# PREDICATIVITY AND FEFERMAN

LAURA CROSILLA

*Dedicated to Professor Solomon Feferman*

Abstract. Predicativity is a notable example of fruitful interaction between philosophy and mathematical logic. It originated at the beginning of the 20th century from methodological and philosophical reflections on a changing concept of set. A clarification of this notion has prompted the development of fundamental new technical instruments, from Russell's type theory to an important chapter in proof theory, which saw the decisive involvement of Kreisel, Feferman and Schütte. The technical outcomes of predicativity have since taken a life of their own, but have also produced a deeper understanding of the notion of predicativity, therefore witnessing the "light logic throws on problems in the foundations of mathematics." (Feferman 1998, p. vii) Predicativity has been at the center of a considerable part of Feferman's work: over the years he has explored alternative ways of explicating and analyzing this notion and has shown that predicative mathematics extends much further than expected within ordinary mathematics. The aim of this note is to outline the principal features of predicativity, from its original motivations at the start of the past century to its logical analysis in the 1950-60's. The hope is to convey why predicativity is a fascinating subject, which has attracted Feferman's attention over the years.

## 1. Introduction

The distinction between predicative and impredicative definitions has its origins in the writings of Poincaré and Russell and was instigated by the discovery of the set-theoretic paradoxes.[1] According to one characterization of (im)predicativity, a definition is impredicative if it defines an entity by reference to a totality to which the entity itself belongs, and it is predicative

This is a manuscript version of an article to appear in *Feferman on Foundations: Logic, Mathematics, Philosophy*, G. Jäger and W. Sieg (eds.), forthcoming in the series "Outstanding Contributions to Logic", Springer. I thank Andrea Cantini for helpful comments on a draft of this article. I also profited from remarks by Gerhard Jäger and Wilfried Sieg, as well as two anonymous referees. I gratefully acknowledge funding from the School of Philosophy, Religion and History of Science of the University of Leeds and from the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement n. 312938.

[1] See, for example, (Poincaré 1905, Poincaré 1906*a*, Poincaré 1906*b*, Russell 1906*b*, Russell 1906*a*, Russell 1908, Poincaré 1909, Poincaré 1912).

otherwise. Adherence to predicativity was proposed as a way of avoiding vicious circularity in definitions and resulted in the creation, by Russell, of ramified type theory; it also motivated a first development of a predicative form of analysis by Weyl.[2] A new phase for predicativity began in the 1950's, with a logical analysis of predicativity which employed state-of-the-art logical machinery for its analysis. That work culminated with an important chapter in proof theory whose principal outcome was the determination of the limit of predicativity by means of ordinal analysis.[3] In addition, Feferman's work and the so-called "Reverse Mathematics programme" have since clarified that large portions of everyday mathematics can be already carried out in predicative settings.[4]

Feferman's engagement with predicativity extends well beyond his celebrated contributions to the determination of the proof-theoretic limit of predicativity, as over the years he has explored alternative ways of explicating and analyzing this notion, as well as assessing the reach of predicative mathematics. This had two principal purposes: to offer further support for the original logical analysis of predicativity and to highlight the significance of predicative mathematics, both within mathematical logic and ordinary mathematics, with a particular attention to scientifically applicable mathematics. Feferman has also offered unrivalled expositions of predicativity.[5]

One of the difficulties in writing on predicativity is what might be called a lack of consensus on this notion. The early writings on predicativity by Poincaré, Russell and Weyl at the turn of the 20th century are rich of stimulating ideas, and deserve further scrutiny, however, to a contemporary technically-trained eye they often appear as insufficiently clear, opening up the way for a number of possible interpretations of predicativity.[6] The subsequent logical analysis of predicativity of the 1950-60's shed light on important aspects of predicativity, employing an array of precise logical instruments that were unavailable at the beginning of the past century. Notwithstanding that fundamental work, predicativity still raises complex

[2]See (Russell 1903, Russell 1908, Whitehead & Russell 1910, 1912, 1913, Weyl 1918).

[3]See (Kreisel 1958, Feferman 1964, Schütte 1965*b*, Schütte 1965*a*).

[4]See e.g. (Feferman 1988*b*, Simpson 1988, Simpson 1999, Feferman 2004*b*, Feferman 2013).

[5]See, for example, (Feferman 2005), and (Feferman 1964, Feferman 1987, Feferman 1988*a*, Feferman 1988*b*, Feferman 1993*a*, Feferman 1993*b*, Feferman 1996, Feferman 1998, Feferman 2000, Feferman 2004*b*).

[6]Feferman (2005) writes: "Though early discussions are often muddy on the concepts and their employment, in a number of important respects they set the stage for the further developments, and so I shall give them special attention."

questions from historical, philosophical and technical perspectives.[7] In addition, further difficulties are induced by the emergence over the years of a *plurality* of forms *of predicativity*, some within a classical and some within an intuitionistic context.[8]

The aim of this note is to offer an outlook of predicativity, sketching the most important (and hopefully less controversial) features of this notion, the original motivations, as well as the logical work that was inspired by the desire to clarify it. In particular, I shall focus on the *classical* form of predicativity that goes under the name of *predicativity given the natural numbers*; this has been extensively studied mathematically and is at the heart of Feferman's work.[9] I hope to convey why predicativity is a fascinating subject, which has attracted Feferman's attention over the years, and why it is an area of research offering the potential to substantially enrich today's philosophy of mathematics. Like Feferman, I also think that an account of predicativity ought to begin from the early discussions on predicativity, which clarify how we arrived at the notion of predicativity given the natural numbers and its logical analysis.

## 2. Predicativity: the origins

The early debates on predicativity were prompted by the discovery of the set theoretic paradoxes, which gained particular attention after Russell's famous letter to Frege in 1902. The general context of the early discussions on predicativity are renowned reflections by prominent mathematicians of the time on new concepts and methods of proof, which had emerged in mathematics from the nineteenth century.[10] These debates are well-known as they gave rise to influential foundational programs as logicism, formalism and intuitionism. In the case of Poincaré and Weyl's writings on predicativity, one finds severe criticism of the new methodology, and, especially, of the concept

[7]In his introduction to a chapter on the ordinal analysis of predicativity, Pohlers (2009, p. 134) writes: "The notion of predicativity is still controversial. Therefore we define and discuss here predicativity in a pure mathematical – and perhaps oversimplified – setting." See (Kreisel 1960, Kreisel 1970, Feferman 1979, Howard 1996, Weaver 2005) for discussions pertaining to the logical analysis of predicativity. See also the discussion on "metapredicativity" in (Jäger 2005). As to the philosophical and historical aspects of predicativity, see e.g. (Parsons 1992, Feferman & Hellman 1995, Mancosu 1998, Parsons 2002, Hellman 2004, Feferman 2004*b*, Parsons 2008).

[8]Predicativity-related themes have appeared in different forms over the years, both in classical and constructive settings. In fact, predicativity is gaining renewed prominence today especially in the constructive context. I shall postpone to another occasion a discussion of other forms of predicativity, as the constructive predicativity which characterizes Martin-Löf type theory (see e.g. Martin-Löf 1975, Martin-Löf 1984, Martin-Löf 2008) and forms of "strict predicativity" (Nelson 1986, Parsons 1992, Parsons 2008). See also (Crosilla 2014, Crosilla 2015).

[9]In the following, I shall also write "predicativity" to denote predicativity given the natural numbers. See section 3.4 for some remarks on the notion of predicativity given the natural numbers.

[10]See e.g. (Stein 1988, Wang 1954).

of arbitrary set which emerged from Cantorian set theory.[11] Adherence to predicativity offered a way of securing a safe concept of set, one that is not prey to the set-theoretic paradoxes and also avoids the arbitrariness implicit in the new concept of set.

The term predicativity itself emerged in an animated discussion between Poincaré and Russell which spanned from 1905 to 1912. Notwithstanding the remarkably different views of Poincaré and Russell, for instance, on the role of formalization within mathematics, they both converged on holding impredicative definitions the cause of the onset of the paradoxes, and attempted to clarify a notion of predicativity, adherence to which would avoid inconsistencies. Through Russell and Poincaré's confrontation a number of ways of capturing impredicativity and explaining its perceived problematic character emerged.

A first observation was that paradoxes as, for example, that of *Russell's class* of all those classes that are not members of themselves, typically display a form of *vicious circularity*.[12] In modern terminology we may define

$$R = \{x \mid x \notin x\}$$

by application of the *Naive Comprehension schema*: given any formula $\varphi$ in the language of set theory, we form the class of all the $x$'s that satisfy $\varphi$, that is, $\{x \mid \varphi(x)\}$. Then we have that $R \in R$ if and only if $R \notin R$. A circularity arises here as $R$'s definition *refers* to the class of all classes, to which $R$ itself is supposed to belong.

Observations along similar lines gave rise to a characterization of impredicativity as follows: *a definition is impredicative if it defines an entity by reference to a totality to which the entity itself belongs.*[13] In particular, a definition is impredicative if it defines an entity by quantifying over a totality which includes the entity to be defined. A definition is then *predicative* if it is not impredicative. Given this notion of impredicative definition, one may call an entity (e.g. a class) impredicative if it can *only* be defined by an

---

[11]See, in particular, (Poincaré 1912, Weyl 1918). See also (Bernays 1935, Maddy 1997, Ferreirós 2011) for a discussion of arbitrary sets and (Parsons 2002) for an analysis of Weyl's conception of predicative set and its reception by Weyl's contemporaries.

