THE JOURNAL OF SYMBOLIC LOGIC
Volume 39, Number 4, Dec. 1974

STRING THEORY

JOHN CORCORAN, WILLIAM FRANK AND MICHAEL MALONEY

Abstract. For each $n > 0$, two alternative axiomatizations of the theory of strings over n alphabetic characters are presented. One class of axiomatizations derives from Tarski's system of the *Wahrheitsbegriff* and uses the n characters and concatenation as primitives. The other class involves using n character-prefixing operators as primitives and derives from Hermes' *Semiotik*. All underlying logics are second order. It is shown that, for each n, the two theories are synonymous in the sense of deBouvere. It is further shown that each member of one class is synonymous with each member of the other class; thus that all of the theories are synonymous with each other and with Peano arithmetic. Categoricity of Peano arithmetic then implies categoricity of each of the above theories.

Because all mathematically precise results in logic (aside from those concerned with infinitary languages) involve the syntactical interrelations among finite strings of characters over a finite alphabet (Tarski [22, p. 172]; Carnap [1, p. 7]), mathematical logic itself may be said to presuppose the theory of strings[1] (Hilbert [12, pp. 464–465]). The view that logic and mathematics are nothing but string theory seems an exaggeration to the point of caricature. Nevertheless, the foundational importance of abstract syntax seems established. Despite its importance, no current treatises on logic present this theory.[2] Moreover, explicit axiomatic

Received December 21, 1972.

[1] The phrase "the theory of strings" is used in the same informal sense in which the phrases "the theory of sets" and "the theory of numbers" are used. The usage emphasizes the universe of objects under consideration: strings, sets, numbers. The same constellation of ideas has been referred to as "the theory of symbol manipulation" and as "concatenation theory." The first of these emphasizes an important application of the theory much as does the reference to geometry as the theory of land measurement. The second of these emphasizes the intended interpretation of one of the primitives of one formulation of the theory—it is analogous to calling the theory of sets "membership theory" and to calling the theory of numbers "successor theory." Below the theory of strings will be referred to in several ways—none new, and all suggesting various aspects of the theory.

[2] Kleene [13, pp. 246–258] presents a theory which serves some of the purposes served by the theories to be presented below. However, Kleene's theory (called generalized arithmetic) does not admit of a natural interpretation in the universe of strings over n characters. A natural interpretation of Kleene's theory is based on a universe constructed by closing a finite alphabet under the three operations of taking sequences of lengths 1, 2 and 3. Kleene's theory is best thought of as a theory of syntactic structures. Admittedly the importance of theories of the sort envisaged by Kleene has also been overlooked—most notably by the modern mathematical linguists whose grammars are intended to generate syntactic structures (cf. Chomsky [2]).

Martin [15] and Quine [18] both present axiomatizations of the theory of strings over the specific alphabets of their respective object languages. Neither work contains any substantive metamathematical discussion of theories of strings.

Algebraists treat string theories as theories of certain free semigroups.

formulation of abstract syntax has also been neglected by other areas such as information science and linguistics which also presuppose it. The present article includes, for each n, two simple and transparent axiomatizations of the theory of strings over n characters.

Each of the theories axiomatized below has a second-order logic[3] as its underlying logic (in the sense of Church [4, p. 317]). For each positive n, Sn and Cn indicate the two theories in question. $S1$ is essentially identical to (second-order) Peano arithmetic. We show that, for every n, Sn and Cn are synonymous.[4] Moreover, by employing radix-n notation for positive integers, we also show that $S1$ is synonymous with each of the Sn. From the categoricity of $S1$ the categoricity of the other theories is obtained.

Let An be a set of n distinct characters $a_1, a_2, \cdots, a_n$. An^*, the set of all finite strings over An (including the null string 0), is the universe of discourse both of the intended interpretation[5] of Sn and also of the intended interpretation of Cn.

---

[3] All object-language theorems have been formally deduced in a natural deduction system which results from adding to a complete first-order system the following: the obvious introduction and elimination rules for universal and existential second-order quantifiers and the usual rules for handling "permanent" and "ad hoc" definitions. Such a system has been shown (Maloney [14]) to be equivalent in the relevant sense to the system F^{**} of Henkin which Henkin [10] himself had shown equivalent to Church's F^2 (Church [3]). Since the above second-order natural deduction system is in effect ordinary mathematical reasoning, formal description of it is here omitted.

