imply* Zorn's lemma. However, in this article the verb 'implies' never carries the presupposition. Our usage reflects Aristotle's fundamental discovery that the logical consequence relation is separable from issues of the material truth of premises. In effect, Aristotle saw that the so-called ground-consequence relation can be analyzed into a property (being 'grounds') and a relation ('implication').

Likewise, 'to infer' is often used with the presupposition that the subject knows that the prepositional object is true. According to this usage we might assert, "if Zorn inferred his lemma from the axiom of choice, he must have known that the axiom of choice is true *and* he must have discovered that the axiom of choice implies his lemma." However, in this article our use of 'to infer' never carries the presupposition. To infer c from P is simply to deduce c from P, i.e., to discover by logical reasoning that P implies c. (Warning: according to this usage 'incorrect inference' is not inference, just as 'false pregnancy' is not pregnancy.)

My opinion, stemming in part from reading Mates' *Stoic Logic* [12], Bury's translation of Sextus' writings [3], Gould [8] and other works, is that the Stoics did use the distinction between implication and inference. Here we come to the first problem.

Problem 1: (a) To explicate the Stoic analogue of the implication-inference distinction. (b) To determine whether the Stoic usage involved presuppositions. (c) To determine whether the Stoics articulated the distinction (which is much more than simply using it). (d) To develop extensive textual support for the answers to the above.

2. Arguments, their deductions and their counterinterpretations

According to Gould [8] and others [e.g., 12, p. 58], the Stoics had a technical term (*logos*) which translates exactly into our technical term 'argument' in the sense of a set *P* of sentences together with *c*, a single sentence (*P* is the premise set and *c* is the conclusion). Our technical term does not agree with common usage in several respects, the most noteworthy of which is that one can produce an argument (technical sense) without engaging in any argumentation (reasoning, inference). To do this one simply specifies a set of sentences together with a single sentence. In the technical sense, arguments never *express* reasoning. In fact, one must engage in reasoning in order to determine the validity of an argument; therefore, the reasoning is not already expressed *in* the argument. An argument (*P*, *c*) is *valid* if and only if *P* implies *c*, otherwise *invalid*. Another confusion results from the fact that the terms 'premises' and 'conclusion' suggest that someone took the premises as 'his premises' and inferred the conclusion. Resnik [14] and Copi [5] define the term 'argument' in such a way that to call (*P*, *c*) an argument is to *presuppose* that someone took the premises as his premises and inferred the conclusion; but, of course, their subsequent usage accords with the definition, which does not make that logically irrelevant presupposition. Another confusion results from the inclination to regard 'argument' as an honorific term and to refuse to count as arguments certain 'bad' arguments (those which are invalid or which have contradictory premises or which include the conclusion among the premises). This confusion is encouraged to some extent by translating Aristotle's term *syllogismos* as 'argument' because for Aristotle all 'syllogisms' are valid; an invalid argument cannot be a 'syllogism' at all (not even an imperfect one). These reflections bring up the second problem.

Problem 2: (a) What were the non-technical uses of the Stoic terms for 'argument', 'premise', 'conclusion' and 'valid'? (b) What were the common connotations of these words? (c) What kinds of confusions were likely to arise in technical usage because of the non-technical connotations? (d) Which of these confusions actually occurred?

To proceed we need to review the well-known asymmetry between the normal mode of establishing validity and the normal mode of establishing invalidity. For example, in *Prior Analytics* (I, 4, 5, 6) in order to establish that an argument (*P*, *c*) is *valid*, Aristotle produces a deduction, a list of easy logical steps leading (although not necessarily directly) from *P* to *c* and "making clear that the conclusion follows". On the other hand, in order to establish that an argument (*P*, *c*) is *invalid*, Aristotle produces a counter interpretation, i.e., he interprets the non-logical terms in such a way as to verify the premises and falsify the conclusion. It's the same in more complicated cases. To establish that Euclid's postulates (and axioms) imply the Pythagorean Theorem, one produces a step-by-step deduction of the latter from the former. To establish that the fifth postulate does not follow from the others one produces a counterinterpretation making the others true and the fifth false.

The asymmetry between Aristotle's method of establishing validity and his method of establishing invalidity is more than just echoed by modern logicians. Tarski, for example, relegates the two methods to separate (but adjacent) sections of his *Introduction to Logic* [16, §36, §37]. After a brief discussion of deduction within an axiomatic framework, Tarski adds [*op. cit.*, p. 119]

> More generally, if within logic or mathematics we establish one statement on the basis of others, we refer to this process as a *derivation* or *deduction*...

