Apriority of Euclidean Geometry

Boris Čulina

Department of Mathematics, University of Applied Sciences Velika Gorica,

Zagrebačka cesta 5, Velika Gorica, CROATIA

email: boris.culina@vvg.hr

"I come more and more to the conviction, that the necessity of our [Euclidean] geometry cannot be proven, at least not by human understanding nor for human understanding. Perhaps in another life we come to different insights into the essence of space, that are now impossible for us to reach. Until then, we should not put geometry on the same rank with arithmetic, which stands purely a priori, but say with mechanics."

Carl Friedrich Gauss, in a letter to Olbers from 1817

Abstract. An argument is given that Euclidean geometry is a priori in the same way that numbers are a priori, the result of modelling, not the external world, but the internal world of our activities. The argument also shows that exactly Weyl's system of axioms reflects a priori intuition (hence mathematical intuition) about Euclidean geometry while the original Euclidean system of axioms reflects a posteriori intuition (hence physical intuition) about Euclidean geometry. Consequently, the argument explains the equal status of number systems and Euclidean geometry in modern mathematics as well as why in modern mathematics the Weyl's system of axioms is dominant to Euclid's system of axioms.

keywords: philosophy of geometry, symmetry, apriority, Euclidean geometry, axioms, Weyl's axioms

Until the appearance of non-Euclidean geometries, Euclidean geometry and numbers had an equal status in mathematics. Indeed, until then, mathematics was described as the science of numbers and space. Whether it was thought that mathematical objects belong to a special world of ideas (Plato), or that they are ultimate abstractions drawn from the real world (Aristotle), or that they are a priori forms of our rational cognition (Kant), mathematical truths were considered, because of the clarity of their subject matter, a priori objective truths that are not subject to experimental verification. Descartes in Meditations (1641) writes: "I counted as the most certain the truths which I conceived clearly as regards figures, numbers, and other matters which pertain to arithmetic and geometry, and, in general to pure and abstract mathematics.". Even Hume considered mathematics to be a non-empirical science that deals not with facts but with relations of ideas.

It seems that Euclid himself, judging by the way he formulated the fifth postulate, considered that the postulate does not have the status of obvious truth like other postulates.

The reason is that the postulate speaks of very distant parts of the plane about which we have no clear idea. Unsuccessful attempts to prove the fifth postulate from the remaining postulates (Saccheri, Lambert), which would establish its unquestionability, eventually resulted in the development of non-Euclidean geometries (Lobachevsky, Bolyai, Gauss) (see, for example, [Tor78]). In the 19th century, it became clear that non-Euclidean geometries were equal to Euclidean geometry both in their internal consistency and as candidates for the “true” geometry of the world. Mathematics ceases to be a set of unquestionable truths about the world and an era of a different understanding of its nature begins. Euclidean geometry loses its mathematical status equal to numbers. Plato’s motto "God geometrizes" turns into Dedekind’s motto "Man arithmetizes" [Ded88]. The period of arithmetization of mathematics begins – the differential and integral calculus is separated from the hitherto dominant geometric intuition and is logically based on number systems and sets. In Riemann’s work [Rie67], geometry is also arithmetized. It is no longer seen as a priori truth but as a mathematical basis for examining real physical space. Arithmetic and set theory are becoming mathematically well-founded theories, primarily in the works of Dedekind and Cantor. The numbers themselves are beginning to be seen as human creations, and the mathematics based on sets is beginning to be considered a priori in a new sense, as a free creation of the human mind (Dedekind [Ded88], Cantor [Can83]). The beginning of this transformation of mathematics as well as the view of its nature can be most strongly connected with the Göttingen circle (Gauss, Dirichlet, Riemann, Dedekind, and even Cantor - see [Fer07]). In parallel, geometry is increasingly seen as part of physics, the science of real space (Riemann [Rie67], Helmholtz [Hel68], Clifford [Cli73]) in which mathematics has the same role as it does in all natural sciences – it considers various mathematical models for the theoretical description of physical phenomena. Thus Euclidean geometry loses the status of a priori mathematical theory and becomes only one of the possible models for physical space distinguished only by the fact that it is a good approximation of the space in which practical science takes place.