[4] Let T and S be two theories having disjoint sets of primitives. T and S are *synonymous* (deBouvere [7]) if there exists a theory TS in the language whose primitive set is the union of that of T with that of S and which can be obtained *both* by adding to T definitions D_{st} of the primitives S in terms of those of T *and* by adding to S definitions D_{ts} of the primitives of T in terms of those of S. Synonymy is *not* the sàme as mutual interpretability in the sense of Tarski [20, p. 20]. For example *if* M is the set of logical truths involving a single monadic primitive and R is the set of logical truths involving a single diadic primitive *then* M and R are mutually interpretable (by arbitrary definitions) whereas they are not synonymous. According to deBouvere [8, p. 403], D. Kaplan has also constructed a counterexample to the proposition that mutual interpretability entails synonymy. The fact that synonymy is an equivalence relation (deBouvere [7]) is used below.

[5] The terms 'character' and 'string' correspond to undefined primitives in the formal theories developed below and it is very likely the case that their current technical meanings cannot be explicated without using equally problematic notions. However, the following informal comments may be useful to some readers. First, by a character we mean an abstract object, a "character-type," which has concrete instances called character-tokens or character-inscriptions. The latter may be destroyed (by fire, e.g.) but the former cannot. Relative to a given system of characters, a character is not decomposable into characters; a character is an atom. By a string we mean a "string-type" which is completely decomposable into characters which *occur* in it. A string-type has instances which are string-tokens or string-inscriptions composed of instances (not occurrences) of characters; the string-tokens are ultimately composed of character-tokens. Hermes and Tarski are both inclined to regard characters as somehow reducible to their instances which are in turn reducible to objects of physics. Thus, for Hermes and Tarski, the axioms are to be verified by "scientific experimentation." The present authors agree that knowledge of the truth of the axioms is to be derived from experience but they doubt that "scientific experimentation" is relevant. In any case, the question of the philosophic status of strings seems open. Finally, by means of human conventions a string may come to denote; but, the potential symbolic use of strings plays no role whatever in the formal theories developed below (although without the symbolic use of characters no written communication is possible).

Cn, "the concatenation theory," has as primitives (besides 0, $a_1, a_2, \cdots, a_n$ which denote themselves under the intended interpretation) the monadic predicate A, intended to indicate the alphabet, and the binary function symbol $+$ which is intended to indicate the operation ("concatenation") of patching one string directly onto the front of another. Thus $a_1 + a_2$ is simply a_1a_2 and, of course, $+$ is associative, satisfies both cancellation laws, has 0 as a null element, etc. *Sn*, "the successor theory," has as primitives (besides 0) n unary function symbols, $s_1, s_2, \cdots, s_n$, the ith of which indicates the operation of prefixing a_i to the front of a string. Thus, combining the two languages and the two intended interpretations we have $s_ix = a_i + x$ for all strings x in An^*. *Cn* is due in all important respects to Tarski[6] whereas Hermes[7] first presented a theory which embodied the main ideas of *Sn*. *Sn* has been mentioned by Kleene [13, p. 246] as a "generalized arithmetic."

As is the case above, n is a parameter which when combined with S or C determines a set of primitives and an intended interpretation. It also determines a set of axioms. Accordingly, the axioms are presented using metalinguistic devices which presuppose in each case prior choice of n. In particular, when Fi is a metalinguistic expression involving i and indicating a formula involving i, $[\&i]Fi$ and $[\bigvee i]Fi$ indicate respectively the conjunction and disjunction of the n formulas indicated by Fi (as i takes values between 1 and n). Similarly, where Fij is a metalinguistic expression involving i and j, $[\&i < j]Fij$ and $[\bigvee i < j]Fij$ indicate respectively the conjunction and disjunction of the $n(n - 1)/2$ formulas indicated by Fij (as i and j take ordered values between 1 and n). When $n = 1$, $[\&i]Fi = [\bigvee i]Fi = F1$ and any expression involving $[\&i < j]$ or $[\bigvee i < j]$ is to be ignored. Any axiom involving free occurrences of variables is assumed to be universally quantified as usual.

§1. Concatenation axioms *Cn* [Tarski].

CA1: $$[\&i < j](a_i \neq a_j)$$

The n characters are distinct.

CA2: $$Ax \equiv [\bigvee i](x = a_i)$$

The alphabet consists exactly in the characters.

CA3: $$\sim A0$$

CA4.1: $$A(x + y) \supset ((Ax \,\&\, (y = 0)) \vee (Ay \,\&\, (x = 0)))$$

CA4.2: $$(0 = x + y) \supset (x = 0 \,\&\, y = 0)$$

[6] Tarski is almost certainly the first person to axiomatize a theory of strings [22, p. 172]. Although Tarski clearly recognized the foundational importance of string theory for logic, for historical accuracy it must be noted that Tarski's intended interpretation involves the class of strings over a countably infinite alphabet.

[7] The idea of using successor functions instead of concatenation derives from Hermes [11] although the system of that work is based on a single second-order primitive (our primitives are all first order). Hermes' work was completed without knowledge of Tarski's so he did not think of himself as giving an alternative axiomatization. By modern standards Hermes' system would probably be adjudged unnecessarily succinct, especially in comparison to those given here—but one should realize that in former times the *number* of primitives of a system was thought to be a measure of its complexity.