A few pages later [p. 124], he takes up the problem of showing that a certain sentence does not follow from certain premises. Here he discusses a reinterpretation of the basic terms in a manner that will leave the premises true while making the conclusion false.

Because the dichotomy of methods may not have been emphasized sufficiently in recent literature, it may appear to persist only in a somewhat muted form. However, I think that a case can be made for the historical thesis that what we now call 'proof theory' has its roots in the method of establishing validity whereas what we now call 'model theory' is rooted in the method for establishing invalidity.

Our main concern here is with the Stoic method for establishing validity, but we can still wonder about the Stoic method for establishing invalidity. As far as I have been able to determine, very little has been written about the latter and it may well be the case that the Aristotelian dichotomy was *not* preserved by the Stoics. They may have been concerned only with establishing validity. If this conjecture seems strange we may note that there is nothing about establishing invalidity either in *The Port-Royal Logic* or in Boole's *The Laws of Thought*. Moreover, long before the method of counterinterpretations was *used to establish* the invalidity of the argument from the other postulates of geometry to the parallel postulate, the argument was *widely assumed* to be invalid [cf. 4, p. 328].

Notice that a deduction is a piece of extended discourse consisting of several sentences over and above the premises and conclusion. As an aside we might point out that our term 'lemma' which usually indicates an especially important intermediate line in a long deduction was used by the Stoics to indicate a premise [*loc. cit.*]. As another aside which may be relevant to avoiding confusion we might note that some recent writers have used the terms 'a deduction' and 'an implication' interchangeably, sometimes using 'an implication' to indicate a valid argument (but, of course, for some older writers an implication is just an if-then sentence!).

It is useful to imagine that the deductions and the counterinterpretations all exist prior to being 'produced' so that 'production' is really only exhibition. If this is too much, just imagine that all deductions and all counterinterpretations potentially exist. In any case think of both classes of 'objects' as 'there'. Now we can ask interesting questions about the completeness of the method for establishing validity and about the completeness of the method of establishing invalidity. First, does every valid argument have a corresponding deduction? Second, does every invalid argument have a corresponding counterinterpretation?

Notice that only one of these questions can be trivial. *If* valid *means* having a deduction *then* the first question is trivial but the second is significant. On the other hand, *if valid means* having no counterinterpretations *then* the second question is trivial while the first is significant. Standard practice seems to be to take the latter point of view, i.e., to assume that valid means having no counterinterpretations. The significant question, then, is whether to every argument lacking counter interpretations there corresponds a deduction (to establish its validity). If not, then there are valid arguments whose validity cannot be established.

In any case we are led to consider three large classes: the class of arguments, the class of deductions and the class of interpretations. In the balance of this note we focus on the class of deductions; but, of course, the class of arguments and the class of interpretations are both continually in the background.

3. Aims of theories of deduction

It is unlikely that God gave men language and left it to Aristotle or to the Stoics to invent deductions. When Aristotle began his work there was an extant corpus of deductive discourses and a well-established activity of producing deductions. In fact, historians believe that there were at least two axiomatizations of geometry which existed prior to Aristotle's time.

This situation leaves Aristotle with three options as far as the aim of his theory of deduction is concerned. He could have had a *descriptive* aim or a *prescriptive* aim or a *conventionalistic* aim. That is, roughly, he could have set himself the task of *describing* the class of deductions (by cataloging the rules according to which they had been produced) or he could have *prescribed* the rules which *should* be used to produce ideally 'correct' deductions or he could have *devised* rules which would produce discourses which would serve the same purpose that ordinary deductions serve (viz. establishing that conclusions follow from premise-sets). There seems to be a tendency among mathematicians to assume that the descriptive approach is the dominant one not only in Aristotle but even in modern logic. Bourbaki [2, p. 1], whose foundational writings have been influential, has said,

> Proofs had to exist before the structure of a proof could be logically analyzed; and this analysis, carried out by Aristotle, and again and more deeply by the modern logicians, must have rested then, as it does now, on a large body of mathematical writing.

Indeed, on reading the *Analytics*, it is hard to escape the conclusion that Aristotle's *aim* was descriptive. However, as Mueller [13] has shown, Aristotle's final product fell far short of success *as a descriptive effort* because even the most elementary deductions in Euclid cannot be produced by Aristotle's rules. Here we come to another problem.

