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$\mathfrak{L}_\alpha$: A Modal Logic to Reason about Analogical Proportion

ABSTRACT. In [Prade and Richard, 2009] a restricted study of analogy was developed through the notion of analogical proportions, i.e. sequences of the form "*a is to b as c is to d*". They define three kinds of *analogical proportions*: analogy, reverse analogy, and paralogy. In [Prade and Richard, 2013] and [Prade and Richard, 2014] many kinds of analogy are defined but we highlight four: analogy, reverse analogy, paralogy, and inverse paralogy. In all of these works analogy is analyzed in a Boolean sense taking an account of analogy in a logical terms.
Our hypothesis is that if we take a restricted notion of analogy in the sense of the mentioned works, analogy could be seen as a modal operator. We proceed as follows. In the first section we present a background of the notion of analogical proportion, we take the main thesis of Henri Prade and Gilles Richard in the mentioned works. Later, in the second part of the paper we present the basic system of analogical proportions: the logic $\mathfrak{L}_\alpha$. We define a modal propositional language with four basic modal operators, then, we present a model based on a relational structure with two types of relations defined as two kinds of accessibility relations between states. Our technique is to interpret analogical proportions as dyadic relations between pairs of objects holding an inclusion relation. In this sense, the formulas related by the analogical modal operators are truth in states that hold some analogical proportion.

KEY WORDS: modal logic, analogical proportion, homogeneous analogy, classical propositional logic.

Introduction

In [Prade and Richard, 2009] a restricted study of analogy was developed through the notion of analogical proportions, i.e. sequences of the form "*a is to b as c is to d"*. They define three kinds of *analogical proportions*: analogy, reverse analogy, and paralogy. In [Prade and Richard, 2013] and [Prade and Richard, 2014] many kinds of analogy are defined but we highlight four: analogy, reverse analogy, paralogy, and inverse paralogy. In all of these works analogy is analyzed in a Boolean sense taking an account of analogy in a logical terms.

Our hypothesis is that if we take a restricted notion of analogy in the sense of the mentioned works, analogy could be seen as a modal operator. We proceed as follows. In the first section we present a background of the notion of analogical proportion, we take the main thesis of Henri Prade and Gilles Richard in the mentioned works. Later, in the second part of the paper we present the basic system of analogical proportions: the logic $\mathfrak{L}_{\alpha}$. We define a modal propositional language with four basic modal operators, then, we present a model based on a relational structure with two types of relations defined as two kinds of accessibility relations between states. Our technique is to interpret analogical proportions as dyadic relations between pairs of objects holding an inclusion relation. In this sense, the formulas related by the analogical modal operators are truth in states that hold some analogical proportion.

One of the main results of our approach is that we could dualize the analogical proportions and define strong notions of analogy, paralogy, reverse analogy, and inverse paralogy, respectively. That means that there could be not only four modal operators of analogical proportions but eight. Related to the previous issue, we can consider what are the advantages of a semantics based on the notion of analogical proportion, and also how we can construct a logical calculus adequate to the remaining semantics. Another result is given by the properties of the four analogical proportion, i.e. reverse reflexivity, odd permutation, symmetry, bi-reflexivity, even permutation, etc. These properties define some characteristic theorems of the logic of analogical proportions, we analyze these issues in the final section.

Background on Analogy

My aim in this part is to offer a restricted notion of analogical proportions. I follow Henri Prade and Gilles Richard in three of theirs works:

"Analogy, Paralogy and Reverse Analogy: Postulates and Inferences"; "From Analogical Proportion to Logical Proportions"; and "From Analogical Proportion to Logical Proportions: A Survey". I only focus on the intuitive notion of analogical proportion and its Boolean interpretation, and only in four kinds of analogical proportions, namely *homogeneous analogies*. For this reason, I only take in account the definition of the four analogical proportions.

The first paper ("Analogy, Paralogy and Reverse Analogy: Postulates and Inferences") develops a three-sided view of analogy, in the author's words:

> (...) we investigate constitutive notions of analogy and we highlight the existence of two relations beside standard analogical proportion, namely paralogical proportion and reverse analogical proportion (...) [Prade and Richard, 2009, p. 307]

Their starting idea is that "analogy is a matter of *similarity* and *difference*" [Idem.], this idea is the core of a definition of three types of analogy with its respective "postulates". The basic definition of analogical proportion given by them is "statements of the form a is to b as c is to d, usually denoted $a : b :: c : d$" [Idem.]; for example “"numeral" is to "two" as "solid" is to "cube"”; the words "numeral" and "two" are similar in the same sense as "solid" and "cube" are, the first and the third refers to a conceptual entity (a notion of numeral and a notion of solid) in this reference lies the similarity, but the difference lies in the fact that one notion refers to an arithmetical concept and the other refers to a geometrical concept.

In this sense analogy is a binary relation between pairs of objects that hold at the same time relations of similarity and dissimilarity, Prade and Richard say that we may have to put two situations in parallel and compare these situations by establishing a correspondence between them. We may extend this correspondence to take a general intuitive definition of analogy: "the way a and b differ is the same as the way c and d differ". This definition of analogy is the base of the remaining definitions of paralogy and reverse analogy, we put them together and get:

a) *Analogy* between $abcd$: the way a and b differ is the same as the way c and d differ,

b) *Paralogy* between $abcd$: what a and b have in common, c and d have it also,

c) *Reverse analogy* between $abcd$: the way a and b differ is the same as the way d and c differ.

This three kinds of operations are studied in the first paper and Prade and Richard give an interesting analysis of them, but in a later paper they introduce many kinds of analogical proportions (in specific 120). In the second citied paper Prade and Richard [2013, p. 445] resort to the notion of "indicator" to define a group of four kinds of analogical proportions. An indicator is a conjunction of two Boolean literals, holding some combination of negation and conjunction in its definition, giving rise to four different combinations of which we have two types: *similarity* and *dissimilarity* indicators. The four combinations are the following:

1) $a \wedge b$ and $\tilde{a} \wedge \bar{b}$ are similarity indicators,
2) $\bar{a} \wedge b$ and $a \wedge \bar{b}$ are dissimilarity indicators.

Prade and Richard take in account the properties and restrictions of this indicators, but we only focus on the notions of similarity and dissimilarity. Later in the paper they introduce the *homogeneous analogies*, proportions that "do not mix different types of indicators" [Ibid.], these are: analogy, reverse analogy, paralogy and inverse paralogy. The new element of the group is the inverse paralogy. To the previous recapitulation of the analogical proportions we introduce the new definition of inverse paralogy as follows:

d) *Inverse paralogy* between $abcd$: what a and b have in common, c and d miss it.