Simplicity of the alphabetic characters and the null string.

CA5.1: $$x + 0 = x$$

CA5.2: $$x + 0 = 0 + x$$

The null element is 0.

CA6: $$(x + y) = (x' + y') \equiv \exists z(x' = x + z \,\&\, z + y' = y) \vee \exists z(x = x' + z \,\&\, z + y = y')$$

Tarski's law. This axiom characterizes the conditions under which "different compounds" give the same string. See Figure 1 below. In such situations, z is informally called "the interpolant for the pairs x, y and x', y'."

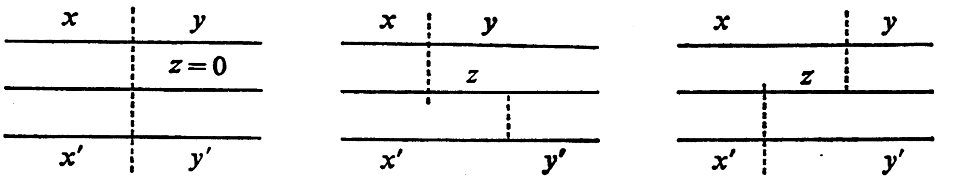

FIGURE 1

CA7: $$\forall P([P0 \,\&\, \forall x(Px \supset \forall y(Ay \supset P(y + x)))] \supset \forall z Pz)$$

String induction.

1.1. THEOREMS OF *Cn*. Let a, b and c be arbitrary. The string b is by construction an interpolant for the pairs a, $b + c$ and $a + b$, c. By Tarski's law, therefore, $a + (b + c) = (a + b) + c$. This proves:

CT1: $$(x + (y + z)) = ((x + y) + z)$$

By induction we get

CT2: $$x = 0 \vee \exists yz(Ay \,\&\, x = y + z)$$

CT3: $$[\&i](a_i + x = a_i + y \supset x = y)$$

To see this let b and c be arbitrary and suppose that $a_i + b = a_i + c$. By Tarski's law $\exists z(a_i = a_i + z \,\&\, z + c = b)$ or $\exists z(a_i = a_i + z \,\&\, z + b = c)$. Suppose the first. Then, for a particular z_0, $a_i = a_i + z_0$ and $z_0 + c = b$. But, by CA2, Aa_i; so, by CA3 and CA4, $z_0 = 0$. By CA5, $0 + c = c$, so $b = c$. In the second case the reasoning is the same. Q.E.D.

CT4: $$(z + x = z + y \supset x = y)$$

By induction. Let a and b be arbitrary and define P as the property which holds for z if $z + a = z + b \supset a = b$. $P0$ is obtained by CA5 and the induction step follows by CT1 and CT3. Q.E.D.

For $n = 1$, *Cn* is (in the same logical form as) second-order arithmetic where 0 denotes 0, a_1 denotes 1, + is addition and A is the property of being 1. In this case, no useful purpose is served by including A as a primitive.

§2. **Successor axioms** *Sn* **[Hermes].**

SA1: $$[\&i < j](s_i x \neq s_j y)$$

The successors have disjoint ranges.

SA2: $$[\&i](s_i x \neq 0)$$

SA3: $$[\&i](s_i x = s_i y \supset x = y)$$

SA4: $$\forall P[(P0 \ \& \ \forall x(Px \supset [\&i]Ps_i x)) \supset \forall y Py]$$

Successor induction.

2.1. *THEOREMS OF Sn.*

ST1: $$(x = 0 \vee \exists y[\bigvee i](x = s_i y)).$$

ST2: $$\exists! f(\forall x(f0x = x) \ \& \ \forall xy[\&i](fs_i yx = s_i fyx)).$$

For $n = 1$, *Sn* is (in the same logical form as) second-order Peano arithmetic and ST2 is the theorem which justifies taking the ordinary recursive definition of addition as a definition, properly so-called. The proof of ST2 mirrors the proof of the analogous theorem in arithmetic and the significance of ST2 here is the same as that of its analogue in arithmetic. In particular, ST2 implies that the extension of *Sn* effected by inclusion of the recursive "definition" (S^+D3 below) is a definitional extension (cf. Corcoran [5]). Actually, the proof of ST2 exhibits the necessary explicit definitions.

§3. **Synonymy of** *Cn* **and** *Sn*. In this section we produce Cn^+ and Sn^+ which are definitional extensions respectively of *Cn* and *Sn*. It can be verified that the axioms of Sn^+ are theorems of Cn^+ and that the axioms of Cn^+ are theorems of Sn^+.

