Chapter 1

Exploring Predicativity

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Prominent constructive theories of sets such as Martin-Löf type theory and Aczel and Myhill constructive set theory, feature a distinctive form of constructivity: predicativity. This may be phrased as a constructibility requirement for sets, which ought to be finitely specifiable in terms of some uncontroversial initial "objects" and simple operations over them. Predicativity emerged at the beginning of the 20th century as a fundamental component of an influential analysis of the paradoxes by Poincaré and Russell. According to this analysis the paradoxes are the resulg of a vicious circularity in definitions; adherence to predicativity was therefore proposed as a systematic method for preventing such problematic circularity. In the following, I sketch the origins of predicativity, review the fundamental contributions by Russell and Weyl and look at modern incarnations of this notion.

1. Introduction

Since recent years the word "constructive" is typically employed as synonym of "using intuitionistic logic". Indeed, influential constructive mathematical theories, as Martin-Löf type theory and Aczel and Myhill constructive set theory employ intuitionistic logic.[a] However, there is a more fundamental

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[a]See, for example, [1–5]. For surveys see [6, 7].

sense of constructivity that these theories also aim at capturing, which is deeply rooted in the mathematical tradition, and is commonly expressed by stating that they are *predicative*. According to one way of spelling out the notion of constructivity, this relates to a *finitary process of "construction" or specification* of a mathematical entity. For example, the Oxford English Dictionary so defines the word "constructive" (in the mathematical case): "*Relating to, based on, or denoting mathematical proofs which show how an entity may in principle be constructed or arrived at in a finite number of steps.*" There is a sense in which an intuitionistic approach to mathematics satisfies this notion of constructivity, as it assigns a fundamental role in mathematics to the availability of explicit or constructive proofs of mathematical statements. For instance, an intuitionistic proof of an existential statement is usually read as embodying an algorithm which provides (at least in principle) a witness to the statement in question and a proof that the witness does satisfy the relevant condition expressed by the statement. In addition, an intuitionistic proof of a disjunction ought to offer the mean for deciding which of the disjuncts holds true.[b] The requirement that an intuitionistic proof of an existential statement also ought to include a witness is a paradigmatic example of the constructivity of the resulting proof, as the latter shows how to "construct", or specify, a witness. Similarly, the requirement of availability of a decision procedure for a disjunction ensures that a constructive proof is fully explicit.

The issue that predicativity rises is distinct from, though related to, that of the availability of an intuitionistic proof, and may be seen as directly pertaining to the question of how we specify domains of quantification.[c] For this reason, debates on predicativity have been traditionally perceived as disputes pertaining the concept of set.[d] The thought is that an analysis of constructivity, which relates to "how an entity may in principle be constructed or arrived at in a finite number of steps", ought to include also a clarification of the concept of set, in as much as sets are domains of quantification. It is then clear that the case of universal quantification on infinite domains is particular problematic, as according to this constructive per-

[b]This understanding of the notion of intuitionistic proof is manifested, for example, by the so-called Brouwer-Heyting-Kolmogorov (BHK) interpretation of intuitionistic logic. See e.g. [8, 9]. The ideas underlying the BKH interpretation are made more precise in Martin-Löf type theory and in models of intuitionistic theories, as realizability models [10].

[c]The interrelation between predicativity and intuitionistic logic is complex, and for this reason an analysis of this issue is postponed to subsequent work.

[d]See the final section for a brief discussion of this point.

spective we require that a domain of quantification be *finitarily* specified.

If sets are to be understood constructively, then two crucial issues need to be examined: which methods of "construction" of sets are considered admissible, and which initial entities can be taken as legitimate starting points of the construction process. As I clarify below, the predicative literature witnesses a number of distinct answers to these questions.

The principal aim of this article is to survey and discuss predicativity.[e] In the following, I outline the origins of predicativity, and review an informal characterization of this notion. I then briefly sketch the main traits of two fundamental predicativist proposals by Russell and Weyl, respectively [14, 15]. Finally, I delineate the principal steps of a logical analysis of predicativity that began in the 1950's, and conclude with a comparison between two forms of predicativity: predicativity given the natural numbers and strict predicativity.

2. The emergence of predicativity

The notion of predicativity has its origins within a remarkable exchange between Poincaré and Russell at the beginning of the last century.[f] The wider context of Poincaré and Russell's debate on predicativity are well-known reflections by prominent mathematicians of the time on the new concepts and methods of proof which had made their way in mathematics from the 19th century.[g] Predicativity specifically was forged as part of Poincaré and Russell's attempts to analyse and counter the paradoxes that afflicted the foundations of mathematics.[h] Russell's influential analysis of the paradoxes imputes them to the illegitimate assumption that *any propositional function gives rise to a class*, the class of all the objects which satisfy it.[i] Russell therefore introduced the term **predicative** to denote those *propositional functions* which define a class and distinguish them from the non-predicative or **impredicative** ones, which do not define a class [20].

[e]This note is part of a wider project of clarification and assessment of predicativity. See also [11–13].
[f]See [14, 16–22].
[g]See e.g. [23–25].
[h]See below for an emblematic example of paradox: Russell's paradox.
[i]In the present context for simplicity we may identify a propositional function with an open formula, i.e. a formula with a free variable (see e.g. [26]). Note, however, that the interpretation of the notions of proposition and propositional function in Russell is complex. See e.g. [27]. Note also that the term "class" is here used as in the original literature, to refer to a collection of elements. Therefore it should be carefully distinguished from the notion of "proper class" that is often used in contemporary set theory.

