Kant's Views on Non-Euclidean Geometry

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Abstract. Kant's arguments for the synthetic a priori status of geometry are generally taken to have been refuted by the development of non-Euclidean geometries. Recently, however, some philosophers have argued that, on the contrary, the development of non-Euclidean geometry has confirmed Kant's views, for since a demonstration of the consistency of non-Euclidean geometry depends on a demonstration of its equi-consistency with Euclidean geometry, one need only show that the axioms of Euclidean geometry have 'intuitive content' in order to show that both Euclidean and non-Euclidean geometry are bodies of synthetic a priori truths.

Friedman has argued that this defence presumes a polyadic conception of logic that was foreign to Kant. According to Friedman, Kant held that geometrical reasoning itself relies essentially on intuition, and that this precludes the very possibility of non-Euclidean geometry. While Friedman's characterization of Kant's views on geometrical reasoning is correct, I argue that Friedman's conclusion that non-Euclidean geometries are logically impossible for Kant is not. I argue that Kant is best understood as a proto-constructivist and that modern constructive axiomatizations (unlike Hilbert-style axiomatizations) of both Euclidean and non-Euclidean geometry capture Kant's views on the essentially constructive nature of geometrical reasoning well.

Résumé. Les arguments de Kant en faveur d'un statut synthétique a priori pour la géométrie sont généralement considérés comme ayant été réfuté par le développement des géométries non-euclidiennes. Récemment, cependant, certains philosophes ont soutenu, qu'au contraire, le développement de la géométrie non-euclidienne a confirmé la position de Kant, car puisqu'une démonstration de la consistance logique de la géométrie non-euclidienne dépend d'une démonstration de son équiconsistance avec la géométrie euclidienne, il suffit de démontrer que les axiomes de la géométrie euclidienne ont de la «teneur intuitive» afin de démontrer que les géométries euclidienne et non-euclidiennes sont des corpus de vérité synthétique a priori.

Friedman soutient que cette défense présume une conception polyadique de la logique, ce qui était étranger à Kant. Selon Friedman, Kant a jugé que le *raisonnement* géométrique repose essentiellement sur l'intuition, et que ceci exclut toute possibilité de la géométrie non-euclidienne. Quoique la caractérisation de Friedman à propos du point de vue de Kant sur le raisonnement géométrique soit convenable, j'affirme que sa conclusion, selon laquelle les géométries non-euclidiennes sont logiquement impossible pour Kant, ne l'est pas. Je soutiens que Kant est mieux vu comme un proto-constructiviste et que les axiomatisations constructives modernes (contrairement aux axiomatisations style-Hilbert) de la géométrie euclidienne et non-euclidienne représentent bien les vues de Kant sur le caractère essentiellement constructif du raisonnement géométrique.

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Introduction

Kant's theoretical philosophy enjoyed a resurgence in the mid 19th century. Prompted by what were perceived as the mistakes of Kant's idealist successors, many thought it was time to, in Otto Liebmann's words, go 'back to Kant!'[1] The 'Neo-Kantian' movement in philosophy began to gain steam—so much so that by the turn of the century the Marburg and Southwest schools of Neo-Kantianism exerted a powerful influence on philosophical thought in continental Europe.

The same period also witnessed, ironically, developments that were destined to deal a heavy blow to the popularity of Kantian philosophy; these were the revolutionary developments in logic and geometry—two sciences that up until then had evolved very little, in substance, since their inception over two thousand years before. Pure Logic, Kant had argued, could never give one synthetic, i.e., expansive, knowledge; yet close to the turn of the century, Frege developed an early version of the modern predicate calculus. Using the new methods, Frege was able to prove non-trivial theorems (e.g., the ancestral relation). His Logicist project was underway. Hilbert, on the other hand, took advantage of the new methods in order to provide an axiomatization for Geometry; Geometry—Kant's paradigm example of synthetic a priori cognition—was shown, by Hilbert, to follow *analytically* from a few simple axioms. In this same period, *non-Euclidean* geometries (those geometries which deny Euclid's parallel postulate) were rigorously developed and shown to be consistent,[2] and a physical application for non-Euclidean geometry was found with the publication of Einstein's General Theory of Relativity in 1916[3]—not only was was the synthetic status of Euclidean geometry now in doubt, but also its *a priori* status. By the middle of the 20th century, Kant's philosophy had become very unattractive indeed. With respect to the situation in geometry, Carnap summed up the prevailing attitude towards Kant's philosophy at that time as follows:

> It is necessary to distinguish between pure or mathematical geometry and physical geometry. The statements of physical geometry hold logically, but they deal only with abstract structures and say nothing about physical space. Physical geometry describes the structure of physical space; it is a part of physics. The validity of its statements is to be established empirically—as it has to be in any other part of physics—after rules for measuring the magnitudes involved, especially length, have been stated. (In Kantian terminology, mathematical geometry holds indeed *a priori*, as Kant asserted, but only because it is analytic. Physical geometry is indeed synthetic; but it is based on experience and hence does not hold *a priori*. In neither of the two branches of science which are called "geometry" do synthetic judgements *a priori* occur. Thus Kant's doctrine must be abandoned.) [10, p. vi].