Contrary to the proclaimed separation of the mathematical status of Euclidean geometry and numbers, in real mathematical practice they still have equal status. Geometric intuition is still an inexhaustible source of mathematical ideas, and it is certainly not the intuition of non-Euclidean but of Euclidean geometry. Euclidean structures permeate modern mathematics as much as number structures. For example, to visualize the various abstract spaces of functions, we look for an Euclidean structure in them rather than a structure of non-Euclidean geometry. Note that this is the Euclidean structure determined not by the Euclid’s system of axioms but by the Weyl’s system of axioms which describes it as an affine space with a positively definite scalar product on the associated vector space [Wey18]. This geometry, which permeates the mathematical way of thinking itself and which is an internal mathematical means of modelling, I will call *mathematical geometry*. It should be distinguished from *physical geometry* as the science of real physical space. Of course, mathematical models of physical geometry are very important, but here Euclidean geometry is only one of the models. Contrary to this, Euclidean geometry forms the very core of mathematical geometry. I consider that we cannot explain this by the fact that very important notions of linearity and approximation have a simple formulation in Euclidean structures, nor by Poincare’s conventionalism according to which Euclidean geometry has a prominent role because it is the simplest geometry. In my opinion, such a situation can arise only if Euclidean geometry is a priori in some sense. Furthermore, in mathematical Euclidean geometry Weyl's system of axioms is dominant to Euclid's system of axioms. Although Euclid's system of axioms is taught in school, Weyl's system of axioms is used in modern mathematics, physics and engineering.[1] Today, Weyl's system of axioms is one of the essential synthesizing tools of modern mathematics while Euclid's system is of a secondary importance. My explanation of this phenomenon, which I will defend below, is that this happens because the original Euclidean system of axioms reflects a posteriori intuition (hence physical intuition) about Euclidean geometry while Weyl's system of axioms reflects a priori intuition (hence mathematical intuition) about Euclidean geometry.

I will argue below that Euclidean geometry is a priori in the same sense in which numbers are a priori, and that its a priori nature is expressed precisely by Weyl's axioms. This will explain why the core of mathematical geometry is precisely Euclidean geometry, why Euclidean geometry is equal to number systems in modern mathematics, and why it is structured by Weyl and not Euclidean axioms. I will argue that just as number systems are idealized conceptions derived from intuition about our internal activities of counting and measuring, so too is Euclidean geometry an idealized conception derived from intuition about our internal spatial activities. By our internal activities, I mean activities that we organize and design according to our human measure, and over which we have strong control (e.g., movements in space, grouping and arranging small objects, writing on paper, painting, playing music, ...). By our internal spatial activities I mean our inborn ability to move and construct in space, regardless of the environment in which we find ourselves. Here by "a priori" I mean "based on our internal activities".

The cognitive science also deals with the question of whether humans possess an innate mathematical intuition, and thus an innate geometric intuition. It seeks answers in the experimental study of animals, children and adults from the positions of anthropology, developmental psychology and neuroscience. Concerning geometry, analysing the latest findings on this topic, De Cruz in [DC07] provides arguments for the following claims: "(1) that humans have innate, evolved and species-universal cognitive adaptations to deal with space, and (2) that these intuitions constrain and govern the development of formal geometry". My analysis of our internal spatial activities will not be based on the scientific study of the human mind but on my human experience of the human mind. However, the obtained results can be verified experimentally.

My argument that Euclidean geometry is a priori has two parts. The first part concerns our internal spatial activities. Internal spatial activities should be distinguished from external spatial activities. The former are conditioned by our human nature, the latter additionally by the world around us. At first glance, it seems difficult, almost impossible, to draw a clear line between these two types of activities, However, for some activities we can clearly determine that they belong to external spatial activities. For example, mountain climbing is an external spatial activity because it involves orientation in a given landscape and taking care of the configuration of the terrain on which we move. Of course, there is also the ubiquitous gravitational force that makes us one direction in space prominent, which we especially have to take care of. However, the vary fact that we consider this direction to be prominent

[1] It follows from this fact that an appropriate reform of school geometry needs to be made.