With this fourth type of analogical proportion we complete the framework used to analyze the notion of analogy in modal terms. We continue in the next section with the definition of the logic of analogical proportions.

The Logic $\mathfrak{L}_\alpha$

This section presents the basic ingredients of the logic of analogical proportions. This logic can be defined in an abstract sense by the structure $\mathfrak{L}_\alpha = \langle L_\alpha, Cn \rangle$ where L_α is a structure and Cn an operation $Cn: P(L_\alpha) \longmapsto P(L_\alpha)$. The structure $\mathfrak{L}_\alpha = \langle L_\alpha, Cn \rangle$ is called sometimes a consequence system [Carneli, Coniglio, Gabbay, Goubeia & Sernadas, 2008, p. 4]. We present an alternative characterization of this logic focused on the semantic elements of a relation of logical consequence, but we show later how the relation of logical consequence induces the operation of consequence and vice versa. Later we see how we may construct a logical calculus based on the semantics defined here. Fist we present some basic definitions of the language and some comments to the notation.

Definition 2.1 (*Relational structure*) A *relational structure* is a tuple $\mathfrak{F}$ whose first component is a non-empty set W called *the universe* of $\mathfrak{F}$, and whose remaining components are relations on W.

Definition 2.2 A modal similarity type is a pair $\tau = (O, \rho)$ where O is a non-empty set and ρ is a function $\rho: O \longmapsto \mathbb{N}$. The elements of O are called *operators*. The function ρ assigns to each operator $\Delta \in O$ a finite *arity*, indicating the number or arguments Δ can be applied to.

Definition 2.3 (*The language* L_α) Let $\tau = (\langle : \rangle, \langle ; \rangle, \langle ! \rangle, \langle ¡ \rangle, \rho)$ a modal similarity type (with $\rho(\langle : \rangle) = \rho(\langle ; \rangle) = \rho(\langle ! \rangle) = \rho(\langle ¡ \rangle) = 4$), $C = \{\neg, \wedge, \vee, \rightarrow, \bot\}$ a set of logical connectives, and A a non-empty set of proposition atoms. An alphabet is a set $\Sigma = \tau \cup C \cup A$ of symbols. A formula φ is a sequence of symbols of the alphabet Σ closed by the following production rule:

$$\varphi := \neg\varphi \mid \bot \mid \varphi \vee \psi \mid \varphi \rightarrow \psi \mid \varphi \wedge \psi \left| \binom{\alpha}{\beta} \langle : \rangle \binom{\varphi}{\psi} \right| \binom{\alpha}{\beta} \langle ; \rangle \binom{\varphi}{\psi} \left| \binom{\alpha}{\beta} \langle ! \rangle \binom{\varphi}{\psi} \right| \binom{\alpha}{\beta} \langle ¡ \rangle \binom{\varphi}{\psi}$$

The language L_α is the set of all formulas.

Let's remark on some aspects of the language starting with the alphabet. The novelty of the language is the introduction of a set of operators of analogical proportions: analogy $\langle : \rangle$, paralogy $\langle ; \rangle$, reverse analogy $\langle ! \rangle$, and inverse paralogy $\langle ¡ \rangle$. A formula like "$\binom{\alpha}{\beta}\langle : \rangle\binom{\varphi}{\psi}$" could be read as "$\alpha$ and β are analogous to φ and ψ", the remaining formulas are read in an "analogous" way: "$\binom{\alpha}{\beta}\langle ; \rangle\binom{\varphi}{\psi}$" may be read as "$\alpha$ and β are paralogous to φ and ψ", $\binom{\alpha}{\beta}\langle ! \rangle\binom{\varphi}{\psi}$ may be read as "α and β are reverse analogous to φ and ψ", and "$\binom{\alpha}{\beta}\langle ¡ \rangle\binom{\varphi}{\psi}$" may be read as "$\alpha$ and β are inverse paralogous to φ and ψ". The main difference with another modal operators is that their *arity* is equal to four, that is, they range over four arguments. In this sense they are applied to four formulas. We could write $\langle ¡ \rangle(\alpha, \beta, \varphi, \psi)$ instead of $\binom{\alpha}{\beta}\langle ¡ \rangle\binom{\varphi}{\psi}$, and in fact we must write so if we want to be strict with the concatenation notation of the modal operators, but we think that it is more convenient to take our "binomial" notation at least for two simple reasons. First, as we want to represent relations of pairs of elements, we think that the binomial notation represents perfectly the visual interaction between the elements in relation, i.e. two items related. Second, we may exploit this *two-sided* representation to manipulate pairs of formulas in a context of binary relations. Despite this, we must not forget that we are facing a quaternary relation.

Definition 2.4 (τ*-frame*) Let be $\mathfrak{F}$ a relational structure, we call $\mathfrak{F}$ a τ-*frame* where $\mathfrak{F}$ be a tuple consisting of the following:

a) A non-empty set S,

b) A relation $\leq \subseteq S^2$,

c) A relation $\approx \subseteq (\mathfrak{S} \times S)^2$.

We write $\mathfrak{F} = (S, \leq, \approx)$ to denote a τ -frame.

The set $\mathfrak{S}$ is called the set of *meta-states* and is defined as $\mathfrak{S} = \{x \in P(S): if\ y \in x\ and\ z \in x, then\ y \leq z\ or\ z \leq y\}$, that is, the meta-states are subsets of S in which its elements hold the contention relation $\leq$. The elements of $\mathfrak{S}$ are called meta-states (m-states in the following). The relation $\approx \subseteq (\mathfrak{S} \times S)^2$ is a relation between m-states, but is hold by pairs which first element is a m-state and its second element is a state belonging to the m-state in question. In symbols $\langle \mathfrak{b}, s\rangle \approx \langle \mathfrak{b}_n, s\rangle$, where $\mathfrak{b}$ and $\mathfrak{b}_n$ are m-states belonging to $\mathfrak{S}$, and s is some states present in both $\mathfrak{b}$ and $\mathfrak{b}_n$. We consider also another kind of states that we will call *complement* states. We define a complement state as follows. Let be $s \in S$ some state, its complement state is the set $\bar{s} = S - s$. Intuitively we may understand a complement state $\bar{s}$ as the set formed by all the states without s.

Definition 2.5 A τ-*model* for L_α is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ where $\mathfrak{F}$ is a τ-frame and V is a valuation $V: A \longmapsto P(S)$.

