LOGIC, MEANING AND COMPUTATION

Essays in Memory of Alonzo Church

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SECOND-ORDER LOGIC

Abstract. This expository article focuses on the fundamental differences between first-order logic and second-order logic. It employs second-order propositions and second-order reasoning in a natural way to illustrate the fact that second-order logic is actually a familiar part of our traditional intuitive logical framework and that it is not an artificial formalism created by specialists for technical purposes. To illustrate some of the main relationships between first-order logic and second-order logic, this paper introduces basic logic, a kind of zero-order logic, which is more rudimentary than first-order and which is transcended by first-order in the same way that first-order is transcended by second-order. The heuristic effectiveness and the historical importance of second-order logic are reviewed in the context of the contemporary debate over the legitimacy of second-order logic. Rejection of second-order logic is viewed as involving radical repudiation of part of our scientific tradition. But even if genuine logic comes to be regarded as excluding second-order reasoning, which is a real possibility, its effectiveness as a heuristic instrument will remain and its importance for understanding the history of logic and mathematics will not be diminished. Second-order logic may some day be gone, but it will never be forgotten. Technical formalisms have been avoided entirely in an effort to reach an interdisciplinary audience, but every effort has been made to limit the inevitable sacrifice of rigor.

No matter what human action you consider, if everyone does it to everyone doing it to them, then everyone has it done to them by everyone to whom they do it. For example, if everyone teaches everyone who teaches them, then everyone is taught by everyone they teach. Likewise, if everyone helps everyone who helps them, then everyone is helped by everyone they help. The same holds for "encourages", "hinders", "supports", "opposes", "ignores", and the rest.

Each of the above propositions is actually a *tautology*, a proposition implied by its own negation. In fact, each of them can be proved to be true by logical reasoning alone; e.g., by deducing them from their own negations. Since every proposition in the same form as a tautology is again a tautology, a discourse formally similar to that expressed above obtains in every universe of discourse, not just in the universe of humans.

In metalogic, for example, we often discuss the universe of propositions in so far as various logical relations are concerned. By a *logical relation* I mean relations such as implication, consequence, contradiction, compatibility, independence, etc. More specifically, I mean what are called binary relations on the universe of propositions. If R indicates such a relation and if a and b are each individual propositions, then aRb can be used to express the propo-

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sition that the first proposition a is related by R to the second proposition b. It is not excluded, of course, that a and b are the same proposition. For example, every proposition implies itself and some but not every proposition contradicts itself.

Now logical relations are certainly not actions. Saccheri's Postulate contradicts the Parallel Postulate but there is no action that Saccheri's Postulate could perform. Nevertheless, as we have just seen, *relation verbs* function grammatically in certain contexts in a manner similar to the function of *action verbs*. The relation verbs significant in the universe of humans include the following: outweighs, outlives, succeeds (in several senses), precedes (in several senses), equals (in many senses), and many others. The action verbs significant in the universe of humans include the following: calls (in at least one sense), serves (in at least one sense), teaches, commands, obeys, and many others.

In normal English some of the logical relations are expressed by relation verbs, as we have seen. For example, implication is expressed by 'implies' and contradiction is expressed by 'contradicts'. However, some of them are expressed by *relation nouns*. For example, consequence is expressed by the relation noun 'consequence'. The law of transitivity of consequence is that every consequence of a consequence of a proposition is again a consequence of that proposition. Moreover, there are logical relations expressed by *relation adjectives*. Compatibility and independence are expressed by 'compatible' and 'independent'. Aristotle's fundamental law of compatibility of truth is that every two true propositions are compatible with each other. Using 'independent' in the most widely accepted sense we can say that every proposition which is independent of a given proposition is neither implied by nor contradicted by the given proposition, and conversely, every proposition which is neither implied by nor contradicted by a given proposition is independent of the given proposition.