Poincaré and Russel's debate witnesses the difficulties involved in spelling out the notion of impredicativity, and clarifying its perceived problematic status. Russell, in particular, turned to devise formal ways of capturing the distinction between predicative and impredicative propositional functions. A clarification of the notion of predicativity became a fundamental component of a thorough analysis of the foundations of mathematics, and adherence to predicativity the main instrument for blocking the paradoxes. Russell's efforts culminated in his ramified type theory, as further discussed in section 3.1.

2.1. *Circularity and Russell's VCP*

A crucial feature of Russell's type theory is its implementation of the so-called Vicious Circle Principle (VCP), introduced to ban vicious-circular definitions (see below). The VCP was prompted by the fundamental observation by Richard, further developed by Poincaré, that paradoxes typically manifest a form of *vicious circularity, or self-reference* [16, 18, 28]. An example may help clarify this point.

Richard's paradox. This paradox arises form the definition of the least non–definable real number, r, by reference to the class of all definable real numbers. More precisely, let us consider all the real numbers which are definable in English by a finite number of words and let D be their collection. D is countable. We can then list all the elements of D, and mimic Cantor's diagonal proof of the non-denumerability of the real numbers to produce a new real number, r, which is different from each element of D. However, one can easily express in English a rendering of the "algorithm" that allows for the definition of r, so that r turns out to be a definable real number after all, and a contradiction arises.

According to the present analysis, Richard's paradox is engendered by a form of circularity: we define r by reference to the *whole* D, and therefore, so it is claimed, by reference to r itself. In fact, Russell introduced the VCP to prevent the formation of collections as D, and claimed that these are *ill-formed.* He gave a number of variants of the VCP. For example:

"no totality can contain members defined in terms of itself" [14, p. 237]. Another version is to be found in [29, p. 198]:

> ... whatever in any way concerns *all* or *any* or *some* of a class must not be itself one of the members of a class.

The latter rendering of the VCP clearly highlights the fundamental link between impredicativity and quantification.[j]

2.2. *Characterising predicativity*

Having introduced the VCP and explained its motivation with an example, I can now review an informal characterisation of predicativity which figures in the early foundational debates. Predicativity is typically presented negatively: we specify what is impredicativity and term predicativity the negation of impredicativity. We say that *a definition is* ***impredicative*** *if it defines an entity by reference to a class to which the entity itself belongs.* In particular, a definition is impredicative if it defines an entity by *quantifying over* a class which includes the entity to be defined. Given this notion of impredicative definition, one can further specify a notion of *impredicative entity*: this is an entity which can *only* be defined by an impredicative definition. We can hence talk of impredicative classes, propositions, properties. A definition or entity is ***predicative*** *if it is not impredicative.*[k]

2.2.1. *Examples*

In the previous section, I have reviewed Richard's paradox, which arises from an impredicative definition, as we define a new element r of D by reference to the whole D. Further examples may help clarify the notion of impredicativity. In presenting these examples I closely follow [14, 34], but also [35, 36], as their analysis clearly highlights a number of reasons that are typically adduced against impredicativity.

1. *Russell's paradox.* The first example of impredicative definition I consider is that of **Russell's "set"**, R. This can be so defined in modern terminology:

$$R = \{x \mid x \notin x\}.$$

Here R arises from an application of the **Unrestricted Comprehension schema**: given any formula φ in the language of set theory, we form the

[j] This is perhaps the best know expression of the VCP. However, Russell gave other renderings, some of which, like the first one above, do not directly involve quantification over, but reference to a class that includes the definiendum. The plurality of formulations of the VCP induces difficulties for an exegesis of Russell's thought, and, indeed, for a thorough clarification of the notion of predicativity, as already noted by Gödel [30].

[k] This negative characterisation of impredicativity is clearly unsatisfactory. As discussed below, subsequent technical work, starting from Russell's type theory, aimed at offering more informative characterisations of predicativity. Note also that the late writings by Poincaré [21, 22] feature another, apparently distinct, characterisation of predicativity which does not appeal to circularity. A predicative set is "invariant under extension": the addition of new elements does not "disorder" the set itself. See [12, 13, 31–33].

set of all the x's that satisfy φ, that is, $\{x \mid \varphi(x)\}$. In R's definition, in particular, one takes φ to be $x \notin x$.

In his analysis of this paradox Russell observes that R is defined impredicatively as it *refers* to the class of all classes [14, p. 225]. He also states that if we tried to block the paradox by deciding that no class is a member of itself, then R would become the class of all classes. But then the question arises whether R is a member of itself, and we have to conclude that R is not a member of itself, that is, that R is not a class. Russell therefore draws the conclusion that there is no class of all classes, since if we supposed there is, then this very assumption would give rise to a new class lying outside the presumed class of all classes.