Definition 2.6 Let be $\mathfrak{M} = (\mathfrak{F}, V)$ a model for L_α, we define a formula φ *satisfied* at a m-state $\mathfrak{b} \in \mathfrak{S}$ and at a state $s \in S$ in a model $\mathfrak{M} = (\mathfrak{F}, V)$ as follows:

a) $\mathfrak{M}, \mathfrak{b}, s \Vdash p$ iff $s \in V(p)$

b) $\mathfrak{M}, \mathfrak{b}, s \Vdash \bot$ never

c) $\mathfrak{M}, \mathfrak{b}, s \Vdash \neg p$ iff not $\mathfrak{M}, \mathfrak{b}, w \Vdash p$

d) $\mathfrak{M}, \mathfrak{b}, s \Vdash p \vee \psi$ iff $\mathfrak{M}, \mathfrak{b}, s \Vdash \varphi$ or $\mathfrak{M}, \mathfrak{b}, s \Vdash \varphi$

e) $\mathfrak{M}, \mathfrak{b}, s \Vdash \varphi \rightarrow \psi$ iff not $\mathfrak{M}, \mathfrak{b}, s \Vdash \varphi$ or $\mathfrak{M}, \mathfrak{b}, s \Vdash \varphi$

f) $\mathfrak{M}, \mathfrak{b}, s \Vdash \varphi \wedge \psi$ iff $\mathfrak{M}, \mathfrak{b}, s \Vdash \varphi$ and $\mathfrak{M}, \mathfrak{b}, s \Vdash \varphi$

g) $\mathfrak{M}, \mathfrak{b}, s \Vdash \binom{\alpha}{\beta} \langle : \rangle \binom{\varphi}{\psi}$ iff $\exists \mathfrak{b}_n \in \mathfrak{S}$ with $\langle \mathfrak{b}, s\rangle \approx \langle \mathfrak{b}_n, s\rangle$ and,

$\exists s_1 s_2 \in \mathfrak{b}, \exists s_1 s_2 \in \mathfrak{b}_n$ with $\begin{cases} s \leq s_1 \leq \bar{s_2} \\ s \leq s_1 \leq \bar{s_2} \end{cases}$; such that $\begin{cases} \mathfrak{M}, \mathfrak{b}, s_1 \Vdash \alpha \\ \mathfrak{M}, \mathfrak{b}, s_2 \Vdash \beta \\ \mathfrak{M}, \mathfrak{b}_n, s_1 \Vdash \varphi \\ \mathfrak{M}, \mathfrak{b}_n, s_2 \Vdash \psi \end{cases}$

h) $\mathfrak{M}, \mathfrak{b}, s \Vdash \binom{\alpha}{\beta} \langle ; \rangle \binom{\varphi}{\psi}$ iff $\exists \mathfrak{b}_n \in \mathfrak{S}$ with $\langle \mathfrak{b}, s\rangle \approx \langle \mathfrak{b}_n, s\rangle$ and,

$\exists s_1 s_2 \in \mathfrak{b}, \exists s_1 s_2 \in \mathfrak{b}_n$ with $\begin{cases} s \leq s_1 \leq s_2 \\ s \leq s_1 \leq s_2 \end{cases}$; such that $\begin{cases} \mathfrak{M}, \mathfrak{b}, s_1 \Vdash \alpha \\ \mathfrak{M}, \mathfrak{b}, s_2 \Vdash \beta \\ \mathfrak{M}, \mathfrak{b}_n, s_1 \Vdash \varphi \\ \mathfrak{M}, \mathfrak{b}_n, s_2 \Vdash \psi \end{cases}$

i) $\mathfrak{M}, \mathfrak{b}, s \Vdash \binom{\alpha}{\beta} \langle ! \rangle \binom{\varphi}{\psi}$ iff $\exists \mathfrak{b}_n \in \mathfrak{S}$ with $\langle \mathfrak{b}, s \rangle \approx \langle \mathfrak{b}_n, s \rangle$ and,

$$\exists s_1 s_2 \in \mathfrak{b}, \exists s_1 s_2 \in \mathfrak{b}_n \text{ with } \begin{cases} s \leq s_1 \leq \overline{s_2} \\ s \leq \overline{s_1} \leq s_2 \end{cases}; \text{ such that } \begin{cases} \mathfrak{M}, \mathfrak{b}, s_1 \Vdash \alpha \\ \mathfrak{M}, \mathfrak{b}, s_2 \Vdash \beta \\ \mathfrak{M}, \mathfrak{b}_n, s_1 \Vdash \varphi \\ \mathfrak{M}, \mathfrak{b}_n, s_2 \Vdash \psi \end{cases}$$

j) $\mathfrak{M}, \mathfrak{b}, s \Vdash \binom{\alpha}{\beta} \langle ¡ \rangle \binom{\varphi}{\psi}$ iff $\exists \mathfrak{b}_n \in \mathfrak{S}$ with $\langle \mathfrak{b}, s \rangle \approx \langle \mathfrak{b}_n, s \rangle$ and,

$$\exists s_1 s_2 \in \mathfrak{b}, \exists s_1 s_2 \in \mathfrak{b}_n \text{with } \begin{cases} s \leq s_1 \leq s_2 \\ s \leq \overline{s_1} \leq \overline{s_2} \end{cases}; \text{ such that } \begin{cases} \mathfrak{M}, \mathfrak{b}, s_1 \Vdash \alpha \\ \mathfrak{M}, \mathfrak{b}, s_2 \Vdash \beta \\ \mathfrak{M}, \mathfrak{b}_n, s_1 \Vdash \varphi \\ \mathfrak{M}, \mathfrak{b}_n, s_2 \Vdash \psi \end{cases}$$

Definition 2.7 A formula φ is global satisfied (or global true) in a model $\mathfrak{M}$ (notation $\mathfrak{M} \Vdash \varphi$) if it is satisfied in all states of all m-states in $\mathfrak{M}$ (that is $\forall \mathfrak{b} \in \mathfrak{S}$ and $\forall s \in S$ we have $\mathfrak{M}, \mathfrak{b}, s \Vdash \varphi$). A formula φ is satisfied in a model $\mathfrak{M}$ if it is satisfied in some state in a m-state in $\mathfrak{M}$, it is refuted in a model if its negation is satisfied. A set of formulas Γ is global satisfied in a model $\mathfrak{M}$ if $\mathfrak{M}, \mathfrak{b}, s \Vdash \Gamma$ for all m-states and all states in $\mathfrak{M}$.

