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VARIABLE BINDING TERM OPERATORS

by JOHN CORCORAN, WILLIAM HATCHER, and JOHN HERRING
in Buffalo, New York and Quebec (Canada)

Introduction

The study of variable binding term operators (vbtos) in logic dates from the beginning of the modern period. Perhaps because the RUSSELL device of contextual definition was used to establish eliminability-in-principle of the common vbtos, full deductive and semantic treatment of them was not regarded as important. However, the fact that symbols of a certain kind are eliminable-in-principle does not by itself imply that the best framework for logic is achieved by renouncing use of such symbols. Adoption of function symbols as primitives is now widespread though, of course, they are eliminable-in-principle in a logic with identity. Moreover, the expedient of introducing vbtos by contextual definition leads quickly to elaborate technical maneuvers involving numerous metatheorems in order to justify ther use in a natural and easy way (cf. QUINE [6], pp. 133, 140ff, for example). The system presented in this paper provides the basis for a standard handling of vbtos.

Given the semantics of variable binding term operators, it is seen that deductive completeness can be achieved by addition of a single axiom scheme, here called the *truth set principle* (see below). The whole situation is analogous to the relationship between the predicate logics with and without identity. Once a semantics for logic with identity is given, deductive completeness is achieved by addition of schemes to the deductive system of logic without identity. The proof of completeness of logic with identity can be obtained from completeness of logic without identity by showing that every model of the theory got by taking the added schemes as proper axioms in logic without identity is equivalent to some interpretation in the new semantics (cf. HENKIN [3], pp. 64–65).

Similar observations hold for soundness. The soundness and completeness proofs for logic with identity and vbtos given below exploit the analogy.

Of course, in the case both of logic with identity and of logic with identity and vbtos, the well-known HENKIN-HASENJAEGER methods can be used to construct direct completeness proofs not based on completeness results for the simpler systems. This type of completeness proof does not indicate, however, the exact interrelation between the original system and its extension. In case this exact interrelation is of interest, a proof based on the completeness of the original system is to be preferred.

1. The logics $\mathfrak{L}K$ and $\mathfrak{L}(K + V)$

Let LK be a first order language with identity where K indicates the set of non-logical constants possibly including individual constants and function symbols. Let ΔK be some deductive system which is sound and complete with respect to some standard semantic system ΣK in which an interpretation (or structure) i is a pair (D, m) with D a non-empty set and m a function assigning appropriate set-theoretic structures to the constants in K.

For simplicity we assume that each set D is accompanied by a set of symbols D^*, disjoint with K and put into canonical one-one correspondence with D. An element of D^* is said to *name* (or to *be the name of*) the unique object of D assigned to it by the canonical bijection. D^* is adjoined to K so that the denotation function, d^i, and the truth-valuation function, E^i, are defined (on closed terms and sentences respectively) without reference to sequences or assignments of values to variables. SHOENFIELD [5] gives a detailed treatment of such a semantic system. Examples of suitable deductive systems can be found in many places, specifically see HATCHER [2], MENDELSON [4], and SHOENFIELD [5].

Because statements of rules involving identity and terms can vary in ways which are crucial to the concerns of this article we explicitly assume that ΔK contains the following schemes wherein $t_1, t_2, \ldots, t_n$ and $t'_1, t_2, \ldots, t'_n$ are terms and F' is a result of replacing zero or more free occurrences of any t_j in F by free occurrences of t'_j and/or vice versa ($\&(t_j = t_j)$ is the conjunction of the formulas $t_j = t'_j$):

1.0 $t = t$; 1.1 $\&(t_j = t'_j) \supset F \equiv F'$.

Any logic $\mathfrak{L}K$ as described above which does not have these as axioms will have them as theorems since $\mathfrak{L}K$ is sound and complete and the schemes 1.0 and 1.1 are valid.

Let V be a new set of symbols called *variable binding term operators* (vbtos). Given $\mathfrak{L}K$ we form a new logic $\mathfrak{L}(K + V)$ which is designed to account for the logical behavior of what are normally called "variable binding term operators" in mathematics. We do this by amending the components of $\mathfrak{L}K$ as follows. $L(K + V)$ is formed by adjoining V to the set K of non-logical constants of LK and adding the following grammatical rule to the recursive definition of "term" for LK:

1.2 (Grammatical Rule). If v is a vbto, x is a variable, F is a formula having a free occurrence of x, then (vxF) is a term in which all occurrences of x are bound. Otherwise, the free variables of (vxF) are the same as the free variables of F.