Attitudes towards Kant began to shift again in the latter half of the 20th century. By then, Logicism was in decline. In the philosophy of geometry, some philosophers argued (See, e.g., [1]) that, far from refuting Kant's philosophy of geometry, the development of

[1] The phrase is used repeatedly by Liebmann [6] to end the various sections of his book.

[2] Properly speaking, they were shown to be *equi-consistent* with Euclidean geometry.

[3] In General Relativity, spacetime is modelled as a four-dimensional Riemannian manifold.

non-Euclidean geometries only served to *confirm* it. For, since a demonstration of the consistency of non-Euclidean geometry depends on a demonstration of its *equi-consistency* with Euclidean geometry, one need only show that the *axioms* of Euclidean geometry depend for their validity on the so-called 'pure intuition of space' in order to show that both Euclidean *and* non-Euclidean geometry, which depend on these axioms (the latter indirectly via the equi-consistency proof), are bodies of synthetic a priori truths.

Michael Friedman [3] has argued that a defence of this sort, on Kant's behalf, relies on a modern conception of logic that was foreign to Kant.[4] According to Friedman, Kant's views on the synthetic nature of geometry amount to much more than the mere claim that the axioms of geometry need to be grounded in pure intuition. For Kant, according to Friedman, geometrical *reasoning* is synthetic; geometrical reasoning must necessarily appeal to *construction* in pure intuition in order for geometrical cognition to be possible. According to Friedman, Kant is simply wrong, therefore, about the synthetic a priori status of geometry in light of the modern (analytical) methods of geometrical reasoning. According to Friedman, Kant is not to be blamed, however, for he could not have foreseen the developments in logic that allowed for the possibility of the modern axiomatizations of geometry. Rather, he is to be commended for comprehending the limitations of the logic of his own time and for tracing through the implications of these limitations for mathematics. But as a consequence of these considerations, Friedman argues, non-Euclidean geometries are, for Kant, *logically impossible*; for geometrical concepts *themselves* then require construction in pure intuition. And since it is impossible to construct, for instance, two straight lines in pure intuition that enclose a space, such a figure is, for Kant, not only indemonstrable, it is *unthinkable*.

Friedman's characterization of Kant's views on geometrical reasoning (indeed, on mathematical reasoning in general) is correct. I will argue, however, that these views *do not* imply the logical impossibility of non-Euclidean geometries. I will argue that constructive axiomatizations of both Euclidean and non-Euclidean geometries do exist, and that in spite of the abstract symbolization techniques involved, that they do (unlike Hilbert-style axiomatizations) capture Kant's views on the essentially constructive nature of geometrical reasoning well. Further, even if such axiomatizations did not exist, I will argue that it is still not the case that non-Euclidean geometries would be unthinkable for Kant. I will argue that it only follows from this that they are not cognizable and that Friedman does a disservice to Kant by conflating Kant's distinction between thinking and cognizing.

Kant's framework

For Kant, all experience involves both a conceptual and an intuitive part. Intuition, on the one hand, is that which relates directly to the object of experience (the 'this', the 'that', etc., of experience), and there are two forms of intuition which, according to Kant, "[allow] the

[4]Recently, Friedman has changed his views on Kant's philosophy of geometry, so much so that the extent to which I now disagree with his conclusions is no longer clear. I am still in the process of assimilating these changes of viewpoint into a longer version of this paper which I intend to publish elsewhere. This said, the changes in Friedman's views do not negate or make redundant what I take to be the main contribution of this paper in its current form: the elucidation of Kant's views on geometry in a way that emphasises the link between his methods and intuitionistic methods in general, and the claim that non-Euclidean geometry, at least from the constructive point of view, is in fact compatible with Kant's framework for mathematics. For these purposes, Friedman's views (in their earlier incarnation) are used essentially as a foil.

manifold of appearance to be intuited as ordered in certain relations" (A20/B34).[5] These forms of intuition are *space* and *time*: space, the form of outer appearances, and time, the form of both inner and outer appearances. These forms lie in the mind *a priori*, according to Kant, since "that within which the sensations can alone be ordered and placed in a certain form cannot itself be in turn sensation" (A20/B34). Thus, Kant calls them *pure*, since, in them, "nothing is to be encountered that belongs to sensation." (A20/B34).

