# A Simple Interpretation of Quantity Calculus

Boris Čulina
Department of Mathematics
University of Applied Sciences Velika Gorica
Zagrebačka cesta 5, 10410 Velika Gorica, Croatia
e-mail: boris.culina@vvg.hr

*"Entities are not to be multiplied without necessity"*

attributed to William of Ockham

**Abstract.** A simple interpretation of quantity calculus is given. Quantities are described as functions from objects, states or processes (or some combination of them) into numbers that satisfy the mutual measurability property. Quantity calculus is based on a notational simplification of the concept of quantity. A key element of the notational simplification is that we consider units intentionally unspecified numbers that are measures of exactly specified objects, states or processes. This interpretation of quantity calculus combines all the advantages of calculating with numerical values (since the values of quantities are numbers, we can do with them everything we do with numbers) and all the advantages of calculating with classically conceived quantities (calculus is invariant to the choice of units and has built-in dimensional analysis). This also shows that the whole metaphysics of the common concept of quantities and their magnitudes is irrelevant to quantity calculus. As an application of this interpretation of quantity calculus an easy proof of dimensional homogeneity of physical laws is given.

**Keywords.** quantities, units, quantity calculus, dimensional homogeneity

Quantity calculus is a relatively easy calculus but with unclear interpretation. We calculate with quantities (some prefer to say the magnitudes of quantities) and units in the same way as with numerical variables. However, the problem is how to justify this calculus, and, generally, how to interpret quantities and units, as well as operations with them. A clear historical survey of the problem is given in [dB95]. However, in my view, the survey also shows that the various offered answers involve unnecessary metaphysics and mathematics. In this article, the concept of quantity is analysed and a simple interpretation of quantity calculus is given: quantities are functions from objects, states or processes (or some combination of them) into numbers that satisfy the mutual measurability property (defined below), while units are intentionally unspecified numbers that are measures of exactly specified objects, states or processes. Consequently, only ratios of values of a quantity function are determined without reference to units. This interpretation has three straightforward but significant consequences:

1. Quantities of objects, states or processes are not (additional) metaphysical entities - they are just numbers associated to objects, states and processes by definite functions (quantities or quantity functions). Consequently, there is no need for additional mathematical objects – the so-called magnitudes of quantities.

2. We can calculate with quantities of objects, states or processes and with units as with numbers, because they are numbers.

3. Because units are unspecified numbers, we can only find ratios of the values of a quantity. However, this is just a proper level of abstraction, because only ratios of the values are important. Everything else would be unwanted overspecification.

An analysis of the concept of quantity follows. The analysis gives the previously described interpretation of quantity calculus as a notational simplification of the concept of quantity. Finally, it is shown how this interpretation solves the standard requirements on quantity calculus. This also shows that the usual metaphysical concepts related to quantity calculus are unnecessary for its explanation.

Already in primary school we got instructions how to manipulate units when calculating physical quantities: *calculate with units in the same way as you calculate with variables in algebraic expressions.* For example, if I drive uniformly at a speed of $v = 90\ \dfrac{\text{km}}{\text{h}}$ for $t = 10$ min then I will cover a path whose length $s$ is

$$s = v \cdot t = 90\ \frac{\text{km}}{\text{h}} \cdot 10\ \text{min} = 900\ \frac{\text{km} \cdot\ \text{min}}{\text{h}}$$

(to calculate further I must know that $\text{h} = 60$ min)

$$= 900 \frac{\text{km} \cdot\ \text{min}}{60\ \text{min}}$$

(we can cancel min now)

$$= 15\ \text{km}$$

We manipulated with length $s$ of the path, velocity $v$ of the car and time $t$ of the motion, which are usually called physical quantities or magnitudes (I will use the term *values of quantities*), and with units km, h and min as they are all unknown numbers. The manipulation is the same as with variables which are also considered as unknown numbers. For example, we cannot simplify $\dfrac{a}{b}$ because we do not know which numbers they name, but we can simplify $\dfrac{a}{a} = 1$ although we do not know which non-zero number is named by variable $a$. In the same way, we cannot simplify $\dfrac{\text{km}}{\text{h}}$ but we can simplify $\dfrac{\text{min}}{\text{min}} = 1$. The main goal of this article is to show that we can manipulate the values of quantities and units as numbers precisely because they are numbers.