3.1. Cn^+: *Interpreting Sn in Cn.* Here we need only define each s_i in terms of the primitives of *Cn*. The obvious choice is as follows.

$C^+D1.i$: $$s_i x = a_i + x$$

From the axioms of *Cn* and the above definitions it is easy to prove the axioms of *Sn*.

3.2. Sn^+: *Interpreting Cn in Sn: The Definitions.* For each i we define a_i as follows:

$S^+D1.i$: $$a_i = s_i 0$$

The primitive A of *Cn* is defined in *Sn* explicitly.

S^+D2: $$Ax \equiv ([\bigvee i]x = s_i 0)$$

Concatenation is defined recursively:

S^+D3: $$\forall x(0 + x = x) \ \& \ \forall xy([\&i](s_i y + x) = s_i(y + x))$$

3.3. Sn^+: *The Theorems.* For each i we get the following directly from the appropriate clause of S^+D3 together with S^+D1 and the first clause of S^+D3.

$S^+T1.i$: $$a_i + x = s_i x$$

Now CA1, CA2, CA3, and CA4 obviously follow. CA5 is obtained by induction using S^+D3 twice. CA6 is proved using the following lemmas.

S^+T2: $$[\&i]((s_i y = x + x') \supset (\exists z(x + x' = s_i z + x') \vee (x = 0 \,\&\, \exists z(x' = s_i z))))$$

S^+T3: $$[\&i < j](x + a_i \neq y + a_j) \,\&\, [\&i]((x + a_i = y + a_i) \supset x = y)$$

S^+T4: $$(x + z = y + z \supset x = y)$$

Finally CA7 is obtained using SA4 and the definitions.

The results of 3.1 and 3.3 do not imply that Sn and Cn are synonymous but only that they are mutually interpretable. It remains to prove the definitions of Cn^+ as theorems of Sn^+ and to prove those of Sn^+ in Cn^+. We omit the details.

§4. Synonymy of every *Sn* to Peano arithmetic. Standard second-order Peano arithmetic is nothing but $S1$ (cf. Montague [16, pp. 131, 135]). In this section we want to show that, for each n, Sn is synonymous with $S1$. We want to conclude that *all* of the above considered theories are synonymous.[8]

But this program reveals a minor short-coming with above accounts of deBouvere's concept of synonymy and Tarski's notion of interpretability. The trouble is that because $S1$ and Sn ($n \geq 2$) share the function symbol s_1 used "in different senses" they are jointly inconsistent. For example, in $S1$ every nonnull object has the form $s_1 x$ but in Sn ($n \geq 2$) some nonnull objects do not have this form (e.g. $s_2 0$). This obstacle is handled in the obvious way using the traditional notion of logical form (cf. Corcoran [6]). Two theories T and T' are defined to be in the *same* (*logical*) *form* if there is a one-one, category-preserving function from the content words of one onto the set of content words of the other which translates the theorems of one into theorems of the other and vice versa. Then define one theory to be *interpretable* in another if some theory in the same logical form as the first can be obtained in a definitional extension of the other. Finally, define two theories to be synonymous if each is in the same logical form as one of a pair of synonymous theories. In order to carry out the above purpose we assume a new theory $S0$ which results from $S1$ by replacing all occurrences of s_1 by the new function symbol s.

4.1. $S0^+n$: *Interpreting Sn in S0.* For purposes of terminology think of $S0$ as an arithmetic.

[8] The connection between string theory and arithmetic was first noticed by Hermes [11]. In effect he did the following two things. First he observed that $S1$ and Peano arithmetic are synonymous. Second he showed that, for every n, Sn is *relatively* interpretable in arithmetic. The latter result was slightly more difficult to obtain than our result (which is somewhat stronger) because Hermes chose to be guided by Gödelization rather than by radix notation.

In order to construct $S0^{+}n$ as a definitional extension of $S0$ which contains Sn we assume that the usual orderings ($<$ and $\leq$), addition $\oplus$, multiplication and exponentiation are all defined relative to $S0$ as usual. In addition we assume for each integer j that $\bar{j}$ is the $S0$-numeral associated with j, i.e., that $\bar{j}$ is a string of j successor symbols s followed by 0. Finally we need to assume that a binary operation has been defined so that $[x]n$ denotes the length of the least series of powers of n containing x; more precisely, so that

$$(1) \quad z = [x]n \quad \text{iff} \quad \begin{cases} \text{if } x \neq 0,\ z \text{ is the least number such that } \sum_i^z n^i \geq x, \\ \text{if } x = 0,\ z = 0. \end{cases}$$

It may be helpful to note that if *abcd* is a numeral in ordinary decimal notation and none of the digits are zero the following holds.

$$abcd = a \cdot 10^{[bcd]10} + b \cdot 10^{[cd]10} + c \cdot 10^{[d]10} + d.$$

Given that these definitions have been added to $S0$ we construct $S^{+}0n$ by adjoining the following definitions, one for each i between 1 and n.