Kant associates concepts, on the other hand, with rules for the subsumption of intuition. The concept 'horse', for example, corresponds to a rule according to which this bushy tail, that long nose, that mane, and those hoofs can be associated together in one representation. When we synthesize, i.e., combine, some particular manifold of intuition according to the particular rule for a concept, we say that this manifold of intuition has been subsumed under the concept. A *pure* concept of the understanding, i.e., a category, is one of a set of 'meta'-concepts that all empirical concepts necessarily presuppose. Like the pure forms of intuition, these categories are a priori. There are twelve categories in all; three each falling under the four headings of *Quantity*, *Quality*, *Relation*, and *Modality* (Cf. A80/B106). When *pure* concepts of the understanding are applied to *pure* intuitions, we say, then, that we obtain *synthetic a priori cognition*: synthetic because it synthesizes the manifold of intuition; a priori because it is cognition from sources that are *pure*.

Mathematics, according to Kant, is a body of synthetic a priori knowledge. And because the pure intuitions, space and time, are the formal conditions for all possible experience, mathematical knowledge—which deals with the combinations of these forms according to reproducible schema (i.e., rules)—is objectively valid for all possible experience. In particular, Arithmetic is concerned with the pure intuition of time; Geometry, with the pure intuition of space. Focusing on geometry, now, the question that is relevant to our concerns, here, is the question of to the extent to which Kant is committed to the a priori status of specifically *Euclidean* geometry. Kant does not argue explicitly that Euclidean geometry is necessarily the metric according to which we intuit. However it does seem to be implied that Kant believed Euclidean geometry to be the pure a priori form of outer intuition in light of the following considerations. First, it is safe to say that geometry *simply was* Euclidean geometry in Kant's time (i.e., the late 18th century). Thus, a proposal to insert the phrase 'Euclidean geometry' in place of 'geometry' everywhere one sees the former term in Kant's writings (in order to 'translate' it into contemporary speak, as it were) cannot be outlandish. Second, Kant argues, in the Transcendental Aesthetic section of the *Critique*, that his (intuitive) construal of the nature of space is what alone allows us to make sense of the synthetic a priori status of geometrical cognition. He writes: "... our explanation alone makes the **possibility** of geometry as a synthetic *a priori* cognition comprehensible. Any kind of explanation that does not accomplish this, even if it appears to have some similarity with it, can most surely be distinguished from it by means of this characteristic." (A25/B41). Thus according to Kant, the fact that geometry is a body of synthetic a priori truths is what shows us that space must be the pure form of intuition (as opposed to a concept). And if geometry just is Euclidean geometry (for Kant) then *Euclidean* geometry (for Kant) is a body of synthetic a priori truths.

[5]References to the *Critique of Pure Reason* are to the Guyer-Wood translation [5]. Page numbers are as in the standard German edition of Kant's works. "A" denotes the first and "B" the second edition of the Critique.

Some considerations arguing against such an interpretation, however, are the following. First: curiously, nowhere (to my knowledge) in his discussions on geometry (in fact nowhere in the entire first *Critique* or in any of his other major writings on geometry) does Kant mention Euclid by name. Second, even in Kant's day, it was widely recognized that Euclid's fifth postulate was not as self-evident as the other four. In fact, almost since Euclid's original formulation of it, attempts had been made to recast it as a theorem, provable by the other four postulates. Many such proofs were attempted; none of these were successful. Third, the idea of replacing the parallel postulate with an alternative postulate *had*, in fact, been proposed by Kant's time. In the 18th century, Giovanni Girolamo Saccheri, for instance, developed a system of Hyperbolic geometry.[6] In light, especially, of these last two considerations it is not surprising that Kant's terminology seems so cautious to modern ears.

But at any rate, let us assume, for the time being, the stronger interpretation of Kant's use of the term 'geometry' as simply meaning 'Euclidean geometry', and let us consider whether Kant, despite himself, has any resources within his philosophy for answering the challenge posed by non-Euclidean geometries. I suggested one possible answer in the introduction to this essay. One can, as a Kantian in the face of non-Euclidean geometry, simply appeal to the equi-consistency proofs and argue as follows: since the consistency of non-Euclidean geometries depends on the consistency of Euclidean geometry, the Euclidean *axioms* guarantee the consistency of both Euclidean and non-Euclidean geometry. Now the theorems of both non-Euclidean and Euclidean geometry are arrived at by some process of analytic reasoning from axiom to theorem; however, the axioms themselves are exhibitable in pure intuition—indeed, this is why they are self-evident—and this is enough to show that geometry *as such* is a body of synthetic a priori knowledge, for it is a body of knowledge that is *grounded* in intuition.