Because of this similarity in manipulation of quantities and units with variables, the significance of variables in thinking will be briefly explained. The attention here will be restricted to numbers but the observation is general. Variables are names of intentionally unspecified numbers. In this way we gain generality in thinking. For example, we use variable "$x$" to denote an unspecified number $x$. Whatever we conclude about $x$, because we do not use anything specific about $x$, is true "for all $x$". Thanks to this unspecified part, the mechanism of variables allows the transition in thinking from statements about concrete numbers to thinking with universal laws about numbers, keeping the simplicity of thinking with concrete numbers. It will be shown that a similar kind of abstraction resolves the problem of quantity calculus.

Analysis of quantity calculus must first answer the question of what quantities are. Einstein wrote in [Ein36] that "The whole of science is nothing more than a refinement of everyday thinking". We can also apply this to the concept of measurement. The model for each measurement is the measurement of lengths of geometrical segments. That is why I will start the analysis with this paradigmatic example. We always measure a segment $S_1$ (what we measure) by comparing it with another segment $S_2$ (by which we measure). The result is a positive number which will be denoted $l(S_1, S_2)$. The function $S_1, S_2 \mapsto l(S_1, S_2)$ will be termed the length function. It is a function $l : \mathcal{S} \times \mathcal{S} \longrightarrow \mathbb{R}^+$, where $\mathcal{S}$ is the set of all segments and $\mathbb{R}^+$ is the set of positive real numbers. From an analysis of the process of measurement there follows the characteristic property of the length function, that it is a linear function in the second argument in the following sense: for segment $U$ which is $\alpha$ times greater than segment $V$ ($\alpha = l(U, V)$) the value $l(S, V)$ of the length function is also $\alpha$ times greater then the value $l(S, U)$: $l(S, V) = \alpha l(S, U)$. If we substitute $l(U, V)$ for $\alpha$, we get a simple expression for linearity:

$$l(S, V) = l(S, U)l(U, V)$$

Following this paradigmatic example, we should look at all other quantities, as numerical functions that are characterized by the aforementioned property. This view of quantities differs from the usual view that is officially expressed in the *International Vocabulary of Metrology—Basic and general concepts and associated terms* (VIM3) [VIM12]: "**quantity**: property of a phenomenon, body, or substance, where the property has a magnitude that can be expressed as a number and a reference". This view is based on the appropriate metaphysics of properties that exist independently of us, and which is very questionable. However, I will not deal with a critique of that metaphysics here[1], but my goal in this article is to show that this metaphysics is not needed at all for quantity calculus. VIM3 also reflects the common confusion present in describing these terms. What defines quantity as a special property in the previous description is that it has its own magnitude.

[1] The interested reader can read [Č20]

And what is magnitude? It is what can be expressed by number and reference. Let's leave aside the inaccuracy of this formulation and ask ourselves what the reference is. In VIM3 under NOTE 2 it says: "A reference can be a measurement unit, a measurement procedure, a reference material, or a combination of such. And what is a measurement unit? It says in VIM3: measurement unit: real scalar quantity, defined and adopted by convention, with which any other quantity of the same kind can be compared to express the ratio of the two quantities as a number. Thus in VIM3 the term quantity is defined[2] by the term quantity. If we discard all unnecessary elements in this whole description, which is ultimately vague and logically unacceptable, what remains important and unquestionable is reality and numbers by which we measure reality. The part of the reality that is being measured and the part of the reality that we are measuring participate in the measurement, and the measurement gives a number for the result. We also know that there are various measurements applied to different parts of reality. The same type of measurement can be applied to different parts of reality and it determines the function from such parts of reality into numbers. It is these functions that are important. We determine them by measuring processes or we postulate them within the framework of some physical theory. They connect parts of reality and numbers. Further analysis will show that everything needed for quantity calculus is in these functions - neither quantities nor magnitudes of quantities are needed, as described in VIM3. I will call these functions quantities or quantity functions, the name which allows readers who prefer the metaphysical term quantity to avoid collisions of names. I find that the previous analysis justifies the following definitions of terms positive quantity and general quantity (quantity function in alternative terminology).