SODi: $$s_i x = (\bar{i} \cdot \bar{n}^{[x]\bar{n}} \oplus x).$$

The details of verifying that the axioms of Sn are provable in $S0^{+}n$ are omitted here. The idea behind the definitions SODi will be discussed below in §5.

4.2. $Sn^{+}0$: *Interpreting $S0$ in Sn.* For purposes of terminology think of Sn as a theory of strings over the finite alphabet $a_1, a_2, \cdots, a_n$ given in alphabetical order. Assume that lexicographic order ($\lessdot$) has been defined so that for any two strings x and y we have $x \lessdot y$ iff, for x and y of different lengths, x is shorter than y or, for x and y of the same length, in the first (= left-most) place where they differ the character in x is alphabetically prior to the corresponding character in y. In addition, we assume the proof of a theorem to the effect that for each string x there is a unique lexicographically next string y, i.e., that for each string x there is a unique string y where $x \lessdot y$ and y lexicographically precedes all other strings lexicographically later than x. Thus we assume that it has been proved that "immediate" lexicographic order is a functional relation. This justifies the following definition.

SnD1: $$sx = y \equiv (x \lessdot y \,\&\, \forall z((z \neq y \,\&\, x \lessdot z) \supset y \lessdot z)).$$

The details of proving the Peano axioms are here omitted.

To establish synonymy of $S0$ and Sn we have to prove (1) the definitions SODi in $Sn^{+}0$ and (2) the definition SnD1 in $S0^{+}n$. These details are omitted.

4.3. $S0^{+}1$ *and* $S1^{+}0$.

$S^{+}01$: In order to get $S^{+}01$ from the result of adding the usual definitions to $S0$ only one definition was added, viz., the following definition of the "new" successor operator.[9]

SOD1: $$s_1 x = (s0 \cdot s0^{s[x]s0} \oplus x),$$

[9] The basic idea involved in the definition occurs in several places, notably (for the present context) in Quine [17]. There Quine wanted to establish two things: first (in effect) that Peano arithmetic is weakly interpretable (cf. Tarski [20, p. 29]) in each Cn ($n \geq 2$) and second (in effect) that Peano arithmetic is interpretable in each Cn. Quine's work differs from ours in

where s is ordinary successor, $s0$ denotes one and $\oplus$ is ordinary addition. In case $x = 0$ we have $[0]s0 = 0$; so using ordinary arithmetic we get

$$s_1 0 = (s0 \cdot s0 + 0) = s0. \tag{2}$$

In case $x \neq 0$ we have $[x]s0 = x$; again using ordinary arithmetic we get

$$s_1 x = (s0 \cdot s0^x \oplus x) = (s0 \oplus x) = sx. \tag{3}$$

Thus as expected we simply repeat in the definitional extension another theory in the same logical form as $S0$, viz. $S1$.

$S1^+0$: In order to get $S1^+0$ we first define lexicographic order in $S1$ and then add SnD1.

At this point, we obtain the theorem

S1$^+$T1.1: $$sx = s_1 x.$$

This "means" that concatenating the single character as a prefix is interpreted as successor. The lexicographic order over a unit alphabet is simply order by length: $0, a_1, a_1a_1, a_1a_1a_1$, etc. Thus the new successor s is the same as the old one and again we have repeated in the definitional extension a new theory in the same form. Moreover, addition turns out, as expected, to be simply concatenation. The idea, alluded to above, used for interpretating Sn in $S0$ amounts to the idea of the relationship of addition to concatenation in Sn. This idea hinges on what is here called radix-n notation for positive numbers. The latter degenerates in the case of $n = 1$ to representing a number m by a string of m 1's.

§5. **The radix-*n* notation for positive numbers.** Assume that we have characters $(a_1, a_2, \cdots, a_{n-1}, 0)$ to be used in constructing notation for the natural numbers. Without loss imagine that the first 9 of these are $1, 2, 3, \cdots, 9$.

The usual radix-n notation for the natural numbers uses all nonnull strings over all n characters and involves the following denotation function d for a string of $m + 1$ characters (since normally d is not defined on the null string).

$$db_m b_{m-1} \cdots b_1 b_0 = db_m \cdot n^m \oplus db_{m-1} \cdot n^{m-1} \oplus \cdots \oplus db_1 \cdot n^1 \oplus db_0 \cdot n^0 \tag{4}$$

where db_i is the denotation of the ith character in the string.

Given a theory of strings themselves interpretable as numerals, one would like to interpret $S0$ in the theory by simply looking at the strings as numerals and seeing what successor would be. But radix-n notation for natural numbers will not do if only because its denotation function is not one-one, it is not totally defined and it is not onto.