suggests that a priori all directions are the same to us. When we are on different parts of a mountain road, different spatial situations will require different responses. However, as far as our ability to react itself is concerned, it is the same in all places and in all directions. If we have to light a fire by placing twigs so that they form a cone, or make a shelter out of wooden bars and reeds, our approach to geometric construction will be the same, whether we are making a small or large fire. This shows that our ideas of spatial constructions are independent of the units we use in the construction. If for some choice of units there is some change that affects the construction, we will attribute it to some external factor. If in this way we try to identify the nature of our internal spatial activities, as invariants to the different spatial situations in which we find ourselves, then we are very close to Delboeuf's analysis [Del60]. He considers what remains when we ignore all differences of things caused by their movements and mutual interactions. According to Delboeuf, in the ultimate abstraction from all diversities of real things we gain the homogeneous (all places are the same), isotropic (all directions are the same), and scale invariant (geometric constructions are independent of size) space – the true geometric space which is different from the real space. However, for Delboeuf this geometry is the background geometry of real space while for me it is the geometry of our a priori activities in space. We can come to the same conclusion if instead of an external argument, seeking common ground in all our external spatial activities, we use an internal argument, analysing directly our internal spatial activities, independently of the external world. A simple introspection shows that we do not distinguish different places, different directions and different units for spatial constructions until the outside world forces us to distinguish them. To eliminate the presence of gravity on the Earth's surface we must look for examples where it is negligible. In addition to the extravagant situation of free fall, these can be examples of activities that take place approximately in the horizontal plane or three-dimensional examples in which gravity is not important. For example, a child will make the same construction from the Lego bricks of his imaginary monster, regardless of where he worked on the construction, how he oriented the construction in space and what dimension of the basic Lego bricks he used. The same indifference to location, direction, and size is present when we rearrange Rubik's cube. Our most basic approach to space, the approach inherent to us, is an *a priori ignorant approach to space*: all places are the same to us (the homogeneity of space), all directions are the same to us (the isotropy of space) and all units of length we use for constructions in space are the same to us (the scale invariance of space). These three principles of symmetry express our basic a priori intuition about our internal spatial activities. Any deviation from these symmetries we attribute to the external world. Thus, it is precisely these principles of symmetry that determine a clear boundary between our internal and external spatial activities.

The importance and validity of the three symmetry principles, the homogeneity of space, the isotropy of space, and the scale invariance of space, has been recognized a long time ago. William Kingdon Clifford in [Cli73] and [Cli85] considers the three symmetry principles as the most essential geometrical assumptions. He considers that the principles are based on observations of the real space. Hermann von Helmholtz has the same opinion for the first two symmetry principles which he unifies in his principle of the free mobility of rigid bodies ([Hel68]). Henri Poincaré, in his analysis of the real space [Poi02], comes to the conclusion that the first two symmetry principles are the most essential properties of the so called geometric space which for him is not the real space but a "conventional space" – the most convenient description of the real space. An interesting explanation of the validity of the three symmetry principles comes from Joseph Delboeuf ([Del60]), as explained above. However, for my argument about the a priori nature of Euclidean geometry, it is crucial that my interpretation of these principles is a different one: they are not a posteriori principles, the result of analysing the real space, but they are a priori principles, the result of analysing our internal activities in space – they express our a priori ignorant approach to space.

The second part of my argument that Euclidean geometry is a priori is a mathematical result. In [Čul17], an elementary system of axioms of Euclidean geometry is developed. The system on the one hand is directly founded on the three principles of symmetry described above, while on the other hand, through the process of algebraic simplification, gives an equivalent Weyl's system of axioms of Euclidean geometry. In this way, Euclidean geometry is characterized by these three principles of symmetry without any additional assumptions (except the idea of continuity). Also, the background symmetry of Weyl axioms is explicated. The system is briefly described in the Appendix.

The connection of Euclidean geometry with the three symmetry principles has a long history. In 17th century John Wallis proved, assuming other Euclid's postulates, that the scale invariance principle "For every figure there exists similar figure of arbitrary magnitude." is equivalent to the Euclid's fifth postulate [Wal99]. Wallis considered his postulate to be more convincing than Euclid's fifth postulate. Tracing back to the famous Riemann lecture Über die Hypothesen welche der Geometrie zu Grunde liegen" ([Rie67]) at Göttingen in 1854, it is well known that among all Riemann manifolds Euclidean geometry is characterized by the three symmetry principles. However, this characterisation is not an elementary one because it presupposes the whole machinery of Riemann manifolds. As I am aware, there is no an elementary description (a description in terms of intuitive relations between points) of Euclidean geometry that is based on the three symmetry principles. The system of axioms developed in [Čul17] provides such an elementary description.

Showing that (i) our a priori approach to space, the world of our internal spatial activities, is characterized by the three principles of symmetry described above, (ii) that the three principles of symmetry immediately support the system of axioms described in the Appendix, and that (iii) the described axioms, through algebraic simplification, entail Weyl's axioms of Euclidean geometry equivalent to them, I gave the argument that Euclidean geometry is a priori and that Weyl's axioms express its a priori nature.