Considered this way, the development of non-Euclidean geometry, far from refuting Kant's view, only confirms it, for the priority of Euclidean geometry is affirmed in its role as the guarantor of the consistency of non-Euclidean geometry. Further, this accords with Hilbert's own views on the nature of mathematical axioms. In the epigraph to his *Foundations of Geometry*, Hilbert quotes Kant: "All human knowledge begins with intuitions, thence passes to concepts and ends with ideas." [4].

Reasoning by construction

Not all of those who are sympathetic to Kant are sympathetic to this line of argument, however. Michael Friedman has made an important contribution to the literature on Kant by pointing out that the synthetic a priori character of geometry, for Kant, entails more than merely that its axioms must be grounded in pure intuition. For Kant, geometrical *reasoning* must necessarily be synthetic; in other words, geometrical proofs must consist of *constructions* in pure intuition. Friedman writes: "What is most striking to me about Kant's theory, as it was to Russell, is the claim that geometrical *reasoning* cannot proceed "analytically according to concepts"—that is, purely logically—but requires a further activity called "construction in pure intuition." [3, p. 56]. Friedman quotes the following passage from the "Doctrine of method" section of the *Critique* in support of his interpretation:

[6]Saccheri's (unsuccessful) intention was to show the inconsistency of such a geometry.

> [Philosophy] confines itself solely to general concepts, [mathematics] cannot do anything with the mere concepts but hurries immediately to intuition, in which it considers the concept *in concreto*, although not empirically, but rather solely as one which it has exhibited *a priori*, i.e., constructed, and in which that which follows from the general conditions of the construction must also hold generally of the object of the constructed concept. Give a philosopher the concept of a triangle, and let him try to find out in his way how the sum of its angles might be related to a right angle. He has nothing but the concept of a figure enclosed by three straight lines, and in it the concept of equally many angles. Now he may reflect on this concept as long as he wants, yet he will never produce anything new. He can analyze and make distinct the concept of a straight line, or of an angle, or of the number three, but he will not come upon any other properties that do not already lie in these concepts. But now let the geometer take up this question. He begins at once to construct a triangle. Since he knows that two right angles together are exactly equal to all of the adjacent angles that can be drawn at one point on a straight line, he extends one side of his triangle, and obtains two adjacent angles that together are equal to two right ones. Now he divides the external one of these angles by drawing a line parallel to the opposite side of the triangle, and sees that here there arises an external adjacent angle which is equal to an internal one, etc. In such a way, through a chain of inferences that is always guided by intuition, he arrives at a fully illuminating and at the same time general solution of the question. (A715-717/B743-745).

Such a conception of geometrical proof is not in line with mainstream modern views on geometrical reasoning. Certainly, no mainline mathematician today would maintain that geometrical constructions are necessary constituents of geometrical proofs. Modern algebraic geometry is able to function perfectly well without appealing to constructions in intuition. Such constructions are, at the most, aids to our understanding of a proof. On a mainstream modern view of geometrical reasoning, it follows *analytically* from Euclid's *axioms* that, for example, the internal angles of a triangle add up to two right angles.

Far from criticizing Kant for not realizing this, however, Friedman points out that Kant was operating with a logic that was far more limited than our own. Kant's logic was syllogistic logic; what we would now call monadic logic. In the monadic predicate calculus, atomic propositions take only one variable, i.e., they are all of the form P_x. For example, "All men are mortal": $(\forall x)M_x$, "Some men are tall": $(\exists x)T_x$, "Joanne is a brunette": B_j, etc. Polyadic predicate calculus, on the other hand, allows for relational predicates, e.g. "Everybody loves someone": $(\forall x)(\exists y)L_{xy}$ (loves: a 2-place predicate), "George lends his car to Jim": L_{gcj} (lends: a 3-place predicate). In general, the polyadic predicate calculus allows for n-place predicates, as opposed to the restriction on 1-place predicates for the case of the monadic calculus.