2. *Burali-Forti paradox.* This is so described by Russell: we can show that every well-ordered class has an ordinal number, and that the ordinal of the class of ordinals up to and including any given ordinal exceeds the given ordinal by one. But "on certain very natural assumptions" the class of all ordinals itself is well-ordered and has an ordinal number, say Ω. However, the class of all ordinals including Ω turns out to have ordinal number $\Omega+1$, contradicting the assumption that Ω is the ordinal of the class of all ordinals. Russell's assessment of this paradox is that it shows that "all ordinals" is an "illegitimate notion; for if not, all ordinals in order of magnitude form a well-ordered series which must have an ordinal number greater than all ordinals." [14, p. 225]

3. *The logicist definition of natural number.* The logicist definition of natural number may be so expressed: n is a natural number if it satisfies all properties which hold of 0 and which are closed under the successor operation. In modern terminology:

$$N(n) := \forall F[F(0) \wedge \forall x(F(x) \rightarrow F(Suc(x))) \rightarrow F(n)].$$

The predicate N expressing the property "to be a natural number" is here defined by reference to all predicates, F, expressing properties of the natural numbers. A circularity arises as the predicate N itself is within the range of the first quantifier: the definition is impredicative, as N is defined by reference to all predicates expressing properties of the natural numbers and thus, so it is contended, to itself.[1]

In the following, I draw on Carnap's clear analysis of this example in [35]. Carnap rightly stresses the importance of this example, which shows

[1]Poincaré is well-known for having noted this difficulty with this definition of natural number, and for suggesting that the principle of mathematical induction and the natural numbers can not be reduced to something more primitive.

that impredicativity affects not only paradoxical cases, but one of the most fundamental concepts in mathematics, that of natural number.[m] Carnap remarks that the above definition induces difficulties which are best seen if we consider a *specific* natural number, say 3, and check whether it satisfies this definition. In order to do so, we need to check if *each* property of the natural numbers holds of 3, that is, if: $\forall F[F(0) \wedge \forall x(F(x) \rightarrow F(Suc(x))) \rightarrow F(3)]$. However, the property "to be a natural number", which is expressed by the predicate N, is one of the properties of the natural numbers. That is, to find out whether $N(3)$ holds, we need to be able to clarify whether the following holds:

$$N(0) \wedge \forall x(N(x) \rightarrow N(Suc(x))) \rightarrow N(3).$$

Hence it would seem that we need first to ascertain whether the property of being a natural number holds of 3, in order to assess whether it holds of 3. Carnap concludes that this definition of natural number is therefore "circular and useless" [35, p. 48] .[n]

4. *Napoleon's qualities.* Another example of impredicative definition which does not involve a paradox is given by the sentence: ***Napoleon had all the qualities that make a great general*** [34, p. 59]. We might wish to compare the expression above with the following: *Napoleon was Corsican*, or *Napoleon was brave.* These are utterly unproblematic, as the properties expressed by "being Corsican" and "being brave" do not refer to other properties. However, the property expressed by "having all the qualities that make a great general" would seem to be itself a quality of a great general, and therefore refer to itself.[o]

5. *Least Upper Bound principle.* Finally an example from analysis: the **Least Upper Bound principle** (LUB). This states that:

> Every bounded, non–empty subset $\mathcal{M}$ of the real numbers has a least upper bound.

[m]A worry for Carnap is that the impredicativity of the logicist definition of natural number seems to compromise the logicist programme.

[n]Carnap [35] hints at the possibility of an alternative reading of universal quantification, which would vindicate the usefulness of impredicative definitions as the one above. The idea is to consider a reading of universal quantification which avoids the presupposition that a universal quantification also entails reference to each individual element of a domain of quantification. Carnap's proposal is briefly discussed by Gödel [30]. See also [37] for a contemporary perspective into this issue inspired by Carnap's discussion. [12] also analyses, under the light of [38], the prospects of eliminating these difficulties by reading universal quantification intuitionistically.

[o]This example is often used by Russell to explain the workings of "ramification" in ramified type theory, as discussed below.

To explain the impredicativity of the (LUB) I follow [36, p. 2]. In a standard way, we can codify rational numbers by suitable natural numbers and identify real numbers with certain sets of rational numbers, via Dedekind sections or Cauchy sequences. If we identify real numbers with the upper parts of Dedekind sections, we see that the least upper bound of a bounded, non-empty set $\mathcal{M}$ of real numbers is given by $S = \cap X[X \in \mathcal{M}]$. The difficulty with this definition can be seen as follows. The set $\mathcal{M}$ will be typically given by a condition, $\mathcal{C}(X)$, such that

$$\forall X[X \in \mathcal{M} \iff \mathcal{C}(X)].$$

Now S above is such that

$$\forall x[x \in S \iff \forall X(\mathcal{C}(X) \to x \in X)].$$

Feferman writes:

> However, to answer the question "What are the members of S?" we would, in general, first have to know what sets X satisfy $\mathcal{C}(X)$, and in particular whether or not $\mathcal{C}(S)$ holds; this would, in turn, in general depend on knowing what members S has. [36, p. 2]

As further discussed below, if impredicativity is seen as problematic, this particular example is critical, as it goes at the very heart of mathematics, affecting a core discipline as analysis.

To conclude this section, I outline Russell's verdict on examples as 1 and 2 above. As already anticipated in [20], Russell claims that a solution to the paradoxes lies in countering the assumption that *any* propositional function gives rise to a set. More precisely, Russell observes that a typical common feature of the paradoxes is that they involve a form of "self-reference of reflexiveness". "In each contradiction something is said about *all* cases of some kind, and from what is said a new case seems to be generated, which both is and is not of the same kind as the cases of which *all* were concerned in what was said." [14, p. 224] Russell concludes that viciously circular definitions do not give rise to a class, on pain of contradiction, and therefore are illegitimate:

> Thus all our contradictions have in common the assumption of a totality such that, if it were legitimate, it would at once be enlarged by new members defined in terms of itself. This leads us to the rule: "Whatever involves *all* of a collection must not be one of the collection"; or, conversely; "If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total."