After the appearance of non-Euclidean geometries, there were several attempts to re-establish Euclidean geometry as a priori geometry. They differ from the approach presented in this article both in the meaning of the term *a priori*[2] and in the extent to which it was possible to explicitly identify Euclidean geometry as a priori (in the given meaning of the word) geometry. Most of these approaches are attempts to modify Kant's conception of mathematics to take into account the changes brought about by modern mathematics [Fol18]. Among them, Cassirer's modification of Kant based on Dedekind's philosophy of mathematics stands out, a modification in which, as in this paper, numbers and geometry

[2] The very concept of a priori knowledge as the independence of that knowledge of experience, when it comes to specifying, leads to a whole range of concepts [Jen08].

have the same a priori basis (in Cassirer it is Dedekind's logicism), and the same status in mathematics [Car10, Hei11]. However, in Cassirer's philosophy, Euclid's geometry is not at all prominent in relation to other geometries. As far as I know, there is only one elaborate attempt to establish the a priori nature of Euclidean geometry. This is the protogeometry of Lorenzen that has its roots in Dingler's ideas. This geometry is conceived "as a theory about the conditions under which spatial measurements are possible" [Lor87]. Protogeometry is also based on symmetry. However, this symmetry is not an expression of basic geometric intuition, which is equal in its depth to arithmetic intuition, but a special intuition about plane, parallelism and orthogonality. Lorenzen shows how plane, parallelism, and orthogonality can be defined using symmetry, and how these concepts lead to Euclidean geometry. However, in his derivation, he uses existence axioms: "For each plane E and each point P there exists a unique plane parallel to E through P, and similarly there exists a unique line through P and orthogonal to E." [Lor87]. These axioms are not justified by anything, and that devalues his a priori foundation of Euclidean geometry. Furthermore, Lorenzen himself writes that "other standards of length measurement are possible" [Lor87], ie that other (non-Euclidean) a priori constructions are also possible as preconditions for spatial measurements. The conclusion is that, even if the a priori nature of Euclidean geometry were shown in such a way, that a priori nature would be of a specialized nature for mathematics and not the core of mathematical geometry.

I hope that the argument developed in this article that Euclidean geometry is a priori could satisfy Gauss who expressed in the quote from the beginning of the article his dissatisfaction with the epistemic status of Euclidean geometry.

Appendix

The axiom system presented here have an immediate support (i) in intuitive ideas about a relation between two points, (ii) in the three symmetry principles, and (iii) in the idea of continuity of space.

The primitive terms of the system of axioms are (i) equivalence of pairs of points (arrows): $AB \sim CD$, with the intuitive meaning that the position of the point B relative to the point A is the same as the position of the point D relative to the point C, (ii) multiplication of a pair of points (an arrow) by a real number: $\lambda, A, B \ \mapsto\ \lambda \cdot AB$, with the intuitive meaning of stretching the arrow and of iterative addition of the same arrow, and (iii) distance between points: $A, B \ \mapsto\ |AB| \in \mathbb{R}$. The multiplication could be avoided. Although, from the point of view of the foundation of the theory, it is better to define multiplication, the procedure is somewhat lengthy and I prefer to introduce the multiplication as a new primitive term. Also, it is more simple to introduce the distance function (to add an arbitrary unit of measurement) as a new primitive term than to introduce congruence between pairs of points as a new primitive term and define the distance function relative to the choice of a unit of measurement.

Axioms and definitions with brief comments follow.

By the very idea *to be in the same relative position*:

Axiom (A1). $\sim$ *is an equivalence relation.*

Concerning the relative positions of points to a given point A we can easily describe the equivalence relation $\sim$: by the very idea of the relative position of points, different points have different relative positions to A:

Axiom (A2). $AB \sim AC \;\rightarrow\; B = C.$

Basic operations with arrows are to invert an arrow and to add an arrow to another arrow. The definitions follow:

inverting arrow: $\quad AB \;\mapsto\; -AB = BA$

addition of arrows; $\quad AB,\ BC \;\mapsto\; AB + BC = AC$

Because of axiom A2 we can extend addition of arrows:

generalized addition of arrows; $\quad AB + CD = AB + BX$, where $BX \sim CD$, under the condition that there is such a point X.