One reason for reviewing the various ways that logical relations are expressed in English is to point out what all creative writers already know, viz. that knowledge of the conventional rules of English should enhance but not inhibit English writing. For example, my very first sentence uses the plural pronoun 'them' as coreferential with 'everyone', which is singular. Even worse, from the point of view of conventional rules, is my use of the filler 'you consider'. The proposition being expressed is not a prediction of what will happen if *you* consider something. The proposition is not about you *per se* at all. The sentence expresses a *general proposition* predicating a certain complex property of every action on the universe of humans. The phrase 'no matter what human action you consider' is just a heuristically effective way of expressing a universal quantifier. From a logical point of view the following would do just as well: 'every human action is one such that', 'every human action is one where', 'with every human action', etc. At any rate, a sentence that violates the conventional rules of English applicable to the expression of a given proposition is sometimes nevertheless a perfectly acceptable and effective way of expressing that very proposition.

Whether a sentence is an acceptable and effective expression of a given proposition is a matter of how readers take it, and not a matter of conventions established in the past. Now we are ready to present a discourse obtaining in the universe of propositions and formally similar to the one which began this essay.

No matter what logical relation you consider, if every given proposition bears it to every proposition bearing it to the given proposition, then every given proposition is borne it by every proposition the given proposition bears it to. For example, if every proposition contradicts every proposition contradicting it, then every proposition is contradicted by every proposition it contradicts. Likewise, if every proposition implies every proposition implying it, then every proposition is implied by every proposition it implies. The same holds for "is a consequence of", "is compatible with", "is logically equivalent to", "is independent of", "is a contradictory opposite of", and the rest.

The propositions expressed in the above paragraph are all tautologies *and* they are all laws of logic. The propositions in the first paragraph of this essay are all tautologies but none of them are laws of logic because they are not about a logical subject-matter. The proposition "Every proposition implies every proposition implying it" is about a logical subject-matter but it is not a law of logic because it is false. For example, "Every proposition is true" implies "Every false proposition is true", but not conversely. The proposition "Every proposition contradicts every proposition contradicting it" is a law of logic, of course, but it is *not* a tautology because it is in the same form as a proposition considered just above and found to be false. By a law of logic I mean a true proposition about a logical subject-matter, e.g., about propositions, about arguments, about argumentations, etc.

The two properties, being tautologous and being a law of logic, are *orthogonal* in the sense that each of the four combinations of the two is exemplified. We have seen above that some but not every tautology is a law of logic and that some but not every non-tautology is a law of logic. There is much confusion concerning this elementary point. Some but not all of the confusion is more or less deliberately nurtured in the service of various dogmas which, happily, are waning in popularity.

The proposition, "Every proposition contradicts every proposition contradicting it", is the law of symmetry (or reciprocity) of contradiction and "Some proposition implies some proposition not implying it in return" is the law of non-symmetry (or non-reciprocity) of implication. "No contradictory opposite of a contradictory opposite of a proposition is a contradictory opposite of that proposition" is the law of antitransitivity of contradictory opposition. A *contradictory opposite* of a proposition is, of course, a proposition logically equivalent to the negation of that proposition. For example, "Some true proposition is not tautologous" and "Not every true proposition is tautologous" are both contradictory opposites of "Every true proposition is tautologous". In order to avoid confusion it should be noted that, although every two propositions that are contradictory opposites of each other contradict each other, not every two propositions that contradict each other are contradictory opposites. To take an extreme example, "No proposition implies itself" contradicts "Some proposition implies every proposition". The same example illustrates another point that clarifies things and helps to avoid confusion, viz. that although no true proposition contradicts a true proposition, some false propositions contradict false propositions. In fact, some false propositions contradict themselves. Thus although no two contradicting propositions are both true, some two contradicting propositions are both false. In such cases, i.e., when two contradicting propositions are both false, they are not contradictory opposites because every two contradictory opposites have different truth-values.