In the semantic system $\Sigma(K + V)$ (essentially the system of HATCHER [2] as modified by CORCORAN and HERRING [1]) an interpretation i is a pair (D, m) where D is as above and m is a function defined on $K + V$ with m the same as above on K and, for v in V, with mv a function from PD (power set of D) to D. The denotation function, d^i, is defined as for $\mathfrak{L}K$ with one additional clause (1.3 below), and the recursive definition of the truth valuation function E^i is exactly as in $\mathfrak{L}K$. In order to state the definition of d^i we define, for each variable x, formula F containing exactly x free, and interpretation i, the *truth set* (or *solution set*) of F relative to i as follows. $T(x, F)^i$ is the set of objects d in D which satisfy F when F is interpreted according to i except that x is taken to denote d. More formally, $T(x, F)^i$ is the set of all d in D such that $E^i(F(x/d^*)) =$ true, d^* being the name of d. We state:

1.3 (Definition of Denotation for Variable Bound Terms).

$$d^i((vxF)) = mv(T(x, F)^i).$$

The deductive system $\Delta(K + V)$ is the result of adding a new scheme (1.4 below) to the axiom schemes of ΔK. This scheme, called the *truth set principle* (TSP) is due to CORCORAN and HERRING [1] who conjectured that its addition to ΔK is complete relative to $\Sigma(K + V)$.

1.4. Let y and y' be distinct variables. Let F (respectively F') be a formula having y but not y' free (respectively y' but not y free). The following is a truth set axiom:

$$\forall y \forall y' (y = y' \supset F \equiv F') \supset (vyF) = (vy'F').$$

Our main theorem, which is immediate from the lemmas of Section 3 below, is the following:

Theorem. *Let F be a sentence of $L(K + V)$. Then F is true in every interpretation in $\Sigma(K + V)$ if and only if F is the last line of a formal deduction in $\Delta(K + V)$.*

2. The pseudo logic $\mathfrak{L}K/V$

In order to study the relationship between $\mathfrak{L}K$ and $\mathfrak{L}(K + V)$ it is convenient to consider an intermediate system whose language is that of $\mathfrak{L}(K + V)$ but whose deductive system is that of $\mathfrak{L}K$, the simpler logic. Let LK/V be $L(K + V)$ and let $\Delta K/V$ be the result of deleting the truth set principle from $\Delta(K + V)$. The behavior of variable bound terms (vbts) in $\Delta K/V$ is very limited in comparison with their behavior in $\Delta(K + V)$ which takes account of their entire inner structure. More precisely, $\Delta K/V$ treats each vbt as if it were obtained by substituting terms for variables in a certain vbt which behaves like a term composed of a function symbol applied to variables. In particular, some vbts (e.g. $(vxPx)$) behave like individual constants, some (e.g. $(vxRxyz)$) behave like terms made with a single function symbol (i.e. fyz) and some (e.g. $(vxRx(vyRyzu)(vwRx'wy'))$ like terms made with more than one function symbol (i.e. $f(fzu, hx'y')$). But there is much inner structure to vbts which cannot be accounted for in this way and this additional inner structure is ignored by $\Delta K/V$. In particular, in $\Delta K/V$ 2.0 and 2.1 below are provable whereas 2.2 and 2.3 are not, where v is in V:

2.0. $\forall xy(fxy = (vzRxyz)) \supset \forall u(fuu = (vzRuuz))$,

2.1. $d = (vxSx) \supset (vzTzd) = vzTz(vxSx)$,

2.2. $\forall xy((vzRxyz) = (vuRxyu))$,

2.3. $\forall x(Px \equiv Qx) \supset (vxPx) = (vxQx)$.

To be more precise concerning $\Delta K/V$, a few definitions are needed. Let t be a vbt containing no constant terms. If t has exactly n free occurrence of variables, let $(x_1, x_2, \ldots, x_n)$ be the sequence of free variables of t in their exact order of occurrence. (Thus, there are as many as n and as few as zero distinct variables in $(x_1, x_2, \ldots, x_n)$.) If the following two conditions hold, then t is a *canonical vbt* (cbvt): (1) $(x_1, x_2, \ldots, x_n)$ is the sequence of the first n (distinct) variables of the language not occurring bound in t in numerical order. (We presuppose a standard enumeration of the variables.) (2) Every term having a free occurrence in t is a variable.

Every vbt is obtainable from a cvbt by replacement of terms for variables. More importantly, every vbt is so obtainable from exactly one cbvt, as is clear from their syntactical structure. Let c be the function defined on the set of vbts and such that ct is the cvbt from which t is obtained by replacement.