[12]See (Poincaré 1905, Poincaré 1906*b*, Poincaré 1906*a*, Russell 1906*b*, Russell 1908). Note that the term "class" is used here as in Russell and Poincaré's texts, that is, to refer to a generic collection. Hence it should be carefully distinguished from the notion of proper class that is found in contemporary set theory. In the original literature one frequently finds also the word "totality". In this section I shall try to avoid the use of the term "set", since the latter has in the meantime acquired additional connotations (as set in e.g. ZFC) that should not be presupposed in this discussion.

[13]See (Poincaré 1905, Poincaré 1906*a*, Poincaré 1906*b*, Russell 1906*b*, Russell 1906*a*). See e.g. (Gödel 1944, p. 455) for discussion. Note also that today the distinction between predicative and impredicative definitions is typically framed as relating to sets. However, Russell and Poincaré's discussions are concerned with definitions of different kinds of entities, including propositions, properties, etc.

impredicative definition.[14] Russell famously introduced his *"Vicious–Circle Principle" (VCP)* to ban impredicative definitions. This had a number of formulations, like, for example: *"no totality can contain members defined in terms of itself"* (Russell 1908, p. 237). Another is to be found in (Russell 1973, p. 198):

> [...] whatever in any way concerns *all* or *any* or *some* of a class must not be itself one of the members of a class.

The latter formulation forbids definitions which quantify over a totality to which the definiendum belongs, and is at the heart of Russell's implementation of the VCP in his type theory (see section 2.1).

Two examples may better clarify the notion of impredicative definition and its perceived difficulties; the first one is given by the logicist definition of natural number, and the second by the Liar paradox. The first example is significant not only because it is of central importance for the logicist project pursued by Russell, but as it clarifies that the discussion on impredicativity, which originated from an analysis of paradoxes of various kind, extended quickly beyond the case of the paradoxes. Let

$$N(n) := \forall F[F(0) \wedge \forall x(F(x) \rightarrow F(Suc(x))) \rightarrow F(n)].$$

According to this definition, the concept of natural number is defined by reference to all properties $F$ of the natural numbers. A circularity arises here as the property $N$ itself is within the range of the first quantifier. As a consequence, $N$ is defined by reference to itself. The difficulty with this definition is typically explained as follows:[15] suppose we wish to determine whether the predicate $N$ holds for a specific natural number, say 3. It would seem that we need to check for *every* property of the natural numbers, $F$, whether $F$ holds of 3, that is, whether:
$\forall F[F(0) \wedge \forall x(F(x) \rightarrow F(Suc(x))) \rightarrow F(3)]$.

However, the property "to be a natural number", which is expressed by the predicate $N$, is one of the properties of the natural numbers. That is, to find out whether $N(3)$ holds, we need to be able to clarify whether the following holds:

$$N(0) \wedge \forall x(N(x) \rightarrow N(Suc(x))) \rightarrow N(3).$$

Therefore it would seem that we need to determine whether $N(3)$ holds prior to determining whether $N(3)$ holds.[16]

---

[14] The issue of how we establish whether an entity is impredicative (and in which context) is more complex than this coarse characterization of impredicativity may suggest. This complexity was further addressed by the development of Russell's type theory, Weyl's (1918) and the logical analysis of predicativity to be discussed below.

[15] Here I shall follow (Carnap 1931, p 48).

[16] Carnap (1931, p. 48) concludes that this definition of natural number is "circular and useless". It is worth recalling that (Carnap 1931) also hints at a form of platonism, attributed to Ramsey (but not endorsed by Carnap), which finds no fault with impredicative definitions. See also (Poincaré 1912, Gödel 1944) for further discussion.

Let us consider another example of impredicative definition that is discussed by Russell (1908): the Liar paradox. In this case it is instructive to see how Russell himself analyzed the paradox. Russell first of all observes that the sentence "I'm lying" is the same as: "There is a proposition which I am affirming and which is false." He also notices that this, in turn, can be rephrased as: "It is not true for all propositions $p$ that if I affirm $p$, $p$ is true." He then concludes that "[t]he paradox results from regarding this statement as affirming a proposition, which must therefore come within the scope of the statement." Russell's conclusion is that the notion of *all* propositions is *illegitimate*, "for otherwise, there must be propositions (such as the above) which are about all propositions, and yet can not, without contradiction, be included among the propositions they are about." In fact, Russell further claims that "[w]hatever we suppose to be the totality of propositions, statements about this totality generate new propositions which, on pain of contradiction, must lie outside the totality." The worry here is that an impredicative definition of an entity (e.g. a proposition) would seem to *generate* a new element of the very class that was employed to define it. As a consequence, "there must be no totality of propositions", and statements such as "all propositions" must be *meaningless.*

A second characterization of predicativity emerged from Poincaré's renewed analysis in (Poincaré 1909, Poincaré 1912). Central to this characterization is the thought that an impredicative definition seems to *generate* new elements of a class which is used (e.g. as a domain of quantification) within that definition. Here Poincaré's criticism of impredicativity is deeply interrelated with a reflection on infinity and the role of definitions in mathematics. For the French mathematician a *definition* is a *classification*: it separates the objects which satisfy, from those which do not satisfy that definition, and it arranges them in two distinct classes. Poincaré also highlights a sort of incompleteness of infinite classes: they are open-ended and unfinished, so that definitions which refer to their totality might become problematic. For example, in the case of the definition of Russell's class, $R$, above, it would seem that we need first of all to *fix* the class of all classes, say $C$, *prior* to defining $R$ by reference to $C$. But then the definition of $R$ would seem to extend $C$ by a new class, $R$ itself. And this process may be repeated at will.[17]

Poincaré's discussion hints towards a distinction between predicative and impredicative classes that appeals to a form of "invariance" of predicative definitions: *a predicative classification is one that can not be "disordered" by the introduction of new elements.* This gives rise to a new characterization of predicativity which does not directly appeal to circularity, and can be so

[17]Poincaré's texts make use of other examples, more directly drawn from the mathematical practice. See also (Dummett 1991, Dummett 1993) for a similar reading of Russell's paradox. (Cantini 1981) proposes a detailed analysis of Poincaré's ideas.

expressed in modern terminology: *a definition is predicative if the class it defines is invariant under extension.*[18]

Poincaré (1909, p. 463) writes:

> Hence a distinction between two species of classifications, which are applicable to the elements of infinite collections; the *predicative* classifications, which can not be disordered by the introduction of new elements; the *non predicative* classifications which are forced to remain without end by the introduction of new elements.[19]

For Poincaré impredicative definitions are problematic as they treat as completed (French "arrêté") infinite classes which are instead "in fieri", open-ended or incomplete by their very nature. Predicative definitions, instead, guarantee that the classes so defined are stable or invariant. Poincaré does not spell out this notion of invariance in any detail, being very critical of formal endeavors; however, he indicates that the relations between the elements of the class and the class itself should not admit of change as we progress introducing new elements through our definitions. He also points towards a kind of genetic construction of predicative classes, which are built up from some initial elements step by step: we construct new elements of a predicative class by defining them in terms of the initial elements, we then define further new elements from the latter, and so on. In the case of infinite classes, this process is without end. A related but more precise account of a predicative conception of set is to be found in (Weyl 1918), as further discussed in section 2.2.

2.1. **Russell's way out.** The analysis of the paradoxes and their relation with impredicativity turned out to be extremely fruitful for the development of mathematical logic, starting from Russell's own implementation of the vicious circle principle through his type theory.[20] Russell's way out from the paradoxes is well-known, as it introduced a regimentation of classes through a hierarchy of types and orders.[21] For Russell the paradoxes were due to the assumption that any propositional function gives rise to a class: the class of all the objects that satisfy it.[22] As discussed above, of particular concern

[18] See (Kreisel 1960) for discussion of this characterization from a modern logical perspective. See also (Feferman & Kreisel 1966, Feferman 1968*a*).

[19] My translation; italics by Poincaré. The word "disordered" translates the French "bouleverseé".

[20] See (Cantini 2009) for a rich discussion of the impact of the paradoxes on mathematical logic.

[21] Russell's ideas on type theory appeared first in an appendix to (Russell 1903), and were further developed (with ramification) in (Russell 1908) and then in (Whitehead & Russell 1910, 1912, 1913).

[22] In the present context we may follow (Feferman 2005), and identify the notion of propositional function with that of open formula, i.e. a formula with a free variable, say $\varphi(x)$. Note, however, that the interpretation of the notions of proposition and propositional function in Russell is complex. See e.g. (Linsky 1988).

were classes defined impredicatively. To avoid impredicativity, in setting up ramified type theory Russell (1908) made two distinguishable moves as follows. The first move amounts to associating a *range of significance* to each propositional function, that is, a collection of all arguments to which the propositional function can be meaningfully applied. In Russell's terms: "within this range of arguments, the function is true or false; outside this range, it is nonsense." (Russell 1908, p. 247) The ranges of significance then form types, and these are arranged in **levels**: first we have a type of individuals, and then types which are ranges of significance of propositional functions defined on the individuals, and so on. The crucial point is that as a consequence of this regimentation of classes, expressions such as $x \in x$ and $x \notin x$ are simply ill–formed, since in $z \in w$, $w$ must be of the next-higher level than $z$. Accordingly, Russell's paradox (and other set-theoretic paradoxes) do not carry through.

