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Epistemic truth and excluded middle*

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Abstract: Can an epistemic conception of truth and an endorsement of the excluded middle (together with other principles of classical logic abandoned by the intuitionists) cohabit in a plausible philosophical view? In PART I I describe the general problem concerning the relation between the epistemic conception of truth and the principle of excluded middle. In PART II I give a historical overview of different attitudes regarding the problem. In PART III I sketch a possible holistic solution.

Part I
The Problem

§1. The epistemic conception of truth.

The epistemic conception of truth can be formulated in many ways. But the basic idea is that truth is explained in terms of epistemic notions, like *experience*, *argument*, *proof*, *knowledge*, etc. One way of formulating this idea is by saying that truth and knowability coincide, i.e. for every statement *S*

i) *it is true that* S *if, and only if, it is (in principle) possible to know that* S.

Knowledge that *S* can be equated with possession of a correct (or ideal) finite argument for *S* in an optimal (or ideal) epistemic situation. For example, mathematical knowledge can be equated with possession of a proof of a mathematical statement. But the notion of ideal argument should be wider than the notion of proof: for empirical statements like "strawberries contain sugar" an ideal argument should contain also some sensory evidence. In terms of an appropriate notion of 'ideal argument' the epistemic conception of truth can be formulated as follows:

i) it is true that* S *if, and only if, there is an ideal argument for the statement that* S.

This is a generalization of Dag Prawitz's proposal to identify mathematical truth with the existence of a proof. If Prawitz's view is generalized to all statements, ''there is'' in *i** ought to be interpreted in an abstract sense of 'is'[1] according to which the existence of an ideal argument does not imply that such an argument is actually found by someone at some time. One can argue that *i* and *i** are not equivalent,[2] but the difference is irrelevant here, because the problem with the excluded middle arises for both, as it does for other formulations of epistemic conceptions of truth.

§2. Can a supporter of an epistemic conception of truth accept the excluded middle?

The principle of the excluded middle affirms that for every statement *S*, it is true that

ii) S *or* not S.

If we equate the falsity of a statement *S* with the truth of its negation *not S*, and accept the equivalence thesis:

iii) it is true that S *if, and only if,* S,[3]

then from the excluded middle, by classical (and intuitionistic) logic, we can derive the principle of bivalence, i.e.

iv) it is true that S *or it is false that* S .[4]

Hence, from the epistemic conception of truth and from the excluded middle we can derive the principle that for every problem concerning the truth or falsity of a statement there is (in principle) a solution. If we adopt version *i* of the epistemic conception, we can derive, for every statement *S*, that

v) it is possible to know that S *or it is possible to know that* not S.

Similarly from formulation *i** of the epistemic conception of truth we can derive that

v) there is an ideal argument for the statement that* S *or there is an ideal argument for the statement that* not S.

Let us call *v* and *v** "principles of existing solution for '*S*'". Assuming an epistemic conception of truth, the excluded middle implies the validity *for every statement* of some principle of existing solution. The tenet that, for each statement *S*, we can find a solution of the problem whether *S* is true or false does not entail that there is a *uniform* method for solving every problem of this kind. Different problems can be solved in completely different ways. However, for some statements, the claim that there is a solution can be controversial. Reasonings *i-v* and *i*-v** highlight a relation between the excluded middle and such a controversial claim. As far as mathematics is concerned, this is more or less what Brouwer emphasized in his "The Unreliability of the Logical Principles" (1908), the famous essay which started the intuitionistic attack on classical logic:

> the question of the validity of the principium tertii exclusi is equivalent to the question whether unsolvable mathematical problems can exist.

But Brouwer added:

> There is not a shred of a proof for the conviction which has sometimes been put forward that there exist no unsolvable mathematical problems.[5]

To be precise, by *intuitionistic* logic we *can* prove that there does not exist any unsolvable mathematical problem, contrary to what Brouwer explicitly declares. Suppose that there is an unsolvable problem concerning whether a mathematical statement is true or false. Intuitionistically, this supposition amounts to the assumption that there is a *proof* that such an unsolvable problem exists. According to the intuitionistic proof-conditions of existentially quantified statements, the latter proof should contain a method for finding a particular problem of which it should be correct to say that it is unsolvable. Thus our supposition implies that we can find a particular statement and a proof *P* that the statement in question is neither provable nor refutable. Such a proof *P* should contain a proof that it is impossible to prove the statement and a proof that it is impossible to refute the same statement. But, according to the intuitionistic meaning of negation, a proof that it is impossible to prove that *S* amounts to a proof that *not S* and thus is a refutation of *S*. If we prove that a statement is not provable, we thereby intuitionistically refute it, and thus prove that it is refutable. Hence our proof *P* should contain a proof that the statement is refutable *and* a proof that it is *not* refutable, so it would be a proof of a contradiction, which is absurd. Thus our original supposition that some unsolvable mathematical problem exists has led us to an absurd consequence. The foregoing constitutes an intuitionistically acceptable proof of the *negative* claim: "there does not exist any unsolvable mathematical problem". However, in "The Unreliability of the Logical Principles", Brouwer seems to be aware of such a proof.[6] Indeed, from the context one can gather that in the above quoted passage Brouwer's target is Hilbert's *affirmative* thesis that every mathematical problem is solvable, a thesis that Hilbert called "Axiom von der Lösbarkeit eines jeden Problems".[7] This so called "axiom", which is intuitionistically unwarranted, corresponds to a generalized principle of existing solution. Hilbert's affirmative "axiom" is *classically* equivalent to the negative thesis that "there exist no unsolvable mathematical problems", but the equivalence does not hold *intuitionistically*, because of the different meanings of intuitionistic quantifiers. In the quoted passage Brouwer addresses himself to the standpoint of a classical mathematician; thus, it is not very surprising that he treats the two theses as equivalent. But his real point is that "there is not a shred of a proof" for Hilbert's tenet that each particular mathematical problem can be solved, in the sense that the question under consideration can either be affirmed or refuted.[8] Brouwer thinks that if we consider specific mathematical questions concerning the truth of particular given statements, we sometimes do not have a guarantee that the questions can be answered. A formulation of Brouwer's view, applied also to non-mathematical statements, is the following "thesis of the missing guarantee for '*S*'":

vi) there is no guarantee that either there is a correct argument for the statement that S *or there is a correct argument for the statement that* not S.

"S" in *vi* can be replaced by statements such that we presently do not know whether they are true or false, nor do we know whether a decision method exists which would make us, in principle, capable of verifying or falsifying them. In natural language there are various ways of forming such statements. Dummett gives the following list: "the use of quantification over an infinite or unsurveyable domain (e.g. over all future times); the use of the subjunctive conditional, or of expressions explainable only by means of it; the possibility of referring to regions of space-time in principle inaccessible to us".[9] Let us call such statements "problematic statements". A famous mathematical example of a problematic statement is Goldbach's conjecture, claiming that every even number greater than two is the sum of two primes. It is still unproved and unrefuted, and no method is known which could make us in principle capable of finding a proof or a refutation. A non-mathematical example is: "there were strawberries in this wood on June 25, 1896", uttered in a wood among the mountains near Mauls, a village in the Eisack valley.

If we, in accordance with common use and with the majority of philosophers, take the equivalence thesis for granted, our reasonings *i*-*v* and *i**-*v** start from two assumptions: the epistemic conception of truth and the excluded middle. The conclusion is a principle of existing solution for any arbitrary statement *S*. Thus, if one accepts the aforementioned reasonings, then either, first, one ought to abandon *i*, the epistemic conception of truth, or, secondly, one ought to refrain, for some statement *S*, from endorsing *ii*, the principle of excluded middle applied to *S*, or, thirdly, one should accept a principle of existing solution for every given *S*.[10]

The *first* option is chosen by a non-epistemic realist, like Frege in *Grundgesetze*.[11] The non-epistemic realist adopts a transcendent, non-epistemic notion of truth, expressed by the *transcendency thesis*:

vii) *It may be true that* S *even if no correct argument for the statement that* S *exists (and thus its truth is not even in principle knowable).*

Here I am not going to deal with arguments against non-epistemic realism. I just call attention to the crucial problem for non-epistemic realism: how is it possible that we attach to our sentences transcendent truth conditions? Transcendent truth conditions can be satisfied even if nothing exists which establishes that they are satisfied according to our criteria of correct assertion in current, and even in ideal, epistemic situations. Thus transcendent truth conditions and our criteria of correct assertion are entirely unrelated. How can *we* connect our sentences with truth conditions that go beyond our assertoric practice in such a way?