In addition to allowing for n-place predicates, another characteristic feature of polyadic logic, that is especially relevant to our discussion, is *quantifier dependence*. Unlike syllogistic logic, we cannot, in polyadic logic, "drive quantifiers in". For example, in syllogistic logic,

$$(\forall x)(\exists y)(F_x \rightarrow G_y) \iff (\forall x)F_x \rightarrow (\exists y)(G_y).$$

In polyadic logic, on the other hand, the existential quantifier in a formula such as $(\forall x)(\exists y)(\mathscr{P})$ depends upon the universal quantifier in a manner that allows us to express functional relationships. In particular, this allows us to model our intuitive idea of an iterative process of generation. Consider the formula:

$$(\forall x)(\forall y)(\exists z)(x < y \rightarrow x < z < y). \tag{1}$$

This formula expresses the intuitive idea of 'denseness'. For each value of x and y, this formula tells us that we can generate some z that falls in between them, which we can then use to replace x (or y) in the formula in order to generate a new value, z'. For example, suppose we start with $x = 1$, $y = 3$. The formula tells us that we can generate some z that falls in between them, and in fact we can: $z = 2$. Now replace x with this value. We now have $x = 2$, $y = 3$, and the formula tells us that we can find a z in between *these* two numbers. We can find such a number, e.g., $z = 2.8$. In fact, the formula tells us that we can do this for *any* two numbers, x and y. Thus, the formula gives us a way to capture the idea of an infinity of objects, *purely logically.* [3, p. 63].

But this is not possible for Kant. Kant's logic is *monadic*, and in monadic logic quantifiers are independent; this leaves us no way to capture, in one concept, the intuitive idea of an infinite number of objects. Thus, Kant writes: "... one must, to be sure, think of every concept as a representation that is contained in an infinite set of different possible representations (as their common mark), which thus contains these **under itself**; but no concept, as such, can be thought as if it contained an infinite set of representations **within itself**." (B40). According to Friedman, in order to represent an infinite number of objects, Kant must appeal to an iterative process of *construction in intuition.* For Kant,

> The notion of infinite divisibility or denseness, for example, cannot be represented by any such formula as [(1)]: this logical form simply does not exist. Rather, denseness is represented by a definite fact about my intuitive capacities: namely, whenever I can represent (construct) two distinct points a and b on a line, I can represent (construct) a third point c between them. Pure intuition—specifically, the iterability of intuitive constructions—provides a uniform method for instantiating the existential quantifiers we would use in formulas like [(1)]; it therefore allows us to capture notions like denseness without actually using quantifier-dependence. Before the invention of polyadic quantification theory there simply is no alternative. [3, p. 63].

This, in Friedman's view, is what lays at the heart of Kant's distinction between synthetic and analytic judgements. To clarify, for Kant, in order for a judgement to be synthetic it must involve an appeal to intuition (B73). If the intuition is empirical, the judgement is synthetic a posteriori; if the intuition is pure, the judgement is synthetic a priori. Thus, in order to cognize two determinations of a thing as pertaining to the same thing, for example, one must cognize these as changes of the thing *in time*; i.e., we must connect them in time. Similarly, geometrical reasoning is synthetic because it exhibits the connections between things *in space.* Now pure intuition, on Friedman's interpretation, amounts to the possibility of iterating this process of construction in intuition indefinitely; a judgement is synthetic if it makes use of such constructions; a judgement is *a priori* synthetic if it makes use merely of the (formal conditions for the) *possibility* of such constructions. "For Kant, this procedure of generating new points by the iterative application of constructive functions takes the place, as it were, of our use of intricate rules of quantification theory such as existential instantiation. Since the methods involved go far beyond the essentially monadic logic available to Kant, he views the inferences in question as synthetic rather than analytic." [3, p. 65].

Kant's argument for the intuitive nature of space, according to Friedman, is based on his observation of the method of geometrical reasoning. Geometrical demonstrations *require* that space consist of an infinite number of parts; but that being the case, space cannot be a concept, for no *monadic* concept—and these were all that were available to Kant—is suitable for the representation of space "as an infinite **given** magnitude." (A25/B39).

The upshot of this, according to Friedman, is that for Kant, intuition is required just in order *to think* mathematical concepts, for if we cannot conceive (since no monadic concept can capture this) of an infinitely dense series of points except by actually connecting them together with an intuitive construction, then it follows that: "We cannot think of a line without **drawing** it in thought, we cannot think of a circle without **describing** it, we cannot represent the three dimensions of space at all without **placing** three lines perpendicular to each other at the same point ..." (B154). In that case, it also follows, according to Friedman, that non-Euclidean geometries are logically impossible, for Kant, for these are not constructible in the space of pure intuition. Friedman writes:

> ... there can be no question of non-Euclidean geometries for Kant. Non-Euclidean straight lines, if such were possible, would have to possess at least the order properties—denseness and continuity—common to all lines, straight or curved. And, on the present interpretation, the only way to represent (the order properties of) a line—straight or curved—is by drawing or generating it in the space (and time) of pure intuition. But this space, for Kant, is necessarily Euclidean ... It follows that there is no way to draw, and thus no way to represent, a non-Euclidean straight line, and the very idea of a non-Euclidean geometry is quite impossible. [3, p. 82].