Problem 3: (a) To decide whether the Stoic logicians had set themselves descriptive or prescriptive or conventionalistic aims. (b) If the first, to decide whether their 'data' included the mathematical and scientific deductions available to them or whether they restricted their data so as to include only 'philosophical' discourse. If the second, to discover the criterion of correctness used to ground the 'should' of the prescriptions. If the third, to discover the reason they abandoned (or overlooked) the first two goals. (c) In any case to adduce persuasive philological arguments for the above.

4. Sentential and argumental systems of deduction

There are many different *styles* of systems of deduction and it is historically important to know the exact style that the Stoic system exemplified. Here we will characterize two styles which seem relevant to discussion of Stoic deduction. In order to determine the style of the latter it may be necessary for the historian to first construct an exhaustive survey of the extant styles and even then there is no reason to think that the Stoic system will necessarily conform to one of them.

When a person first starts to think about deductions he often conceives of a deduction of *c* from *P* as a list of *sentences* beginning with those of *P*, having intermediate sentences added according to rules and ending with *c*. A deduction whose 'lines' are all sentences is called a *sentential deduction*. A *direct*, *linear* sentential deduction is one of the sort described above – one goes from the premises step-by-step directly to the conclusion.

As I have suggested, I think that there is an inclination to think at first that all deductions are direct, linear and sentential. But this would be to overlook the indirect, linear, sentential deductions which proceed from *P* to *c* by assuming sentences in *P*, supposing also 'the denial' of *c* and then adding immediate inferences until one arrives at a sentence and its own denial. Aristotle's deductive system is a linear sentential system with direct and indirect deductions.

In regard to style the systems of Boole and Hilbert are more primitive than that of Aristotle because their deductions are *all direct and linear*. Systems of direct, linear, sentential deductions can have binary rules (which proceed from two local premises to a local conclusion, e.g. *modus ponens*) unary rules (which proceed from a single local premise to a local conclusion, e.g. universal instantiation) and nullary rules (which need no local premises and produce a local conclusion *ab initio*). Nullary rules are commonly referred to as logical axiom schemes.

In addition to linear rules which proceed from finitely many local premises to a local conclusion, a sentential system can also have suppositional rules which correspond to inference of a local conclusion (not *from* local premises but) *on the basis of* a 'pattern' of reasoning. For example conditionalization can be stated as a suppositional rule which proceeds to a conditional on the basis of a pattern of reasoning from the antecedent to the consequent. Thus the class of sentential deductive systems is quite diverse. It includes systems of direct linear deductions (Boole and Hilbert), systems of direct and indirect linear deductions (Aristotle) and systems of suppositional deductions (Jaskowski, Fitch, etc.). Many (but by no means all) of the so-called natural-deduction systems are sentential (cf. [6, III]).

Opposed to the sentential deductions (which are lists of sentences) there are those which are lists of arguments. Systems which consist entirely of lists of arguments are called *argumental deductive systems*. The systems of Lemmon [9], Suppes [15] and Mates [11] are in this style. In creating an argumental deduction one does not start with premises and proceed to a conclusion but rather one takes *ab initio* certain simple arguments and constructs from them, line-by-line, increasingly complex arguments until the argument with desired premises and conclusion is reached. In argumental systems the rules produce arguments from arguments (not sentences from sentences).

Given a certain minimal clear-headedness about the notion of a deduction, the problem of determining the exact nature of the Stoic deductive system (or systems) emerges. Let us put this down with a little care.

Problem 4: (a) To describe the class(es) of discourses which the Stoic logicians regarded as deductions, i.e., which were taken to establish the validity of arguments. (b) For the (each) Stoic deductive system we need both an exact description of the rules and also an account of how the rules were used to produce extended discourses (deductions).

5. The Stoic Fragments

The main purpose of this section is to review and interpret some of the available information concerning Stoic deduction in order to contribute toward a solution of the problem of discovering the style of the Stoic system.

It has been suggested that the theory of deduction may have been of minor importance in Stoic logic because, since the Stoics had truth-tables, they could establish the validity of arguments by a computational rather than discursive means. Two points are relevant here. First, Mates claims that there is no evidence that the Stoics *used* any computational means for establishing validity. Apparently the fact that truth-functional validity admits of a computational decision procedure, as embarrassingly trivial as it is, had to wait until 1920 to be noticed. Second, the existence of truth table methods should not disguise the fact that validity is always established by a deduction – to compute a truth-table for a truth-functionally valid argument is nothing more (or less) than writing a deduction-by-cases in tabular form.