The *second* option, *logical revisionism*, is chosen by Brouwer and his followers, who adopt intuitionistic logic. Though their meaning-theoretic arguments are different from those of the original intuitionists, also Dummett and Prawitz sympathize with logical revisionism.[12] The problem with logical revisionism is that classical logical principles not accepted by the intuitionists, like the excluded middle, double negation elimination or *reductio ad absurdum*, are deeply rooted in our inferential practice (not only in mathematics), and failure to endorse them involves an undesirable conflict with such a practice, which is epistemically much firmer than the epistemic conception of truth. The conflict cannot be solved by adopting one of the so-called "translations" of classical logic into intuitionistic logic. These so-called "translations" are one-to-one mappings which preserve theoremhood. But they "translate" atomic formulas into their double negations and instances of the excluded middle into instances of other logical laws, e.g. the principle of non-contradiction. Now, if one follows classical logic, one employs atomic formulas differently from their double negations, and one views the excluded middle as a principle which is different from the principle of non-contradiction. In classical logic, to be sure, the "translated" formulas are equivalent to the formulas which correspond to them according to the translation, but logical equivalence is not sameness of use. Hence if one follows classical logic, one will not accept such mappings as genuine and faithful translations of one's practice.

Is philosophy entitled to dictate to mathematics and science – only on the basis of a philosophical conception of truth – laws which mathematics and science ought to obey against their internal tendencies? The supporter of an epistemic conception of truth who does not consider philosophy such a supreme law-giver had better avoid demanding the abandonment of the excluded middle for intuitionistic reasons. But how can a supporter of the epistemic conception of truth accept the excluded middle? There is a *third option* besides non epistemic realism and logical revisionism. The third option is precisely to accept both an epistemic conception of truth and the excluded middle, but then a principle of existing solution for every given statement S ought also to be endorsed. Reasoning i-v (or i^*-v^*) is a deduction of the principle of existing solution from the excluded middle and the epistemic conception of truth, which, if true, both can be taken to be *a priori* truths. Thus, also every instance of the principle of existing solution should be taken as an *a priori* truth. For example, the third option seems to involve the view that the prephilosophically unwarranted statement "the problem whether Goldbach's hypothesis is true or false is solvable" is an *a priori* truth. This is a conclusion which sounds rather implausible, if we recall that a solution of the problem concerning Goldbach's hypothesis has to be a finite argument. What entitles us *a priori* to assert that either there is a finite proof or a finite refutation of Goldbach's statement?

The epistemic philosopher faces an embarassing dilemma between the third and the second option, which both have some undesirable facet. How can the dilemma be solved? I shall now survey the different attitudes of some major epistemic philosophers towards the problem.

Part II
Historical Overview

§ 3. Kant on truth and excluded middle.

Although Kant does not give a general formulation like i or i^* of any epistemic conception of truth, he can be rightly considered one of the fathers of such a view. In the "Introduction" to the "Transcendental Logic", the second part of the *Critique of Pure Reason*, Kant gives a "nominal" definition of truth as "the agreement of knowledge with its object".[13] This version of the correspondence theory of truth – according to Kant – is only a nominal definition: it does not really explain how truth is possible, because it does not explain how the agreement of knowledge with its object is possible. But this is precisely what ought to be explained. Kant does not take for granted the notions of "truth", "agreement", or "object". He tries to explain these notions in terms of a theory abou t the ways in which we know the truth of a judgement and thereby know the object (or objects) to which the subject of the judgement refers.

About analytical truths, Kant endorses very explicitly an epistemic conception of truth: analytical truths can always be known by means of the principle of non-contradiction.[14] What is more difficult to explain is how synthetical truths are possible. Synthetical truths can be denied without contradiction because they add to the concept of the subject a predicate which was not already contained in it. In order to acquire this new knowledge one has to go outside the concept of the subject: one has to address oneself to objects. Therefore synthetical truths more genuinely depend on the agreement of judgements with objects, while analytical truths depend only on the internal agreement of judgements with themselves. [15]

Kant tries to solve the problem concerning the nature of synthetical truths – that is, how synthetical truths are possible – by trying to clarify how objects are possible. An object, according to Kant, is the result of an act of joining different given representations, intuitions, to each other under a concept. This act is called *synthesis*. In the "Transcendental Analytic", by clarifying the preconditions of a synthesis in general, Kant explains at the same time: 1) how objects are possible; 2) how (synthetical) knowledge and experience are possible; 3) how concepts can refer to objects and thereby have "meaning"; 4) how the agreement between a piece of knowledge and its object is possible, i.e. how (synthetical) truth is possible. He explains all four notions of "object", "knowledge", "reference" and "truth" by means of his explanation of how a synthesis is possible. Synthetic truth is possible only within the boundaries of possible knowledge (possible experience). Therefore, also for synthetical truths, Kant has an epistemic conception of truth: if a synthetical judgement is true, then there is a possible knowledge that corresponds to it.

What about the excluded middle? In Kant's *Logic*, edited by Jäsche, the principle of excluded middle is listed among the three universal, merely formal or logical criteria of truth (together with the principle of contradiction and identity and the principle of sufficient reason).[16] Some passages suggest that Kant did not really distinguish between excluded middle and bivalence.[17] Can we then conclude that Kant accepted the principle of excluded middle and the principle of bivalence unrestrictedly?[18]

Three possible objections can be raised. *First objection*: Kant affirms that the thesis and the antithesis of the first and of the second Antinomy of pure reason are both false (e.g. in the first antinomy the thesis "the world is finite " and the antithesis "the world is infin ite" are both false). Isn't this a violation of the principle of excluded middle? *Second objection*: the acceptability of proofs by *reductio ad absurdum*, which Kant calls "apagogical proofs" rests upon the principle of excluded middle.[19] However, in the "Transcendental Doctrine of Method" Kant writes that "[transcendental] proofs must never be apagogic [or indirect], but always ostensive [or direct]".[20] This seems to show that in transcendental philosophy (differently from natural science or mathematics)[21] the principle of excluded middle does not hold. *Third objection*: from the excluded middle one can be easily led to infer the validity of the principle of complete determination: "every thing, as regards its possibility, is likewise subject to the principle of complete determination, according to which, if all the possible predicates of things be taken together with their contradictory opposites, then one of each pair of contradictory opposites must belong to it".[22] This principle seems to follow from the excluded middle. But in the "Transcendental Dialectic" Kant comments that an experience *in concreto* can never correspond to the concept of complete determination. Such a concept is therefore only an idea, and it can only have a regulative role. Is Kant saying that the principle of excluded middle is not true and that it has only a regulative role?

If that were the case (but, as we shall see, it is not), Kant's attitude would superficially resemble a view recently advocated by Richard Tieszen. Tieszen indeed maintains that problematic mathematical instances of the principle of excluded middle have a regulative role. He explicitly mentions Kant's treatment of the ideal of pure reason and the principle of complete determination. But the resemblance would be only superficial and misleading, because Tieszen seems to think that the principle of excluded middle is a true statement (though merely regulative).[23] He takes from Husserl the "conception of ideal or maximal truth [which] serves as a regulative idea, as an ideal or norm toward which knowledge aims if it is to be more fully determined and perfected".[24] For Kant, on the contrary, a merely regulative principle cannot be true (because it would be a synthetic truth with which no object of possible experience can agree, something which contradicts Kant's conception of truth). On the other hand, any instance of the excluded middle is, for Kant, a legitimate and true judgement (simply because it is a merely formal principle, and thus an analytical truth). Therefore Kant does not think that the excluded middle is a merely regulative principle. Differently from Kant and following Husserl, Tieszen claims that the regulative principle of excluded middle is an "epistemically illegitimate (when applied beyond certain bounds) but unavoidable postulation of human reason".[25] Husserl's view will be considered in §5, where an argument will be presented against the "unavoidability" of the excluded middle and of the principles of existing solution. Here, let me try to describe Kant's position.

Coming back to our third objection, the answer is rather easy. The principle of complete determination is based upon an idea which can have only a regulative role, and is not a simply formal logical principle, because the principle of complete determination presupposes *the totality of all predicates* as something given,[26] but such a totality can never be given in experience. On the other hand, the different principle according to which, for any given object and for each pair of *given* contradictory predicates, one (and only one) of the two contradictory predicates applies to the object, is for Kant a correct logical principle.[27] If you allow the anachronism, we might say that here Kant is criticizing not the excluded middle but second order impredicative quantification.