It was subsequently realized by Chwistek and Ramsey that if one implements only this restriction, then one obtains a formalism that is interesting in its own right.[23] Today this goes under the name of simple type theory and its formulation was subsequently simplified by Church (1940). Simple type theory seems sufficient to block all set theoretic paradoxes; however, it does not eliminate all impredicativity. The second move, ramification, has the effect of eliminating all impredicativity.[24] As discussed above, for Russell one of the lessons of the paradoxes was that impredicative totalities, as, for example, the totality of all propositions, are illegitimate; hence quantification over them makes no sense. He therefore introduced, alongside a notion of *level* for ranges of significance of propositional functions, a notion of **order** for propositional functions, and required that *a propositional function can only quantify over propositional functions of lower order than its own.* Thus in ramified type theory, one has *first order* propositional functions, *second order* ones, etc.; in addition, the second order propositional functions can quantify on the first order ones, but not on propositional functions of order higher than one, and so on.

In this way one apparently blocks not only the set theoretic paradoxes, but semantic paradoxes as the Liar, too. This is analyzed as follows by Russell (1908, p. 238):

> if Epimenides asserts "all first-order propositions affirmed by me are false", then he asserts a second order proposition; he may assert this truly, without asserting truly any first order proposition, and thus no contradiction arises.

While ramified type theory fully complies with predicativity, it also turns out to make the development of mathematics awkward. This may be seen by considering again the definition of natural number discussed above, which

[23]See (Chwistek 1922, Ramsey 1926).

[24]See (Hodes 2015) for a discussion of the reasons that might support Russell's (and Whitehead's) choice of a ramified type theory over a simple type theory.

requires a universal quantifier over properties of the natural numbers. When appropriately re-formulated in a ramified context, this definition gives rise to only partial renderings of the notion of natural number, one for each order of propositional functions, and therefore it does not offer a general definition of the concept of natural number. As a consequence, many proofs by induction do not carry through in their usual form, as they would require the full generality of universally quantified statements; for example, in ramified type theory we can not prove in full generality that if $m$ and $n$ are finite numbers, so is $m+n$.[25] These difficulties prompted Russell (1908) to introduce the axiom of reducibility, which, however, has the effect of reinstating impredicativity. Reducibility is so presented in (Russell 1908, p. 242-3): "every propositional function is equivalent, for all its values, to some predicative function", where a function $\varphi$ of one argument $x$ is predicative if it is "of the order next above $x$".[26] This axiom was strenuously criticized for being introduced for purely pragmatic reasons and for being ad hoc.[27] For example, Weyl (1949, p. 50) wrote:

> Russell, in order to extricate himself from the affair, causes reason to commit harikari, by postulating the above assertion [the axiom of reducibility] in spite of its lack of support by any evidence.

2.2. **Das Kontinuum.** With (Weyl 1918) we have another approach to predicativity which also played a significant role for the subsequent logical analysis of predicativity, and especially Feferman's work. Weyl's (1918) aim was to develop a predicative form of analysis, founded on a concept of set which is immune from paradoxes and vicious circularity. Weyl's concern was that impredicativity affected not only set theory in general, but it was to be found already at the heart of analysis, as the Least Upper Bound

[25]See (Russell 1908). See also (Myhill 1974). See (Feferman 2000, Coquand 2015) for introductory expositions of the ideas underlying ramified type theory and the difficulties it encounters.

[26]Russell (1908, p. 243) also writes: "Thus a predicative function of an individual is a first-order function; and for higher types of arguments, predicative functions take the place that first-order functions take in respect of individuals. We assume, then, that every function is equivalent, for all its values, to some predicative function of the same argument."

[27]Wilfried Sieg has informed me about perceptive discussions by Hilbert and Bernays on predicativity and Russell's logicism, including the axiom of reducibility. See e.g. Hilbert's lecture notes from 1917/18 entitled "Prinzipien der Mathematik" and those from 1920 entitled "Probleme der mathematischen Logik" published in (Ewald & Sieg 2013). See also (Sieg 1999) for discussion. Sieg (1999) also draws important correlations between (Weyl 1918) and Hilbert and Bernays' work around 1920. As suggested by Sieg, the relations between Hilbert and Bernays' writings and Weyl's (1918) deserve more thorough investigations.

principle (LUB) requires impredicative reasoning.[28] One of Weyl's fundamental achievements was to show how to circumvent this difficulty without resorting to ramification or reducibility.[29] The result is a predicative (in fact, arithmetical) treatment of large portions of 19th century analysis.

Weyl (1918) expounds in detail a concept of *predicative set*: a set is the *extensional counterpart of a property* and may be seen as if it were constructed step by step from some primitive domain of objects by application of elementary operations over it. The "production" of sets from an initial domain is expressed first in full generality, and then specialized to the particular case of the natural numbers as starting domain, which is of relevance for the development of analysis. One begins with an initial domain (or basic category) of objects, and "certain individual, immediately exhibited 'primitive' properties" which apply to the objects of this domain (Weyl 1918, p. 28).[30] One then considers derived properties which arise from the primitive ones (as clarified below) and takes sets to be the extensional counterparts of primitive and derived properties. Weyl (1918, p. 20) writes: "to every primitive or derived property $P$ there corresponds a set (P)", the set of all the objects which have the property $P$. Crucially sets are identified extensionally, that is, "the same set corresponds to two such properties $P$ ad $P'$ if and only if every object (of our category) which has the property $P$ also has the property $P'$." (Weyl 1918, p. 20) The step from primitive properties to derived ones is discussed in the first section of "Das Kontinuum", where Weyl describes the formation of judgments.[31] The starting point is once more a given basic domain of objects and some primitive properties which apply to the objects of that domain. One then forms simple (i. e. atomic) judgments affirming that the primitive properties hold of the objects of the basic domain. The next step is given by taking combinations of these judgments by means of the ordinary logical operations, but with the crucial constraint that *quantifiers are only allowed to range over the basic domain.*[32] In this way one essentially obtains first-order definable properties of the objects of the initial domain; sets then arise as extensions of such properties (modulo extensionality). Weyl calls *"mathematical process"* (Weyl 1918, p. 22) the formation described above of a new "system" of sets from a basic initial domain and certain primitive properties of its objects.

[28]The (LUB) states that every bounded, non–empty subset $M$ of the real numbers has a least upper bound. See (Feferman 1964, Feferman 1988*b*) for discussion.

[29]Weyl, in particular, made use of sequential rather than Dedekind completeness, the first being amenable to predicative treatment. See also (Feferman 1988*b*).

[30]In addition to properties, Weyl (1918) also considers relations, here omitted for simplicity.

[31]The notion of "judgment" is so clarified by (Weyl 1918, p. 5): a "*judgment* affirms a *state of affairs*".

[32]Weyl also considers a principle of substitution (Weyl 1918, p. 10). In addition, in the paradigmatic case of the natural numbers as basic domain, one also applies a principle of iteration, as further discussed below.

A particularly important application of the mathematical process arises when the initial domain is the natural numbers. Here, from the contemporary logician's perspective, Weyl's concept of set gives rise to subsets of the natural numbers obtainable by application of the comprehension schema restricted to *arithmetical formulas*, that is, to those formulas that do not quantify over sets (but may quantify over natural numbers). This restriction to number quantifiers in the comprehension principle aims at preventing vicious–circular definitions of subsets of the natural numbers.[33]

An aspect of particular foundational interest is that Weyl, as Poincaré before him, takes the natural numbers with *full mathematical induction* as starting point, as intuitively given.[34] The comparison with Russell is instructive, as Russell aimed at a *definition* of the natural number concept, to witness its logical nature. Instead both Poincaré and Weyl criticized any attempt at founding the concept of natural number (and in particular the principle of mathematical induction) on logic or on the concept of set, given the fundamental role the natural numbers play within all of mathematics. Weyl (1918, p. 48) wrote: " *the idea of iteration, i.e., of the sequence of the natural numbers, is an ultimate foundation of mathematical thought, which can not be further reduced*". The natural numbers, for Weyl, are *"individuals"*, in the sense that they can be characterized uniquely by means of their properties: starting from an initial element, the iteration of the successor operation allows us to *characterize uniquely each natural number in elementary terms, and by exclusive appeal to its predecessors.* Weyl (1918, p. 15) also writes that "it is impossible for a number to be given otherwise than through its position in the number sequence, i.e. by indicating its characteristic property." This also justifies Weyl's adoption of bivalence for statements on the natural numbers and their assumption as paradigmatic initial domain for the mathematical process. The latter now gives rise to a system of sets as extensions of arithmetical properties of the natural numbers.