Radix-n notation for positive numbers does, however, have a one-one, total, onto, denotation function. Here the null string denotes zero and each positive

---

several respects. In the first place his underlying logics are all first-order (most of our results do not hold when the underlying logics are first order). In the second place he does not consider specific axiomatizations of any of the theories involved. In the third place, as is intimated by his title, his concern is with interpretability and weak interpretability—not with synonymy. Although none of the authors had seen Quine's paper until after this paper was written, some ideas involved in this research are already in Quine's paper.

number is denoted by a unique nonnull string over $\{a_1, a_2, \cdots, a_n\}$. Given that $da_i = i$, equation (4) above still defines the required denotation function. Moreover, because all digits are nonzero, for all strings x, the length of x is simply $[dx]n$, and, *contrary to the case of radix-n notation for natural numbers*, when the ith successor prefixes the ith digit we have

$$(5) \qquad ds_i x = (i \cdot n^{[dx]n} \oplus dx)$$

For example, using ordinary decimal notation to discuss ordinary decimal notation:

$$(2 \cdot 10^{[d00]10} + d00) = (2 \cdot 10^0 + 0) = 20$$

But of course $d200 = 200$, and therefore $d200 \neq (2 \cdot 10^{[d00]10} + d00)$. The point is, of course, that in order to know the significance of the left-most digit it is not sufficient to know the value of the numeral represented by the remaining digits. Zeros to the right of a digit contribute to its significance but they do not contribute to the size of the number represented by the remaining digits. In particular, equation (5) above does not hold in decimal notation for numerals having an initial nonzero digit followed by zeros.

§6. **Categoricity of all theories.** Categoricity of $S0$ (cf. Robbin [19, pp. 161–163] for recent proof) was first proved by Dedekind in the last century [9, pp. 92–96]. Any theory synonymous with a categorical theory is itself categorical. To see this let M and N be models of an arbitrary theory T synonymous with a categorical theory S. Using the definitions D which extend T to S, M and N can be expanded both in a unique way to M' and N' which are both models of $T + D$ (the composite theory). Now consider M'/S and N'/S the reducts of M' and N' to the language of S. Since S is categorical there exists a 1-1 structure preserving function between the universes of M'/S and N'/S. The composite theory $T + D$ must contain explicit definitions of the primitives of T in terms of those of S. Thus the isomorphism extends back to M' and N'.

Although $S1$ is *monotransformable* (any two of its models have only one isomorphism between them) this is obviously not the case for any Sn and Cn ($n \geq 2$). The reason for this, vis-a-vis the above proof, is that if T is one of the above-mentioned theories then there are several nonequivalent choices of definitions D so that $S + D$ includes T.

Synonymy with $S0$ of all theories Sn and Cm ($m, n > 0$) follows from the above results, since synonymy is an equivalence relation. Thus all of the theories Sn and Cm ($m, n > 0$) are categorical.

§7 **Unnamed characters: The theories** $C(n)$. To some it may seem superfluous that the characters in A are given special names in formulating the theory of strings in A^*. To eliminate the possibly extraneous names from the theory, eliminate them from the language and then replace CA1 and CA2 by a single axiom which asserts that there are exactly n elements in the alphabet. The following will do for $n > 1$.

CA1&2: $\exists x_1 x_2 \cdots x_n([\&i < j](x_i \neq x_j) \;\&\; \forall x(Ax \equiv [\bigvee i](x = x_i)))$.

For $n = 1$, take

CA1&2: $\exists x \forall y(Ay \equiv y = x)$.

For each n we define $C(n)$ as the theory whose primitives are simply 0 and + and whose axioms are those of Cn with CA1 and CA2 replaced by the appropriate CA1&2. From the forms of the added axioms one can see that, for each n, if S is in the language of $C(n)$ then S follows from $C(n)$ if and only if S follows from Cn. Since Cn is complete $C(n)$ is also complete. Moreover, since any model of $C(n)$ can be expanded to a model of Cn, the categoricity of $C(n)$ follows from that of Cn: In addition for $n = 1$, a_1 can be defined as the unique member of the alphabet giving an interpretation of Cn in $C(n)$. Thus also taking the null set of sentences as a set of definitions we have the synonymy of $C1$ and $C(1)$. However, since each model of $C(n)$ can be expanded in factorial n different ways to get a model of CA1 and CA2, it is clear (by Padoa's test) that $a_1, a_2, \cdots, a_n$ cannot be defined in $C(n)$ relative to Cn. The latter fact by itself does not establish that Cn and $C(n)$ are not synonymous; a rather trivial and involved argument is needed.