As to the first objection, Mirella Capozzi[28] has remarked that Kant himself offers an answer in the "Dialectic"[29] by distinguishing between "denying the existence of an infinite world" (i.e. affirming "*mundus non est infinitus*", the contradictory opposite of the antithesis) and "affirming the existence of a finite world" (i.e. affirming the thesis: *mundus est finitus*). One may add that such a distinction is a particular instance of the distinction between negative judgements (S non est P) and infinite judgements (S est non P), which is explained in the *Logic* with the following words:

> In negative judgements the negation always affects the copula; in infinite judgements not the copula but the predicate is affected by the negation, which can best be expressed in Latin.[30]

An infinite judgement is a judgement which ascribes to the subject a negative predicate. In general logic (which abstracts from the content of judgements and "enquires only whether the predicate be ascribed to the subject or opposed to it")[31], affirmative and infinite judgements are not distinguished. But transcendental logic, which considers the content of subjects and predicates, must distinguish between affirmative and infinite judgements.[32] Moreover, transcendental logic must distinguish between analytical and synthetical judgements. If a judgement is an analytical affirmative (infinite) truth, then the (negative) predicate which is already contained in the concept of the subject is merely extracted from it – and the concept of the subject can also be an empty concept. But if the affirmative (infinite) judgement is a synthetical judgement, its truth requires the existence of an object under the concept of the subject: the synthetical judgement is true if, and only if, there are objects under the concept of the subject and such objects fall within the (unlimited) sphere of the (negative) predicate. Corresponding synthetical affirmative (S est P) and infinite (S est non P) judgements share a common condition: they both imply that the concept of the subject S is non-empty. Hence they are not contradictory opposites: if the condition that the subject be non-empty is not satisfied (i.e. if the subject is empty), then the two judgements are both false. Kant suggests an analogy with the somewhat different example concerning a disjunctive judgement that might be affirmed of each given body: "this body has a good smell or it has a smell which is not good". Here both disjuncts imply that the body in question has a smell. And if it has no smell, then both disjuncts are false.

The difference between a negative synthetical judgement "S non est P" and an infinite synthetical jud gement "S est non P" is that the former simply denies that there is a non-empty extension of S included in the extension of P (thus leaving open the question whether the extension of S is empty or not), while the latter affirms that the extension of S is non-empty and that such a non-empty extension is included in the extension of the negative predicate non-P, i.e. in the infinite (unlimited) sphere of all the things that are not P.[33] If we assume that the extension of S is non-empty, "S non est P" and "S est non P" are equivalent. But if, on the contrary, the extension of the concept of the subject is empty, synthetical infinite judgements are false, whereas synthetical negative judgements are true.

If S is "th e world", the extension of S is empty. The concept of world as the totality of all experience is void, i.e. no object can belong to its extension, because experience is never complete. The totality of all experience can never be given. The world as the unconditioned totality of experience does not exist. Thus the synthetical negative judgement "*mundus non est infinitus*" is true, because the extension of the subject is empty, but, for the same reason, the synthetical infinite judgement "*mundus est non-infinitus*" i.e. " *mundus est finitus*" is false. So, the instance of the excluded middle "*mundus est infinitus aut mundus non est* *infinitus*" is true, because the second disjunct is true, whereas the disjunction "*mundus est infinitus aut mundus est finitus*" is false, because both disjuncts (which are both synthetical) are false. But the latter disjunction is *not* an instance of the excluded middle.

This answers also the second objection. Proofs by *reductio ad absurdum* are admissible, Kant says in the "Doctrine of Method", only in those sciences where it is impossible to mistake a subjective representation for a concept which really refers to objects. This mistake (*Subreption*) is impossible in mathematics, because in *mathematics* we can always construct *a priori* an object corresponding to the mathematical concept; so Kant says, "it is there, therefore, that apagogical proofs have their true place".[34] In *natural science* "where all our knowledge is based upon empirical intuitions, the subreption can generally be guarded against through repeated comparison of observations";[35] thus, indirect proofs are acceptable, even if, according to Kant, they are not very useful in this sphere of knowledge. But in *transcendental philosophy* we can easily mistake a subjective representation for a concept which really refers to objects. If we make this mistake, and for example think that a concept S has an objective content, but, in reality, it is an empty concept which cannot refer to any object, we can wrongly believe that a certain synthetical infinite judgement "S est non P" is equivalent to the (negative) contradictory opposite of a synthetical affirmative judgement "S est P"; thus, by proving that one of the two judgements (the infinite or the affirmative judgement) implies something false, we can wrongly conclude that the other is true. But such a conclusion would be wrong, because, if S is empty, both judgements are false. This general description corresponds very well to the particular situation of the first two antinomies, where the thesis is proved by *reductio ad absurdum* of the antithesis and *viceversa*. In conclusion, Kant is not really saying that proofs by *reductio ad absurdum* are not valid in transcendental philosophy, so that if we, for example, proved that "A non est B" is false, this would not be a proof that "A est B" is true.[36] What he is saying is that in transcendental philosophy one ought not to use this method of proof, because there is a danger of applying it wrongly, since it is easy to mistake infinite synthetical judgements for negative ones.[37] If we were to commit this error, then, by applying wrongly the indirect or apagogical method of proof, we could perform wrong inferences from «"A est B" is false» to «"A est non B" is true» or from «"A est non B" is false» to «"A est B" is true». Kant is aware that what he takes to be "the logical form" of a judgement is not always completely manifest. It can be hidden by the grammatical form of

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particular languages. Judgements may contain negations and affirmations in a covert way (*in versteckter Weise*). This is specially clear, if we consider that it is necessary to resort to Latin in order to distinguish clearly an infinite from a negative judgement. So we can easily mistake the one for the other. This mistake would be harmless if the judgements were analytic or if the concept of the subject were a mathematical or an empirical concept, for which, according to Kant, it is always possible to construct a corresponding object in pure intuition or to check whether the concept is empty by means of observation. But if the judgements are synthetic and the concept of the subject is a concept like the concept of "the world", the emptiness of which can be established only by an intricate philosophical theory, the logical mistake can lead us into deep philosophical errors. Therefore Kant thinks that transcendental proofs must never be indirect.

According to §2, since Kant upholds an epistemic conception of truth and accepts bivalence unrestrictedly, he ought to accept a principle of existing solution for *every* given problem. However, he does not draw this conclusion. He maintains that for problems within natural science we have no guarantee that there is a solution. There are three sciences in the sphere of which every problem is necessarily solvable and all questions have answers, "although, up to the present, they have perhaps not been found". [38] The three sciences are transcendental philosophy, pure mathematics and pure ethics. They are disciplines in which an answer comes from the same sources from which the question arises, i.e. from our pure thinking. In other words, they are *a priori*. That is why every question which arises in their sphere is necessarily solvable. (This inference from aprioricity to solvability is suspect: Brouwer would have objected that even if arithmetical concepts like "prime number" or "even number" are *a priori* and we can construct corresponding objects in pure intuition, we still lack a guarantee that we can construct a proof of Goldbach's conjecture.) However, natural science is *a posteriori*. In natural science the solutions of our problems depend also on those representations which we receive from outside us, empirical intuitions. Of course empirical intuitions are epistemically accessible, but with respect to them we are receptive, passive. Therefore in natural science – according to Kant – "many questions must remain insoluble"[39]:

> The key to the solution of such questions cannot [...] be found in us and in our pure thinking, but lies outside us, and for this reason is in many cases not to be discovered.[40]

Kant accepts the principle that every problem is solvable for *a priori* sciences like mathemathics and transcendental philosophy, but denies it for natural science. For this reason Carl Posy writes that Kant "might be an intuitionist for empirical science" but "would simultaneously advocate classical logic for mathematical science".[41] This is slightly misleading. Kant would accept the thesis of the missing guarantee for empirical science, as Brouwer accepted it for mathematics. Therefore Kant *ought* to abandon the principle of excluded middle for empirical judgements as the intuitionists did in mathematics, or at least ought not to regard it as a logical law *a priori*. But, as we have seen, he does not abandon the excluded middle for empirical judgements, nor does he abandon bivalence, nor the method of proof by *reductio ad absurdum*. On the contrary, he considers the excluded middle one of the fundamental logical criteria of truth, which are *a priori* valid for all sciences. It is easy to suggest that Kant simply did not realize that the epistemic conception of truth and the thesis of the missing guarantee are incompatible with the excluded middle and bivalence; it was Brouwer who understood it more than hundred years later. But this diagnosis is not completely satisfactory. Kant was a great mind, and there must be something that prevented him from discovering the inconsistency between his metaphysical and epistemological theses and the formal logical principle of excluded middle. It is well known that Kant thought that formal logic since Aristotle had been a complete and definitive science to which nothing could be added and from which nothing could be withdrawn. This privileged status of formal logic depends, according to Kant, on its being a very limited science. Regarding synthetical truths, the formal logical principles are *conditio sine qua non*, but not the determining ground of truth. However, Kant not only thinks that Aristotelian logic is valid. He maintains, actually without real proof, that the principles of formal Aristotelian logic are the absolutely necessary laws of *thinking*, without which no thinking (and therefore no knowledge and no truth of any kind) is possible. Whatever violates them must be false. The doctrine that Aristotelian logic captures definitively the general laws of thinking was also crucial for Kant's "metaphysical deduction" of the categories or pure concepts of understanding. Kant's *Leitfaden*, his clue to the discovery of an exaustive list of the categories, was based on a classification of all the possible logical forms of judgements according to the Aristotelian logic of his time, a classification that had to be definitive. The idea that some important feature of Aristotelian logic, not a mere subtlety, could be changed because of epistemological considerations concerning a particular field of knowledge, would have had disastrous consequences in Kant's system, and in any case such an idea was very far from Kant's mind. My conjecture is that this was the reason why he did not think of rejecting the excluded middle for natural science. A substantial part of his philosophy was based on the idea that the fundamental principles of Aristotelian logic would be sacrosanct: this was the obstacle that prevented Kant from acknowledging the tension between his theory of truth, his conception of problems in natural science and his unrestricted endorsement of the principles of bivalence and excluded middle.