A *contradiction* (or a self-contradiction) is a proposition that contradicts itself, i.e., that implies its own negation. Every contradiction is a contradictory opposite of a tautology and every tautology is a contradictory opposite of a contradiction. A proposition is said to be *contradictory* (or self-contradictory) if it is a contradiction. Every two contradictory propositions contradict each other but no two contradictory propositions are contradictory opposites of each other. The expression 'two contradictory propositions' means "two propositions each of which is self-contradictory" whereas 'two contradicting propositions' means "two propositions contradicting each other" which, in view of the symmetrical nature of "contradicts", amounts to "two propositions one of which contradicts the other".

In ordinary technical English, 'is contrary to' and 'is a contrary to' are ambiguous. Sometimes, "contradicts" is meant and sometimes "is a contradictory opposite of" is meant. Surprisingly, the ambiguity does not seem to be troublesome. However, in former times logicians had attached a third technical meaning that did lead to confusion. Two propositions were said to be *contraries* (sc of each other) if they contradict each other but their negations do not contradict each other. For example, "Every number is prime" and "Every number is non-prime" are contraries. It is easy to prove that every two contradicting propositions that are not contradictory opposites are contraries, and vice versa. In modern logic, 'contrary' is rarely used in the obsolete technical sense.

We have had occasion just now to state several laws of logic and to mention (or talk about) several laws of logic. As indicated above, by a *law of logic* I mean a true proposition about a logical subject-matter (propositions, arguments, argumentations, etc.). The most basic laws of logic are the laws of excluded middle, non-contradiction, and truth and consequence: "Every proposition is either true or false", "No proposition is both true and false" and "Every proposition implied by a true proposition is true".

The laws of logic, in fact all propositions about logical subject-matter are in some sense *second-level* (or *meta-level*) propositions in the sense that they are about things that are themselves about things (usually, of course, non-logical things). Some people, either ignorant of or in opposition to logical tradition, call such propositions 'second-order'. This is *not* how 'second-order' is used in this essay although some second-level propositions are also second-order. A proposition is classified as *basic, first-order, second-order*, etc., not on the basis of what it is about but rather on the basis of its logical structure. The very first proposition of this essay is second-order. The second proposition is first-order. Every proposition to the effect that one named proposition is in a mentioned logical relation to another named proposition is basic, e.g., "Saccheri's Postulate contradicts the Parallel Postulate". It will become obvious that the two properties, being second-level and being second-order, are orthogonal.

The *basic propositions*, very roughly speaking, are those without common nouns. It is perhaps easiest to describe the *basic propositions of arithmetic* (*BPA*). Actually, instead of describing the *BPA* outright, it is convenient to describe the *basic sentences of arithmetic* (*BSA*) and then to say that the basic propositions of arithmetic are the propositions expressed by the basic sentences when the sentences are understood in their intended interpretations. Now, the substantives of the basic sentences of arithmetic are exclusively *numerals* (number-names) in the wide sense: 'zero', 'one', 'two', 'three', ..., 'zero plus one', 'zero plus two', ..., 'two plus (zero times one)', ... Among the numerals I also intend: 'two-squared', 'two-cubed', etc. The *atomic* sentences of arithmetic include, in the first place, all so-called equations: 'one plus one is two', 'one plus two is one', etc., in other words any sentence in the pattern *numeral is numeral.* The 'is' here, of course, is intended to express numerical identity which is often improperly called *equality* and expressed by 'equals'. Next we have the sentences that normally attribute a quality to a number, e.g., 'one is even', 'one is odd', 'two is prime', 'five is perfect'; and so on. Next we have the sentences that normally relate one number to another, e.g., 'two exceeds three', 'three divides two', etc. These include the identities (or equalities, equations) already mentioned. Next we have the sentences that normally indicate that three numbers are in a ternary relation, e.g., 'two is between one and three', etc.