The thinkability of non-Euclidean geometry

On the issue of whether, for Kant, construction in intuition plays a critical role in mathematical reasoning, Friedman is certainly correct. In ascribing, to Kant, the view that non-Euclidean geometries are not even thinkable, however, Friedman does not do justice to Kant's notion of 'thinkability'. To start with, Kant himself admits of the logical possibility of two straight lines enclosing a space:

> Thus in the concept of a figure that is enclosed between two straight lines there is no contradiction, for the concepts of two straight lines and their intersection contain no negation of a figure; rather the impossibility rests not on the concept in itself, but on its construction in space, i.e., on the conditions of space and its determinations; but these in turn have their objective reality, i.e., they pertain to possible things, because they contain in themselves *a priori* the form of experience in general. (A220-21/B268).

Friedman suggests a way of reconciling these remarks with Kant's requirement of constructibility in intuition by distinguishing two notions of possibility: "What produces confusion here is the circumstance that Kant is operating with two notions of possibility: "logical possibility," given by the conditions of thought alone; and "real possibility," given by the conditions of thought plus intuition" [3, p. 93]. This distinction is a promising—indeed, the correct—way to make sense of Kant's seemingly contradictory statements. But then, curiously, Friedman reiterates his line on the logical impossibility of such figures: "... while there may be no (monadic!) contradiction in the concept of a non-Euclidean figure ... this does not mean that there is a possible non-Euclidean structure containing such a figure. ... There is only one way even to think such properties: in the space and time of *our* (Euclidean) intuition. Considered independently of *our* sensible intuition, then, the concept of a non-Euclidean figure remains "empty" and lacks both "sense and meaning ..."" [3, pp. 93–94].

Friedman is conflating, however, Kant's distinct notions of 'thinking' and 'cognizing'. In Friedman's terminology, thinking refers to a merely logical possibility; cognizing refers to a real possibility. While it is perfectly admissible to *think* without reference to possible experience, it is nevertheless impossible to cognize anything without there being a possibility of exhibiting this something in the space and time of (our) intuition.

> The pure concepts of the understanding are related through the mere understanding to objects of intuition in general, without it being determined whether this intuition is our own or some other but still sensible one, but they are on this account mere **forms of thought**, through which no determinate object is yet cognized. (B150).

Yet our inability to cognize by means of such concepts *does not* imply that such concepts lack *meaning*, at least not for Kant. Such concepts (what Kant calls noumena) *do* have a use; on the one hand, in what Kant calls the negative sense,[7] they function as boundary conditions for sensible experience and serve to define the limits of objective cognition.

> I call a concept problematic that contains no contradiction but that is also, as a boundary for given concepts, connected with other cognitions, the objective reality of which can in no way be cognized. The concept of a **noumenon**, i.e., of a thing that is not to be thought of as an object of the senses but rather as a thing in itself (solely through a pure understanding), is not at all contradictory; for one cannot assert of sensibility that it is the only possible kind of intuition. Further, this concept is necessary in order not to extend sensible intuition to things in themselves, and thus to limit the objective validity of sensible cognition (A254/B310).

In the *positive* sense, on the other hand, a noumenon corresponds to what Kant calls an idea of reason: the concept of an object of nonsensible intuition that is thinkable, but not cognizable, due to our inability to exhibit the object corresponding to such a concept in spatio-temporal intuition. In this positive role, the noumena are thought of as regulative ideals or archetypes that guide us in our quest for knowledge (*Cf.* A508-515/B536-544). By conflating Kant's distinction between thinking and knowing, Friedman misrepresents Kant's philosophical enterprise. There is much more to Kant's philosophical universe than mathematical or scientific cognition.

[7]Kant distinguishes between noumena in the negative sense: "objects that are not of sensible intuition" and noumena in the positive sense: "objects of nonsensible intuition".

Constructive axiomatizations

Friedman's account of Kant's *reasons* for appealing to construction in intuition as the necessary condition for the objectification of mathematical concepts is also in error. Friedman seems to think that Kant would not have felt the need to appeal to construction in intuition at all had it not been in order to transcend the limitations of his own monadic logic. The implication seems to be that polyadic logic obviates this need. Yet Kant had independent reasons for requiring that all cognition involve a reference to intuition (pure intuition, for mathematics; empirical intuition, for empirical science); for, as I alluded to above, it was Kant's desire to *limit* objective cognition to appearances, and in so doing, limit the pretensions of speculative reason so that it could not pretend to speculate on the nature of things as they are in themselves. Indeed, Kant's practical philosophy, which arguably was his main philosophical concern, is grounded on this limitation of theoretical reason.