§ 4. The thesis that every given problem is solvable based upon a restricted notion of meaningfulness.

The intuitionistic abandonment of the principle of excluded middle is based on the thesis of the missing guarantee (as we called it in §2), according to which for some *S* we are not entitled to assert that the problem whether *S* or *not S* has a solution. But other supporters of an epistemic conception of truth and reality, unlike intuitionists, accepted the excluded middle and, unlike Kant, also the view that every problem has a solution (i.e. some generalized principle of existing solution). However, this move can be performed by an epistemic philosopher in various ways. One way is to embrace a narrow conception of meaningfulness, according to which genuinely meaningful sentences are only those sentences for which we know a decision method (or at least have some specific evidence that such a method exists). This is the drastic strategy of logical positivists who (echoing a famous proposition of Wittgenstein's *Tractatus*)[42] in their *manifesto* of 1929 declared that the scientific conception of the world does not admit insoluble riddles.[43] They followed in Ernst Mach's footsteps. In 1872, in an influential lecture entitled "Über die Grenzen des Naturerkennens" Emil du Bois Reymond had maintained that certain problems are scientifically unsolvable; about such problems we must humbly admit: *Ignoramus et Ignorabimus*. Mach objected that, when a problem turns out to be in principle unsolvable, it must be wrongly formulated.[44] To deny that one can indicate a *particular* unsolvable problem without thereby showing that the indicated problem is meaningless is not yet to affirm in general that every meaningful problem is solvable. But the logical positivists at the beginning of the Thirties made this further step: they held that a statement is meaningful if, and only if, a decision method is known for that statement. The decision method the logical positivists had in mind was *observation*. A lucid formulation of this view is in "Die Wende der Philosophie", a famous essay by Moritz Schlick:

> Wherever a meaningful problem presents itself, it is always possible, in theory, to indicate the road leading to its solution, for it turns out that the indication of this road is basically equivalent to stating its meaning [...] The act of verification, in which the road to solution finally terminates, is always of the same kind: it is the occurrence of a particular state of affairs, ascertained by observation and immediate experience. [45]

An analogous idea can be applied to mathematics: one can consider meaningful only sentences which are decidable by a finite *computation*. This is what Hilbert did in the Twenties by distinguishing the real (contentual) sentences of finitary mathematics from the ideal (non-contentual) sentences of transfinite mathematics.[46] He regarded only the former as genuinely meaningful sentences. There is a clear Kantian influence on Hilbert. Already in 1900, at the International Conference of Mathematics in Paris, Hilbert pronounced his "axiom" of the solvability of every mathematical problem:

> There is the problem: seek its solution. You can find it by pure thinking, for in mathematics there is no *ignorabimus*.[47]

Hilbert's phrase "pure thinking" (*reines Denken*) is borrowed from Kant (see §3). In this context, solvability by means of "pure thinking" means solvability *a priori*.[48] The aprioricity of mathematics is precisely the reason why, for Kant, every mathematical problem is solvable. On the other hand, Hilbert's use of the expression "*Ignorabimus*" shows clearly that Hilbert is implicitly contrasting mathematical problems and the non-mathematical problems which Du Bois Reymond – using the same latin word – had declared unsolvable in 1872.

Wittgenstein in the *Tractatus*, the logical positivists, and, at least as far as mathematics is concerned, also Hilbert: many important philosophers adopted the narrow conception of meaning, according to which only sentences for which we *in advance* know a decision procedure have meaning. If the epistemic philosopher takes this view he/she can accept the excluded middle and a generalized principle of existing solution. However, this kind of strategy is very drastic because, if consistently pursued, it leads to the consequence that a very great number of sentences which are actually used (and apparently understood) in mathematics, in natural science and in the language of daily life are in reality only meaningless concatenations of signs. Therefore, it does not appear to be an attractive option.

§ 5. The thesis that every given problem is solvable based upon a transcendental argument.

Another way of upholding a principle of existing solution for any given statement without declaring most of our sentences meaningless is to transform such a principle into a transcendental principle. Already with his "How to Make Our Ideas Clear" (1878) Charles Peirce defended the idea that every problem has a solution:

> though in no possible state of knowledge can any number be great enough to express the relation between the amount of what rests unknown to the amount of the known, yet it is unphilosophical to suppose that, with regard to any given question (which has any clear meaning), investigation would not bring forth a solution of it, if it were carried far enough.Who would have said, a few years ago, that we could ever know of what substances stars are made whose light may have been longer in reaching us than the human race has existed? Who can be sure of what we shall not know in a few hundred years?[49]

With the qualification "which has any clear meaning" Peirce could be taken to banish from the domain of meaningful statements those problematic undecided statements for which we do not know whether a decision method exists (as the logical positivists did). But is "In the star Beta there is the substance XYZ" a statement for which we know a decision method like computation or direct observation? Clearly it is not; indeed, in the tenth volume of his *Cours de Philosophie Positive*, Auguste Comte had taken the problem concerning the chemical composition of stars to be unsolvable.[50] Here, mentioning its solution, Peirce probably refers to the analysis of the spectrum of a star, and the reliability of such a method for determining whether a substance is present in the star depends on the truth of problematic theoretical sentences of physics. Anyway, meaningfulness for Peirce is given by the "maxim of pragmatism"[51], and that maxim does not exclude problematic statements from the domain of meaningfulness, on the contrary, it "allows any flight of imagination [risky scientific hypothesis], provided this imagination ultimately alights upon a possible practical effect".[52] It is clear that Peirce does not limit his doctrine that every problem is in principle solvable to cases for which decision procedures are already known. He often applies the doctrine to scientific hypotheses, and he stresses that, even if we are bound to assume that we can solve any given scientific problem by "guessing" the right hypothesis, we assume it "independently of any evidence", and this is, of course, a situation very different from the situation with those statements for which we know a decision procedure which, if applied, would in principle lead us

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to a solution. On the other hand, the assumption that a solution exists is for Peirce a "primary" and "fundamental" precondition of enquiry.

> Underlying all such principles [which should guide us in abduction] there is a fundamental and primary abduction, a hypothesis which we must embrace at the outset, however destitute of evidentiary support it may be. [...] for the same reason that a general who has to capture a position or see his country ruined, must go on the hypothesis that there is some way in which he can and shall capture it [...] we are [...] bound to hope that [...] our mind will be able, in some finite number of guesses, to guess the sole true explanation. That we are bound to assume independently of any evidence that it is true. Animated by that hope, we are to proceed to the construction of a hypothesis.[53]

Perhaps his adherence to this principle of "hope"[54] was the reason why Peirce apparently did not mind accepting the excluded middle also for problematic mathematical statements concerning actual infinity.[55]

Edmund Husserl as well defended an epistemic notion of truth as a possibility of evident judgement, and he accepted bivalence (called by him "the excluded middle"[56]) for judgements that are not "senseless in respect to content".[57] In a section of *Formal and Transcendental Logic* entitled "The idealizing presuppositions contained in the laws of contradiction and excluded middle" he writes:

> The law of excluded middle [...] decrees not only that if a judgement can be brought to an adequation [...] then it can be brought to either a positive or a negative adequation; but [...] it decrees also [...] that every judgement necessarily admits of being brought to an adequation. -"Necessarily" being understood with an ideality for which, indeed, no responsible evidence has ever been sought. We all know very well how few judgements anyone can in fact legitimate [...] and yet it is supposed to be a matter of *a priori* insight [und doch soll es apriori einsehbar sein] that there can be no non-evident judgements that do not "in themselves" admit of being made evident in either a positive or negative evidence.[58]

For Husserl the idealizing presupposition contained in the principle of bivalence is that in principle every judgement can be verified or refuted. This presupposition corresponds to a precondition of scientific enquiry: "the belief in truth-in-itself".