A *positive quantity* (or *positive quantity function*) is any function from some non-empty set $\mathcal{W}$ of objects, states ad processes (or some combination of them) to positive real numbers, the function $Q : \mathcal{W} \times \mathcal{W} \longrightarrow \mathbb{R}^+$ such that for all $W, V, U \in \mathcal{W}$

$$Q(W, V) = Q(W, U)Q(U, V)$$

I will term this property the *mutual measurability property.* The value $Q(W, V)$ will be termed the *relative measure* of $W$ in respect to $V$, or more

[2]The authors of VIM3 consider this to be a definition and not a description of the concept of quantity.

simply the *value of the quantity*. For simplicity, I will call the objects, states, and processes belonging to the quantity function domain its arguments or parts of reality.

The definition of any quantity function, whose value can be any real number (not necessarily positive), due to the presence of zero requires some modification. The existence of zero value means that some arguments of this function cannot measure other arguments, so they cannot be on the second input of the function $Q$. Therefore, we define a general *quantity* (*quantity function*) as a function $S \times S^1 \longrightarrow \mathbb{R}$, where $S^1$ is a nonempty subset of $S$, so that in addition to the measurability property, a special condition on the arguments of measure zero also applies:

$$Q(W,V) = 0 \text{ for some } V \in S^1 \leftrightarrow W \notin S^1$$

Argument $W$ such that $Q(W,V) = 0$ for some $V$, that is, the argument that can occur only at the first input of the function $Q$, we will call *null carrier*, and the other arguments we will call *unit carriers*. When we write Q(W,V) there is a tacit agreement that this expression makes sense, that is, that $V$ is necessarily a unit carrier, so this will not be particularly emphasized.

From the measurability property it easily follows that for a null carrier $W$ is $Q(W,U) = 0$, for each unit carrier $U$. Indeed, by definition of null carrier , there is $V$ such that $Q(W,V) = 0$. Thus, for an arbitrary unit carrier $U$ from the measurability property it follows

$$0 = Q(W,V) = Q(W,U)Q(U,V)$$

Since $U$ is a unit carrier, so $Q(U,V) \neq 0$, it follows from the above equation that $Q(W,U) = 0$.

Also, using measurability property it is easy to show that for each unit carrier $U$ what we expect is valid: $Q(U,U) = 1$. Namely, if we put in measurability property $W = V = U$, we get

$$Q(U,U) = Q(U,U)Q(U,U)$$

Since $U$ is a unit carrier, $Q(U,U) \neq 0$, then by cancelling $Q(U,U)$ on both sides of the equation we get that $Q(U,U) = 1$.

From the very definition of a quantity function (from measurability property) it follows that only relative measures make sense. If we compare the measurements with two unit carriers $U$ and $V$, it is easy to see that the relative measurements in relation to these two carriers always differ by the same multiplicative factor. $k = Q(U, V)$:

$$Q(W, V) = k \cdot Q(W, U)$$

ie that the ratios of relative measures are independent of the choice of unit carriers:

$$\frac{Q(W_1, V)}{Q(W_2, V)} = \frac{Q(W_1, U)}{Q(W_2, U)}$$

Although relative measures are numbers, in order to have a simple quantity calculus as we use it in practice, we need additional notational simplification that will hide the fact that one argument has infinitely relative measures. If we were completely explicit, then, for example, we would have to write $v(P, U)$ for the velocity of the particle $P$ in relation to the referent object in motion $U$. We should describe each value in quantity calculus in the same way and it would be unnecessarily cumbersome. True, we usually imply arguments, so we could simplify the notation:

$$Q(P, U) \mapsto Q_U$$

But even then, the notation is unnecessarily bulky because we are constantly pulling a reference in the notation. Of course, if we chose one unit carrier as the standard then each argument would have a unique measure. We could then remove the selected reference from the notation and get a simple record: $Q_U \mapsto Q$. But it is an overspecification that we want to avoid, Although we need a unit carrier for measuring, there is no theoretical reason to prefer any unit carrier. We want to work in a simple notation simultaneously with all the measures of a given argument. We want a simple quantity calculus invariant to unit carriers. And we can achieve it because the measurability property gives us a simple connection between unit carriers. Let $U$ be some salient unit carrier, and $V$ any other unit carrier. The relative measures of any measured $W$ in relation to these two unit careers differ, by measurability property, up to a multiplicative constant:

$$Q(W, V) = Q(W, U)Q(U, V)$$

Not writing arguments gives the first simplification. We will also reduce the role of unit carriers by placing them in indices, and we will denote the relative measure of U in relation to V by $u_V$:

$$Q_V = Q_U \cdot u_V$$

Since this relation is valid for any unit carrier $V$, we will forget about it in the notation:

$$Q = Q_U \cdot u$$

The position $Q_U$ next to $u$, the measure of $U$, carries the information that this number depends on $U$, so we do not have to emphasize this, ie we can remove the index $U$ from the notation. The relative measure of $Q$ in a given unit of measure is usually denoted by $\{Q\}$. Thus we get the standard notation of quantity calculus:

$$Q = \{Q\} \cdot u$$

Let us emphasize once again its interpretation: the relative measure $Q$ of the object $W$ in relation to any unit carrier $V$ is equal to the product of the relative measure $\{Q\}$ in relation to a salient unit carrier $U$ and the relative measure $u$ of the unit career $U$ relative to $V$. It is nothing but a measurability property in a simplified notation. The difference from the standard interpretation of quantity calculus is that they are all numbers, not magnitudes of quantities, as described in VIM3. It follows from the nature of measurement that only $\{Q\}$ is a definite number while $Q$ and $u$ are indeterminate up to the choice of unit of measurement. Thus $u$, like any other unit, is an unspecified value (number) of a precisely specified unit carrier: $u = Q(U, V)$, where $U$ is a specified unit carrier while $V$ is an unspecified " any " unit carrier. Just as we consider variables to be the names of intentionally unspecified objects, so we can think of units as constants that are intentionally unspecified measures (because we did not specify the relative measure) of precisely specified unit carriers. Just as the mechanism of variables allows us abstraction in thinking, so the mechanism of unit constants allows us the right level of abstraction for a simple quantity calculus. It allows us to simultaneously calculate with all the relative measures of the arguments given.

Let us illustrate this interpretation on the example of measuring the length of a segment using some standard unit carrier, e.g. the prototype of the metre which is kept at the International Bureau of Weights. Let's call $\mathrm{m} > 0$ (metre) the length of the prototype (telative to any other unit carrier). Then we can express the length $l$ of any segment using $m$. For example, if in measuring a segment $S$ by a carrier of metre, the carrier can be posit just 3 times on $S$ then the length $l$ of $S$ is always $l = 3 \cdot \mathrm{m}$ whatever segment we take for the "official" unit segment (whatever value for $m$ we use). The choice of an official unit segment determines only what number is m. If we take the metre carrier as the official unit carrier then $\mathrm{m} = 1$ and $l = 3$. If we take the foot carrier as the official unit carrier then $\mathrm{m} \simeq 3.28$ and $l = 3 \cdot 3.28$, because we can posit a carrier of foot approximately 3.28 times on a metre carrier. However, it is not important at all what number is m. Knowing that m is a number associated with the definite segment (a metre carrier ) is enough: then we know exactly how much is $3\,\mathrm{m}$ – it is the length of the segment in which a metre carrier posits exactly three times. Because of this we do not need to choose any segment as an official unit segment – we can work with "any" official unit segment. How many times we can posit such a chosen unit segment on a metre carrier will be denoted as m, on a foot carrier as ft, etc. lt is not important at all what numbers these are, because we can express all lengths of segments by them. Also, we have formulas to transform these units, independently of their values in the chosen unit segment. It is always $\mathrm{m} \simeq 3.28 \cdot \mathrm{ft}$ as well as $l = 3 \cdot \mathrm{m} \simeq 9.84 \cdot \mathrm{ft}$. In this way we get the simple unit invariant theory just by associating constant symbols to various unit carriers. The values of these constants are measures of specified unit carriers although these values are unspecified. And this is just a proper level of abstraction, because only relative measures (ratios) are significant. The specification of these constants (the choice of definite unit carriers) is an unnecessary specification which destroys the nature of measuring.