It is important to clarify why Weyl takes sets as extensions of first-order definable properties. Weyl (1918, p. 20) writes:

[33]As remarked by Feferman (1988*b*) (see also Feferman 2000) it is not completely clear how strong is the system Weyl sketches in (Weyl 1918). Feferman has, however, verified that system $W$ of (Feferman 1988*b*), which is inspired by (Weyl 1918), suffices to carry out all of Weyl's constructions in "Das Kontinuum". System $W$ is a conservative extension of Peano Arithmetic, PA (Feferman & Jäger 1993). As clarified in section 3.3, Feferman has also shown that $W$ allows for the development of a more extensive portion of contemporary analysis, compared with (Weyl 1918).

[34]Poincaré (see e.g. Poincaré 1906*b*) states that mathematical induction is synthetic a priori. Note also that Weyl expresses mathematical induction by appeal to a principle of iteration. See (Feferman 1998, p. 264-5) for discussion. Poincaré and Weyl fully realized the significance of the assumption of unrestricted mathematical induction. This is further clarified by a comparison with approaches to predicativity which instead introduce restrictions on induction (Nelson 1986, Parsons 1992).

> Finite sets can be described in two ways: either in *individual* terms, by exhibiting each of their elements, or in *general* terms, on the basis of a rule, i.e., by indicating properties which apply to the elements of the set and to no other objects. In the case of infinite sets, the first way is impossible (and this is the very essence of the infinite).

He also writes (Weyl 1918, p. 23):

> No one can describe an infinite set other than by indicating poperties which are characteristic of the elements of the set. And no one can establish a correspondence among infinitely many things without indicating a *rule*, i.e. a relation, which connects the corresponding objects with one another.

Weyl's "arithmetical" sets are the extensional counterparts of arithmetical rules or laws, and may be seen as if they were obtained through application of a fixed set of elementary operations starting from the natural numbers (with iteration). This is contrasted by Weyl with the concept of *arbitrary* set which had recently emerged within set theory, and which is characterized by the absence of any requirement of law or rule of formation. In his criticism of the concept of arbitrary set Weyl is once more in agreement with Poincaré, who drew a direct connection between the debate on impredicativity and the lack of explicit definability of impredicatively defined sets (Poincaré 1912). An arbitrary set for Weyl (1918, p. 23) is "a 'gathering' brought together by infinitely many individual arbitrary acts of selection, assembled and then surveyed as a whole by consciousness" and, as such, it is "nonsensical". (Weyl 1918) instead shows how predicative sets may be "produced" step by step from the safety of the natural numbers by application of a rule or a uniform condition.

Weyl's remarkable achievement in the second part of (Weyl 1918) was to show that his arithmetical concept of set suffices to develop a fundamental portion of 19th century analysis. (Weyl 1918) is also particularly interesting from a philosophical perspective, as it clearly puts forth a predicativist position: Weyl is adamant that what can not be predicatively accounted for, needs to be relinquished.

## 3. The logical analysis of predicativity given the natural numbers

Interest in predicativity declined after (Weyl 1918) for a number of reasons, like, for example, the realization that simple type theory was apparently sufficient to block the set-theoretic paradoxes.[35] In addition, the rapid accreditation of impredicative set theory as standard foundation, especially

[35]See (Chwistek 1922, Ramsey 1926).

in the form of the Zermelo-Fraenkel system with choice, ZFC, played a crucial role in the downfall of predicativity.[36]

The technical results obtained in (Russell 1908, Whitehead & Russell 1910, 1912, 1913, Weyl 1918), however, paved the way for subsequent work in mathematical logic which eventually gave rise to a new phase for predicativity starting in the 1950's: a *logical analysis of predicativity.* A crucial point to note is that the motivation prompting the new discussions on predicativity differed profoundly from the ones which had given rise to the first debates on predicativity outlined above. In this respect, already Gödel proposed a shift of attitude in his influential appraisal of Russell's contribution to mathematical logic in (Gödel 1944). There Gödel clearly expressed the view that predicativity is a fruitful concept which can give rise to mathematical progress, but that it should be pursued "independently of the question whether impredicative definitions are admissible."[37] Gödel's observations mark the beginning of a study of predicativity which, although of relevance for the philosophical debates on the foundations of mathematics, is carried out independently of predicativism; its principal aims are no more to secure the ultimate justification of (a portion of) mathematics, but to draw a clearer demarcation of the *boundary* between predicative and impredicative mathematics. We may distinguish two main objectives: (1) the determination of a theoretical *limit* of predicativity; and (2) the clarification of the *extent* of predicative mathematics.

It is important to recall the notion of predicativity that has been so investigated. This takes inspiration from Poincaré and, especially, Weyl's writings, and is characterized by the assumption, at the start, of the natural numbers with full mathematical induction.[38] For this reason it has been termed *"predicativity given the natural numbers"*, and, as in Weyl, it uses classical logic.[39] A difference with Weyl's approach is that the new logical analysis of predicativity also aims at exploring *how far* can we extend beyond the natural numbers in a predicatively justified way; it therefore focuses on a notion of predicativity given the natural numbers that stretches beyond Weyl's arithmetical predicativity. In fact, Russell's original idea of ramification and a distinctive use of ordinals (and ordinal notations) played a crucial role in setting out this form of predicativity.

3.1. **The limit of predicativity.** The literature from the 1950's and 1960's witnesses the complexity of the task of clarifying the limit of predicativity, which saw the involvement of a number of prominent logicians, as Feferman,

[36]See also (Feferman 2005) for additional thoughts on what "pushed predicativity to the sidelines."

[37]Gödel (1944) also mentions a prominent example of the fruitfulness of predicativity: the constructible hierarchy (Gödel 1938, Gödel 1940), inspired by Russell's idea of ramification.

[38]See also section 3.4 for more on the notion of predicativity given the natural numbers.

[39]The use of classical logic marks a crucial difference with the form of predicativity that is to be found in e.g. Martin-Löf type theory (Martin-Löf 1975).

Gandy, Kleene, Kreisel, Lorenzen, Myhill, Schütte, Spector and Wang. The first attempts at a logical analysis of predicativity focused on issues of definability of sets of natural numbers and highlighted a connection between predicativity and the recently developed concept of the hyperarithmetical hierarchy.[40] This consists of a hierarchy of sets of natural numbers which can be equivalently characterized in a number of ways. The simplest characterization is in terms of definability, and sees the hyperarithmetical sets as those sets of natural numbers that can be defined equivalently by a $\Sigma^1_1$ and by a $\Pi^1_1$ formulas, also called the $\Delta^1_1$ sets.[41] Given this characterization of the hyperarithmetical sets, the relation between the hyperarithmetical hierarchy and predicativity might at first seem problematic, as a hyperarithmetical set is defined by formulas with unrestricted set quantifiers. However, a more constructive rendering of the hyperarithmetical sets was given by Kleene in terms of iteration of the so-called Turing jump through the recursive ordinals.[42]

Another way of bringing the relation with predicativity to light is by drawing a correlation between the hyperarithmetical hierarchy and (a fragment of) the ramified analytic hierarchy. The latter essentially represents an implementation of Russell's idea of ramification to the particular case of second order arithmetic, now, however, with orders extending into the transfinite. The idea is to define a hierarchy analogous to Gödel's constructible hierarchy (Gödel 1938, Gödel 1940), but at successor steps to collect definable *subsets of the natural numbers*. More precisely, let $Def^2(X)$ be the set of all those $A \subseteq N$ such that $A$ is definable over $X$ in second order arithmetic, that is, there is a formula $\varphi(x)$ of second order arithmetic such that for all $n$, $n \in A \iff (\varphi(n))^X$. Here the notation $(\varphi(n))^X$ indicates that all second-order quantifiers in $\varphi$ range over $X$. Then we let $R_0 := \emptyset$, and $R_{\alpha+1} := Def^2(R_\alpha)$; at limit ordinal $\lambda$, we take $R_\lambda := \bigcup_{\xi<\lambda} R_\xi$. It is clear that the step from each level of the ramified analytic hierarchy to its successor is predicatively justified, as all second order quantifiers range over previous levels of the hierarchy. However, the ramified analytic hierarchy *as a whole* is problematic from a predicative perspective, since it presupposes

[40]The hyperarithmetical hierarchy has a central place in the development of mathematical logic because of its prominence within a number of fundamental areas in mathematical logic: definability theory, recursion theory and admissible set theory. This witnesses the centrality within logic of themes that pertain to the predicativity debate, and further explains the interest of this notion form a logical point of view.

[41]In the language of second order arithmetic, a $\Sigma^1_1$ formula is one of the form: $\exists X\, \varphi(X)$, with $\varphi$ an arithmetical formula, that is, a formula that does not quantify over sets (but may quantify over natural numbers). Note that here the upper case letter $X$ denotes a second order variable, standing for a set of natural numbers. A $\Pi^1_1$ formula is one of the form $\forall X\, \psi(X)$, with $\psi$ an arithmetical formula.