3. Shedding light on predicativity: Russell's ramified type theory and Weyl's "Das Kontinuum"

The above characterisation of impredicativity and the examples suffice to convey a general idea of this notion. However, the above characterisation is insufficiently precise to fully clarify what counts as predicative and what does not. In fact, as already noted by Zermelo, mathematical notions are typically defined in a number of alternative but equivalent ways [39]. It is therefore to be expected that some mathematical notions which are prima facie impredicative may, under closer scrutiny, turn out to be predicative after all. A detailed assessment of this issue requires therefore a more sophisticated approach which makes use of a precise logical machinery.

Two main lines of research arise from this observation. First of all, from a perspective that finds faults with impredicativity, one would like to have some general criteria which systematically guarantee that when developing mathematics we do not introduce impredicative notions or entities. One way to achieve this is to develop a suitable foundational theory (e.g. a set theory) which complies with predicativity, so that working within it fully guarantees adherence to predicativity.

Secondly, as the examples of the logicist definition of natural number and the LUB principle clarify, impredicativity is to be found in everyday mathematics. Therefore, from both a perspective that favours and one that objects to predicativity, it becomes crucially important to assess what is the impact of complying with predicativity. A clarification of the latter point turns out to be more complex than the above informal characterisation of predicativity may suggest, especially because of the possibility of developing a portion of mathematics in a number of alternative ways. In the following, I briefly review two particularly significant steps in the development of these two lines of research: Russell's type theory and Weyl's predicative analysis. In Section 4, I discuss more recent developments that directly tackle the second issue.

3.1. *Russell's type theory*

The need to replace the purely negative characterisation above with a positive one was fully acknowledged by [14], who claimed that "our positive doctrines [...] must make it plain that 'all propositions' and 'all properties' are meaningless phrases." [14, p. 226] Russell's "positive doctrine" finds full expression in his ramified type theory [14, 34]. This is a careful and complex formulation of a concept of set which introduces a number of restrictions compared with the naive concept of set given by unrestricted comprehension. Retrospectively, Russell [14] introduces *simultaneously* two kinds of regimentation: a type restriction and an order restriction for propositional functions.

Type restrictions have the effect of producing a hierarchy of "levels" or "types". As a consequence, membership is now a relation between, on the one side, members of a type and, on the other, the type itself. A type is "[...] the range of significance of a propositional function, i.e. [...] the collection of arguments for which the said function has values." [14, 236] The underlying idea is that we start from a type of individuals, and then consider types which are ranges of significance of propositional functions defined on the individuals, and so on. This apparently suffices to block set-theoretic paradoxes as, for example, Russell's paradox: expressions as $x \in x$ or $x \notin x$ are simply ill-formed.[p]

In addition to type restrictions, Russell also introduced further constraints on propositional functions, whose effect is to block impredicativity more generally. Example 4 above highlights the difference between expressions as "being Corsican" and "being brave" on the one hand, and expressions as "having all the qualities that make a great general" on the other hand. The first are utterly unproblematic, while the latter refers to "all qualities" of a great general, including the property it refers to, and therefore is problematic from a predicative perspective. The strategy underpinning ramification may be concisely summarised as follows: to avoid defining a property in terms of an expression which refers to "all properties", we subdivide the propositional functions (referring to properties) in different "orders". First order propositional functions are those, as "being Corsican" and "being brave" which do not refer to other propositional functions. At the second order we have propositional functions which quantify over all propositional functions of the first order, and so on.[q] This is accounted for by introducing a hierarchical structure, ramification, at the level of propositional functions; quantification is then constrained to range over propositional functions of lower orders. Consequently, cases of impredicativity as that involved in Example 4 and in the definition of the natural numbers discussed above can now be eliminated. However, as quickly realised by Russell [14], the device of ramification causes difficulties as soon as

[p] Set-theoretic paradoxes are paradoxes, as Russell's and Burali-Forti's, that relate to the concept of set. See also page 14.
[q] On ramification see e.g. [40–44].

we attempt to prove statements by induction on the natural numbers. For example, we can not refer to all the properties of the natural numbers, but only to those of some given order. These difficulties propagate to the case of the real numbers, as witnessed by the impredicativity of the LUB. As a consequence, many "fundamental theorems not only could not be proved but could not even be expressed." [35, p. 46] In fact, "many of the most important definitions and theorems of real number theory are lost" [35, p. 49]. Russell [14] felt compelled to introduce the axiom of reducibility, "by means of which the different orders of a type could be reduced in certain respects to the lowest order of the type." [35, p. 46] A frequent criticism of this axiom today, is that from an extensional point of view it has the effect of reintroducing impredicativity. Soon after its introduction, reducibility was met by stark criticism for its lack of satisfactory justification: "[t]he sole justification for this axiom was the fact that there seemed to be no other way out of this particular difficulty engendered by the ramified theory of types." [35, p. 46] More forcefully Weyl wrote:

> Russell, in order to extricate himself from the affair, causes reason to commit hara-kiri, by postulating the above assertion [the axiom of reducibility] in spite of its lack of support by any evidence. [45, p. 50]

3.2. *Weyl's "Das Kontinuum"*

A fresh attempt at developing mathematics from a predicative point of view was proposed by Weyl in his book "Das Kontinuum" [15]. Weyl may be seen as contributing in original ways to both the above mentioned lines of research: he put forth a concept of predicative set, that is, a systematic predicative foundation, but also explored the mathematical extent of predicativity, developing (a portion of) analysis on the basis of this predicative concept of set.