Incidentally, I find it very difficult to understand how anyone could believe that the Stoics *knew* that their deductive system was complete when there is no evidence that they availed themselves of truth-table methods for establishing validity. Indeed, as has been pointed out elsewhere, if the Stoics had demonstrated completeness then surely they must have worked on the problem and, yet, there seem to be no fragments which admit of interpretation either as deliberation on the problem of demonstrating completeness or as alluding to such deliberation. In my opinion, it is not even clear that the Stoics *believed* their system complete (cf. [12, pp. 81–82]).

(A) *Language*: The Stoics analyzed sentences as truth-functional combinations of atomic sentences using as connectives: the conditional, conjunction, exclusive disjunction, and negation. Here we use $\supset$, &, v and $\sim$.

(B) *Sentential rules*: There were evidently five rules which 'produced' a single sentence from a pair of sentences and it is clear in each case that whenever the operands are true the resultant is true. Thus these five rules *could* serve as immediate *sentential-inference rules* (SIR, plural: SIRs) These can be written as follows:

(SIR1) $p \supset q, p/q,$
(SIR2) $p \supset q, \sim q/\sim p,$
(SIR3) $\sim(p \,\&\, q), p/\sim q,$
(SIR4) $p \text{v} q, p/\sim q,$
(SIR5) $p \text{v} q, \sim q/p.$

(C) *Argumental rules*: There were evidently four rules which produced an argument from a pair of arguments or (in at least one case) from a single argument. It is clear in the three known cases that whenever the operands are valid the resultant is also valid. Thus these rules could serve as immediate *argumental-inference rules* (AIR, plural AIRs). This concept will be discussed below but, for the present, we will write these rules using a symbolic notation. For later reference we will quote the rule before symbolizing it. In symbolizing the argumental rules we use the arrow to separate premises from conclusion and we use the double slant line to separate operands from resultant (just as we used the single slant line to separate operands from resultant in the sentential rules).

(AIR1) *If from two propositions a third is deduced, then either of the two together with the denial of the conclusion yields the denial of the other.*

This evidently gives two subrules.

(AIR1.1) $p, q \rightarrow r // p, \sim r \rightarrow \sim q,$
(AIR1.2) $p, q \rightarrow r // \sim r, q \rightarrow \sim p.$

Here it should be noted that the Stoics could have been using the term 'the denial' ambiguously to indicate *either* the result of adding a negation to a sentence *or* the result of deleting a negation from a sentence (which stands with a negation). If this is so, one would get sixteen subrules (when r is a negation, when p is a negation and when q is a negation).

(AIR2) *Whenever we have premises from which a certain conclusion can be validly deduced, potentially we have also that conclusion among the premises, even if it is not stated explicitly.*

To symbolize this let S be a set of premises and let $S+p$ be the result of adding p to S.

(AIR2) $S \rightarrow p; S+p \rightarrow r // S \rightarrow r.$

Today this rule is sometimes called 'the cut rule'; but there are other 'cut' rules as well.

(AIR3) *Whenever from two premises a third is deduced and other propositions from which one of the premises is deducible are assumed, then from the other premise and those other propositions the same conclusion will be deducible.*

(AIR3) $p, q \rightarrow r; S \rightarrow p // q+S \rightarrow r.$

This is another 'cut' rule. A modern logician might be baffled by the presence of two cut rules. That the 'force' of (AIR2) is so close to that of (AIR3) causes some speculation concerning the accuracy of the sources.

It is not known what the fourth rule is but it has been alleged that the Stoics 'had conditionalization'. One AIR version of conditionalization can be written as follows.

(AIR4) $S+p \to q \,//\, S \to (p \supset q)$.

Incidentally, it is important to distinguish between having a rule of conditionalization and knowing the principle of the corresponding conditional (which is semantic). The latter can be stated: an argument is valid if and only if the corresponding conditional (if 'conjunction-of-premises', then 'conclusion') is logically true. A rule of conditionalization is a rule for constructing deductions whereas the principle of the corresponding conditional is a semantic metatheorem. Obviously one could have either without the other. As far as I have been able to tell the Stoics knew the principle of the corresponding conditional but there is no evidence to indicate that they employed a deductive rule of conditionalization. (Note that the rule of conditionalization does not mention the conjunction connective.)

Another possibility for the fourth rule is one which would permit something like indirect deductions. One way of putting this is as follows.

(AIR5) *A set of premises implies a conclusion if the premises together with the denial of the conclusion imply a contradiction.*

(AIR5) $S+p \to q, \quad S+p \to \sim q \,//\, S \to \sim p$.