Definition 2.8 (*Logical consequence*) Let τ be a modal similarity type and $\mathbb{M}$ a class of τ-models $\mathfrak{M}$. Let be Γ a set of formulas and φ a formula of L_α, we say that φ is a logical consequence of Γ over $\mathbb{M}$ (in notation $\Gamma \Vdash_{\mathbb{M}} \varphi$) if $\forall \mathfrak{M} \in \mathbb{M}$, $\forall \mathfrak{b} \in \mathfrak{S}$ and $\forall s \in S$ if $\mathfrak{M}, \mathfrak{b}, s \Vdash \Gamma$ then $\mathfrak{M}, \mathfrak{b}, s \Vdash \varphi$.

A logic may be defined as a pair $\mathfrak{L}_\alpha = \langle L_\alpha, \Vdash \rangle$ where L_α is a structure and $\Vdash$ is a relation $\Vdash \subseteq P(L_\alpha) \times L_\alpha$. We show that this relation induces a consequence operation on the same universe and vice versa. Consider a logical consequence relation $\Vdash_{\mathbb{M}}$ defined as above and a consequence operation $Cn: P(L_\alpha) \longmapsto P(L_\alpha)$, we say that a consequence relation Cn *induces* a logical consequence relation $\Vdash_{\mathbb{M}}$ such that for every $\Gamma \subseteq L_\alpha$ and every $\varphi \in L_\alpha$:

$$\Gamma \Vdash \varphi \text{ iff } \varphi \in Cn(\Gamma)$$

On the other side, we say that a logical consequence relation $\Vdash$ induces a consequence operation Cn such that for every $\Gamma \subseteq L_\alpha$ and every $\varphi \in L_\alpha$:

$$Cn(\Gamma) = \{\varphi \in L_\alpha : \Gamma \Vdash \varphi\}$$

In this sense a logic may be defined also as a pair $\mathfrak{L}_\alpha = \langle L_\alpha, Cn\rangle$. Let´s turn to the meaning of the operators of analogical proportion defined here, in the next section we analyze this questions in detail.

Some questions about the Logic $\mathfrak{L}_\alpha$

In this section we analyze some issues concerning the meaning of the operations defined in the previous part. In the first place, what does it mean that some formulas hold an analogical proportion relation? Specifically, how does the semantics works. In the second place, we consider the option to *dualize* the four operators to get the "strong" operations of analogical proportions.

A similarity type is a tuple with a number of operations and a function that assigns to all operators its arity. When we want to define an operator semantically we use the arity of the operator to assign a relation with a $n + 1$ arity, when the arity of the operator is n. In our case the relational structure (*τ-frame*) has two dyadic relations although the modal operators are tetradic. Strictly speaking we must assign a pentadic relation to an operator with tetradic arity, but the application of the operators does not meet a pentadic relation.

We believe that the best image representing the behavior of operators is two dyadic relations interacting. The main reason is related with the meaning of the notion of “analogical proportion”. This operation is executed by pairs of objects which in turn are pairs of other objects, this operation is not carried out by four objects related simultaneously with a fifth object, thus a pentadic relation do not represent this operator. Instead, we believe that the best image of an analogical proportion relation in a modal semantics is given by a two dyadic relations of different level.

The first relation ($\leq$) ranges over objects of the universe S (states), the second relation ranges over sets of states $\mathfrak{b}_n \in \mathfrak{S}$ (m-states). We consider the first relation as a "partial contention" or "preservation of information" between states, that is, a state x is partially contained in a state y, or a state y preserve all the information that preserve the state x (in symbols $x \leq y$). The second relation "$\approx$" is similar to the first but it satisfies some restrictions. It is a relation of partial order but the objects over the relation applies to are neither states nor meta-states, but pairs composed by one meta-state and a state, in this order. Intuitively we say that two m-states are related (in symbols $\langle \mathfrak{b}, s \rangle \approx \langle \mathfrak{b}_n, s \rangle$) if and only if they contain the same information "until" s, where s is some state. As the states and meta-states satisfy a partial order, a state may serve as "separator"[1] of identical m-states generating disjoint (forked) meta-states. In this sense, we consider the relation between m-states as a connection, that is, we say that the m-states are connected by a state. Let $\mathfrak{b}$ and $\mathfrak{b}_n$ be two meta-states, we say that $\mathfrak{b}$ and $\mathfrak{b}_n$ are connected if and only if they have the same information until s "is given" and "beyond" s they differ at least in one portion of information (a state).

The next issue is connected with the following section; I refer to the dualization of the operators. As we can see in the definition we have two quantified parts, and the question is in which quantifier we apply the dualization? Our thesis is that we must apply the dualization to the quantifier that operates on the part of the definition that describes the behavior of the m-states. We have three main reasons to maintain this idea and we explain each one in detail.

In the first place, as we say we interpret analogy as a dyadic relation between pairs, in this sense analogy must be, in our interpretation, a relation (let say "(dis)similarity") between m-states composed by states related by another relation (let say "contention"). The main relation in this ap-

[1] Or an "identifier" of non-identical m-states, as we will see in the final example of Central Permutation Theorem.

proach is (dis)similarity and the objects related are m-states, therefore we conclude that we must change the quantifier that operates on the m-states in the dualization process. In the second place, when quantifying universally on m-states we include states as elements of the objects in which quantifiers operate (m-states). The opposite does not hold, if we operate only in states, we not obey m-states. Finally, in the operator's definition we use two dyadic relations chained with a link. This link indicates that the relation of the first pair of items (the states) define the other relation between pairs of states (in the binomial notation the link between relations is clear). In this two relational link we have a dominant relation and a derivate relation. The dominant relation is referred to the link between pairs of states (or in the definition between pairs of m-states/states), and the derivate relation is a basic order relation between states. In this sense is natural to think that the quantifier affected by the dualization is that represent the dominant relation. On the contrary, if we take the quantifier that represents the derivate relations we do not have a very important property of duality i.e. transposition; therefore, we must change the quantifier that ranges over m-states. In the next section we follow with this argument considering how to define a tableaux calculus for the logic $\mathfrak{L}_\alpha$.