By the homogeneity principle, the operations are invariant under the equivalence of arrows:

Axiom (A3.1). $AB \sim A'B' \;\rightarrow\; BA \sim B'A'.$

Axiom (A3.2). $AB \sim A'B' \;\wedge\; BC \sim B'C' \;\rightarrow\; AC \sim A'C'$

Until now, we know only that AB is equivalent to itself (reflexivity of $\sim$) and to no other arrow from the point A (axiom A2). All other axioms are conditional statements. It remains to describe the equivalence of arrows originating from different points. **Multiplication of an arrow by a real number** will give us a description of the equivalence of arrows originating from different points. It is a new primitive operation based on an idea of stretching arrows and of an idea of iterative addition of the same arrow (numbers will be labelled with letters from the Greek alphabet):

$$\cdot : \mathbb{R} \times S^2 \rightarrow S^2 \qquad \lambda, AB \;\mapsto\; \lambda \cdot AB$$

Sometimes, since it is a common convention, we will not write the multiplication sign at all.

The very idea of the multiplication as stretching arrows is formulated in the next axiom:

Axiom (A4). $\forall \lambda, A, B \;\exists C \quad \lambda \cdot AB = AC.$

By the homogeneity principle, multiplication of an arrow by a number is invariant under the equivalence of arrows:

Axiom (A5). $AB \sim CD \;\rightarrow\; \lambda AB \sim \lambda CD.$

For a point C such that $AC = \lambda \cdot AB$ we will say that it is **along** AB. Also, for arrow AC we will say that it is **along** AB.

The very idea of the multiplication as addition of the same arrow leads to the next axiom:

Axiom (A6.1). $1 \cdot AB = AB.$

By the homogeneity principle, we can translate any arrow along AB to any point along AB. So, we can add such arrows. Specially, we can add $\lambda \cdot AB$ and $\mu \cdot AB$ and the result will be $\lambda \cdot AB + \mu \cdot AB = \nu \cdot AB$ for some number ν. Moreover, by the very idea of the multiplication as iterative addition of the same arrow, $\nu = \lambda + \mu$. This is the content of the next axiom:

Axiom (A6.2). $\lambda \cdot AB + \mu \cdot AB = (\lambda + \mu) \cdot AB.$

Let's note that with this equation we postulate also that the left side of the equation is defined.

If we stretch an arrow along AB the result will be an arrow along AB, too. So, $\lambda \cdot (\mu \cdot AB) = \nu \cdot AB$, for some number ν. Moreover, from the very idea of the multiplication as iterative addition of the same (stretched) arrow it follows that $\nu = \lambda \cdot \mu$. This is the content of the next axiom:

Axiom (A6.3). $\lambda \cdot (\mu \cdot AB) = (\lambda \cdot \mu) \cdot AB.$

Let's note that with this equation we postulate also that $\lambda \cdot (\mu \cdot AB)$ is along AB.

The last axiom (and the most important one) expresses the scale invariance principle.

Axiom (A7). *(the scale invariance axiom)*

If $AC = \lambda \cdot AB$ *and* $AC' = \lambda \cdot AB'$ *then* $CC' \sim \lambda \cdot BB'$. (Fig.1)

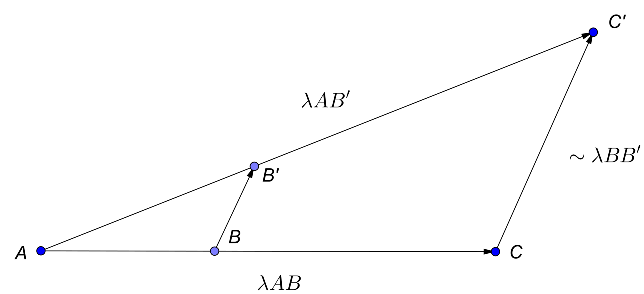

Figure 1:

Of the special interest is a somewhat modified special case of the scale invariance axiom, for $\lambda = 2$:

Theorem (A'5). *(the elementary scale invariance law)*

$AB \sim BC$ *and* $AB' \sim B'C' \;\rightarrow\; \exists\, P \;\; CP \sim PC' \sim BB'$. (Fig.2)

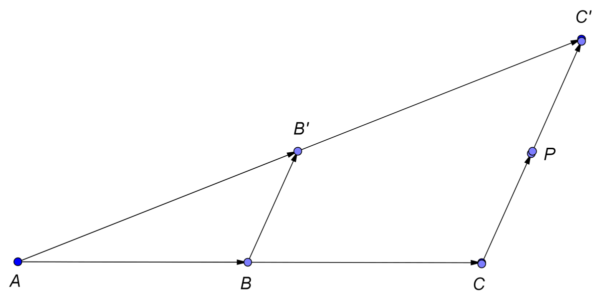

Figure 2:

This theorem has two important consequences:

Theorem (T3). *(the unique translation of arrows law)*
$\forall B, B', C \ \exists! P \quad BB' \sim CP.$

Figure 2 gives a hint for the construction of point P.