Let T and S be theories with primitives t and s respectively and, without loss, assume that t and s are disjoint. Let i_t and i_s be interpretations of the respective languages and let i_{ts} be an interpretation of the combined language. If i_{st} has the same universe as i_s (and i_t) and agrees with i_s (respectively i_t) on s (respectively t) then (1) i_s (i_t) is the reduct of i_{st} to s (t) and (2) i_{st} is an expansion of i_s (i_t) to $s + t$ (the union of s and t). Let i be an interpretation with universe u. Let m be a member of u, let f be a function from u^n into u, and let r be a subset of u^n. Following Tarski [21] we say that (a) m is *definable in* i when there is a formula in the language of i having exactly one free variable and which holds exactly of m, (b) f is *definable in* i when there is a formula $F(x_1, x_2, \cdots, x_n, y)$ in the language of i having exactly the indicated variables free and which hold of exactly the $(n + 1)$-tuples $m_1, m_2, \cdots, m_n, fm_1m_2\cdots m_n$ and, finally, (c) r is definable in i when there is a formula $F(x_1, x_2, \cdots, x_n)$ in the language of i having exactly the indicated variables free and which holds exactly of the n-tuples $m_1, m_2, \cdots, m_n$ in r. Given these definitions the following is obvious.

THEOREM. *If T is synonymous with S then every model i_t of T has an expansion i_{st} which satisfies S and, when i_s is the reduct to s, exactly the same entities of the universe (individuals, functions or relations) are definable in i_s as in i_t.*

The following can also be shown. Let S be a complete theory. Let i_s be a model of S and let e be an entity definable in i_s. Let i_s be expanded to i_{s^+} by letting the new constant $\bar{e}$ denote e in the usual way. Then $\bar{e}$ is definable relative to S^+, the set of truths of i_{s^+}.

Suitably changing primitives, take Cn for T, $C(n)$ for S, i_t any model of T and da_i for the object denoted by a_i, we have that the da_i are all definable in i_t. By the theorem every da_i is definable in some expansion i_{st} with reduct i_s. But since $C(n)$ is complete, if we denote da_i by $\bar{a}_i$ we have, in effect, that the a_i are definable relative to Cn, contradicting an observation above based on Padoa's test.

All this was designed to show that Cn and $C(n)$, $n > 1$, are not synonymous; something obvious enough in itself.

§8. Unnumbered alphabets: The theory *C*. In many contexts not only are the names of the alphabetic characters irrelevant but it is also not to the point to consider the size of the alphabet. Thus one seeks a set of axioms involving 0, *A* and + which has as consequences exactly those sentences which are true in every interpretation *i* with universe A^*, *A* an arbitrary alphabet, where 0 denotes the null string, the symbol *A* indicates *A* and + indicates concatenation. Deletion of CA1 and CA2 from the axioms of *Cn* yields a plausible candidate for such a set. Call the desired theory *C*. *C* is necessarily not complete (so not categorical and not synonymous with Peano arithmetic nor with any of the above theories).

Bourbaki and others have found it heuristically useful to distinguish formal theories into two classes on the basis of their usual mathematical significance. A theory which is intended as an axiomatic codification of the truths of a science studied antecedent to the axiomatization belongs to the first class which contains geometry, set theory, arithmetic, and the like. A theory which comes into existence only as the consequences of a set of sentences (e.g., chosen because they occur together in various contexts) belongs to the second class which contains the theory of equivalence relations, the theory of associative systems (semigroups), the theory of partial order, and the like. Categoricity (or at least completeness) is regarded as a desirable property for theories of the first class whereas theories of the second class are normally expected to be inherently incomplete. The incompleteness of (first-order) arithmetic is regarded as unfortunate whereas the incompleteness of the theory of equivalence relations is regarded as neutral or even as desirable. For example, some writers have "blamed" the incompleteness of first-order arithmetic on "inadequacy of first-order expressive power" while emphasizing that second-order arithmetic is categorical. The points seem to be: First, that one *can* ask of a given axiomatization of the first class whether "enough" axioms have been given; *but*, second, that such a question is inherently meaningless when applied to a member of the second class. Obviously our theories *Sn*, *Cn* and *C*(*n*) are theories of the first class whereas *C* is not in the first class. However, *C* does not seem to belong to the second class either.

Theories in the first class can be called *individual* because each seems to presuppose an (essentially unique) intended interpretation. Those of the second can be called *abstract* because they are formulated without reference to any intended interpretations. It would seem that heuristic purposes would be served by distinguishing a third class of theories each of which is intended as a codification of the truths common to a "general class" of interpretations. The term *generic* may be applied to theories of the third class. The question of "enough axioms" is meaningful applied to generic theories but categoricity (or completeness) is not to be expected.