There are also quaternary relational sentences, e.g., 'one is to three as three is to nine'. And so on. Anything of this sort is countenanced as long as there are no common nouns. Even common nouns are allowed as long as they are understood as nominalized adjectives (e.g., 'two is a prime' means "two is prime") or as nominalized relatives (e.g., 'two is a divisor of four' means "two divides four"), etc. Once the atomic basic sentences of arithmetic have been determined, the basic sentences can be defined as the so-called truth-functional combinations of atomic sentences, the atomic sentences plus what can be obtained from atomic sentences by negations, conjunctions, disjunctions, conditionals, bi-conditionals, etc. It should be explicitly mentioned in this connection that passives (or converses) of binary relation verbs are again binary relational verbs and thus sentences such as 'two is divided by four', 'two is exceeded by four', etc. are included. Likewise included are sentences involving the so-called modified relation verbs: 'properly divides', 'immediately precedes', 'immediately exceeds', etc.

Basic logic is the logic of basic propositions. Basic logic is concerned fundamentally with the question of how we determine the validity or invalidity of an argument whose premises and conclusion are exclusively basic propositions. As you know, an argument is determined to be valid by giving a derivation (or a deduction) of its conclusion from its premises. This means giving an extended discourse, normally much longer than the premises-plus-conclusion which shows step-by-step how the conclusion can be seen to be true were the premises true. The rules for making up these derivations are obtained by looking at what people do with basic propositions when they are reasoning correctly. In order to deduce from any set of basic premises any basic conclusion that actually follows, it is sufficient to use rules from a very small set. These include the usual rules of propositional logic, the rule of substitution of identities, the rule of conversion (the active and passive are interdeducible) and the logical axioms of identity ("one is one", etc.). To show that a given basic conclusion does not follow from a given basic premise-set, it is sufficient to produce a *counterargument*, i.e., a conclusion and a premise-set together in the same form and having false conclusion and true premises. For example, to show that the argument on the left below is invalid it is sufficient to notice that the argument on the right is in the same form and has true premises and false conclusion.

Two is not three.	One is not two.
Three is not two plus two.	Two is not one times one.
? Two is not two plus two.	? One is not one times one.

The argument on the right is obtained in three steps from the argument on the left. First 'one' is substituted everywhere for 'two' on the left. Then in the "new" left argument (in which 'two' no longer occurs), 'two' is substituted everywhere for 'three'. Then in the "second new" left argument, 'times' is substituted for 'plus'.

Strictly speaking an *argument* (more properly, *premise-conclusion argument*) is a two part system composed of a set of propositions called the *premise-set* and a single proposition called the *conclusion*. To represent or express an argument we use an argument-text which is a list of *sentences* (not propositions) followed by a single sentence somehow marked as the conclusion-sentence. Some logic books use a line above the conclusion-sentence, but it is easier and less messy to use a question-mark as above. The method outlined above of transforming one argument-text into another argument-text in such a way that the argument represented by the second is in the same form as the argument represented by the first works only when the argument-texts are written in a so-called *logically perfect language* in which the outer grammatical form of the sentences mirrors exactly the inner logical form of the propositions. When logical issues are important, the language in question is *regimented* (*normalized*) so that it becomes logically perfect (or approximately so). This is why I write 'two plus (zero times one)' instead of 'two plus zero times one'. Logicians typically go immediately to a symbolic language carefully constructed to be logically perfect but for many purposes, especially that of exposition, this method, though virtually essential for some purposes, can be counter-productive.

Basic logic can be called *finite logic* because every finite invalid argument of basic logic is refutable by a counterargument whose propositions have reference only to a finite number of individuals. By a finite argument I mean an argument having only a finite number of premises and by reference only to a finite number of individuals I mean not only that the propositions refer only to finitely many individuals (which is obvious) but also that the functions referred to are all defined on one and the same finite universe of discourse.

By the way, this includes the so-called zero-premise arguments (arguments having the null premise-set) which are valid when and only when the conclusion is a tautology. Some examples follow.