A Calculus for the Logic $\mathfrak{L}_\alpha$

The basic rules are the usual rules for classical propositional logic. We extend the calculus of classical propositional logic adding sixteen rules of tableaux, two for each operator, one for the affirmative operator and one for the negative version of the operator. In the explication of the rules we proceed as follows. We present the two rules of weak and strong analogy operator and based on this presentation we explain the restrictions of the rules of the other analogical proportion operators, which in fact are "analogous" to these two first rules.

$$1.\ \binom{\alpha}{\beta}\langle : \rangle\binom{\varphi}{\psi} \quad (hip.), \mathfrak{b}_0, s_0$$

$$\downarrow$$

$$\langle \mathfrak{b}_0, s_0 \rangle \quad \approx \quad \langle \mathfrak{b}_1, s_0 \rangle$$

$$s_0 \leq s_1 \leq \bar{s}_2$$

$$\swarrow \quad \searrow$$

$$2.\ \alpha\ (1), \mathfrak{b}_0, s_1 \quad \varphi\ (1), \mathfrak{b}_1, s_1$$

$$3.\ \beta\ (1), \mathfrak{b}_0, s_2 \quad \psi\ (1), \mathfrak{b}_1, s_2$$

Fig. 1

The first two rules are of the weak operator of analogy, the first is the rule of the affirmative analogy operator (Fig. 1). The rule has three components as it is common in the tableaux: the numeration of the sequences of formulas, the sequences of formulas properly said, and the justification of the sequences of formulas. In this case the novelty of our rules lies in the justification. As in the case of basic modal logic we add to the justification the "possible world" in which the justified formula is true. In our rules we have states (s_0) and m-states ($\mathfrak{b}_0$) instead of "worlds", and by this reason we add also the m-state in which the formula and the state belongs. For example, the first formula is true at a state s_0 in the m-state $\mathfrak{b}_0$.

Our rules are divided in two groups, the rules that separate the component formulas and the rules that send the external negation of the main formula to the component formulas without separating. This example is of the kind of rules that separate the formula to which we apply the rule in its component formulas, and as in the case of basic modal logic (again) this separation generates an interaction between the entities in which the formula is true, "possible worlds" in the case of basic modal logic and states and m-states in this case.

When we remove the operator of the formula the first thing to do is to relate the m-states, as we see in the rule ($\langle \mathfrak{b}_0, s_0 \rangle \approx \langle \mathfrak{b}_1, s_0 \rangle$). This link is stated by the relation "≈" between pairs composed by a m-state and a state (in this order). When this link has been stated we need a relation between states, but we need some restrictions to the states related. First we need that all the states related include the state by which the m-state is related, that is the state until the two m-states coincide, we will call this state "the actual state". In our example the actual state is s_0. In the second place the states must be related by the relation of inclusion. The states are divided in pairs, belonging to each m-states as in the example see with $s_1 \leq \overline{s_2}$ in both sides, the former belongs to $\mathfrak{b}_0$ and the latter belongs to $\mathfrak{b}_1$. The first specification of this rule, which establishes the variation between conditions stated by each rule, is that the two states related must be a state that includes the actual state and a complement state that is equal or include the actual state. In other words, in this rule the relation between states is satisfied by pairs "state/complement state" in this order, as we can see in the example with s_1 and $\overline{s_2}$. Finally, the formulas are "sent" to the corresponding states when we disjoin the formula.

In this rule also we restrict the use of states (and m-states). In the case of the weak operators the restriction consists in using a new m-state to relate them with the actual m-state, and therefore we use new states belonging to the new m-state to generate the inclusion relations. That is, the m-states and states related in the proof should not appear previously. Now let's move on to the second rule (Fig. 2).

$$\mathbf{1.}\quad \neg\left(\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\langle : \rangle\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right)\quad (hip.), \mathfrak{b}_0, s_0$$

$$\downarrow$$

$$\mathbf{2.}\quad \begin{pmatrix}\neg\alpha\\ \neg\beta\end{pmatrix}[:]\begin{pmatrix}\neg\varphi\\ \neg\psi\end{pmatrix}\quad (1), \mathfrak{b}_0, s_0$$

Fig. 2

This rule takes the same operations as the basic modal logic rules of equivalence between the diamond and the box (for example$\neg\Box p \equiv \Diamond \neg p$), the external negation becomes internal and the operator changes by its dual. In our rule the external negation is transferred to the component formulas, the negation should be transferred to each one of the four formulas related by the operator because the operator is tetradic and no dyadic as we mistakenly may assume. As we say in the presentation of the syntax of our system the interpretation of the formulas with binomial notation is only a symbolic resource, and we do not forget that we are facing with a quaternary relation. The last feature of the rule is the change of the operator by his dual. As this rule has no interaction between m-states and states, the resulting formula of the application of the rule does not change from state to state (and the same with m-states). Now we present the strong versions.

The two rules are very similar to the previous one but only satisfy the next restrictions. In the case of the rule for the positive version of strong analogy operator the interaction states' and m-states' interaction must be previously generated, and the states used in the previous part of the proof may be present in the application of this rule. As we may have seen in the Fig 3, the vertical arrow states that the nexus between m-states and states has been made and only we sent the formulas to the states in question.

$$
\begin{array}{c}
1.\ \binom{\alpha}{\beta}[:]\binom{\varphi}{\psi} \quad (hip.), \mathfrak{b}_0, s_0 \\
\langle \mathfrak{b}_0, s_0 \rangle \quad \approx \quad \langle \mathfrak{b}_1, s_0 \rangle \\
s_0 \leq s_1 \leq \overline{s_2} \\
\downarrow \\
\swarrow \quad \searrow \\
2.\ \alpha\ (1), \mathfrak{b}_0, s_1 \quad \varphi\ (1), \mathfrak{b}_1, s_1 \\
3.\ \beta\ (1), \mathfrak{b}_0, s_2 \quad \psi\ (1), \mathfrak{b}_1, s_2
\end{array}
$$

Fig. 3

On the other side, in the rule of the operator with external negation the application of the rule has the same properties as in the case of the weak version as we see in the (Fig. 4). The external negation is sent to the component formulas and the operator is changed by its dual, and again there is no interaction between states. We finalize this section with a presentation of the remaining twelve rules pointing out the pattern followed by the interaction between states in each rule and the behavior of the branches, inasmuch as, this is the main differences between all the rules.

$$1.\ \neg\left(\binom{\alpha}{\beta}[:]\binom{\varphi}{\psi}\right) \quad (hip.), \mathfrak{b}_0, s_0$$

$$\downarrow$$

$$2.\ \binom{\neg\alpha}{\neg\beta}\langle:\rangle\binom{\neg\varphi}{\neg\psi} \quad (1), \mathfrak{b}_0, s_0$$