Instead of explaining what is meant by a "general class" of interpretations consider the following examples. Define a *string structure* as quadruple $\langle u, A, 0, +\rangle$ where *u* is the class of strings over a given alphabet, *A* is the set of characters in *u*, 0 is the null string and + is concatenation on *u*. Define a *class structure* as a quintuple $\langle u, 1, 0, +, -\rangle$ where *u* is the class of subclasses of a given class, 1 is the given class, 0 is the null class, + is union on *u* and − is complementation on *u*. Define a *permutation structure* as a quadruple $\langle u, 0, +, -\rangle$ where *u* is the class of permutations of a given class, 0 is the identity permutation on the given class, + is composition on u and $-$ is the inverting function on u. An arbitrary class of interpretations of a given language will not serve as a "general class" of interpretations. All members of a given general class must have universes taken from a common homogeneous class of objects and the special individuals, functions and relations must be uniformly defined and "natural." Moreover, a general class must satisfy certain strong closure conditions.

Acknowledgments. The authors are grateful to Herbert Bohnert (Michigan State University) for reading and criticizing an earlier version. John Herring and Terry Nutter (SUNY at Buffalo) deserve thanks for checking the deductions and for proof-reading. Useful suggestions were received from Gabor Herman, John Kearns and Nicolas Goodman (all of SUNY at Buffalo) as well as from George Weaver (Bryn Mawr).

REFERENCES

[1] RUDOLF CARNAP, ***Logical syntax of language,*** 1934; translation by Amethe Smeaton, New York, 1959.

[2] NOAM CHOMSKY, ***Syntactic structures,*** The Hague, 1957.

[3] ALONZO CHURCH, ***Introduction to mathematical logic*** (based on notes by C. A. Truesdell), Princeton, 1944.

[4] ———, ***Introduction to mathematical logic,*** Princeton, 1956.

[5] JOHN CORCORAN, *A semantic definition of definition*, presentation to Chicago meeting of Association for Symbolic Logic (1967) abstracted in this JOURNAL, vol. 36 (1971), pp. 366–367.

[6] JOHN CORCORAN, *Conceptual structure of classical logic*, ***Philosophy and Phenomenological Research,*** vol. 33 (1972), pp. 25–47.

[7] K. L. DEBOUVERE, *Logical synonymity*, ***Indagationes Mathematica,*** vol. 27 (1965).

[8] ———, *Synonymous theories*, ***The theory of models (Proceedings of the 1963 International Symposium at Berkeley),*** North-Holland, Amsterdam, 1965.

[9] RICHARD DEDEKIND, *The nature and meaning of numbers* (written before 1893), ***Essays on the theory of numbers;*** translation by W. W. Beman, New York, 1963.

[10] LEON HENKIN, *Banishing the rule of substitution for functional variables*, this JOURNAL, vol. 18 (1953), pp. 201–208.

[11] HANS HERMES, *Semiotik, Eine Theorie der Zeichengestalten als Grundlage für Untersuchungen von formalizierten Sprachen*, ***Forschungen zur Logik und zur Grundlage der exakten Wissenschaften,*** n.s. no. 5, Leipzig, 1938.

[12] DAVID HILBERT, *The foundations of mathematics*, 1927; reprinted in VAN HEIJENOORT, ***From Frege to Gödel,*** Cambridge, Mass., 1967.

[13] STEPHEN COLE KLEENE, ***Introduction to metamathematics,*** New York, 1952.

[14] MICHAEL J. MALONEY, *Logical and axiomatic foundations for the study of formal languages and symbolic computation*, unpublished Ph.D. dissertation, University of Pennsylvania, 1969. Available from University Microfilms, Ann Arbor, Michigan.

[15] RICHARD MARTIN, ***Truth and denotation,*** London, 1958.

[16] RICHARD MONTAGUE, *Set theory and higher order logic* in ***Formal systems and recursive functions*** (Crossley and Dummett, Editors), North-Holland, Amsterdam, 1965.

[17] WILLARD QUINE, *Concatenation as a basis for arithmetic*, this JOURNAL, vol. 11 (1946).

[18] ———, ***Mathematical logic,*** revised edition, Cambridge, Mass., 1951.

[19] JOEL ROBBIN, ***Mathematical logic,*** New York, 1969.

[20] ALFRED TARSKI, ***Undecidable theories*** (in collaboration with A. Mostowski and R. Robinson), North-Holland, Amsterdam, 1953.

[21] ———, *On definable sets of real numbers* (1931), ***Logic, semantics and metamathematics,*** Oxford, 1956.

[22] ———, *The concept of truth in formalized languages* (1934), ***Logic, semantics and metamathematics,*** Oxford, 1956.

STATE UNIVERSITY OF NEW YORK AT BUFFALO
AMHERST, NEW YORK 14226