? One is one.

? If one is two then two is one.

? If (if one is not two then one is two) then one is two.

? If one exceeds two then two is exceeded by one.

It follows from what was said above that every basic proposition that is not a contradiction is in the same logical form as a true basic proposition having reference only to a finite number of objects. This means that among the basic propositions there are no so-called *infinity propositions.* In order for a proposition to be an *infinity proposition* it is necessary and sufficient that it be non-contradictory and for every proposition in the same form having reference only to a finite number of individuals to be false. In other words an infinity proposition is a proposition expressed by a sentence which is "satisfiable" only in infinite universes of discourse.

Basic logic covers most of the arithmetic reasoning done by school children, all of the logic "done" by computers (though in a sense computers can *simulate* finite stretches of higher logics), and much of the logic on the normal aptitude test.

In a sense, *first-order logic* (*FOL*) begins when we generalize basic propositions. In fact, it is not stretching things to say that basic tautologies are tautologies *because* they are instances of first-order tautologies. When you prove a basic tautology you feel that you have not exhausted your reasoning in that direction. To illustrate this I will give a basic tautology and then give four first-order generalizations.

? If three exceeds two then two is exceeded by three.

? Every number exceeding two is a number that two is exceeded by.

? Every number that three exceeds is exceeded by three.

? Every number exceeding a given number is a number the given number is exceeded by.

? Every number that a given number exceeds is exceeded by the given number.

We are inclined to think that the basic tautology is logically derived from its generalization, e.g., that "if three exceeds two then two is exceeded by three" is true because "every number exceeding two is one that two is exceeded by" is true ... thus emphasizing the fact that the former is no peculiarity of three. Likewise we are inclined to think that the latter generalization is true because of the truth of *its* generalization, viz. "Every number exceeding an arbitrary number is one that the arbitrary number is exceeded by" ... thus emphasizing that no peculiarity of two is involved.

The *first order sentences of arithmetic* (*FOSA*) are the sentences obtainable from the basic sentences by quantification and taking truth-functional combinations. It is important that these operations are taken recursively, e.g., a basic sentence can be generalized and then combined with other generalizations by truth-functional combinations and then generalized again before taking further truth-functional combinations. The *first-order propositions of arithmetic* (*FOPA*) are the propositions expressed by the first-order sentences interpreted in the usual way. Below is an example of one of the simplest valid arguments in first-order logic.

Every number is either even or odd.

No number is both even and odd.

Every number which is odd is one whose square is odd.

? Every number whose square is even is itself even.

There is a radical increase in expressive power of first-order languages as compared to basic languages. For example, even the first premise in the above argument implies infinitely many basic consequences but it is not implied by any number of its basic consequences, not even by all of them together. The idea that a generalization is logically equivalent to the set of its singular instances is but one of the fallacies that is to be confronted by those seeking to reduce first-order logic to basic logic. Below are a few of the singular instances of the proposition under discussion.

One is either odd or even.

Two is either odd or even.

Three is either odd or even.

As mentioned above basic logic is sometimes called finite logic because each of its consistent (or non-contradictory) propositions is finitely satisfiable. This is no longer true of first-order logic. Indeed, the conjunction of the following two propositions is not satisfiable in any finite universe of discourse.

Zero is not the successor of any number.

Every two numbers which are successors respectively of distinct numbers are themselves distinct.

It is known, however, that every first-order proposition which is consistent is satisfiable in a countable universe of discourse. In fact, every consistent first-order proposition that is not satisfiable in a finite universe of discourse is, like the above conjunction, satisfiable in the universe of natural numbers. For this reason, first-order logic can be called *countable logic.*

Just as every valid basic argument is deducible using a small set of axioms and rules of inference, the same is true of valid first-order arguments. This means that as far as knowledge of validity of first-order arguments is concerned, human knowing faculties are equal to the task. The so-called principle of sufficiency of reason, viz. that every true proposition can be known to be true, can be shown to be false. Human faculties of knowing truth are not equal to the task of knowing truth—truth outruns knowledge. With *validity of first-order arguments*, reason is sufficient—every valid first-order argument can be known to be valid. Whether *every* valid argument (whatever the order) can be known to be valid is a question of considerable complexity and well beyond the scope of this elementary exposition.