On grounds of common sense one would be inclined to accept the hypothesis that the Stoics had a rule for constructing 'indirect deductions'. However, there seems to be no textual evidence to corroborate that hypothesis.

(D) *The Stoic System*: Because of the existence of the argumental rules it is impossible that the Stoics had a sentential system. On the other hand, a sentential rule can easily be adapted for use as an *ab initio* (nullary) argumental rule. For example, *modus ponens* can be adapted to the following nullary argumental rule.

(AIR5) $//\, p, \; p \supset q \to q$.

Thus it seems possible that the Stoic system was an argumental system. Taste for simplicity tends toward this conclusion. However, it may have been the case that the argumental rules were thought of as rules for producing sentential rules from sentential rules so that the Stoics had a double-tiered sentential system: a kind of argumental system for producing sentential rules which were then incorporated into a sentential system for producing sentential deductions.

To exemplify the idea of producing sentential rules from sentential rules by means of argumental rules we offer the following.

(SIR6) $p, \sim q / \sim (p \supset q)$ (from (SIR1) by (AIR1.1)),
(SIR7) $\sim \sim q, p / \sim (p \vee q)$ (from (SIR4) by (AIR1.2)).

In order to settle these questions it is necessary to review the extant corpus and isolate all passages which are expressions of deductions. One must then try to discover the kind of rules which would best account for each passage. As far as I can see we still do not know exactly what the rules are because one cannot know what a rule is unless one knows how it is used.

There is a final consideration which may be important. Imagine that a deductive system emerges from a kind of operational conception. For example if we think of a logical consequence of a set of sentences as being somehow 'contained in' the set then we are inclined to view deduction as an operation of 'analyzing' a set of sentences to find out what is 'contained in' it. From this conception the linear, direct, sentential systems emerge (logical axioms will have to be thought of as catalysts which may be added in an analytic process without adding to the 'content' of the set of sentences being analyzed). An argumental system, especially those of Lemmon [9], Mates [11] and Suppes [15], may be seen as emerging from a constructional or synthetic conception; one starts with trivially valid arguments and uses them to synthesize increasingly complex arguments.

According to Mates [12, pp. 64, 77] the Stoics spoke of *analyzing* complex arguments and of *reducing* complex arguments to simple arguments. If this is to be taken literally then we can assume that the Stoics thought of complex arguments as some how 'composed of' simple arguments and that they used the argumental rules backward, so to speak, i.e. that they established the validity of a given argument by first finding simpler arguments which could be synthesized to yield the given argument, then doing the same thing to the simpler arguments, and so on until a set of 'simple arguments' was reached. If this is so then the Stoic 'deductions' were actually tree diagrams fanning out to simpler arguments from the given argument and having simple arguments at the extremities.

This conclusion seems to be compatible (at least) with the evidence that Mates cites but it goes counter to Mates' own conclusion. However, Mates' own account of the Stoic deductive process [12, p. 78] does not involve the argumental rules at all.

State University of New York at Buffalo

BIBLIOGRAPHY

[1] Aristotle, *Prior Analytics*.
[2] Bourbaki, N., 'Foundations of Mathematics for the Working Mathematician', *Journal of Symbolic Logic* **14** (1949), 1–8.
[3] Bury, R. G. (transl.), *Sextus Empiricus*, Vols. 1 and 2, Cambridge, Mass. 1933.
[4] Church, A., *Introduction to Mathematical Logic*, Princeton 1956.
[5] Copi, I., *Symbolic Logic* (2nd ed.), New York 1965.
[6] Corcoran, J., 'Discourse Grammars and the Structure of Mathematical Reasoning, I, II, III', *Journal of Structural Learning* **3** (1972).
[7] Frege, G., *On the Foundations of Geometry and Formal Theories of Arithmetic* (transl. by E. W. Kluge), New Haven 1971.
[8] Gould, J., 'Deduction in Stoic Logic', this volume, p. 151.
[9] Lemmon, E. J., *Beginning Logic*, London 1965.
[10] Łukasiewicz, J., *Aristotle's Syllogistic* (2nd ed.), Oxford 1957.
[11] Mates, B., *Elementary Logic*, New York 1965.
[12] Mates, B., *Stoic Logic*, Berkeley 1961.
[13] Mueller, I., 'Greek Mathematics and Logic', this volume, p. 35.
[14] Resnik, M., *Elementary Logic*, New York 1970.
[15] Suppes, P., *Introduction to Logic*, Princeton 1957.
[16] Tarski, A., *Introduction to Logic* (3rd ed.), New York 1965.