> [...] a fundamental conviction [...] guides every scientist in his province: his settled belief in truth in itself and falsity in itself. For us, the legitimacy of many judgements remains undecided. And most of the judgements that are somehow possible can never be evidently decided in fact; but, in themselves, they can be. In itself every judgement is decided. [...] This surely signifies: by a "method", by a

> course of cognitive thinking, a course existing by itself and intrinsically pursuable, which leads to [...] a making evident of either the truth or the falsity of any judgement. All this imputes an astonishing Apriori to every subject of possible judging [...] - astonishing: for how can we know *a priori* that courses of thinking with certain final results "exist in themselves"; paths that can be, but never have been trod; actions of thinking that have unknown subjective forms and that can be, though they never have been, carried out?[59]

Peirce's and Husserl's positions would deserve a more detailed analysis, but it is interesting how they both think that the idea that any given problem is in principle solvable is a fundamental idea that must guide every scientist in his cognitive enterprise, and, therefore, they accept it, even if they would both agree with Brouwer that "there is not a shred of a proof" to support it. Husserl writes that no responsible evidence has ever been sought and that it is supposed to be a matter of *a priori* insight ("an astonishing Apriori"). Peirce thinks that it is a necessary hope "destitute of evidentiary support", without which we would not even try to solve problems. Though Peirce's "necessary hope" sounds weaker (and thus less implausible) than Husserl's "*a priori* insight", Peirce's and Husserl's attitude is essentially the same. They both accept the excluded middle and bivalence. They both accept the principle that every problem has a solution without adopting a restricted notion of meaningfulness; they both admit that the latter principle is not supported by any evidence, but they consider it a necessary precondition of enquiry, a sort of transcendental principle.

A reconstruction of the transcendental argument for the principle of existing solution is the following.

> 1) Even though it is not supported by any specific evidence, the conviction that a given problem is solvable is a belief that *must* guide every scientist in his attempt to solve the problem. Without this conviction the scientist would not investigate the problem.
>
> 2) Thus a principle of existing solution for S holds *a priori*, as a necessary precondition (foundation) of an enquiry concerning the problem whether S is true or not.

However, this "transcendental" way of dealing with the dilemma concerning the epistemic conception of truth and the principle of excluded middle cannot resist critical examination. One might object that the step from (1) to (2) is a *non sequitur*, because "it is necessary that x believes that Q" does not imply Q. This would be an objection to many

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transcendental arguments. But even if we accept the argument (1-2), this is not sufficient to accept the conclusion, for the simple reason that premise (1) is false. In order to pursue an investigation one must believe that there is an interesting problem which deserves to be investigated. One cannot be absolutely convinced that no solution exists, otherwise one would not even try to find a solution. But this is fully compatible with being agnostic as to whether a solution exists or not. Take, for example, the question: "is there a cure for lung cancer?", our enquiry about this problem is motivated by our conviction that it is an important problem, even though we do not know whether we can solve it. Even if we are not firmly persuaded that a solution exists, we can engage in the investigation, if we are interested in the problem: *perhaps* we shall find a solution. Therefore, the acceptance of a principle of existing solution is not a necessary precondition lying at the foundation of every enquiry.

§ 6. The thesis that every given problem is solvable as an analytical truth, based upon a linguistic convention.

In two essays written in the Eighties Crispin Wright described various ways in which one can be an "antirealist", i.e. can reject a transcendent notion of truth, without falling into revisionism with respect to classical logic.[60] Wright's aim was to show that "an inexorable course [...] from antirealism to revisionism has yet to be mapped out".[61] To this aim is perhaps sufficient to show that some variety of non revisionary antirealism apparently remains a consistent option. The views pictured by Wright seem, indeed, consistent, but in my opinion they are implausible. Some of the non revisionary antirealists imagined by Wright refrain from endorsing bivalence and thus, in the light of the reasoning exhibited in §2, they accept classical logic without accepting the equivalence thesis:[62] this is an undesirable facet of such positions because the equivalence thesis, as many philosophers (Frege, Ramsey, Wittgenstein, Tarski, Quine) have variously emphasized, is a fundamental principle on which our use of the word "true" is based. Other non revisionary antirealists accept bivalence without a satisfactory justification: "on the say-so of an oracle" or "as an expression of [...] confidence that human ingenuity can surmount all obstacles".[63] But another view considered in Wright's essays is more interesting. It is "an antirealism which regards the global acceptability of classical logic as *conventional*".[64]

Wright's antirealist conventionalist sees "certain principles of classical logic simply as *implicitly definitional* of the concepts which feature in them and so as immune to revision".[65] This kind of non revisionary antirealist arbitrarily lays down that "*S* or *not S*" is always assertible and immune to revision because its assertibility is part of a speaker's understanding of disjunction and negation. Though Wright did not develop his idea in this direction, such a stance *can* lead to an acceptance of bivalence and of a principle of existing solution: the conventionalist can accept the equivalence thesis as "implicitly definitional" of the meaning of "true" an d thus can derive bivalence from classical logic and the equivalence thesis (as in §2); then, from bivalence and an epistemic conception of truth, the conventionalist can deduce a (generalized) principle of existing solution. For this epistemic philosopher, a principle of existing solution would be *a priori* valid in virtue of the arbitrary linguistic conventions that fix the meanings of the logical constants and of the word "true". The latter claim is clearly implausible: how can arbitrary linguistic conventions be enough to justify the claim that we can either prove or refute Goldbach's conjecture? How can arbitrary linguistic conventions make such a claim *a priori* valid and immune to revision? However, this is only a particular instance of the main problem raised by the conventionalist's view.

Wright's conventionalist espouses the meaning-theoretical thesis that any given set of logical principles can, by itself, determine the meanings of the involved logical constants. Dummett claimed that this thesis is "a form of linguistic holism", i.e. of the implausible doctrine that "no one sentence of the language can be fully understood unless the entire language is understood".[66] Rejection of linguistic holism, Dummett concluded, implies rejection of the thesis in question. Wright expressed doubts that his conventionalist should be necessarily a holist in this sense.[67] The reader can find a confirmation of Wright's doubts in my doctoral dissertation, where a detailed description of a non-holistic theory of meaning and understanding is presented, one based on the idea that we can confer sense to a word by associating with it a finite set of (mostly implicit) argumentation rules which concern that word (without any restriction on the form of such argumentation rules).[68] Thus, though my point in the present paper is not meaning-theoretical, I sympathize with the thesis that we can give meaning to the logical constants by associating them with a finite set of logical principles of any form. So, I agree that the excluded middle can be taken to be constitutive of the meaning of classical disjunction and negation. However, in my view, the position of the conventionalist described above is vitiated by a wrong assumption, which is rejected in my dissertation,[69] but is shared by both Dummett and Wright:

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the assumption that if a logical or inferential principle determines the meanings of certain involved words, and thus is constitutive of our *understanding*, nothing else is necessary to justify our acceptance of that principle, because its being constitutive of our understanding is sufficient to make the principle *valid* and to show its validity to us. This assumption amounts to an adherence to the doctrine of *analytical validity*: if a logical or inferential principle is constitutive of our understanding of some words (and hence part of their meanings), then the principle is valid in virtue of the meanings of those words, i.e. analytically valid, and immune to rational revision.

The meaning-theoretical thesis that any set of logical principles can determine the meanings of the logical words involved, taken together with the doctrine of analytical validity, leads to the consequence that we can arbitrarily decide the validity *a priori* of logical principles and make them immune to rational revision. We can decide that the excluded middle is valid. But we can equally well decide that its negation is valid. This simply shows that we give different meanings to our logical constants. To quote Carnap: "in logic there are no morals". [70]

This is the main problem for the conventionalist's view: it implies that we can arbitrarily decide the validity of logical principles. Such a tenet clashes with our linguistic and scientific practice. The epistemic philosopher is troubled by the difficulty that motivates this paper just because classical logic plays an important role in our current epistemic situation, so that its abandonment may involve heavy epistemic losses. Hence our acceptance (or rejection) of classical logic is not at all arbitrary. The choice of a logic is not a matter beyond rationality. To adopt one logic instead of another can have enormous consequences on mathematics, physics, other sciences and everyday life. Therefore, if one favours the idea that any set of logical principles can determine meanings of corresponding logical constants, one had better drop the doctrine of analytical validity. One should maintain that the mere fact that a logical principle is a linguistic convention constitutive of our understanding certain logical words is not sufficient to make that principle acceptable as valid. If a logical principle is viewed as a linguistic convention, the convention should not be considered arbitrary: there are criteria for judging whether such a convention, together with other related conventions, is justified or not. The act of associating certain conventions with certain signs constitutes a language, but such an act leaves open the question whether the language is rationally acceptable. The supporter of this view should try to detect the criteria which bear upon the rational acceptability in a given epistemic situation of the language (or language fragment) introduced by the conventions in question. These criteria determine whether the meaning-constitutive logical principles can be taken as valid. Thus, the acceptance as valid of meaning-constitutive logical principles does not depend on their being meaning-constitutive, but on whether the resulting language satisfies the above mentioned criteria of rational acceptability in the given epistemic situation.[71] If classical logic, bivalence, and a principle of existing solution are justified in this way, their acceptance is relative to a particular epistemic situation. Since the epistemic situation can change, classical logic, bivalence, and the principle of existing solution are not immune to revision. Though the holistic argument in favour of classical logic which is described in part III is independent of the conception of meaning developed in my dissertation, from the point of view of such a conception the holistic argument can be regarded as an exemplification of the criteria on the basis of which we can rationally accept a language and the associated meaning-giving principles. The reader may think that the latter remark confirms Dummett's charge of linguistic holism. However it will soon be clear that the kind of holism involved in the argument of part III is not linguistic holism, but *epistemological* holism, and it is possible to argue that in this case we can have epistemological holism without linguistic holism.[72]

Part III
Epistemic truth and epistemological holism

§ 7. Quine's rejection of the *a priori*.