There is another much less important fact about first-order and basic logic that is worth mentioning. For this we have to divide the logical concepts into *positive* and *negative.* Without going into the details, let me say that there are no surprises here. "Every", "Some", "Is", "And", "Or", "If", etc. are positive. "Not", "No", "Distinct", "Nor", etc. are negative. The result is that every contradictory first-order proposition involves at least one negative logical concept.

Just as we motivated the transition from basic logic to first-order logic by reflecting on the fact that the reasoning used to establish a basic tautology seems stronger than needed for that purpose and indeed is sufficient (or virtually so) to establish all generalizations of the basic tautology, we use the same sort of insight to transcend first-order logic. Consider the following first-order propositions.

No number divides exactly the numbers that do not divide themselves.

No number precedes exactly the numbers that do not precede themselves.

No number exceeds exactly the numbers that do not exceed themselves.

No number perfects exactly the numbers that do not perfect themselves.

The relation of perfecting arises in connection with the so-called perfect numbers. Every number having proper divisors is perfected only by the successor of the sum of its proper divisors. The other numbers, viz. zero, one and the prime numbers, are not perfected by any numbers at all. Thus four is perfected by three since two is the only proper divisor of four. But six is perfected by itself. In fact, as you may have seen already, every perfect number perfects itself and, conversely, every number perfecting itself is perfect. Now, the reason for introducing the perfecting relation is to give an example of a tautology in the same form as the first three of the above set but not as mathematically trivial. The first of the above propositions is mathematically trivial because zero, which is the only number that does not divide itself, is divided by every other number. The second is trivial because every number precedes other numbers but not itself. The third is trivial for similar reasons.

Now, as you know, each of the above can be deduced from their own respective negations by familiar (but intricate) reasoning. The fact is that the following premise-conclusion argument is valid.

Some number perfects exactly the numbers that do not perfect themselves.

? No number perfects exactly the numbers that do not perfect themselves.

A deduction of this argument, i.e., a deduction of its conclusion from its premise, can easily be transformed into an indirect *proof* of its conclusion. The reason that a deduction of a conclusion from the null-premise set is a *proof* (i.e., a deduction whose premises are known to be true) is because universal propositions with null "subjects" are vacuously true. Every member of the null set of premises is known to be true ... there being no counterexamples.

Once one of these tautologies has been proved to be true by a deduction from the null set of premises the others are also virtually proved to be true also. The reason for this is the principle of form for deductions: every argumentation in the same form as a deduction is again a deduction. Thus a proof of, say, the fourth can be obtained from a proof of, say, the first by substituting in the latter the concept "perfects" for the concept "divides". So it is clear that the reasoning establishing one of the four *virtually* establishes much more.

Now we move to *the* second-order generalization of the above. Actually, the following second-order proposition is at once a generalization of each of the above four first order tautologies and, in a certain reasonable sense, the only generalization.

> No matter which numerical relation you consider, no number bears it to exactly the numbers that do not bear it to themselves.

Once you have seen that this is true you will feel that it is the *ground* of the truth of the previous four propositions, e.g., that the truth of the fourth of them depends on no peculiarity of the perfecting relation.

My main point in this essay is that the reasoning in a given logic achieves more than can be expressed in that logic and that the transcending of a given logic by going to a higher order is one way of reaping the full fruit of one's reasoning in a given logic. This vague principle applies not just to first-order in relation to basic logic and to second-order logic in relation to first-order but in general to any logic in relation to the next lower order.