Fig. 4

As we see, the rules can be divided in two groups: rules of weak and rules of strong operators. In the previous example we take the two types of rule. Now we use another division, the rules that branch the proof when the main formula is disjoined and rules that do not branch the proof. The previous examples were of the first type, i.e. these rules branch the proof. In the following we see that we have rules that do not ramify either. The main difference between the rules is the way in which the states are related and the way in which we separate the component formulas. The way in which the states are related follow some patterns showed in (Fig. 5). In the picture we can see the operator, then down we can see the pattern of the relation between states followed by the rule, and in the bottom we se some arrows that represent if the rule branch or if not.

$$
\begin{array}{cccc}
 & & \langle ! \rangle & \langle ¡ \rangle \\
\langle : \rangle & \langle ; \rangle & s_0 \leq s_1 \leq \overline{s_2} & s_0 \leq s_1 \leq s_2 \\
s_0 \leq s_1 \leq \overline{s_2} & s_0 \leq s_1 \leq s_2 & s_0 \leq \overline{s_1} \leq s_2 & s_0 \leq \overline{s_1} \leq \overline{s_2} \\
\swarrow \quad \searrow & \downarrow & \swarrow \quad \searrow & \downarrow
\end{array}
$$

Fig. 5

We conclude with the rules and some main theorems of this logic. In the appendix we present a proof as example of the use of the rules and we explain some properties of the theorems.

$$
\begin{array}{c}
1.\ \binom{\alpha}{\beta} \langle ; \rangle \binom{\varphi}{\psi} \quad (hip.), \mathfrak{b}_0, s_0 \\
\downarrow \\
\langle \mathfrak{b}_0, s_0 \rangle \quad \approx \quad \langle \mathfrak{b}_1, s_0 \rangle \\
s_0 \leq s_1 \leq s_2 \\
2.\ \alpha\ (1), \mathfrak{b}_0, s_1 \\
3.\ \beta\ (1), \mathfrak{b}_0, s_2 \\
4.\ \varphi\ (1), \mathfrak{b}_1, s_1 \\
5.\ \psi\ (1), \mathfrak{b}_1, s_2
\end{array}
$$

Fig. 6

$$
\begin{array}{c}
1.\ \neg \left(\binom{\alpha}{\beta} \langle ; \rangle \binom{\varphi}{\psi} \right) \quad (hip.), \mathfrak{b}_0, s_0 \\
\downarrow \\
2.\ \binom{\neg\alpha}{\neg\beta} [;] \binom{\neg\varphi}{\neg\psi} \quad (1), \mathfrak{b}_0, s_0
\end{array}
$$

Fig. 7

$$1.\ \binom{\alpha}{\beta}[;]\binom{\varphi}{\psi} \quad (hip.), \mathfrak{b}_0, s_0$$
$$\langle \mathfrak{b}_0, s_0\rangle \quad \approx \quad \langle \mathfrak{b}_1, s_0\rangle$$
$$s_0 \leq s_1 \leq s_2$$
$$\downarrow$$
$$2.\ \alpha\ (1), \mathfrak{b}_0, s_1$$
$$3.\ \beta\ (1), \mathfrak{b}_0, s_2$$
$$4.\ \varphi\ (1), \mathfrak{b}_1, s_1$$
$$5.\ \psi\ (1), \mathfrak{b}_1, s_2$$

Fig. 8

$$1.\ \neg\left(\binom{\alpha}{\beta}[;]\binom{\varphi}{\psi}\right) \quad (hip.), \mathfrak{b}_0, s_0$$
$$\downarrow$$
$$2.\ \binom{\neg\alpha}{\neg\beta}\langle ; \rangle\binom{\neg\varphi}{\neg\psi} \quad (1), \mathfrak{b}_0, s_0$$

Fig. 9

$$1.\ \binom{\alpha}{\beta}\langle ¡ \rangle\binom{\varphi}{\psi} \quad (hip.), \mathfrak{b}_0, s_0$$
$$\downarrow$$
$$\langle \mathfrak{b}_0, s_0\rangle \quad \approx \quad \langle \mathfrak{b}_1, s_0\rangle$$
$$s_0 \leq s_1 \leq s_2$$
$$s_0 \leq \bar{s_1} \leq \bar{s_2}$$
$$2.\ \alpha\ (1), \mathfrak{b}_0, s_1$$
$$3.\ \beta\ (1), \mathfrak{b}_0, s_2$$
$$4.\ \varphi\ (1), \mathfrak{b}_1, s_1$$
$$5.\ \psi\ (1), \mathfrak{b}_1, s_2$$

Fig. 10

$$1.\ \neg\left(\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\langle¡\rangle\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right)\quad (hip.), \mathfrak{b}_0, s_0$$
$$\downarrow$$
$$2.\ \begin{pmatrix}\neg\alpha\\ \neg\beta\end{pmatrix}[¡]\begin{pmatrix}\neg\varphi\\ \neg\psi\end{pmatrix}\quad (1), \mathfrak{b}_0, s_0$$

Fig. 11

$$1.\ \begin{pmatrix}\alpha\\ \beta\end{pmatrix}[¡]\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\quad (hip.), \mathfrak{b}_0, s_0$$
$$\downarrow$$
$$\langle \mathfrak{b}_0, s_0\rangle \quad \approx \quad \langle \mathfrak{b}_1, s_0\rangle$$
$$s_0 \leq s_1 \leq s_2$$
$$s_0 \leq \bar{s_1} \leq \bar{s_2}$$
$$2.\ \alpha\ (1), \mathfrak{b}_0, s_1$$
$$3.\ \beta\ (1), \mathfrak{b}_0, s_2$$
$$4.\ \varphi\ (1), \mathfrak{b}_1, s_1$$
$$5.\ \psi\ (1), \mathfrak{b}_1, s_2$$

Fig. 12

$$1.\ \neg\left(\begin{pmatrix}\alpha\\ \beta\end{pmatrix}[¡]\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right)\quad (hip.), \mathfrak{b}_0, s_0$$
$$\downarrow$$
$$2.\ \begin{pmatrix}\neg\alpha\\ \neg\beta\end{pmatrix}\langle¡\rangle\begin{pmatrix}\neg\varphi\\ \neg\psi\end{pmatrix}\quad (1), \mathfrak{b}_0, s_0$$