Weyl was fully aware of the difficulties introduced by ramification for the development of mathematics, and severely criticized the axiom of reducibility, as also witnessed by the quotation above; he thus refrained from both. As to ramification, in Section 6 of "Das Kontinuum", Weyl does consider this possibility, but concludes: "A 'hierarchical' version of analysis is artificial and useless. It loses sight of its proper object, i.e. number [...]. Clearly, we must take the other path [...] *to abide the narrower iteration procedure*." [15, p. 32] In the following, I present the main characters of Weyl's approach and clarify what is the "narrower iteration procedure" mentioned in the above quotation.[r]

In setting up his predicative analysis, Weyl's starting point are the natural numbers, which are assumed as given together with the principle of mathematical induction.[s] Weyl refers back, approvingly, to Poincaré, who had strongly criticised any attempts to justify the principle of induction as *viciously circular*.[t] As for Poincaré, also for Weyl we have no option but to take the natural numbers with the principle of induction at the start: "*the idea of iteration, i.e., of the sequence of the natural numbers, is an ultimate foundation of mathematical thought, which can not be further reduced*" [15, p. 48]. In fact, Weyl clearly highlights the fundamental role that the principle of induction has for the natural numbers, as it is this principle that allows us to characterize uniquely each natural number in terms of its position in the number sequence.

While the natural numbers are assumed as "given", Weyl imposes restrictions, motivated by predicativity concerns, at the next level of idealization beyond the natural numbers: the continuum. As the real numbers can be represented by sets or sequences of rational numbers, and the rational numbers, in turn, by natural numbers, the question underlying Weyl's predicative approach may be phrased as follows: which sets of natural numbers can be justified predicatively? From an impredicative perspective, Weyl's predicative analysis may be seen as introducing predicative restrictions on the *powerset of the natural numbers*, which is an emblematic manifestation of the concept of "arbitrary" set underpinning ZFC [23, 49, 50]. For Weyl, an arbitrary set is "a 'gathering' brought together by infinitely many individual arbitrary acts of selection, assembled and then surveyed as a whole by consciousness" and, as such, it is "nonsensical" [15, p. 23]. Weyl proposes an alternative concept of set, one which can be portrayed as if it were produced by a step-by-step process from the safety of the natural numbers by application of well-understood logical operations. He is very clear that only by reforming the concept of set so to anchor it to the safe domain of the natural numbers, can we be confident that the edifice of mathematics stands on "pillars of enduring strength". For example, Weyl notes the impredicativity of the LUB and writes:

> But the more distinctly the logical fabric of analysis is brought

[r]See also [46–48], and [13, 26] for overviews.

[s]I shall also write "induction" for "mathematical induction".

[t]See e.g. [18].

> to givenness and the more deeply and completely the glance of consciousness penetrates it, the clearer it becomes that, given the current approach to foundational matters, every cell (so to speak) of this mighty organism is permeated by the poison of contradiction and that a thorough revision is necessary to remedy the situation. [15, p. 32]

The process of "formation" of predicative subsets of the natural numbers suggested by Weyl may be concisely summarised as follows: we start from the natural numbers with full mathematical induction and use the ordinary logical operations to form judgements expressing properties of (and relations between) natural numbers. Crucially, *quantification is restricted to the domain of natural numbers.*[u] Sets are then *extensions* of properties (and relations) expressed by such judgements, modulo extensionality. More precisely, a set is the collection of all and only those objects which satisfy a property affirmed by one such judgement, and sets are identified if and only if they have the same elements. The restriction to quantification on the natural numbers witnesses Weyl's fundamental choice of following the "narrower iteration procedure". One could, in fact, imagine that once a set has been justified in the above manner, it would be legitimate to quantify over it, therefore using it to define new predicatively justified sets, that can again act as domains of quantification, and so on. As mentioned above, Weyl does consider this possibility but suggests that for the purpose of developing real mathematics, it is unnatural. He therefore explores how far can we go by restricting quantification to the natural numbers.

In modern terminology, in the language of second order arithmetic, Weyl introduced restrictions on how we form subsets of the natural numbers, that, in practice, justify only applications of the comprehension schema to *arithmetical formulas*, that is, those formulas which do not quantify over sets (but may quantify over natural numbers). In this way one justifies sets of the form $\{x \mid \varphi(x)\}$ only if φ does not contain set quantifiers. This restriction prevents vicious–circular definitions of subsets of the natural numbers: the restriction to number quantifiers in the comprehension principle does not permit the definition of a new set by quantification over a collection of sets to which the definiendum belongs.

Weyl's fundamental realisation was that adopting this very restrictive

[u]Weyl in [15] takes a more general approach, by proposing a concept of set which is built on any "definite" basic category of object. The natural numbers are a paradigmatic example of basic category, and the fundamental one for analysis.

concept of set does not impair the development of large parts of 19th century analysis. In fact, Solomon Feferman [46] has extended Weyl's work to include large portions of contemporary analysis, as further discussed below.