In basic sentences, there are no common nouns. In first-order sentences, there are common nouns, but no "second-order nouns" such as 'property', 'relation', 'function', etc. The presence of nouns inevitably and automatically entails the presence of quantifiers because nouns require articles and articles express quantifiers. For example, the following sentences express the same proposition.

> Every false proposition implies a true proposition.
>
> Every proposition which is false implies some proposition which is true.
>
> For every proposition which is false there exists a proposition which is true and which is implied by the false proposition.
>
> The same phenomenon can be exemplified in the universe of natural numbers (beginning with zero).
>
> Every odd number exceeds an even number.
>
> Every number which is odd exceeds some number which is even.
>
> For every number which is odd there exists a number which is even and which is exceeded by the odd number.

When we move to second-order by adding second-order nouns we also add second-order adjectives whose ranges of significance are the second-order objects denoted by the second-order nouns. Examples of second-order adjectives are the familiar terms indicating properties of relations: reflexive, symmetrical, transitive, dense, etc. The following are typical second-order sentences involving such expressions.

> Every reflexive relation relates every object to itself.
>
> Every relation that relates every object to itself is reflexive.
>
> Every symmetric relation relates to each other every two objects one of which it relates to the other.
>
> Every relation that relates to each other every two objects one of which it relates to the other is symmetric.

Orthogonality is a second-order relation between properties. In order for one property to be orthogonal to another it is necessary and sufficient that there be four objects, one having both properties, one having the first but lacking the second, one lacking the first but having the second, and one lacking both. These examples show that much of this essay has been written using a second-order language.

Since basic logic is finite and since first-order logic is countable, neither is adequate to axiomatize theories whose universes of discourse are uncountable. The most familiar examples of such theories are calculus and geometry. Now just as first-order logic is not finite, second-order logic is not countable. There are consistent second-order propositions which are not satisfiable in any countable universe. One example is from Hilbert's axiom set for the theory of real numbers (which is foundational for calculus). Another is from Veblen's axiom set for Euclidean geometry. Naturally, second-order logic can be called *uncountable logic.*

First-order logic is not even adequate to axiomatize theories whose universes of discourse are countably infinite. The paradigm case of such a theory is number theory, or the arithmetic of natural numbers, which requires *the principle of mathematical induction (PMI).*

> Every property belonging to zero and to the successor of every number to which it belongs also belongs to every number without exception.
>
> In order for a property to belong to every number it is sufficient for that property to belong to the successor of every number having it and also that zero have it.

Mathematical induction is the second-order generalization of each of the following propositions which are among its first-order instances.

> *If* zero is even and the successor of every even number is even, *then* every number is even.
>
> *If* zero is perfect and the successor of every perfect number is perfect, *then* every number is perfect.

In first-order axiomatizations of arithmetic *PMI*, induction, is replaced by the infinite set of its first-order instances, a set which is insufficient to imply mathematical induction. In fact, no set of true first-order propositions is sufficient to imply *PMI* and therefore no first-order axiomatization of arithmetic adequately codifies our knowledge of arithmetic. Moreover, the ground of our knowledge of the instances is our knowledge of *PMI* itself. Thus infinitely many of the so-called axioms of first-order arithmetic are not axiomatic in the traditional sense. Nevertheless, there are able logicians and mathematicians who reject the traditional second-order axiomatizations due to Dedekind and Peano in favor of first-order axiomatizations which date from the 1930's.

Just as second-order logic is necessary to fully exploit first-order reasoning as well as to understand the ground of first-order tautologies, likewise second-order logic is necessary to fully exploit first-order knowledge in arithmetic as well as to understand the ground of acceptance of first-order axiomatizations of arithmetic. Even logicians who reject second-order axiomatizations of arithmetic admit their historic importance and make heuristic and pedagogical use of such axiomatizations. By the way, the same thing may be said of axiomatizations of set theory, but the technical details involved in set theory require distinctions and principles which go beyond the scope of this essay.