Kant's idea of explicating the notions of truth, reference and object in terms of knowledge is the basic idea of the epistemic conception of truth and reality. Such an idea can be consistently separated from Kant's further idea that we can detect the essential structure of our cognitive faculties *a priori* so as to discover a complete system of synthetic *a priori* judgements which provide a foundation of all knowledge. Since the two ideas are separable, a supporter of the epistemic conception can consistently reject the idea that there are synthetic *a priori* judgements and that knowledge has foundations.

Not only can the epistemic philosopher deny that there are synthetic *a priori* judgements, but he/she can also deny that there are analytic *a priori* judgements. In other words, the epistemic philosopher can reject the thesis that there is *a priori* knowledge in general. The epistemic philosopher can be persuaded that there is no such thing as *a priori*

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knowledge by Quine's critique of aprioricity and analyticity. Quine's holistic picture of knowledge in "Two Dogmas of Empiricism" has brought into focus the fact that *the epistemic properties of any sentence* (i.e. its conditions of verification or falsification, its inferential consequences etc.) *may turn out to depend on any part of the totality of accepted sentences*. I call this fact "epistemological holism". For example, the acceptance or rejection of certain logical laws can turn out to depend on the acceptance of certain physical theories. Thus, no sentence is absolutely independent of experience, and no sentence is absolutely unrevisable. That is why Quine denies that there are analytic or in general *a priori* truths. Logical laws are not justifiable *a priori* on the basis of their meaning; they are accepted in virtue of their contribution to the overall epistemic simplicity and fruitfulness of the system of sentences in which they are framed in a certain epistemic situation (and overall epistemic fruitfulness depends also on the adequacy of the whole system to empirical evidence).

Holistic considerations of this kind underlie Quine's acceptance of classical logic. In the Seventies in *Philosophy of Logic* (1970) he defended "the convenience, the simplicity and the beauty" of classical logic and two-valued semantics against intuitionists.[73] Virtually at the same time, in "The Limits of Knowledge" (1973), discussing Heisenberg's principle of indeterminacy and the view that an elementary particle "has indeed its exact position and velocity, and that these are in principle inscrutable",[74] he commented: "there is reluctance to assign meaning to strictly unanswerable questions [...] If a question could in principle never be answered, then one feels that language has gone wrong...and the question has no meaning".[75] In this passage (which resembles very much Mach's comment about Du Bois Reymond) Quine seems to maintain (like the logical positivists) that only those questions which can in principle be answered are meaningful and on this ground he seems to reject a transcendent notion of truth and to sympathize with an epistemic conception.

Later, however, in "What Price Bivalence?" (1981) Quine accepts bivalence "for the simplicity of theory that it affords", and since he agrees with Dummett[76] that the transcendency thesis can be inferred from bivalence (and, we ought to add, the thesis of the missing guarantee),[77] he is willing to accept, for the same reason, also a transcendent notion of truth and non-epistemic realism. But he admits that transcendent truth is "no small price".[78]

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§ 8. The holistic solution of our problem.

Despite his reluctance to accept a transcendent notion of truth, in "What Price Bivalence?" Quine thinks that the epistemic advantages in favour of classical logic are strong enough to get one to swallow transcendency and reject the epistemic conception of truth. This way of solving the problem by means of something like a *global cost-benefit analysis* reveals the holistic character of Quine's epistemic decision in favour of classical logic. Costs and benefits are considered in the frame of an overall epistemic situation, i.e. within a given set of accepted reasonings, sentences, open problems etc. For Quine, the cost (or "price") of classical logic is transcendency. The benefits are "simplicity and beauty". Also epistemic fruitfulness can be brought into consideration. Hilbert wrote: "taking the principle of excluded middle from the mathematician would be the same, say, as proscribing the telescope to the astronomer or to the boxer the use of fists".[79] One may add, following Burgess, that the price of replacing classical logic in mathemathics with intuitionistic logic would be particularly high if one considers the consequences of such a revision for the applications of mathematics to physics, because certain important physical results cannot be obtained without classical logic.[80] If this is true, abandoning classical logic involves not only a loss of simplicity, but also a loss of epistemic fruitfulness, and the balance of perceived benefits of classical logic over perceived costs is greater than the corresponding balance of any available alternative. From Quine's and Burgess' claims the following picture can be drawn:

CLASSICAL LOGIC

COSTS	BENEFITS
transcendent truth	simplicity and beauty epistemic fruitfulness

My aim here is not to take a stand about the issue whether Burgess' claims regarding the costs of abandoning classical logic are correct. What is essential to my present concern is an answer to the question: is the price to pay for classical logic necessarily the acceptance of a transcendent notion of truth and the rejection of the epistemic conception? And the answer is: no. Differently from Quine, the epistemic philosopher may retain the epistemic conception of truth and consider all the holistic factors in favour of classical logic as reasons for upholding a principle of existing solution. Hence, the price to pay for classical logic can be the endorsement of instances of the principle of existing solution for every given statement,

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even if no *specific* evidence supports the claim that the statement in question can be decided. The holistic cost-benefit analysis on the basis of which the epistemic philosopher can accept classical logic in the present epistemic situation is thus represented:

CLASSICAL LOGIC

COSTS	BENEFITS
principles of existing solution for problematic statements without *specific* warrant	simplicity and beauty epistemic fruitfulness

An endorsement of instances of the principle of existing solution for problematic statements does not conflict with any already acquired piece of knowledge. Of course we know that, at present, some sentences, e.g. Goldbach's hypothesis, are undecided and that no decision method for such sentences is presently known. We know that presently we do not know how to solve the problem. But it is equally clear that a statement to the effect that such a problem is absolutely unsolvable does not express something that we know. In order to know such a thing, we would have to know in advance all our possible means of knowledge, all conceivable epistemic strategies, and we do not possess such epistemological knowledge. Sometimes it is claimed that Gödel's incompleteness theorems show that there are mathematical sentences which are neither provable nor disprovable in an absolute sense.[81] This would be true if it were also true that there is a single formal system which satisfies the conditions specified in the incompleteness theorems and encompasses all our possible means of proof. In themselves Gödel's theorems show only that, given any particular consistent formal system Σ containing a rather weak subtheory of Peano Arithmetic, there is a true sentence which is undecidable in Σ, without ruling out that the same sentence may be proved in some other way. Since it is highly implausible that all our means of mathematical knowledge can be soundly formalized in a single formal system, Gödel's theorems are far from showing that there are unknowable truths or absolutely undecidable sentences. Though the matter is perhaps only of historical importance, it is worth mentioning that Gödel himself drew from his theorems the conclusion that 'the human mind (even within the realm of pure mathematics) infinitely surpasses the powers of any finite machine, or else there exist absolutely unsolvable Diophantine problems".[82] But Hao Wang reported that '[Gödel] proposed what he called a 'rationalistic optimism' to exclude the second alternative of the disjunction. He thought that Hilbert was right in rejecting the second alternative”.[83] Thus, according to Wang, Gödel accepted Hilbert’s view that every mathematical problem is solvable.

Since we have abandoned the Kantian idea that it is possible to detect the essential structure of our cognitive faculties *a priori*, we may claim that no limit can be set in advance to our capacity of devising new languages, new concepts, new instruments and new epistemic strategies (decision procedures, argumentation rules etc.) which in future can enable us to solve problems which are *now* unsolved, and perhaps even unsolvable *relatively to* a fixed set of presently accepted epistemic strategies, but not *absolutely* unsolvable. One might object that if we, for example, succeed in proving Goldbach’s conjecture by means of some newly invented concept or epistemic strategy, the problem we now solve is not the same problem which confronted us before, because the available cognitive means have changed. This would amount to saying that the meaning of Goldbach’s conjecture is modified by the addition of the new concept or epistemic strategy. The latter claim could be advanced on the basis of a form of *linguistic* holism, according to which *all* epistemic properties (i.e., assertability conditions and inferential consequences) of a statement are constitutive of our linguistic *understanding* of that statement. This version of linguistic holism clashes with our pretheoretical criteria of understanding. My friend X is not a mathematician, but the reader would certainly grant that X *understands* Fermat’s last theorem (which until 1993 was another typical example of a problematic undecided statement); however it is clear that X does not know anything about the Shimura-Taniyama conjecture, through which Andrew Wiles in 1993 proved that Fermat’s theorem is true. If the meaning of a statement is that which one knows when one *understands* the statement, then we should conclude that the connection with the Shimura-Taniyama conjecture does not belong to the meaning of Fermat’s statement and this meaning did not change when the connection was established. Thus the statement proved by Wiles is the same statement asserted by Fermat without proof in the seventeenth century. Similarly, though our understanding of a statement like Goldbach’s conjecture requires our knowledge of *some* proof conditions and of *some* inferential consequences of that statement, it does not require our knowledge of *all* its epistemic connections with other statements; hence new connections can be established without changing its meaning. The epistemic philosopher has no need of linguistic holism and thus had better avoid adhering to such an implausible view.