This interpretation of quantity calculus combines all the advantages of calculating only with numerical values in a given selection of units, which dominated physics until the 1920s, and all the advantages of calculating with quantities as products of numerical values and units, which began to dominate physics thanks primarily to Wallot's works [Wal26, Wal57]. This transition lasted for a long time precisely because of the insufficiently clear interpretation of the quantity of calculus.

In addition to a clear interpretation, computing only with numerical values allow any mathematical operation, as opposed to computing with classical quantities. For example, if you need to find a derivative using a logarithm. Calculating with classically conceived quantities does not allow the application of logarithm because it makes no sense to talk about the logarithm of one meter, just as it does not allow many other mathematical operations that occur naturally in mathematical processing of functions and equations that connect numerical values. In the interpretation developed in this article, the values of quantities and units are numbers so that we can do with them everything we do with numbers – there is no additional limitation as with the classically interpreted quantity calculus. Also, when zero is obtained in numerical computation, it is always the same number, while in classical quantity calculus we have infinite zeros. We should even write not only 0, but, if it is not a dimensionless quantity, 0 meters or 0 joules, etc.

On the other hand, calculation with numerical values is connected to a certain choice of units and thus loses a very important property of quantity calculus – the invariance to the choice of units, easy transition from one system of units to another, distinguishing quantities of different types and kinds, and dimensional analysis.[3] The interpretation developed in this article shows that the invariance to the choice of units, as well as the simple transition from one unit to another, can be achieved without introducing the classical concepts of quantity and magnitude. It will be shown below that this interpretation also enables distinguishing quantities of different types and kinds, and dimensional analysis.

Quantities that have the same unit of measure are said to have the *same dimension*. It is easy to see that this is an equivalence relation that does the partition of the set of all quantities into equivalence classes. The *dimension* of a quantity can be defined as the equivalence class to which the quantity belongs. Within the same dimension we can define when two quantities are of the same kind - when they can be measured by the same unit carrier, such

[3]These advantages and disadvantages of computing with numerical values are clearly seen in Bridgman's book [Bri22], which alternates masterful parts, where numerical values are important, and burdened parts, when units of measure must be included in the discussion.

as the potential and kinetic energy of a body. Otherwise, since they have the same unit, this means that, although there are not the same carriers of the unit, there are different carriers of that unit, one of which measures one quantity and the other another, such as energy and moment of force. Of course, this division of quantities into dimensions, as well as the division into the same or different kinds within the same dimension, is relative – it depends on the physical theory and measurement conventions we have set. For example, in the theory of relativity it is natural that spatial and temporal distances are measured by the same measure and not by different measures as in non relativistic physics. Therefore, in relativistic theories, it is natural to assume that these are quantities of the same dimension and type, while in classical physics they are of different dimensions.