In the case of basic logic, as well as that of first-order logic, a small set of simple rules of inferences suffices to enable every valid argument to be deduced. This is no longer the case with second-order logic. In fact, it is a corollary to the famous Gödel Incompleteness Theorem that no simple set of rules is sufficient for this purpose. This means that the principle of sufficiency of reason when applied to second-order validity is false. To be explicit, there are finite valid arguments in second-order logic whose conclusions can not be deduced (in a finite number of steps using simple rules) from their premise-sets. This result is known as the *incompleteness of second-order logic.*

There are logicians who feel that human reasoning must be equal to the task of determining the validity of valid arguments. In most cases such logicians are empiricistically oriented and are fully willing to accept the fact that there are true propositions about the material universe that can not be known to be true. But they feel that validity is intrinsically amenable to analytic *a priori* methods and, in particular, that every valid argument must be deducible. One way out of this quandary is to deny that second-order logic is really logic.

Incidentally, second-order axiomatizations do not evade the incompletability of arithmetic. First-order axiomatizations are deficient because first-order languages are too weak to express our knowledge of arithmetic even though first-order reasoning is adequate to first-order validity. Within second-order the situation is reversed. Second-order axiomatizations are deficient because second-order reasoning is too weak to deduce all of the consequences of second-order axioms even though second-order language is strong enough to express our knowledge of arithmetic. In fact, our second-order arithmetic knowledge implies absolutely every true second-order arithmetic proposition even those that we are powerless to deduce (using any given simple set of rules fixed in advance).

Another phenomenon that gives some logicians doubts about second-order logic is existence of contradictory propositions devoid of negative logical concepts. Recall that in first-order logic every contradictory proposition involves at least one negative logical concept. Below are two second-order propositions the first of which is tautological and the second of which is contradictory, neither of which involve negative logical concepts.

Every object has at least one property.

Every property belongs to at least one object.

The reason that the second proposition is self-contradictory is that it contradicts the following tautology.

No object has the property of being distinct from itself.

We have seen that second-order logic differs radically from first-order. First-order is a logic of countability; second-order is a logic of uncountability. First-order is deductively complete; second-order is deductively incomplete. In first-order every contradiction is negative; in second-order there are self-contradictory propositions which are exclusively positive. The above-mentioned historic examples of axiomatized sciences remind us that higher-order reasoning is not a recent innovation but rather a feature of human thought having a long history. Moreover, it is not the case that logicians started out studying basic logic and then moved on to first-order and then to second-order, etc. In the first place, Aristotle's logic is a fragment of first-order and fundamental aspects of basic logic were not to be discovered for some centuries later. In the second place, in modern times higher-order logics were studied *before* first-order logic was isolated as a system worthy of study in its own right.

After Aristotle's logic had been assimilated by later thinkers, people emerged who could not accept the idea that Aristotle's logic was not comprehensive. These conservative logicians attempted to "reduce" all logically cogent reasoning to Aristotle's syllogistic logic. Likewise, after first-order logic had been isolated and had been assimilated by the logic community, people emerged who could not accept the idea that first-order logic was not comprehensive. These logicians can be viewed not as conservatives who want to reinstate an outmoded tradition but rather as radicals who want to overthrow an established tradition. It remains to be seen whether higher-order logic will ever regain the degree of acceptance that it enjoyed between 1910 and 1930. But there has never been a serious doubt concerning its heuristic and historic importance. In fact, people who do not know second-order logic can not understand the modern debate over its legitimacy and they are cut-off from the heuristic advantages of second-order logic. And, what may be worse, they are cut-off from an understanding of the history of logic and the history of mathematics, and thus are constrained to have distorted views of the nature of the two subjects. As Aristotle first said, we do not understand a discipline until we have seen its development. It is a truism that a person's conceptions of what a discipline is and of what it can become are predicated on a conception of what it has been.

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