[42]See (Kleene 1959), see also (Sacks 1990). See (Kreisel 1960, Feferman 1964) for further clarification of why this may be seen as offering a predicative justification for this kind of second order quantification. Kreisel (1960) offers additional considerations that directly relate to Poincaré's notion of invariance discussed above.

the notion of "arbitrary" ordinal, i.e. "arbitrary" well-ordering relation, which, from a predicative perspective, is "as meaningless as the notion of 'arbitrary' set" (Feferman 1964, p. 9). To overcome this difficulty a first thought was to introduce a "proviso of autonomy" on the ordinals used as indexes of the hierarchy: each ordinal used is to be determined by a well-ordering relation of the natural numbers that, considered as a set of ordered pairs, is already admitted as predicative (Feferman 1964, p. 9). We may call the resulting (suitably specified) ordinals "predicatively definable ordinals" (Feferman 2005); then it turns out that by crucial results by Spector (1955) and Kleene (1959) the predicatively definable ordinals do not go beyond the recursive ordinals. In particular, Kleene (1959) showed that $R_{\omega_1^{CK}} = HYP$, where $\omega_1^{CK}$ is the first non-recursive ordinal. These results brought Kreisel (1960) to tentatively identify the predicatively definable sets with those definable within $R_{\omega_1^{CK}}$ and thus also with the hyperarithmetical sets.[43]

The proposed identification of the realm of predicativity with the hyperarithmetical hierarchy, however, turned out to rely on the assumption of the countable ordinals up to the first non-recursive ordinal, $\omega_1^{CK}$, along which to iterate the construction of the ramified analytic hierarchy. Feferman (2005) writes:

> Though the considerations leading to the identification of the predicative ordinals, resp. sets of natural numbers, with the recursive ordinals, resp. hyperarithmetical sets, have a certain plausibility, they ignored one crucial point if predicativity is only to take the natural numbers for granted as a completed totality, namely that they involve in an essential way [...] the impredicative notion of being a well-ordering relation.

For these reasons a new phase in the logical analysis of predicativity began, which was prompted by another suggestion by Kreisel (1958). Kreisel (1958) put forth a hierarchy of *formal systems* that would canonically represent predicative reasoning and called for the determination of its limit. A remarkable consequence of this new course of inquiry is that it shifted the focus of research from *definability* issues to *provability* issues.[44] The celebrated upshot of that research was the determination of the limit of predicativity by Feferman and Schütte (independently) (Feferman 1964, Schütte 1965*b*, Schütte 1965*a*) by means of proof-theoretic techniques. Russell's original idea of ramification had once more a crucial role, as a transfinite progression of systems of ramified second order arithmetic indexed by ordinals was introduced as a tool for determining a precise limit of predicativity by appeal

[43]See also (Wang 1954).

[44]See also the review by Gandy (1967).

to ordinal analysis.[45] The subsystems of second order arithmetic that make up the levels of the hierarchy, $RA_\alpha$, are characterized by principles of ramified comprehension which express closure under the appropriate ramified definitions and essentially give rise to a formal version of the conditions we saw for the ramified analytic hierarchy. Each level of the hierarchy, therefore, can be seen as predicatively justified, since quantification is suitably restricted to previous levels. Once more, a fundamental issue turned out to be how to specify the iteration that justifies the ascent to higher levels of the hierarchy. Here one introduces a suitable "boot-strapping" condition. A crucial difference with the previous attempts is in that the ordinals indexing the hierarchy are not only those that can be *defined* by well-ordering relations within the hierarchy, but those which can also be *proved* to be such relations at *previous stages* of the hierarchy. That is, one carefully introduces a notion of *predicatively provable ordinal*, which has the purpose of guaranteeing that one progresses up along the hierarchy to a stage $\alpha$ only if $\alpha$ has already been *recognized* as predicative, i.e. if at a previous stage of the hierarchy we have a proof that it is an ordinal. The fundamental contribution of Feferman and Schütte was to determine that the least non-predicatively provable ordinal is an ordinal known as $\Gamma_0$.[46] Therefore, in proof theory $\Gamma_0$ is often referred to as **the limit of predicativity**.

It is important to note that the limit of predicativity so determined is an "external limit". As clearly acknowledged by Feferman (see, e.g., Feferman 1964), one takes an impredicative stance and attempts to clarify the limit of predicativity from "the outside". The convinced predicativist will not recognize the limit $\Gamma_0$, as it lies beyond his reach, its very definition being impredicative. Gandy (1967) writes: "The role played by $\Gamma_0$ for predicative systems is closely analogous to that played by $\epsilon_0$ for finitist systems. $\Gamma_0$ is not a predicatively definable ordinal, but he who understands $\Gamma_0$ understands the consistency, the potentialities and the limitations of predicative proof." This once more clarifies the deep change in attitude between

[45]Here ordinals are not to be considered set-theoretically, rather as notations from a suitable ordinal notation system. See (Pohlers 2009) for details on ordinal notation systems.

[46]See e.g. (Pohlers 2009, Ch. 1) for details. In the branch of proof theory known as ordinal analysis, suitable (countable) ordinals, termed "proof-theoretic ordinals", are assigned to theories as a way of measuring their consistency strength and computational power. The "proof-theoretic strength" of a theory is then expressed in terms of such ordinals. The countable ordinal $\Gamma_0$ is the proof-theoretic ordinal assigned to the progression of ramified systems mentioned above. It is relatively small in proof-theoretic terms. As a way of comparison, it is well above the ordinal $\epsilon_0$ which encapsulates the proof-theoretic strength of Peano Arithmetic, but it is much smaller than the ordinal assigned to a well–known theory, called $ID_1$, of one inductive definition. The latter ordinal is known in the literature as the Bachmann–Howard ordinal (Buchholz, Feferman, Pohlers & Sieg 1981). The strength of $ID_1$ is well below that of second order arithmetic, which is in turn much weaker than full set theory. For surveys on proof theory and ordinal analysis see, for example, (Rathjen 1998, Rathjen 1999, Rathjen 2006).

the early discussions on predicativity and its logical analysis, as the latter is an attempt at understanding predicativity rather than arguing for it as a foundational stance.

3.2. **Proof-theoretic reducibility.** Feferman has explored a number of alternative perspectives over the years, as a way of corroborating the analysis of predicativity. In particular, one of the aims was to avoid direct appeal to the idea of ramification, which is particularly artificial and distant from mathematical practice. For example, already in the fundamental article (Feferman 1964), Feferman introduced a progression of formal systems which are based on a Hyperarithmetic Comprehension principle, therefore exploiting the previous observation of an important connection between predicative subsets of the natural numbers and hyperarithmetical sets. In that paper Feferman also introduced a single formal system $IR$ which does not make direct reference to provability or definability.[47] Feferman (1964) then established that also these two approaches give rise to $\Gamma_0$ as limit.[48] The fact that a number of distinct approaches to predicativity converged to the ordinal $\Gamma_0$ was then seen as confirmation of the thesis that $\Gamma_0$ marks the limit of predicativity, in a similar way as the convergence of different characterizations of computability are usually taken to support Church's thesis. The study of alternative routes to predicativity was also suggested by the desire to clarify which parts of mathematics can be given predicative form. As ramified systems are cumbersome to work in, one needs a way of assessing the predicativity of other systems which are better suited to the practical needs of a codification of ordinary mathematics. The notion of *proof–theoretic reducibility* was therefore appealed to for this purpose.[49] In order to assess the predicativity of a formal system $T$ it suffices to appropriately "translate" it in (that is, proof-theoretically reduce it to) one of the ramified systems. The latter, thus, act as *canonical systems of reference* in terms of which the predicativity of other systems can be assessed. The outcome is a notion of *predicative justification*: a formal system is considered predicatively justified if it is proof–theoretically reducible to a system of ramified second order arithmetic indexed by an ordinal less than $\Gamma_0$. In addition, a notion of *locally predicative justification* was introduced, which applies to the case in which a system $T$ is proof-theoretically reducible to the union of all the $RA_\alpha$. In this case each theorem in $T$ may be considered predicative, although the system $T$ in its whole is not predicatively justified.

[47]See (Howard 1996, p. 283) for discussion.

[48]Further approaches to predicativity were explored, for example, in (Feferman 1966, Feferman 1968*b*, Feferman 1974, Feferman 1979, Feferman 1982). See also (Feferman 1975). More recently, Feferman has developed the notion of "unfolding"; see (Feferman & Strahm 2000, Feferman & Strahm 2010) and (Strahm 2017). See also (Cantini & Fujimoto & Halbach 2017) for relations between Feferman's work on predicativity and theories of truth.

[49]See e.g. (Feferman 1993*b*) for discussion of this notion, and (Feferman 2005) for an informal account of its application to an analysis of predicativity.

A well-known locally predicatively justifiable system is Friedman's system $ATR_0$, which has been extensively studied in the Reverse Mathematics programme (Friedman 1976, Simpson 1999).

3.3. **Predicativity and ordinary mathematics: the extent of predicativity.** Weyl's aim in "Das Kontinuum" was to clarify how far can we proceed in developing analysis from the bare assumption of the natural numbers with full induction and by iterating elementary properties and relations over them. His work gives a first, partial answer, to the important question of the relation between predicative mathematics with ordinary, or everyday, mathematics.[50]

A number of mathematical logicians in the 1950's felt that the early debates on impredicativity had left unresolved the question of which role impredicativity plays within ordinary mathematics. Wang (1954, p. 244) clearly expressed this concern, when he observed that the use of uncountable (or indenumerable) and impredicative sets "remains a mystery which has shed little light on any problems of ordinary mathematics. There is no clear reason why mathematics could not dispense with impredicative or absolutely indenumerable sets."