Consider the free occurrences of terms in t, a vbt. In some cases one or more occurrences will be proper parts of an occurrence which itself is not a proper part of any other proper free occurrence of a term in t. A free occurrence in t of a term which is within no other free term occurrence in t (other than t itself) is called a *maximal term occurrence* in t. A term which has a maximal term occurrence in t is called a *maximal term* of t. If $(t_1, t_2, \ldots, t_n)$ is the sequence of maximal terms of t in exact order of occurrence as maximal terms of t, and if $(x_1, x_2, \ldots, x_n)$ is the sequence of free variables of ct in their exact order of occurrence, then t is obtained from ct by simultaneously replacing each free occurrence of a variable in ct by the maximal term of t which occurs at the corresponding place in t.

Now let CV be the set of cvbts of $L(K + V)$ and let G be a set of function symbols (including individual constants) disjoint with K and such that r is a bijection from CV onto G which associates n-ary function symbols with cvbts having n free variables. Using r we define a bijection R from the terms and formulas of $L(K + V)$ onto those of $L(K + G)$. On LK, the expressions common to the two languages, R is the identity. If $t = ft_1t_2 \ldots t_n$, set $Rt = fRt_1Rt_2 \ldots Rt_n$. If t is a vbt with maximal terms $(t_1, t_2, \ldots, t_n)$ as above, set $Rt = (rct)\, Rt_1Rt_2 \ldots Rt_n$. Finally, let R preserve sentential connectives, quantifiers and identities. Using R, one can translate sequences of formulas in $L(K + V)$ into sequences of formulas in $L(K + G)$ so that a sequence in $L(K + V)$ is a proof of a formula F in $\Delta K/V$ if and only if its translation is a proof of RF in $\Delta(K + G)$. Moreover, by changing variables if necessary, every proof in $\Delta(K + G)$ of a formula F can be converted into another proof in $\Delta(K + G)$ of F', an alphabetic variant of F, so that the new proof is a translation of a proof in $\Delta K/V$. Thus F is a theorem of $\Delta K/V$ if and only if RF is a theorem of $\Delta(K + G)$, and F is a theorem of $\Delta(K + G)$ if and only if F is equivalent to RH from some theorem H of $\Delta K/V$.

Despite obvious deficiences, $\Delta K/V$ is still a "sensible" deductive system and it is not difficult to provide a natural semantics for it by amending ΣK. Let us define $\Sigma K/V$, the *pseudo semantics for* $\mathfrak{L}K/V$, as follows. A pseudo interpretation i in $\Sigma K/V$ is a pair (D, m) where D is non-empty and m is a function defined on the union of K and the set CV of cvbts such that (1) if $m \mid K$ is the restriction of m to K, then $(D, m \mid K)$ is an interpretation in ΣK and (2) if t is a cvbt having exactly n free variables, then mt is a function from D^n to D (a function from D^0 to D is understood to be a member of D). The definition of denotation of terms and truth valuation of formulas relative to an i of $\Sigma K/V$ is the same as for ΣK except for the following additional clause for defining d^i (the denotation function) on closed vbts.

2.4. Let t be a closed vbt having $(t_1, t_2, \ldots, t_n)$ as its sequence of maximal terms as above. Assuming d^i defined on the maximal terms of t, we pose $d^it = mct(d^it_1, d^it_2, \ldots, d^it_n)$.

The function r above can be used in the obvious way to construct a bijection $\bar{r}$ from the pseudo interpretations of $\Sigma K/V$ onto the interpretations of $\Sigma(K + G)$ so that every i is equivalent to $\bar{r}i$ in the sense that:

(a) for all closed t in $L(K + V)$, $d^i t = d^{\bar{r}i} Rt$ and

(b) for all sentences F in $L(K + V)$, $V^i F = V^{\bar{r}i} RF$.

Since $\mathfrak{L}(K + G)$ is an ordinary sound and (strongly) complete logic with identity, we have the following lemma, in which satisfiability and consistency are defined as usual.

2.5. Lemma. *Let S be a set of sentences of $L(K + V)$. Then S is satisfiable in $\Sigma K/V$ if and only if S is consistent in $\Delta K/V$.*

3. Soundness and completeness of $\mathfrak{L}(K + V)$

3.1. Lemma (Soundness). *Let F be a sentence of $L(K + V)$. If F is the last line of a formal deduction of $\Delta(K + V)$, then F is true in every interpretation in $\Sigma(K + V)$.*

Proof. Let T be the theory of $\mathfrak{L}K/V$ got by taking the truth set principle as proper axioms. Clearly T has exactly the same theorems as $\mathfrak{L}(K + V)$. Thus, by the soundness of $\mathfrak{L}K/V$ (Lemma 2.5), every theorem of $\mathfrak{L}(K + V)$ holds in every pseudo model of T. Thus, if every interpretation in $\Sigma(K + V)$ is equivalent to a pseudo model of T, then the present lemma is proved.