But perhaps most importantly, Friedman ignores the *ongoing* philosophical debates between intuitionist and classical philosophers of mathematics with respect to the validity of certain notions of classical logic for mathematics. Whichever position Kant would have taken in these debates, it is clear from the above considerations that he would not have been on the side of those who affirm the validity of classical (polyadic) logic for the whole domain of mathematical discourse. To this effect, what is particularly relevant to our discussion is the fact that Hilbert-style axiomatizations of geometry are not the only ones possible. Friedman sees the presence of some such axioms as the following to be characteristic of all geometrical axiomatizations:

(1) $\sim (a < a)$
(2) $(a < c \;\&\; c < b) \rightarrow a < b$
(3) $a < b \vee b < a \vee a = b$
(4) $\forall a \exists b (a < b)$
(5) $\forall b \exists a (a < b)$
(6) $\forall a \forall b \exists c (a < b \rightarrow a < c < b)$

There do exist, however, *quantifier-free* geometrical axiomatizations in which the basic axioms of the system are expressed in terms of geometric *constructions*. Nancy Moler and Patrick Suppes' [7] axiomatization of the (Euclidean) geometry of ruler and segment transporter,[8] for example, utilizes two primitive constructions: that of laying off (or transferring) line segments, and that of finding the intersection of two lines. It utilizes three individual constants standing for points: α, β, γ. Segment transport is symbolized by $S(xyuv) = w$, which we read as: "the point w is as distant from u in the direction of v as y is from x." [7, p. 144]. Finding the intersection of two lines is symbolized by $I(xyuv) = w$, which we read as: "the point w is collinear with the two points x and y and also collinear with the two points u and v; in other words, w is the point of intersection of lines xy and uv." [7, p. 144]. The axiomatization consists of four definitions ("Betweenness", "Collinearity", "Noncollinearity of four points", and "Distinctness") and eighteen axioms for the geometry.

Moller and Suppes begin their pioneering work as follows:

> The purpose of this paper is to state a set of axioms for plane geometry which do not use any quantifiers, but only constructive operations. ... as far as we know, no prior set of quantifier-free axioms for plane geometry has

[8]Cf. [4, §§36–39].

> been published. In a way, this omission is surprising, for an emphasis on geometric constructions has existed for a long time. The step of explicitly stating axioms in terms of the familiar constructions seems not to have been taken. In view of the highly constructive character of Euclidean geometry, it seems natural to strive for a formulation that eliminates all dependence on purely existential axioms, but not, of course, by the use of some wholly logical, non-geometric method of quantifier elimination. [7, p. 143].

Constructive axiomatizations have also been given for Hyperbolic geometry. An axiomatization of the hyperbolic plane, utilizing only ternary (3-place) construction operations has been formulated by Pambuccian [8]. Hyperbolic constructions are the analog of Euclidean ruler and compass constructions; however, in Hyperbolic geometrical construction, three specialized compasses are used instead of one:[9] the *H-compass*, *Horocompass*, and *Hypercompass*. Given three non-collinear points,[10] A, B, C that determine a triangle in the Hyperbolic plane (visualized as a disc), ΔABC, and three perpendicular bisectors $l_i; i = 1, 2, 3$, the bisectors may be either: concurrent (in that case they determine a H-circle, constructible by the H-compass); asymptotically parallel (i.e., they meet on the boundary of the plane. In this case they determine a 'Horocycle', constructible by the Horocompass); divergently parallel (i.e., they intersect neither within the plane nor on its boundary. In this case they determine a 'Hypercircle', constructible by the 'Hypercompass').[11]

Constructive axiomatizations such as these require only intuitionistic quantifier-free logic and seem to capture the traditional picture of geometrical reasoning that Kant had in mind. Pambuccian writes:

> ... *postulates* ask for the production ... of something not yet given ... whereas *axioms* refer to ... a given, to insight into the validity of certain relationships that hold between given notions. In traditional axiomatizations, that contain relation symbols, and where axioms are not universal statements, such as Hilbert's, this ancient distinction no longer exists. The constructive axiomatics preserves this ancient distinction, as the ancient postulates are the primitive notions of the language, namely the individual constants and the geometric operation symbols themselves ... whereas what Germinus would refer to as "axioms" are precisely the axioms of the constructive axiom system. [9, p. 25].

It may seem strained to consider even this type of axiomatization as compatible with Kant's views on mathematics, for all of these constructive axiomatizations require the use of abstract symbols to represent operations and points—a very far cry from the 'hands on' method of geometrical proof that Kant describes and that I quoted above in A715-717/B743-745. But that constructive axiomatizations such as the one above capture Kant's views on mathematics well should be clear when one considers Kant's views on algebra (which he views as a generalization of arithmetic).