The value of a given quantity on a given argument is in a simplified notation, by which we avoid the existence of infinitely relative measures, always the product of its numerical value and unit. However, we cannot generally define quantity value (not the value of a quantity!), which should replace the concept of magnitude, as the product of a numerical value and a unit, without invoking the corresponding quantity function. We cannot simply define that quantity value is every product of a number and a unit, because a unit is also a number, although intentionally unspecified. By that definition, any number would be a quantity value and all quantity values would have the same dimension. But the intended concept of quantity value can be obtained by the following construction. For each unit $u$ we define the corresponding *quantity value function* $\alpha \mapsto \alpha u$. For $Q = \alpha u$ we say that it is the *quantity value* associated with the numerical value $\alpha$ in a given unit $u$, or abbreviated, as usual, that $\alpha u$ is a quantity value. Quantity values (more precisely, quantity value functions) can also be classified into dimensions that correspond to the dimensions of quantity functions. Namely, we will define that two quantity values generated by the units $u$ and $v$ are equivalent if those units belong to the same quantity function (differ up to a multiplicative constant). It is easy to see that this is an equivalence relation that classifies quantity values into appropriate classes that can be considered dimensions of quantity values. These dimensions are closely related to the dimensions of quantity functions so we can identify them. Now we can easily show which combinations of quantity values are also quantity values:

1. The product of quantity values is a quantity value. Indeed, $\alpha u \cdot \beta v = \alpha\beta uv$.

Analogously, the quotient is shown to be a quantity value.

2. The sum of quantity values is a quantity value if and only if they have the same dimension. To prove it, consider when it can be $\alpha u + \beta v = \gamma w$. Since we are actually talking about quantity value functions, this must be valid for all $\alpha$ and $\beta$ ($\gamma$ depends on $\alpha$ and $\beta$). If we put $\beta = 0$, we get that $\beta v = \gamma_1 w$. Therefore, $v$ and $w$ differ up to the multiplicative constant. In the same way, putting $\beta = 0$ we get that $u$ and $w$ differ up to the multiplicative constant. This means that the sum of quantity values is also quantity value if and only if they have the same dimension. Then their sum is also of the same dimension. The analogous result is valid for the difference of quantity values.

Thus we see that the interpretation of quantity calculus described here on the one hand allows unlimited application of operations with numbers, and on the other hand retains important dimensional criteria.

As an application of this interpretation of quantity calculus an easy proof will be given of dimensional homogeneity of physical laws which are expressed in the quantity calculus, that is, the laws that are the unit invariant. Let's take a coherent system of units. For example, we can take metre (m), kilogram (kg) and second (s) in classical mechanics which determine the derived units, for example, the unit of force N = kg m s$^{-2}$. Let's denote such unit for quantity $q$ with $u_q$. For the sake of simplicity a simple case will be taken when quantity $y$ depends only on one quantity $x$:

$$y = f(x)$$

It will be shown that function $f$ has the property of dimensional homogeneity, i.e. it obeys the law:

$$f(u_x x) = u_y f(x)$$

Let's remember that units are unspecified numbers, so this relation really gives the scaling factors. The scaling factor for each quantity is exactly its unit of measure! For example, for Newton's second law

$$F = f(m, a) = ma$$

we have the scaling condition

$$f(\text{kg } m, \text{ms}^{-2}\ a) = \text{kgms}^{-2} f(m, a)$$

For example, if we take s = 2, m = 5 and kg = 8, then it means that if we want $\text{ms}^{-2} = \dfrac{5}{4}$ times greater acceleration and kg = 8 greater mass then we must have $\text{kgms}^{-2} = 10$ times greater force.

The proof is easy. Let $y = f(x)$. It means that in the chosen units

$$y_n u_y = f(x_n u_x) \quad (1)$$

where $x_n$ and $y_n$ are numerical values of $x$ and $y$ in the chosen units. However, when we take all basic units to be 1, then all the derived units will be one. So, in this choice $u_x = u_y = 1$ and $y_n = f(x_n)$. Substituting this expression for $y_n$ in (1) we get

$$f(x_n) u_y = f(x_n u_x)$$

i.e.

$$f(u_x x_n) = u_y f(x_n)$$

Because $x_n$ is any number, we get what we want to prove:

$$f(u_x x) = u_y f(x)$$

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