4. The re-emergence of predicativity

The interest in predicativity sharply declined soon after the publication of Weyl's book for a number of reasons, among which, for example, Weyl's brief conversion to Brouwer's intuitionism [**?**].[a] Peharps the most significant circumstance that determined predicativity's loss of appeal was the rapid accreditation of impredicative set theory as standard foundation.[b] In addition to historical and sociological reasons, the widespread neglect for predicativity is also due to two kinds of objections that were quickly leveled against it: one of a mathematical and one of a philosophical nature. The mathematical objection was that adherence to predicativity is unnecessary to avoid the paradoxes that afflict the foundations of mathematics. This followed a proposed distinction between set-theoretic and semantic paradoxes [52, 53]. Set-theoretic paradoxes are those which directly relate to the concept of set, and include e.g. Russell's and Burali-Forti paradoxes; semantic paradoxes involve linguistic or semantic notions, and include e.g. Richard's paradox and the Liar paradox. It was then suggested that only the first kind of paradoxes constitutes a serious threat to the foundations of mathematics [52, 53] (see also [35]). Furthermore, Chwistek and Ramsey [53, 54] observed that Russell's ramified type theory could be simplified by introducing type restrictions without also imposing ramification. The resulting formalism goes under the name of *simple type theory* and its formulation was subsequently simplified by Church [55], among others. Simple type theory does not eliminate all impredicativity, but seems sufficient to block all known set-theoretic paradoxes. This observation was then taken to undermine what is typically seen as the principal motivation for predicativity: to avoid the paradoxes. It certainly intimates more care in construing an argument for predicativism, the philosophical position according to which only predicative mathematics is justified.

As to the philosophical objection, a frequent interpretation of impredicativity is that it becomes a genuine difficulty only if the role of definitions

[a] Note that Weyl's classical approach in "Das Kontinuum" had lasting influence on Lorenzen [51].

[b] See also [26] for additional historical and sociological reasons for the decline of interest in predicativity.

is to produce or create their definiendum, rather than select it from a previously given domain of mathematical entities. In this context, one often mentions [53]'s example: "the tallest man in this room".[c] This is an impredicative definition, but its impredicativity is harmless. Its innocent nature is often explained by claiming that its purpose is *singling out* rather than *creating* a particular individual. Ramsey's objection to predicativity is so summarised, for critical purposes, by [35, p. 50]: "That we men are finite beings who cannot name individually each of infinitely many properties is an empirical fact that has nothing to do with logic." In other terms, it is often argued that it is only from a throughly constructive perspective that impredicativity is problematic; however, a realist attitude to mathematical entities grants the legitimacy of impredicativity. This interpretation of predicativity is very common, so much so that it might be termed the "received view" on predicativity. Its prevalence is probably also due to [30]'s well-known analysis of predicativity, that is often read along these lines. A discussion of these objections is beyond the aims of this note. As to the philosophical objection, I argue elsewhere that both the legitimacy of impredicativity from a realist perspective and its illegitimacy from a constructive perspective require further scrutiny.

4.1. *A new stage for predicativity*

Notwithstanding these objections, renewed interest in predicativity emerged from the 1950's, when fresh attempts were made to obtain a clearer demarcation of the boundary between predicative and impredicative mathematics, by making use of state-of-the-art logical machinery. The literature from the 1950's and 1960's witnesses the complexity of the task of clarifying the limit of predicativity, which saw the involvement of a number of prominent logicians, as Feferman, Gandy, Kleene, Kreisel, Lorenzen, Myhill, Schütte, Spector and Wang. In the following, I sketch the most salient characters of this new phase for predicativity.[d] As argued in [13], a very striking aspect of the logical analysis of predicativity is the radical change in purpose, compared with the first discussions on predicativity. The mathematical logicians who approached predicativity in the second half of the last century typically aimed not at rectifying the foundations of mathematics, but at further clarifying the notion of predicativity and distinguishing it from impredicativity. The principal aim was to establish

[c]See also the discussion of this example in [35].

[d]See e.g. [13, 26] for surveys and further references.

the limit of predicativity, clarifying how far predicativity goes. This was approached by two distinct, but related, strategies. A first objective was to devise formal instruments for capturing the notion of predicativity and establish their theoretical limit. A second purpose was to clarify which parts of contemporary mathematics can be developed without any appeal to impredicativity, by systematically analysing mathematical theorems from a logical perspective.

The efforts to establish the limit of predicativity culminated in a fundamental chapter in proof theory. Here Russell's original idea of ramification had a crucial role, as it lead to the definition of a transfinite progression of systems of ramified second order arithmetic indexed by ordinals. As in [15], the natural numbers were assumed as starting point, and constraints were introduced at the next level, restricting the powerset of the natural numbers. However, there was also a departure from Weyl's "narrower iteration" procedure, as the aim was now to assess *how far* can we go from a predicative perspective: the ascent to sets beyond the arithmetically definable ones was therefore permitted, as long as it could be predicatively justified. The principal difficulty was, however, devising suitable criteria for justifying predicatively the extension beyond Weyl's original approach. Here a notion of "predicative ordinal" played a pivotal role, as ascent along the progression of ramified systems was admitted only along predicative ordinals. The intuition underlying the notion of predicative ordinal is that this is an ordinal which can to be recognized by exclusive appeal to notions that have already been secured. Roughly, one introduces a "boot-strapping" condition, requiring that we progress up along the hierarchy of ramified systems to a stage α only if α has already been recognized as predicative at a previous stage of the hierarchy, i.e. if α has been proved to be an ordinal at a previous stage of the hierarchy. Following a proposal by Kreisel [56], Feferman and Schütte (independently) determined the so-called **limit of predicativity** in terms of the first non-predicative ordinal, known as Γ_0 [36, 57, 58].[e] The claim was that theories whose proof theoretic strength is below Γ_0 could be predicatively justified.