Fig. 13

$$1.\ \binom{\alpha}{\beta}\langle!\rangle\binom{\varphi}{\psi}\quad (hip.), \mathfrak{b}_0, s_0$$

$$\downarrow$$

$$\langle \mathfrak{b}_0, s_0\rangle \quad \approx \quad \langle \mathfrak{b}_1, s_0\rangle$$

$$s_0 \leq s_1 \leq \bar{s_2}$$

$$s_0 \leq \bar{s_1} \leq s_2$$

$$\swarrow \quad \searrow$$

$$2.\ \alpha\ (1), \mathfrak{b}_0, s_1 \quad \varphi\ (1), \mathfrak{b}_1, s_1$$

$$3.\ \beta\ (1), \mathfrak{b}_0, s_2 \quad \psi\ (1), \mathfrak{b}_1, s_2$$

Fig. 14

$$1.\ \neg\left(\binom{\alpha}{\beta}\langle!\rangle\binom{\varphi}{\psi}\right)\quad (hip.), \mathfrak{b}_0, s_0$$

$$\downarrow$$

$$2.\ \binom{\neg\alpha}{\neg\beta}[!]\binom{\neg\varphi}{\neg\psi}\quad (1), \mathfrak{b}_0, s_0$$

Fig. 15

$$1.\ \binom{\alpha}{\beta}[!]\binom{\varphi}{\psi}\quad (hip.), \mathfrak{b}_0, s_0$$

$$\downarrow$$

$$\langle \mathfrak{b}_0, s_0\rangle \quad \approx \quad \langle \mathfrak{b}_1, s_0\rangle$$

$$s_0 \leq s_1 \leq \bar{s_2}$$

$$s_0 \leq \bar{s_1} \leq s_2$$

$$\swarrow \quad \searrow$$

$$2.\ \alpha\ (1), \mathfrak{b}_0, s_1 \quad \varphi\ (1), \mathfrak{b}_1, s_1$$

$$3.\ \beta\ (1), \mathfrak{b}_0, s_2 \quad \psi\ (1), \mathfrak{b}_1, s_2$$

Fig. 16

$$1.\ \neg\left(\binom{\alpha}{\beta}[!]\binom{\varphi}{\psi}\right) \quad (hip.), \mathfrak{b}_0, s_0$$

$$\downarrow$$

$$2.\ \binom{\neg\alpha}{\neg\beta}\langle ! \rangle\binom{\neg\varphi}{\neg\psi} \quad (1), \mathfrak{b}_0, s_0$$

Fig. 17

Theorems

1. $\binom{\alpha}{\beta}\langle : \rangle\binom{\alpha}{\beta}$ Reflexivity
2. $\binom{\alpha}{\alpha}\langle : \rangle\binom{\beta}{\beta}$ Reflexivity
3. $\left(\binom{\alpha}{\beta}\langle : \rangle\binom{\varphi}{\psi}\right) \longrightarrow \left(\binom{\alpha}{\varphi}\langle : \rangle\binom{\beta}{\psi}\right)$ Central permutation
4. $\left(\binom{\alpha}{\beta}\langle : \rangle\binom{\varphi}{\psi}\right) \longrightarrow \left(\binom{\varphi}{\psi}\langle : \rangle\binom{\alpha}{\beta}\right)$ Symmetry
5. $\binom{\alpha}{\beta}\langle ! \rangle\binom{\beta}{\alpha}$ Reverse reflexivity
6. $\binom{\alpha}{\alpha}\langle ! \rangle\binom{\beta}{\beta}$ Reverse reflexivity
7. $\left(\binom{\alpha}{\beta}\langle ! \rangle\binom{\varphi}{\psi}\right) \longrightarrow \left(\binom{\varphi}{\beta}\langle ! \rangle\binom{\alpha}{\psi}\right)$ Odd permutation
8. $\left(\binom{\alpha}{\beta}\langle ! \rangle\binom{\varphi}{\psi}\right) \longrightarrow \left(\binom{\varphi}{\psi}\langle ! \rangle\binom{\alpha}{\beta}\right)$ Symmetry
9. $\binom{\alpha}{\beta}\langle ; \rangle\binom{\beta}{\alpha}$ Bi-reflexivity
10. $\binom{\alpha}{\beta}\langle ; \rangle\binom{\alpha}{\beta}$ Bi-reflexivity
11. $\left(\binom{\alpha}{\beta}\langle ; \rangle\binom{\varphi}{\psi}\right) \longrightarrow \left(\binom{\beta}{\alpha}\langle ; \rangle\binom{\varphi}{\psi}\right)$ Even permutation
12. $\left(\binom{\alpha}{\beta}\langle ; \rangle\binom{\varphi}{\psi}\right) \longrightarrow \left(\binom{\varphi}{\psi}\langle ; \rangle\binom{\alpha}{\beta}\right)$ Symmetry
13. $\left(\binom{\alpha}{\beta}\langle : \rangle\binom{\psi}{\varphi}\right) \longleftrightarrow \left(\binom{\alpha}{\beta}\langle ! \rangle\binom{\varphi}{\psi}\right)$ Equivalence 1
14. $\left(\binom{\alpha}{\psi}\langle : \rangle\binom{\varphi}{\beta}\right) \longleftrightarrow \left(\binom{\alpha}{\beta}\langle ; \rangle\binom{\varphi}{\psi}\right)$ Equivalence 2
15. $\left(\binom{\alpha}{\neg\psi}\langle : \rangle\binom{\neg\varphi}{\beta}\right) \longleftrightarrow \left(\binom{\alpha}{\beta}\langle \text{¡} \rangle\binom{\varphi}{\psi}\right)$ Equivalence 3

Conclusion

We present a brief recapitulation of the notion of analogical proportion, in specific of the four homogeneous analogies, i.e. analogies that not mix different kind of indicators. These ideas extracted from some works of Henri Prade and Gilles Richard served as a base to the presentation of our system of logic. We have presented the language and the needed syntactic elements to understand our modal interpretation. The novelty of our analysis was the use of binomial notation in the representation of analogical relations. Then we continued offering an interpretation of the language and defining the operators of analogy semantically based on two relations that interact simultaneously. Finally, in this section we have presented the relation of logical consequence and we have showed how this relation induce a consequence operation.

In the following section we have analyzed some consequences of our definitions, specifically on the meaning of the operators of analogy and on the possibility of dualize them. In this part we conclude with an effective way to generate the dual operators and we clarify some questions linked to the relations with we define the operators semantically. Finally, we have present a logical calculus based on the semantics defined, the main novelty is the double reference to a one state and a one m-state, and the relation of non-identical m-states.

Although we have analyzed many questions we consider that there are some open questions related with this issue, we mention some of them. In the first place what is the philosophical relevance of the notion of complement state? Is it possible to offer a more restrictive definition that generates different behavior of the operators? Also, how we make more clear the relation between states and m-states? And finally, what versions of the "classical" modal systems may be defined in the logic presented here? These important questions escape the reach of this work and the answers are left to a future research.