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In sum, since our present epistemic situation does not exhaust our cognitive faculties, nothing we now know rules out that there is a solution to a presently unsolved problem, e.g. Goldbach's problem, in some epistemic situation which would result from a rational development of our present epistemic situation and would perhaps involve some completely new concepts. In Peirce's words we cannot assert that, "with regard to any given question (which has any clear meaning), investigation would not bring forth a solution of it, if it were carried far enough". Nothing we presently know entitles us to rule out that there is some yet unknown proof or refutation of Goldbach's sentence which we might discover in the future. Neither, one might object, is there anything which entitles us to claim that there is such a proof or refutation. But the holistic counterobjection can be that (*if* the cost-benefit analysis in favour of classical logic is right) all the advantageous characteristics of classical logic outweigh the present lack of any direct and specific warrant for the thesis that either a proof or a refutation of Goldbach's sentence in some ideal epistemic situation exists, although we have not found it yet. The global benefits of classical logic in the present epistemic situation can thus count as grounds for accepting the principle of existing solution for Goldbach's sentence. My suggestion amounts to saying that the epistemic philosopher can justify the excluded middle and the principle of existing solution in a way that Putnam called "quasi empirical" [84], and which resembles the justification offered by Zermelo of his axiom of choice. In 1908 Zermelo argued that his axiom was "necessary for science" because many "fundamental theorems and problems [...] could not be dealt with at all without the principle of choice"[85]. In the same spirit Gödel maintained that "besides mathematical intuition there exists another (though only probable) criterion of the truth of mathematical axioms, namely their *fruitfulness* in mathematics and, one may add, possibly also in physics"[86]. The same idea can be applied to classical logic, or to any other logic.

One might ask whether the cost-benefit analysis justifies the claim that the *excluded middle* is valid (i.e. that every statement of that form is true) or simply justifies our *belief* in its validity, our *practice* of employing it. The distinction is legitimate, since there may be excellent reasons for making oneself believe a false statement: one's actions may be much more productive if they are based on some illusions.[87] My answer is that the cost-benefit analysis *non-conclusively* justifies the *fallible*, *revisable* assertion that all instances of the excluded middle and of the principle of existing solution are true. As we have seen, nothing justifies the claim that the problem concerning Goldbach's conjecture is unsolvable. Thus, we are not entitled to say that the corresponding principle of existing solution is false and that our belief in its truth is only a useful illusion.

This holistic argument is very different from Peirce's and Husserl's transcendental justification of a principle of existing solution. The transcendental argument makes of the principle of existing solution an *a priori* law lying at the foundation of scientific investigation. The holistic argument is not *a priori* and justifies the principle of existing solution only relatively to a particular epistemic situation in which, after considering other known alternative logics, it turns out that the contribution of classical logic to the whole system of accepted sentences and reasonings is too precious to be renounced. The acceptance of classical logic and of a principle of existing solution for every given statement is based only on such cost-benefit considerations relative to a particular epistemic situation, and it may be withdrawn if some relevant data concerning the epistemic situation (which includes mathematics, physics, and all empirical sciences) modify the results of the cost-benefit analysis. For this *a posteriori*, revisable character, the holistic position is different also from the view that a principle of existing solution is an analytical *a priori* truth based upon arbitrary linguistic conventions. Finally, the holistic acceptance of the principle of existing solution differs widely from the view advocated by the logical positivists in the early Thirties. The logical positivists accepted the principle of existing solution, generalized and extended to all statements *a priori*, because they limited the area of genuinely meaningful statements to the statements for which a decision procedure is known. The holistic view, on the contrary, does not involve any restriction of the area of meaningfulness.

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[1]NOTES

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1 Cf. Prawitz (1987), p.153 and Prawitz (1980), p. 8. This view is developed in Prawitz (1998), which can also serve as an introduction to the epistemic conceptions of truth. Cf. also Martin Löf (1991), p. 141: "Intuitionistically, truth of a proposition is analyzed as existence of proof".

2 The formulations *i* and *i** are not equivalent because – as it is shown by Fitch's paradox of knowability – *i* is incompatible with the thesis: "there are true statements whose truth will remain unknown", whereas *i** is fully compatible with the thesis in question. Therefore, in Cozzo (1994a) it is maintained that the only plausible formulation of the epistemic conception of truth is *i**.

3 I am using the operator "it is true that ..." which yields a sentence "it is true that *S*" if the blank is filled in by a sentence *S*. But an analogous reasoning can be easily developed if one prefers to use a metalinguistic *predicate* "...is true" where the blank should be filled in by *names* of sentences of the object language. In the latter case the equivalence thesis has the form specified in Tarski's condition of material adequacy on a definition of the predicate "... is true" for an object language. Tarski show s that a definition which satisfies such a condition implies bivalence, if the excluded middle is accepted in the metalanguage; cf. Tarski (1935), Satz 2, p. 317. The proposal to call "equivalence thesis" both the principle containing the operator and the principle containing the metalinguistic predicate was advanced in Dummett (1978), p. xx.

4 Bivalence differs from the particular instance of the excluded middle "it is true that *S* or it is not true that *S*", but *if we accept the equivalence thesis* and thus take "it is true that *E*" and *E* to be interdeducible for any statement *E*, we can easily see, by an intuitionistically acceptable reasoning, that "it is not true that *S*" and "it is true that *not S*" (i.e. "it is false that *S*") are also interdeducible; hence, if the equivalence thesis holds, bivalence and the aforementioned instance of the excluded middle are equivalent.

5 Brouwer (1908), p.109.

6 Cf. Brouwer (1908), p.110: "Still, we shall never, by an unjustified application of the principle [of bivalence], come up against a contradiction and *thereby* discover that our reasonings were badly founded. For then it would be contradictory that an imbedding were performed and at the same time it would be contradictory that it were

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contradictory, and this is prohibited by the principium contradictionis". A much more articulated argument for the thesis that there are no absolutely undecidable propositions is developed by Per Martin Löf in the frame of intuitionistic type theory; cf. Martin Löf (1995).

[7] Hilbert (1965), p. 296; cf. §4 below.

[8] Cf. "Statements, joined to the thesis" in Brouwer (1907), Eng. trans., p. 101: 'The following conviction of Hilbert (Göttinger Nachr. 1900, p. 261) is unfounded: 'dass ein jedes bestimmte mathematische Problem einer strengen Erledigung notwendig fähig sein müsse, sei es dass es gelingt, die Beantwortung der gestellten Frage zu geben, sei es dass die Unmöglichkeit der Lösung und damit die Notwendigkeit des Misslingens aller Versuche dargetan wird'.

[9] Dummett (1976), p. 81.

[10] Another possibility is to abandon the equivalence thesis. Crispin Wright in Wright (1981) describes a view in which an epistemic conception of truth cohabits with acceptance of classical logic (and thus of the excluded middle) for problematic statements about the past. Wright's idea is based on a non -standard variety of semantics for which the distributivity of truth over disjunction does not hold and classical logic can be validated without any appeal to bivalence. The supporter of such a view rejects bivalence and the principle of existing solution for undecidable statements about the past; cf. Wright (1987), pp. 322-325. An undecidable statement *S* of the relevant kind would be neither true nor false, whereas «'*S*' is true» would be false; therefore. this view involves abandonment of the equivalence thesis. As Dummett observed, 'if we allow that there are sentences that are neither true nor false, then the equivalence thesis does not hold", Dummett (1978), p. xx, cf. Dummett (1959), pp. 4-5.

[11] Cf. Frege (1903), p.69, where Frege deduces transcendency from the excluded middle. For arguments in favour of transcendency see Putnam (1967), p. 53.

[12] Cf. Dummett (1975), Prawitz (1977).

[13] 'Die Namenerklärung der Wahrheit, daß sie nämlich die Übereinstimmung der Erkenntnis mit ihrem Gegenstand sei, wird hier geschenkt und vorausgesetzt" (B82/A57).

[14] Cf. B190/A151: 'wenn das Urteil analytisch ist, es mag nun verneinend oder bejahend sein, so muß dessen Wahrheit jederzeit nach dem Satze des Widerspruchs hinreichend können erkannt werden".

[15] Cf. B84/A60, B189/A150.