The second component of the logical analysis of predicativity was a detailed logical investigation of the underlying assumptions which are implicit in ordinary mathematics, with the purpose of elucidating the role of impredicativity in ordinary mathematics. The expression "ordinary mathematics" refers to mainstream mathematics, that is, those areas of mathe-

[e]See e.g. [59, 60] for more details.

matics which make no essential use of the concepts and methods of abstract set theory and, in particular, the theory of uncountable cardinal numbers. Weyl's pioneering work in "Das Kontinuum" constituted fundamental reference, especially for Feferman's investigations [26, 46]. Feferman [46] has carefully analysed Weyl's text and proposed a system, W (for Weyl), which can be used to codify the analysis in "Das Kontinuum". He has verified that not only Weyl's analysis, but large portions of contemporary analysis can be carried out on the basis of system W.[f] The significant point is that W is very weak proof-theoretically, as it is no stronger than Peano Arithmetic (and hence well within the Γ_0 limit). Another important source of insight on the mathematical extent of predicativity are the findings obtained within Friedman and Simpson's programme of Reverse Mathematics [62], which also produced important independence results.[g] This research overall confirms that if we confine our attention to ordinary mathematics, then impredicativity is largely unnecessary.[h] The situation bears similarity to the one we encounter in constructive mathematics. Brouwer and, subsequently, Bishop [65] realised that notwithstanding the fact that the principle of excluded middle is extensively used in ordinary mathematics, we can re-develop a large body of interesting and useful mathematics without any appeal to this logical principle. Here it is tempting to draw the moral that at least for a substantial portion of ordinary mathematics, the apparent necessity of certain features of ordinary mathematics, like the use of the principle of excluded middle or impredicativity, turns out to be a by-product of the context in which it is developed, and might also depend on the specific formulation of their statements. In particular, in the constructive case a careful choice of definitions allows for a constructive redevelopment of parts of mathematics that are non-constructive. In the case at hand, many instances of prima facie impredicativity become amenable to predicative treatment once we work within sufficiently weak systems.[i]

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[f] See also [61].

[g] In this context, Weyl's predicative analysis can be recast within the system ACA_0.

[h] See [63] for details and [64] for examples of mathematical theorems that lie beyond predicativity.

[i] There has been extensive cross-fertilisation between reverse and constructive mathematics. Simpson, however, also emphasizes a difference with constructive mathematics, in that the aim in reverse mathematics is "to draw out the set existence assumptions which are implicit in the ordinary mathematical theorems *as they stand*". Bishop's goal, according to Simpson, is instead "to replace ordinary mathematical theorems by their "constructive" counterparts." [63, p. 137]

In addition, like in constructive mathematics, we need to rely on *individual case studies* for our findings, so that any general conclusion can only be achieved on the basis of a thorough investigation of the mathematical practice.

5. Plurality

The logical analysis of predicativity aimed at determining the limits and the extent of a notion of predicativity ***given the natural numbers***. Here one takes an approach to predicativity analogous to Weyl's, in that the natural numbers with full (i.e. unrestricted) induction is assumed at the start, and appropriate predicatively motivated constraints are imposed on the formation of subsets of the natural numbers. In the case of predicativity à la Weyl, an appeal to the natural numbers together with simple logical operations enables for the articulation of a predicative concept of set which offers a secure foundation for mathematics, as its certainty is grounded on the reliability of the natural numbers and those simple logical operations. In particular, by restricting quantification to the natural numbers, we do not appeal to dubious sets, definable only by vicious circles. The legitimate subsets of the natural numbers may be portrayed as if they were the result of a bottom-up procedure, or the repeated application of an arithmetical rule which, starting from the natural numbers, prescribes step-by-step which elements belong to the given set (and when two such elements are equal). The comparison with the full powerset of the natural numbers is instructive, as in that case one collects together *all* the subsets of the natural numbers, irrespective of whether we can offer, even in principle, a finitary rule of formation. The arithmetical sets are therefore a particularly clear exemplification of a predicative and constructive notion of set grounded on the natural numbers (and the first order logical operations).

The assumption of the natural numbers with full induction as starting point for predicativity has, however, not gone unchallenged. Different forms of predicativity have been proposed in the relevant literature. For example, systems as Martin-Löf type theory and Aczel and Myhill constructive set theory embody a particularly generous notion of predicativity, and combine it with the rejection of the principle of excluded middle. The thought here is that compliance with intuitionistic logic makes predicatively legitimate constructions, as generalised inductive definitions, which are problematic from a classical predicativist perspective.[a] Another variant of predicativity instead preserves the adherence to classical logic, but restricts rather than extends the domain of predicatively justifiable mathematics. This variant of predicativity has been called *strict predicativity* [68] and arises from a criticism of the natural numbers from a predicativist perspective. An analysis of these variants of predicativity and their respective relations is beyond the remits of this note. Here I only briefly discuss strict predicativity with the aim of placing predicativity given the natural numbers under perspective.