Let i be an interpretation in $\Sigma(K + V)$, $i = (D, m)$. We construct the pseudo interpretation $j = (D, m')$ as follows. For s in K, let $m's = ms$. For t a cvbt of $L(K + V)$, $m't$ is a function from D^n to D defined by $m't(d_1, d_2, \ldots, d_n) = d^i t'$, where $(x_1, x_2, \ldots, x_n)$ is the sequence of variables free in t in their order of occurrence and t' is the result of replacing the occurrence of x_k in t by the name d_k^* (from D^*) of d_k for $0 \leqq k \leqq n$. Clearly, j is equivalent to i. Since TSP is valid in $\Sigma(K + V)$, j is a pseudo model of T.

3.2. Lemma (Completeness). *Let F be a sentence of $L(K + V)$. If F is true in every interpretation in $\Sigma(K + V)$, then F is the last line of a formal deduction of $\Delta(K + V)$.*

Proof. It is sufficient to show that every pseudo model of T is equivalent to some interpretation of $\Sigma(K + V)$. For then if F is true in every interpretation of $\Sigma(K + V)$ it is true in every pseudo model of T. But by completeness of $\mathfrak{L}K/V$ (Lemma 2.5), if F is true in every pseudo-model of T, then F is a theorem of T and therefore a theorem of $\Delta(K + V)$.

Let $i = (D, m)$ be a pseudo-model of T. We define $j = (D, m')$, an interpretation in $\Sigma(K + V)$ as follows. For s in K take $m's = ms$. For each vbto v in V we want to define mv as a function from PD to D such that all terms have the same denotations and all sentences have the same truth values in both i and j. Let F be a formula having only x free and possibly containing occurrences of names. For each v, set $m'vT(x, F)^i = d^i(vxF)$. Since TSP holds in i, $m'v$ has a unique value on each subset of D definable in i. Let $m'v$ take arbitrary values elswhere on $\mathrm{P}D$. By c struction, closed terms and sentences without vbtos have the same values (denc tions or truth values) in i and j. By course of values induction on the length closed terms and sentences, one extends the equivalence to all of $L(K + V)$.

These two lemmas establish our main theorem which was stated in Sectio The strong completeness of $\mathfrak{L}(K + V)$ is also immediate from the proof of Lemma

4. Generalization

In Section 1 above we have given the details of the treatment of vbtos wh bind one variable. Extension of the treatment to vbtos which bind more than variable is obvious. In particular, terms are formed by the following grammat rule.

4.1 (Grammatical Rule for n-ary vbtos). Let v be an n-ary vbto. $x_1, x_2, \ldots, x_n$ be n distinct variables all of which have free occurrences in the mula F. Then $(vx_1x_2 \ldots x_nF)$ is a term in which $x_1, x_2, \ldots, x_n$ are bound. Ot wise, the free variables of the term are those of F.

If $i = (D, m)$ is an interpretation of a language LK where K contains an n- vbto v, then mv is a function from $\mathrm{P}D^n$ to D. Moreover, if $(vx_1x_2 \ldots x_nF)$ is a clo term, then we pose:

4.2. $d^i(vx_1x_2 \ldots x_nF) = mvT(x_1, x_2, \ldots, x_n, F)^i$

where $T(x_1, x_2, \ldots, x_n, F)^i$ is the truth set (or solution set) of F, i.e. the se n-tuples from D which satisfy F when F is interpreted according to i except t each x_k is taken to denote the k^{th} member of the n-tuple.

Finally, completeness is achieved by addition of the following scheme:

4.3 (Generalized Truth Set Principle). Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be strings of n disti variables. Let $\&(\boldsymbol{x} = \boldsymbol{y})$ indicate the conjunction of the identities between responding components of $\boldsymbol{x}$ and $\boldsymbol{y}$. Let F be a formula having free occurre of all variables in $\boldsymbol{x}$ and none of those in $\boldsymbol{y}$. Let H be a formula having free oc rences of all variables in $\boldsymbol{y}$ but none of those in $\boldsymbol{x}$. Then the universal closure the following are axioms: $\forall \boldsymbol{x}\boldsymbol{y}(\&(\boldsymbol{x} = \boldsymbol{y}) \supset F \equiv H) \supset (v\boldsymbol{x}F) = (v\boldsymbol{y}H)$.

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