[16] Cf. Kant (1800), Introduction, VII, and also Ch. 1, §§ 22, 48, 78. In § 22 Kant formulates the principle of excluded middle as follows: "According to the principle of excluded middle (exclusi tertii) the sphere of a concept, relative to another, is either exclusive or inclusive". The princip le of sufficient reason is so formulated in § 76: "a ratione ad rationatum, a negatione rationati ad negationem rationis valet consequentia".

[17] Cf. Kant (1800) § 48: "According to the principle of the excluded middle not both contradictory judgments can therefore be true; as little, however, can they both be false. If therefore one is true, the other is false, and vice versa". Moreover, as Mirella Capozzi has noticed, in the Logik Dohna-Wundlacken (Ak. XXIV, 731), Kant (1992) p. 467, there is a place where bivalence is asserted in the most direct way: "Objectiv sind alle Sätze gewiß wahr oder gewiß falsch", cf. Capozzi (1991) p.159.

[18] The thesis that Kant accepted bivalence unrestrictedly is mantained in Capozzi (1991) against Posy (1983), p. 127 and Tiles (1980), p.161.

[19] Cf. Kant (1800), § 78.

[20] B817/A789.

[21] Cf. B820/A792.

[22] B600/A572.

[23] Cf. Tieszen (1992), p. 66 and Tieszen (1989), p. 193.

[24] Tieszen (1989), p. 181.

[25] Tieszen (1992), p.66.

[26] Cf. B601/A573.

[27] Cf. B599/A571.

[28] Capozzi (1991), p.162.

[29] Cf. B531/A573.
[30] Kant (1800) 22, note 3.
[31] B97/A72.
[32] Cf. B97/A72.
[33] See Kant (1800), 22, note 1, and the comment after the table of judgments in the "Transcendental Analytic" B97/A72: 'by the proposition 'the soul is not -mortal' I have so far as the logical form is concerned, really made an affirmation. I locate the soul in the unlimited sphere of non–mortal beings".
[34] B820/A792.
[35] *Ibidem.*
[36] See note 17.
[37] Cf. Capozzi (1991), p.163: "The avoidance of apagogic proofs is then a prudential rule which does not depend on the fact that in the 'transcendental enterprises of pure reason' bivalence 'cannot be assumed' Tiles (1980), but on the fact that, due to the 'dialectical illusion', we are liable to mistake true contradiction for dialectical opposition".
[38] B508/A480.
[39] B504/A476.
[40] B508/A480: "weil die Naturerscheinungen Gegenstände sind die uns unabhängig von unseren Begriffen gegeben werden, zu denen also der Schlüssel nicht in uns und unserem reinen Denken, sondern außer uns liegt, und eben darum in vielen Fällen nicht aufgefunden, mithin kein sicherer Aufschluß erwartet werden kann". Kemp Smith gives the following translation: "for the natural appearances are objects which are given to us independently of our concepts and the key to them lies not in us and in our pure thinking, but outside us, and therefore in many cases [...] an assured solution is not to be expected". Kant's expression "pure thinking" (reines Denken) in this context suggests a Kantian influence on Hilbert's "axiom" of the solvability of every mathematical problem. See §4 below and note 47.
[41] Posy (1984), p.128.
[42] Wittgenstein (1921) 6.5.: 'Das Rätsel gibt es nicht".
[43] Hahn, H., Neurath, O. and Carnap, R. (1929), p.15: 'Die wissenschaftliche Weltauffassung kennt keine unlösbaren Rätsel".
[44] "War es doch ein wesentlicher Fortschritt, daß Dubois die Unlösbarkeit seines Problems erkannte, und war diese Erkenntniss doch für viele Menschen eine Befreiung, wie der sonst kaum begreifliche Erfolg seiner Rede beweist. Den wichtigeren Schritt der Einsicht, daß ein prinzipiell als unlösbar erkanntes Problem auf einer verkehrten Fragenstellung beruhen muß, hat er allerdings nicht getan", Mach (1900), p.256. Mach's influence on logical positivists is clear in Richard von Mises' words: "Die Frage, in der Du Bois Reymond die unübersteigbaren Grenzen der Naturerkenntnis erblickte [...] ist uns das typische Scheinproblem" in Mises (1930), p.892. Du Bois Reymond's lecture "Über die Grenzen des Naturerkennens", held in Leipzig in 1872, is reprinted in Du Bois Reymond (1967).
[45] Schlick (1930), p. 7: "Wo immer ein sinnvolles Problem vorliegt, kann man theoretisch stets auch den Weg angeben, der zu seiner Auflösung führt, denn es zeigt sich, daß die Angabe dieses Weges im Grund mit der Aufzeigung des Sinnes zusammenfällt [...] Der Akt der Verifikation, bei dem der Weg der Lösung schließlich endet, ist immer von derselben Art: es ist das Auftreten eines bestimmten Sachverhaltes, das durch Beobachtung, durch unmittelbares Erlebnis konstatiert wird".
[46] On the distinction between real and ideal sentences cf. Hilbert (1925). I give here a very simplified picture of Hilbert's distinction. A more accurate description is in Prawitz (1981), pp. 254-258, or Prawitz (1993), §4.
[47] Hilbert (1965), p.298: 'Da ist das Problem, suche die Lösung. Du kannst sie durch reines Denken finden; denn in der Mathematik gibt es kein Ignorabimus!". I modified the English translation in Hilbert (1976), p.7, where "reines Denken" is translated "pure reason". See note 40.
[48] However in "Naturerkennen und Logik", published in 1930, Hilbert endorses a principle of existing solution generalized to all scientific statements, including empirical ones, cf. Hilbert (1965), p. 387: "Für den Mathematiker gibt es kein

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Ignorabimus, und meiner Meinung nach auch für die Naturwissenschaft überhaupt nicht".

49 Peirce (1931-35), 5.409.

50 Cf. Comte (1842), Leçon. 19, pp. 2-4. Of course, Peirce was aware of Comte's views on this matter.

51 Cf. "Lectures on Pragmatism" VII 1903 in Peirce (1931 -35), 5.196.

52 *Ibidem.*

53 C.S.Peirce, "The Logic of Drawing History from Ancient Documents" 1901, in Peirce (1958), 7.219.

54 Cf. C.S.Peirce, "Review of Royce's The World and the Individual" 1900, Peirce (1958), 8.113: "... the ultimate opinion ...will, as we hope, actually be attained concerning any given question (though not in any finite time concerning all questions)".

55 C.S.Peirce, "Continuity Redefined" (1906), in Peirce (1931 -35), 6.177.

56 Husserl (1929), Eng. trans. by D.Cairns, Section 20, p.66.

57 Husserl's example of judgement senseless in respect to content is: "the sum of the angles of a triangle is equal to the color red": Husserl (1929), Section 90, p.220.

58 Husserl (1929), Section 77, p.193-194.

59 Husserl (1929), Section 79, pp.197-198.

60 Cf. Wright (1981) and Wright (1987a). Cf. Dummett's reply in Dummett (1987).

61 Wright (1987b), p. 328.

62 See note 10.

63 Wright (1987b), p. 347-348.

64 Wright (1987b), p. 353.

65 Wright (1987b), p. 330.

66 Dummett (1977), pp. 365-366.

67 Cf. Wright (1987b), pp. 330-336.

68 The notions of "argumentation rule" and "concerning" are explained in my doctoral dissertation: cf. Cozzo (1994b), ch. 3.

69 Cf. Cozzo (1994b), ch. 5.

70 Carnap (1934), p. 52.

71 Cf. Cozzo (1994b), ch. 5, where a distinction between understandability and correctness of a language is developed.

72 Cf. Cozzo (1994b), ch. 4.

73 Quine (1970), p.87. In the same book (cf. p.12.), though he did not infer bivalence from classical logic and the equivalence thesis, Quine insisted very much on the importance of the equivalence thesis in its Tarskian version.

74 Quine (1973), p.66.

75 Quine (1973), p.67.

76 Cf. Dummett (1976), p. 82, Dummett (1993), p. 46.

77 An objection to the inference from bivalence to transcendency is in Wright (1987b), p. 318 and pp. 343-347. Dummett's reply is in Dummett (1987).

78 Cf. Quine (1981a).

79 Hilbert (1927), p.476.

80 Cf. Burgess (1984), pp. 191-192, Hellmann (1993).

81 Cf. for example Kline (1980), p. 264: "Gödel's incompleteness theorem is to an extent a denial of the law of excluded middle. We believe a proposition is true or false, and in modern foundations this means provable or disprovable [...] But Gödel showed that some are neither provable or disprovable. This is an argument for the intuitionists who argued against the laws [of logic] on other grounds".

82 Cf. Gödel (1995), p. 310.

83 Wang (1993), p. 119; cf. Wang (1976), pp. 324-325.

84 Cf. Putnam (1975), pp. 64-67.

85 Zermelo (1908), p. 188.

86 Gödel (1947/63), p. 485.

87 I owe this question to Peter Pagin.

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