In his fundamental article, Feferman (1964, p. 3-4) writes:

> It is well known that a number of algebraic and analytic arguments can be systematically recast into a form which can be subsumed under elementary (first order) number theory. [...] It is thus not at first sight inconceivable that predicative mathematics is already (formally) sufficient to obtain the full range of arithmetical consequences realized by impredicative mathematics.

As Feferman quickly clarifies, not every elementary statement can be so obtained. The logical analysis of predicativity in (Feferman 1964) readily provides us with a counterexample: the very arithmetical statement expressing the consistency of predicative analysis. However, Feferman suggests that one could argue that "*all mathematically interesting statements about the natural numbers, as well as many analytic statements, which have so far been obtained by impredicative methods can already be obtained by predicative ones*".

The fundamental question of whether predicative mathematics is "already (formally) sufficient to obtain the full range of arithmetical consequences realized by impredicative mathematics" has been addressed by combining an appeal to the notion of proof-theoretic reducibility (that enables us to

[50]The expression "ordinary mathematics" refers to mainstream mathematics, and has been so characterized, for example, by (Simpson 1999, p. 1): "that body of mathematics which is prior to or independent of the introduction of abstract set-theoretic concepts". That is: "geometry, number theory, calculus, differential equations, real and complex analysis, countable algebra, the topology of complete separable metric spaces, mathematical logic and computability theory".

work in syntactically convenient systems) with a careful *case by case* logical analysis of ordinary mathematics. Here Weyl's pioneering work in "Das Kontinuum" has been a fundamental reference, especially for Feferman's investigations (Feferman 1988*b*, Feferman 2005). More precisely, Feferman (1988*b*) has carefully analysed Weyl's text and proposed a system, W, which codifies in modern terms Weyl's system in "Das Kontinuum". System W is particularly weak proof-theoretically, as it is as strong as Peano Arithmetic; as a consequence, it lies well within predicative mathematics. Feferman has verified that large portions of contemporary analysis can be carried out on its basis, in fact "most of classical analysis and substantial portions of modern analysis" (Feferman 2013); therefore he has significantly improved on Weyl's (1918).

Another source of insight is the research carried out within Friedman and Simpson's program of Reverse Mathematics (Simpson 1999), which has analyzed large portions of ordinary mathematics from a logical point of view. The principal outcome of these studies is a further confirmation that large parts of ordinary mathematics can be framed within predicative systems.[51] More surprisingly, it typically turns out that if a theorem can be established predicatively, it can already be carried out within a system not stronger than Peano Arithmetic. In fact, a finitary system suffices for most cases.[52]

The outcome of this research is that, once analyzed in detail, the prima facie necessity of abstract features of ordinary mathematics turns out to be avoidable in many cases. As a consequence, a substantial portion of ordinary impredicative mathematics is eliminable in favor of predicative mathematics. This is a striking result, highly unexpected from the perspective of Weyl's contemporaries. In fact, as suggested by Feferman, these insights have the potential of enriching the philosophy of mathematics in a number of ways. For example, they may have an impact on current discussions on indispensability of mathematics to science. Feferman (1993*b*) has argued that the case can be made that all scientifically applicable mathematics can be codified by predicative theories (in fact, by system $W$). The above mentioned work has brought Feferman to formulate the following "working hypothesis":

*All of scientifically applicable analysis can be developed predicatively.*

If, indeed, weak predicative systems turned out to be formally sufficient to develop all of scientifically applicable mathematics, this would imply the dispensability, at least from a formal point of view, of impredicative mathematics - when we restrict consideration to the mathematics that is required by our best scientific theories. This could then imply that an appeal to indispensability arguments to support the belief in the existence of mathematical entities would only grant, in the most favorable case, a rather limited ontology. As a consequence the above research might contribute to a more

[51]See (Simpson 1999) for details and (Simpson 2002) for independence results.

[52]See (Feferman 1988*b*, Feferman 2005, Simpson 1988) for informal discussions and further references.

careful assessment of the possible outcomes of indispensability arguments, and the kind of platonism they might support, if they were to succeed.[53]

3.4. **Predicativity given the natural numbers.** I conclude with some remarks on the notion of predicativity given the natural numbers that is the focus of the logical analysis of predicativity. At first the logical analysis of predicativity aimed at clarifying the notion *predicatively definable from* the natural numbers. In section 3.1 I have emphasized that a shift of focus occurred at the end of the 1950's, so that predicativity given the natural numbers aimed at explicating a notion of *predicatively provable presupposing* the natural numbers (Gandy 1967). From a philosophical perspective, it is natural to wonder how we should read the presupposition of the natural numbers. As it turns out, the relevant literature provides us with a number of different answers to this question. As reviewed above, Weyl, for example, suggested that the natural numbers, and in particular the idea of iteration, are "an ultimate foundation of mathematical thought", in fact, a "pure" intuition (Weyl 1918, p. 48). In particular, the natural numbers are "individuals", classical logic applies to them and they can act as domain of quantification; therefore they can be employed for building predicative sets step-by-step by repeated application of elementary operations over them.

In Feferman's writings on predicativity one sometimes reads that the natural numbers (with full mathematical induction) can be regarded as "given" in some sense.[54] Sets are instead considered as no more than definitions, *façons de parler*, or convenient idealizations; as such they need to undergo appropriate constraints to avoid vicious circularity in definitions (Feferman 1964, p. 1-2).[55]

Sometimes the distinction between predicative and impredicative definitions or entities is presented in epistemological terms, and predicativity is seen as an instrument for clarifying what is implicit in our understanding of the natural numbers.[56] Feferman (1987, p. 449) writes:

[53]Note that Feferman draws different conclusions on the impact of the logical research on indispensability arguments in the philosophy of mathematics (Feferman 1993*b*).

[54]In the following I shall often omit explicit reference to the unrestricted principle of mathematical induction, and simply write "the natural numbers"; however, I shall presuppose that in the case of predicativity given the natural numbers full induction is also assumed.

[55]Feferman (Feferman 1964, Feferman 2005) also describes the predicativist position as one that takes the natural numbers as a "completed totality", and views the rest in potentialist terms. However, I could find no further elucidation of the notion of complete totality, beyond the claim that we can use classical logic to reason about it. In (Feferman 2004*a*, Feferman 2009), Feferman proposes to read the "giveness" of the natural numbers in terms of realism in truth value (restricted to the natural numbers). A fundamental theme that emerges within Feferman's discussions on predicativity is an opposition, analogous to Weyl's, to arbitrary sets, and in particular to the powerset of an infinite set (see e.g. Feferman 2004*b*).

[56]See e.g. (Kreisel 1958, Feferman 1996).

> That there is a fundamental difference between our understanding of the concept of natural numbers and our understanding of the set concept, even for sets of natural numbers, seems to me undeniable. The study of predicativity, as what is implicit in accepting the structure of natural numbers, is thus of special foundational significance. This is *not* to say that only what is predicative is 'justified'. What we are dealing with here are questions of relative conceptual clarity and foundational status [...].

The latter point is significant, and demonstrates, once more, the crucial difference between the attitude of the logicians that studied predicativity from the 1950's and that of the mathematicians that forged this notion at the turn of the 20th century, in particular Weyl (1918). Perhaps it also explains why there is insufficient clarity in the logical discussion on the philosophical aspects: the aim of the logician is not a defence of predicativism but a clarification of predicativity. In fact, the logician is primarily interested in clarifying the consequences of given assumptions. A crucial question is: which mathematical constructions and which portions of mathematics can we develop from the assumption of certain mathematical entities and operations over them? A conceptual clarification of the mathematical facts is then seen as prior to a clarification of the underlying philosophical stances that determine the choice of certain assumptions; Feferman (2005) writes: "[t]he potential value for philosophy then is to be able to say in sharper terms what arguments may be mounted for or against taking such a stance."

In fact, a number of authors, including Feferman, have suggested that the notion of predicativity is more profitably understood as a relative rather than an absolute notion: we analyze what is predicative given some prior assumption, as, for example, the natural numbers. One might take, however, different starting points. From this perspective Gödel's constructible hierarchy may also be framed as an example of predicativity, one which may be seen as reducing all kinds of impredicativity to one special kind: "the existence of certain large ordinal numbers (or well-ordered sets) and the validity of recursive reasoning for them" (Gödel 1944, p. 464). A weaker form of predicativity, compared with predicativity given the natural numbers, is instead obtained if one considers restrictions to the induction principle as in (Nelson 1986, Parsons 1992, Parsons 2008).

Predicativity may now become a tool for an analysis of mathematics, helping us distinguish different portions of the mathematical landscape, distinct for the assumptions and the methodology they require. In other terms, the logical analysis of (forms of) predicativity becomes an instrument for a finer understanding of contemporary mathematics, which addresses the question of which concepts and methods are necessary for the development of given portions of contemporary mathematics.[57]

## 4. Conclusion

The history of predicativity is witness to a remarkable example of cross-fertilization between philosophy of mathematics and mathematical logic. A critical reflection on the new abstract concepts and methods that were introduced in mathematics in the 19th century gave rise to proposals for the development of mathematics on predicative grounds. Adherence to predicativity was proposed as a way of avoiding vicious circularity in definitions and resulted in Russell's ramified type theory and Weyl's predicative analysis. A clarification of the notion of predicativity and its mathematical implications stimulated further technical advances, and saw the involvement of prominent logicians, especially in the 1950-60's. In particular, Feferman has contributed to the determination of the limit of a notion of predicativity given the natural numbers, and has attempted, over the years, new ways of explicating this notion of predicativity.