> But mathematics does not merely construct magnitudes (*quanta*), as in geometry, but also mere magnitude (*quantitataem*), as in algebra, where it

[9] It has been shown that we cannot construct any more using these instruments than we can by means of a regular ruler and compass. [9, pp. 33-34].

[10] Collinear points are points that lie on the same line segment.

[11] Cf. [2].

> entirely abstracts from the constitution of the object that is to be thought in accordance with such a concept of magnitude. In this case it chooses a certain notation for all construction of magnitudes in general (numbers), as well as addition, subtraction, extraction of roots, etc., and, after it has also designated the general concept of quantities in accordance with their different relations, it then exhibits all the procedures through which magnitude is generated and altered in accordance with certain rules in intuition; where one magnitude is to be divided by another, it places their symbols together in accordance with the form of notation for division, and thereby achieves by a symbolic construction equally well what geometry does by an ostensive or geometrical construction (of the objects themselves), which discursive cognition could never achieve by means of mere concepts. (A717/B745).

Consider, first, basic arithmetic. It is possible to represent the addition of one unit to another directly by recourse to empirical intuition, e.g., counting on one's fingers. What is happening here is that each successive synthesis, i.e., combining, of the manifold of intuition constitutes one instant of time—one unit—which we can then aggregate with other units. Since time is the pure a priori form of all intuition, our ability to count in this way does not depend on any particular empirical combination of the manifold, but only on our ability to synthesize representations in general, and on our ability to give an objective ordering to these representations. In order to count higher numbers, we begin by defining symbolic representations for certain magnitudes, for example: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$. We then define operations that we can perform with these symbols (e.g., addition and subtraction), rules for the addition of larger numerals (e.g., 'carrying' in addition, 'borrowing' in subtraction, etc.), extensions of the basic operations (e.g., multiplication, division, exponentiation, etc.). In this way, we slowly build up a system of arithmetic. In Algebra, we define rules for the addition and subtraction, etc. of numbers *as such*, e.g., $(x-1)^2 = x^2 - 2x + 1$. However algebra is *not* different, at least for Kant, in that it must always be possible, at least in principle, to exhibit whatever we symbolize (using our abstract notation) in our system directly in intuition.[12]

Kant did not foresee the developments that were to take place in geometry. In Kant's time, geometry was much like arithmetic before the advent of algebra. Nevertheless, it should be obvious that Kant would not have barred us from symbolizing our geometrical constructions in much the same way that we symbolize our arithmetical constructions. Constructive axiomatizations, such as have been formulated by Moler and Suppes for Euclidean geometry, or such as have been formulated by Pambuccian for Hyperbolic geometry, can be considered as generalizations of geometrical construction in much the same way that the algebra of Kant's time was seen as a generalization of arithmetic, just so long as in principle it is possible to exhibit whatever is represented by our symbolizations directly in intuition.

To conclude: we have seen that, while Friedman is correct in his interpretation of Kant's views on the nature of geometrical reasoning, he is incorrect in his construal of the implications of this for Kant's notion of logical possibility. We have also seen how Friedman's uncritical exposition of classical logical inference ignores the fact that alternative (constructive) models of inference exist and are applicable to geometrical cognition. Non-Euclidean geometries are thinkable, for Kant, and since these geometries can be axiomatised constructively, their objective validity is cognizable as well.

[12]Classical, but not intuitionistic mathematics, actually fails this test.

References

1. Lewis White Beck, *Can Kant's synthetic judgments be made analytic?*, Kant-Studien **47** (1955), 168–181.
2. Robert R. Curtis, *Duplicating the cube and other notes on constructions in the hyperbolic plane*, Journal of Geometry **39** (1990), 38–59.
3. Michael Friedman, *Kant and the exact sciences*, Harvard University Press, Cambridge, MA, 1992.
4. David Hilbert, *The foundations of geometry*, Open Court Publishing Company, La Salle, Illinois, 1902, Reprint Edition (1950).
5. Immanuel Kant, *Critique of pure reason*, Cambridge University Press, Cambridge, 1998.
6. Otto Liebmann, *Kant und die Epigonen*, Carl Schober, Stuttgart, 1865.
7. Nancy Moler and Patrick Suppes, *Quantifier-free axioms for constructive plane geometry*, Compositio Mathematica **20** (1968), 143–152.
8. Victor Pambuccian, *The simplest axiom system for plane hyperbolic geometry*, Studia Logica **77** (2004), 385–411.
9. ———, *Axiomatizing geometric constructions*, Journal of Applied Logic **6** (2008), 24–46.
10. Hans Reichenbach, *The philosophy of space & time*, Dover Publications Inc., New York, 1958, English edition, with introductory remarks by Rudolf Carnap.

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