Appendix 1

We present a proof of a theorem of the logic $\mathfrak{L}_{\alpha}$, we call this theorem the *Central Permutation Theorem* (CPT) that represents the so called property of the analogy operation. We analyze all the elements of the proof and we explain the main features of the rules defined above.

$$1.\ \neg\left(\left(\binom{p}{q}\langle:\rangle\binom{r}{s}\right)\rightarrow\left(\binom{p}{r}\langle:\rangle\binom{q}{s}\right)\right)\quad hip., \mathfrak{b}_1, s_1$$

$$2.\ \left(\binom{p}{q}\langle:\rangle\binom{r}{s}\right)\quad (1), \mathfrak{b}_1, s_1$$

$$3.\ \neg\left(\binom{p}{r}\langle:\rangle\binom{q}{s}\right)\quad (1), \mathfrak{b}_1, s_1$$

$$4.\ \left(\binom{\neg p}{\neg r}[:]\binom{\neg q}{\neg s}\right)\quad (1), \mathfrak{b}_1, s_1$$

$$\langle \mathfrak{b}_1, s_1\rangle \quad \approx \quad \langle \mathfrak{b}_2, s_1\rangle$$

$$s_1 \leq s_2 \leq \overline{s_3}$$

↙ ↘

$$5.\ p,\ (2), \mathfrak{b}_1, s_2 \qquad r,\ (2), \mathfrak{b}_2, s_2$$

$$6.\ q,\ (2), \mathfrak{b}_1, s_3 \qquad s,\ (2), \mathfrak{b}_2, s_3$$

$$7.\ \neg p,\ (2), \mathfrak{b}_1, s_3 \qquad \neg r,\ (2), \mathfrak{b}_2, s_3$$

↙ ↘ ↙ ↘

$$s_1 \leq \overline{s_3} \leq s_2$$

8.	$\neg p,\ (4), \mathfrak{b}_1, s_2$	$\neg q,\ (4), \mathfrak{b}_2, s_3$	$\neg p,\ (4), \mathfrak{b}_1, s_3$	$\neg q,\ (4), \mathfrak{b}_2, s_2$
9.	$\neg r,\ (4), \mathfrak{b}_1, s_3$	$\neg s,\ (4), \mathfrak{b}_2, s_2$	$\neg r,\ (4), \mathfrak{b}_1, s_2$	$\neg s,\ (4), \mathfrak{b}_2, s_3$
10.	$p,\ (4), \mathfrak{b}_1, s_3$	$q,\ (4), \mathfrak{b}_2, s_2$	$p,\ (4), \mathfrak{b}_1, s_2$	$q,\ (4), \mathfrak{b}_1, s_3$
	×	×	×	×
	(5,8)	(6,8)	(5,9)	(6,9)

This formula is a representation of the property of *central permutation* of the analogy operation [Prade and Richard, 2009a, p. 132]. The property states that if A and B are analogous to C and D, we may conclude that A and C are analogous to B and D. The first line of the proof contains the formula properly said but negated, as this formula is a conditional the following lines 2 and 3, has the antecedent of the formula and the consequent of the formula, respectively. The step 4 is obtained from the application of the weak analogy operator negated. Between lines 4 and 5 are the operator restrictions that states the link between m-states and states.

The lines 5 – 7 contain the resulting formulas from the application of the rule of elimination of the analogy operator affirmed, this rule is applied to the formula in line 2. As we have been mentioned, this rule has as result two branches in which the two pairs of component formulas are sent. In this case p and q are sent to the left branch where p is present in s_2 and q in s_3; and r and s are sent to the right, and r is present in s_2 and s in s_3. An important and restrictive issue of the application of this rule is the presence of the $\neg p$ and $\neg r$ formulas in each branch, this fact is debt to the relation of the states with its complement. In this case s_2 is related with the $\overline{s_3}$ and the information on s_2 (p) is preserved in $\overline{s_3}$, but not in s_3, and for this reason the negation of p is sent to them. The same situation happens with r in s_2 and its negation s_3. This fact causes that in each branch are present three formulas and not only two.

Between the steps 7 and 8 we have a new interaction between states, the reason of this restriction is that the relation of information inclusion between states satisfy properties in the same sense of the modal systems K, T, S4, S5, etc. In this case, the relation is transitive and symmetric, therefore, it is plausible to think that this is a theorem of an extended version of S4. We explain briefly the properties in the example. We have a previous link between s_1 to s_2, and from s_2 to $\overline{s_3}$; and we have a strong operator that "recycles" the mentioned link. We assume the properties of transitivity and symmetry, and we know that the formula with strong operator (step 4) is present in s_1. By transitivity we relate s_1 with $\overline{s_3}$, and with symmetry we relate state $\overline{s_3}$ with s_2, and in this state the formulas of the center of the branches are present (center permutation). Insomuch as this restriction only affects the central branches, and as the formulas in this branches take advantage of the symmetry and transitivity to generate the contradictions needed to close all the branches of the tree, this relation between states is the one that represents central permutation.

We think that this procedure is justified at least in the following idea. When we generate the symmetric and transitive link between the three states, the two m-states become the same m-state, that is, this link serve as "identifier" of m-states. As $s_1$is related with $\overline{s_3}$ state, and s_1 serve as separator of the m-states, we invert the separation and identify the two m-states when we relate $\overline{s_3}$ with s_2. The (Fig. 18) shows how we could think this interaction.

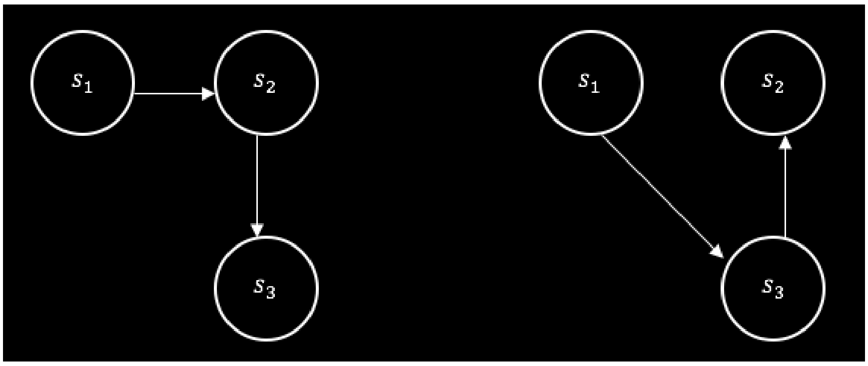

Fig. 18

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