Beyond the purely logical interest of predicativity, this notion may play a role in the philosophy of mathematics. Compliance with predicativity requirements enables us to carve a restricted concept of set; in particular, in the case of predicativity given the natural numbers we have a concept of set that is deeply rooted in the natural numbers. This, in turn, may be used to assess which mathematical concepts and theories can be developed purely on the basis of this more constrained concept of set, and which ones instead require an essential appeal to more abstract and complex notions. The logical analysis of predicativity has particularly highlighted the crucial role for predicativity of two components: some initial entity, e.g. the natural number set with full mathematical induction, and the iteration of elementary operations over it. This opens up the way for a number of notions of predicativity, which may be employed to help us clarify the difference between distinguishable conceptual spheres of mathematical activity.

## References

Benacerraf, P. & Putnam, H. (1983), *Philosophy of Mathematics: Selected Readings*, Cambridge University Press.

Bernays, P. (1935), 'Sur the platonisme dans les mathématiques', *L'Enseignement mathématique* (34), 52–69. Translated in (Benacerraf & Putnam 1983) with the title: On Platonism in Mathematics.

Buchholz, W., Feferman, S., Pohlers, W. & Sieg, W. (1981), *Iterated inductive definitions and subsystems of analysis*, Springer, Berlin.

[57]Wilfried Sieg has drawn my attention to a passage in Hilbert's 1920's lectures (Ewald & Sieg 2013, p. 363-4) which suggests that this view of predicativity is in agreement with a Hilbertian perspective. It is also worth observing that with the shift of the logical analysis of predicativity to proof-theoretic considerations this enterprise gained a clear Hilbertian character.

Cantini, A. (1981), 'Una teoria della predicatività secondo Poincaré', *Rivista di Filosofia* **72**, 32–50.

Cantini, A. (2009), Paradoxes, self-reference and truth in the 20th century, *in* D. Gabbay, ed., 'The Handbook of the History of Logic', Elsevier, pp. 5–875.

Cantini, A. & Fujimoto, K. & Halbach F. (2017), 'Feferman and the Truth', this volume.

Carnap, R. (1931), 'Die Logizistische Grundlegung der Mathematik', *Erkenntnis* **2**(1), 91–105. Translated in (Benacerraf & Putnam 1983). (Page references are to the translation).

Church, A. (1940), 'A Formulation of the Simple Theory of Types', *Journal of Symbolic Logic* **5**, 56–68.

Chwistek, L. (1922), 'Über die Antinomien der Prinzipien der Mathematik', *Mathematische Zeitschrift* **14**, 236–43.

Coquand, T. (2015), Type theory, *in* E. N. Zalta, ed., 'The Stanford Encyclopedia of Philosophy', summer 2015 edn.

Crosilla, L. (2014), Constructive and intuitionistic ZF, *in* E. N. Zalta, ed., 'Stanford Encyclopedia of Philosophy'. http://plato.stanford.edu/entries/set-theory-constructive/.

Crosilla, L. (2015), Error and predicativity, *in* A. Beckmann, V. Mitrana & M. Soskova, eds, 'Evolving Computability', Vol. 9136 of *Lecture Notes in Computer Science*, Springer International Publishing, pp. 13–22.

Dummett, M. (1991), *Frege: Philosophy of Mathematics*, Cambridge MA, Harvard University Press.

Dummett, M. (1993), What is Mathematics About?, *in* A. George, ed., 'The Seas of Language', Oxford University Press, pp. 429–445. Reprinted in (Jacquette 2001), pp. 19–30.

Ewald, W. & Sieg, W. (2013), David Hilbert's Lectures on the Foundations of Arithmetic and Logic, 1917–1933, Heidelberg, Springer.

Feferman, S. (1964), 'Systems of predicative analysis', *Journal of Symbolic Logic* **29**, 1–30.

Feferman, S. (1966), 'Predicative provability in set theory', *Bull. Amer. Math. Soc.* **72**, 486–489.

Feferman, S. (1968*a*), 'Persistent and invariant formulas for outer extensions', *Compositio Math.* **20**, 29–52 (1968).

Feferman, S. (1968*b*), 'Systems of predicative analysis. II. Representations of ordinals', *J. Symbolic Logic* **33**, 193–220.

Feferman, S. (1974), Predicatively reducible systems of set theory, *in* 'Axiomatic set theory (Proc. Sympos. Pure Math., Vol. XIII, Part II, Univ. California, Los Angeles, Calif., 1967)', Amer. Math. Soc., Providence, R.I., pp. 11–32.

Feferman, S. (1975), Impredicativity of the existence of the largest divisible subgroup of an abelian $p$-group, *in* 'Model theory and algebra (A memorial tribute to Abraham Robinson)', Springer, Berlin, pp. 117–130. Lecture Notes in Math., Vol. 498.

Feferman, S. (1979), A more perspicuous formal system for predicativity, *in* M. Boffa, D. van Dalen & K. McAloon, eds, 'Logic Colloquium '78', North Holland, Amsterdam.

Feferman, S. (1982), Iterated inductive fixed-point theories: application to Hancock's conjecture, *in* 'Patras Logic Symposion (Patras, 1980)', Vol. 109 of *Stud. Logic Foundations Math.*, North-Holland, Amsterdam-New York, pp. 171–196.

Feferman, S. (1987), *Proof theory: a personal report*, in , pp. 447–485. Appendix to the Second edition of Takeuti (1987).

Feferman, S. (1988*a*), 'Hilbert's program relativized: Proof–theoretical and foundational reductions', *The Journal of Symbolic Logic* **53**(2), 364–384.

Feferman, S. (1988*b*), Weyl vindicated: Das Kontinuum seventy years later, *in* C. Cellucci & G. Sambin, eds, 'Temi e prospettive della logica e della scienza contemporanee', pp. 59–93.

Feferman, S. (1993*a*), What rests on what? The proof-theoretic analysis of mathematics, *in* 'Philosophy of Mathematics, Part I, Proceedings of the 15th International Wittgenstein Symposium', Verlag Hölder–Pichler–Tempsky, Vienna.

Feferman, S. (1993*b*), Why a little bit goes a long way. Logical foundations of scientifically applicable mathematics, *in* 'PSA 1992', Vol. 2, Philosophy of Science Association (East Lansing), pp. 442–455. Reprinted in (Feferman 1998).

Feferman, S. (1996), Kreisel's 'Unwinding Program', *in* P. Odifreddi, ed., 'Kreiseliana. About and Around Georg Kreisel', A K Peters, pp. 247–273.

Feferman, S. (1998), *In the Light of Logic*, Oxford University Press (New York).

Feferman, S. (2000), The significance of Hermann Weyl's *Das Kontinuum*, *in* V. Hendricks, S. A. Pedersen & K. F. Jørgense, eds, 'Proof Theory', Dordrecht, Kluwer.

Feferman, S. (2004*a*), 'Comments on "Predicativity as a philosophical position" by G. Hellman', *Review Internationale de Philosophie* **229**(3).

Feferman, S. (2004*b*), The development of programs for the foundations of mathematics in the first third of the 20th century, *in* S. Petruccioli, ed., 'Storia della scienza', Vol. 8, Istituto della Enciclopedia Italiana, pp. 112–121. Translated as: Le scuole di filosofia della matematica.

Feferman, S. (2005), Predicativity, *in* S. Shapiro, ed., 'Handbook of the Philosophy of Mathematics and Logic', Oxford University Press, Oxford.

Feferman, S. (2009), 'Conceptions of the continuum', *Intellectica* **51**, 169–189.

Feferman, S. (2013), Why a little bit goes a long way: predicative foundations of analysis. Unpublished notes dating from 1977-1981, with a new introduction. Retrieved from the address: https://math.stanford.edu/∼feferman/papers.html.

Feferman, S. & Hellman, G. (1995), 'Predicative foundations of arithmetic', *Journal of Philosophical Logic* **22**, 1–17.

Feferman, S. & Jäger, G. (1993), 'Systems of explicit mathematics with non-constructive $\mu$-operator, Part I', *Ann. Pure Appl. Logic* **65**(3), 243–263.

Feferman, S. & Kreisel, G. (1966), 'Persistent and invariant formulas relative to theories of higher order', *Bull. Amer. Math. Soc.* **72**, 480–485.

Feferman, S. & Strahm, T. (2000), 'The unfolding of non-finitist arithmetic', *Annals of Pure and Applied Logic* **104**(1-3), 75–96.

Feferman, S. & Strahm, T. (2010), 'Unfolding finitist arithmetic', *Review of Symbolic Logic* **3**(4), 665–689.

Ferreirós, J. (2011), 'On arbitrary sets and ZFC', *The Bulletin of Symbolic Logic* (3).