Edward Nelson [69] has proposed a form of predicative arithmetic which imposes severe restrictions on the principle of induction. Nelson's principal motivation for his predicative arithmetic is the complaint that the natural numbers hide a form of circularity. Charles Parsons has also argued that impredicativity already makes its way at the level of the natural numbers, and that induction is the culprit [67, 68]. Nelson and Parsons' criticism of induction bears similarities with the objection to the impredicativity of the logicist definition of natural numbers which was discussed above, since it highlights a circularity in the definition of natural number. Nelson's complaint with induction is put forth in a dense paragraph at the beginning of [69], whose interpretation is complex. In the following, I propose a possible way of arguing for the impredicativity of induction which is inspired by [69] and [67].

The logicist definition of natural number is impredicative as it has a second order quantifier at the start. It is therefore problematic from a predicativist perspective. A more promising definition of natural number is its inductive specification: a natural number is either 0, or the successor of a natural number, and nothing else. Here the closure condition is given by the induction principle. Induction is expressed as follows in Peano Arithmetic:

$$[\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(Suc(x)))] \to \forall x \varphi(x),$$

where φ is an *arbitrary* formula in the language of PA, and $Suc(x)$ is the successor of x. First of all, one claims that the principle of induction plays a fundamental role in clarifying what are the natural numbers. This point was already stressed by Poincaré and Weyl: the principle of induction is a crucial component of the natural number structure. One could also say that induction is required to determine the extension of the natural number

[a]The proof-theoretic strength of theories of inductive definitions is well above the Kreisel-Feferman-Schütte limit of predicativity mentioned above [66]. See [12, 36, 67] for discussion.

concept (i.e. what falls under that concept). Secondly, one observes that in the induction principle, the induction formula, φ, is arbitrary, so that it might be instantiated by a formula with unrestricted (number) quantifiers. Now a difficulty arises as follows: in order to make sense of an instance of induction with an unrestricted universal quantifier ranging over the natural numbers, we would seem to require a prior clarification of what falls under the natural number concept. That is, it would seem that we need first to have a specification of what belongs to the natural number concept before we can make sense of a statement of the form "for all natural numbers ...". However, if induction plays an essential role in determining the extension of the natural number concept, we end up with a vicious circle: we need induction to clarify what belongs to the natural numbers, but we need to already know what the natural numbers are, in order to make sense of crucial instances of induction.

Nelson's rejection of the induction principle on the grounds of circularity leads him to justify only weak subsystems of Peano Arithmetic. Interpretability in a fragment of primitive recursive arithmetic, Robinson's system Q, seems to be the main criteria Nelson adopts for assessing the predicativity of a formal system. Nelson predicative arithmetic therefore lays within the realm of bounded arithmetic [70].[b]

6. Conclusion

Strict predicativity is philosophically interesting as it suggests that from a thoroughly predicativist point of view already the theory of Peano Arithmetic, with its unrestricted induction, may be problematic. In other terms, Nelsons' argument, if granted, would imply that if we were to appeal to predicativity as a way of avoiding all forms of vicious circularity, we would have to impose restrictions already on the principle of induction. This suggests that a predicativist given the natural numbers would have to offer suitable argumentation for the exemption of the natural numbers from predicativity constraints.[a]

The comparison between strict predicativity and predicativity given the natural numbers is particularly significant also from a perspective that does not attempt to argue for predicativism. From this "external" perspective, predicativity may be viewed as a notion that may be applied to a num-

[b]See [71] for examples of systems which would seem to conform with Nelson's views.
[a] [15] offers a possible strategy.

ber of different contexts.[b] In general, predicativity imposes a requirement on domains of quantification: to be specifiable without vicious circularity. When spelling out the notion of predicative domain of quantification, we might choose, however, different initial assumptions. In the case of strict predicativity we only take a bare minimum. This could be expressed in terms of an initial element, say 0 and a successor operation. We then see how far we can go in building further mathematical constructions from these two initial "ingredients" and the usual first order logical operations. In the case of predicativity given the natural numbers, instead, we take as starting point not only 0 and successor, but also the principle of mathematical induction (unrestricted) and start building new sets from that. In all cases, once a certain initial "base" has been granted (or agreed on), the aim is to proceed without incurring in vicious circularity as far as we can, in predicatively justified ways. The crucial point is that with respect to a certain initial "base", sets are defined explicitly by application of a fixed set of simple operations. We could also say that sets which are so specified are predicative, relative to a certain "base".[c] When it is so formulated, predicativity may become a useful instrument in the philosophy of mathematics, as it may help clarify the assumptions underlying a number of philosophical positions.

I conclude with a remark. During the Autumn School, Professor Schwichtenberg organised a discussion on predicativity and raised a number of stimulating questions, which also go in the direction of broadening standard interpretations of predicativity. The notion of predicativity is customarily seen as pertaining primarily to the concept of set. In addition, the concept of set is often taken as prior to that of function, due to our familiarity with theories as ZFC, in which functions are codified as particular sets: graphs. However, it would be important to assess whether a comparison with other foundational contexts in which functions, possibly partial, are primitive, would help furthering our understanding of predicativity.[d] In fact, the early discussions on predicativity were typically framed within complex contexts, as the underlying systems displayed forms of intensionality or assumed a primitive concept of function. It would seem that a thorough discussion of predicativity would benefit from a deeper analysis of issues of identity and a comparison between different approaches to the notion of function, including partiality.

[b] See also [12, 26, 72].
[c] See also [11].
[d] See e.g. [73]. See also the notion of predicativity in Martin-Löf type theory.

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