Friedman, H. (1976), 'Systems of second order arithmetic with restricted induction, I, II (abstracts)', *Journal of Symbolic Logic* **41**, 557–559.

Gandy, R. O. (1967), 'Review of (Feferman 1964)', *Mathematical Reviews* .

Gödel, K. (1938), 'The Consistency of the Axiom of Choice and of the Generalized Continuum-Hypothesis', *Proceedings of the National Academy of Sciences of the United States of America* **24**, 556–557.

Gödel, K. (1940), *The Consistency of the Axiom of Choice and of the Generalized Continuum-Hypothesis with the Axioms of Set Theory*, Vol. 17, Princeton University Press.

Gödel, K. (1944), Russell's mathematical logic, *in* P. A. Schlipp, ed., 'The philosophy of Bertrand Russell', Northwestern University, Evanston and Chicago, pp. 123–153. Reprinted in (Benacerraf & Putnam 1983). Page references are to the reprinting.

Hellman, G. (2004), 'Predicativism as a philosophical position', *Revue Internationale de Philosophie* **3**, 295–312.

Hodes, H. T. (2015), 'Why ramify?', *Notre Dame J. Formal Logic* **56**(2), 379–415.

Howard, W. A. (1996), Some Proof Theory in the 1960's, *in* P. Odifreddi, ed., 'Kreiseliana. About and Around Georg Kreisel', A K Peters, pp. 274–288.

Jacquette, D., ed. (2001), *Philosophy of Mathematics: An Anthology*, Wiley-Blackwell.

Jäger, G. (2005), Metapredicative and explicit Mahlo: a proof-theoretic perspective, *in* R. C. et al., ed., 'Proceedings of Logic Colloquium 2000', Vol. 19 of *Association of Symbolic Logic Lecture Notes in Logic*, AK Peters, pp. 272–293.

Kleene, S. C. (1959), 'Quantification of number-theoretic functions', *Compositio Mathematica* **14**, 23–40.

Kreisel, G. (1958), Ordinal logics and the characterization of informal concepts of proof, *in* 'Proceedings of the International Congress of Mathematicians (August 1958)', Gauthier–Villars, Paris, pp. 289–299.

Kreisel, G. (1960), 'La prédicativité', *Bulletin de la Societé Mathématique de France* **88**, 371–391.

Kreisel, G. (1970), Principles of proof and ordinals implicit in given concepts, *in* R. E. V. A. Kino, J. Myhill, ed., 'Intuitionism and Proof Theory', North-Holland, Amsterdam, pp. 489–516.

Linsky, B. (1988), 'Propositional functions and universals in principia mathematica', *Australasian Journal of Philosophy* **66**(4), 447–460.

Maddy, P. (1997), *Naturalism in Mathematics*, Oxford University Press.

Mancosu, P. (1998), *From Brouwer to Hilbert. The Debate on the Foundations of Mathematics in the 1920s*, Oxford: Oxford University Press.

Martin-Löf, P. (1975), An intuitionistic theory of types: predicative part, *in* H. E. Rose & J. C. Shepherdson, eds, 'Logic Colloquium 1973', North–Holland, Amsterdam.

Martin-Löf, P. (1984), *Intuitionistic Type Theory*, Bibliopolis, Naples.

Martin-Löf, P. (2008), The Hilbert–Brouwer controversy resolved?, *in* e. a. van Atten, ed., 'One Hundred Years of Intuitionism (1907 – 2007)', Publications des Archives Henri Poincaré , pp. 243–256.

Myhill, J. (1974), The undefinability of the set of natural numbers in the ramified principia, *in* G. Nakhnikian, ed., 'Bertrand Russell's Philosophy', London, Duckworth, pp. 19–27.

Nelson, E. (1986), *Predicative arithmetic*, Princeton University Press, Princeton.

Parsons, C. (1992), The impredicativity of induction, *in* M. Detlefsen, ed., 'Proof, Logic, and Formalization', Routledge, London, pp. 139–161.

Parsons, C. (2002), Realism and the debate on impredicativity, 1917–1944, Association for Symbolic Logic.

Parsons, C. (2008), *Mathematical Thought and Its Objects*, Cambridge University Press.

Pohlers, W. (2009), *Proof Theory: The First Step into Impredicativity*, Universitext, Springer Berlin Heidelberg.

Poincaré, H. (1905), 'Les mathématiques et la logique', *Revue de Métaphysique et Morale* **1**, 815–835.

Poincaré, H. (1906*a*), 'Les mathématiques et la logique', *Revue de Métaphysique et de Morale* **2**, 17–34.

Poincaré, H. (1906*b*), 'Les mathématiques et la logique', *Revue de Métaphysique et de Morale* **14**, 294–317.

Poincaré, H. (1909), 'La logique de l'infini', *Revue de Métaphysique et Morale* **17**, 461–482.

Poincaré, H. (1912), 'La logique de l'infini', *Scientia* **12**, 1–11.

Ramsey, F. P. (1926), 'Foundations of mathematics', *Proceedings of the London Mathematical Society* **25**. Reprinted in (Ramsey 1931).

Ramsey, F. P. (1931), *Foundations of Mathematics and Other Logical Essays*, Routledge and Kegan Paul.

Rathjen, M. (1998), The higher infinite in proof theory, *in* J. A. Makowsky & E. V. Ravve, eds, 'Logic Colloquium '95', Vol. 11 of *Springer Lecture Notes in Logic*, Springer, New York, Berlin.

Rathjen, M. (1999), The realm of ordinal analysis, *in* 'Sets and proofs (Leeds, 1997)', Vol. 258 of *London Math. Soc. Lecture Note Ser.*, Cambridge Univ. Press, Cambridge, pp. 219–279.

Rathjen, M. (2006), 'Theories and ordinals in proof theory', *Synthese* **148**(3), 719–743.

Russell, B. (1903), *Principles of Mathematics*, Routledge.

Russell, B. (1906*a*), 'Les paradoxes de la logique', *Revue de métaphysique et de morale* **14**, 627–650.

Russell, B. (1906*b*), 'On Some Difficulties in the Theory of Transfinite Numbers and Order Types', *Proceedings of the London Mathematical Society* **4**, 29–53.

Russell, B. (1908), 'Mathematical logic as based on the theory of types', *American Journal of Mathematics* **30**, 222–262.

Russell, B. (1973), *Essays in Analysis*, George Braziller, New York. Edited by D. Lackey.

Sacks, G. E. (1990), *Higher recursion theory*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin.

Schütte, K. (1965*a*), 'Eine Grenze für die Beweisbarkeit der Transfiniten Induktion in der verzweigten Typenlogik', *Archiv für mathematische Logik und Grundlagenforschung* **7**, 45–60.

Schütte, K. (1965*b*), Predicative well–orderings, *in* J. Crossley & M. Dummett, eds, 'Formal Systems and Recursive Functions', North–Holland, Amsterdam.

Sieg, W. (1999), 'Hilbert's programs: 1917–1922', *Bull. Symbolic Logic* **5**(1), 1–44.

Simpson, S. G. (1988), 'Partial realizations of Hilbert's program', *Journal of Symbolic Logic* **53(2)**, 349–363.

Simpson, S. G. (1999), *Subsystems of Second Order Arithmetic*, Perspectives in Mathematical Logic, Springer-Verlag.

Simpson, S. G. (2002), Predicativity: the outer limits, *in* 'Reflections on the foundations of mathematics (Stanford, CA, 1998)', Vol. 15 of *Lect. Notes Log.*, Assoc. Symbol. Logic, Urbana, IL, pp. 130–136.

Spector, C. (1955), 'Recursive well-orderings', *J. Symb. Logic* **20**, 151–163.

Stein, H. (1988), Logos, logic and logistiké, *in* W. Asprey & P. Kitcher, eds, 'History and Philosophy of Modern Mathematics', Minneapolis: University of Minnesota, pp. 238–59.

Strahm, T. (2017 ), 'Unfolding schematic systems', this volume.

Takeuti, G. (1987), *Proof Theory*, North Holland, Amsterdam. Second edition.

Wang, H. (1954), 'The formalization of mathematics', *The Journal of Symbolic Logic* **19**(4), pp. 241–266.

Weaver, N. (2005), Predicativity beyond $\Gamma_0$. Preprint submitted to the arXiv repository: arXiv:math/0509244.

Weyl, H. (1918), *Das Kontinuum. Kritische Untersuchungen über die Grundlagen der Analysis*, Veit, Leipzig. Translated in English, Dover Books on Mathematics, 2003. (Page references are to the translation).

Weyl, H. (1949), *Philosophy of Mathematics and Natural Science*, Princeton University Press. An expanded English version of Philosophie der Mathematik und Naturwissenschaft, München, Leibniz Verlag, 1927.

Whitehead, A. N. & Russell, B. (1910, 1912, 1913), *Principia Mathematica, 3 Vols.*, Vol. 1, Cambridge: Cambridge University Press. Second edition, 1925 (Vol 1), 1927 (Vols 2, 3); abridged as Principia Mathematica to *56, Cambridge: Cambridge University Press, 1962.

School of Philosophy, Religion and History of Science, University of Leeds, LS2 9JT, UK, E-mail: Laura.Crosilla@gmail.com