The unique translation of arrows law enables us to add arbitrary arrows, without any condition, as we have done before.

$AB + CD = AB + BX$, where $BX \sim CD$

Theorem (T4). *(the parallelogram law)*
$AB \sim A'B' \ \rightarrow \ AA' \sim BB'.$ (Fig.3)

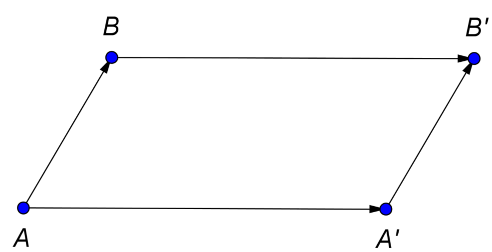

Figure 3:

The basic geometric measure is a measure of the distance between points, the function $| \ | : S^2 \rightarrow \mathbb{R}$. This is the next and the final primitive term. The real number $|AB|$ will be termed the **length** of the arrow AB or **distance** from the point A to the point B.

By the homogeneity of space the length of an arrow must be invariant under equivalence relation $\sim$:

Axiom (A8). $AB \sim CD \ \rightarrow \ |AB| = |CD|.$

By the very idea of measuring distance:

Axiom (A9.1). $|AA| = 0$.

Every point $B \neq A$ determines a direction in which we can go from A. Because of the isotropy of space, the algebraic sign of distance must be always the same – distance must be always negative or always positive or always zero. The zero case gives a trivial measure which does not make any difference between arrows, so, it is a useless measure. Thus, the two other possibilities remain. Technically speaking they are mutually equivalent choices, but by the very idea of measuring it is natural to choose a positive algebraic sign:

Axiom (A9.2). $B \neq A \;\to\; |AB| > 0$. *(positive definiteness)*

By the isotropy of space we also have:

Axiom (A9.3). $|AB| = |BA|$.

For every direction from a point A determined with a point $B \neq A$ we already have a measure of distance. If we take AB as a unit of measure, than we can take the number $\lambda > 0$ as a measure of distance of AC where $AC = \lambda AB$. Note that such a choice of measure along every direction need not be isotropic. However, along every direction the measure of distance $A, B \mapsto |AB|$ must be in accordance with this λ measuring (although it must be more than this):

Axiom (A10). $|\lambda AB| = \lambda |AB|$, *for* $\lambda > 0$,

We can express axioms A9.1, A9.3 and A10 in a uniform way by the next equivalent proposition:

Theorem (7). *(compatibility of distance with multiplication)*

$|\lambda AB| = |\lambda||AB|$, *for every real number* λ.

The description of distance function we have achieved until now enables us to compare distances in a given direction with distances in the opposite direction and with distances in parallel directions. What remains is to solve the main problem: how to compare distances along arbitrary directions in an isotropic way. Let's take, in a given plane, along every direction from a point S, a point at a fixed distance $r > 0$ from S. The set of such points is the **circle** with centre S and radius r, $C(S, r) = \{T : |ST| = r\}$. Let's choose two points A and B on the circle and consider the unique line $p(A, B)$ through these points (Fig.4):

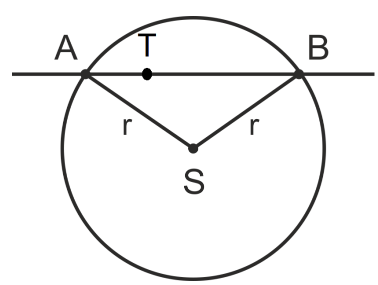

Figure 4:

Let's take an arbitrary point T on the line $p(A, B)$ and consider how the distance $d(T)$ from T to the centre S of the circle varies with the choice of T. Thereby, we will use the idea of continuity of space and of continuity of function $d(T)$. Because of the isotropy of space, the function $d(T)$ must be symmetrical with respect to the relative position of the point T to the points A and B (directions SA and SB). For example, the values of the function in the points A and B are the same (equal to r). Also, the function must have the same value in a point we reach when we move a certain distance from A to B as well as in the point we reach when we move the same distance from B to A (Fig.5):

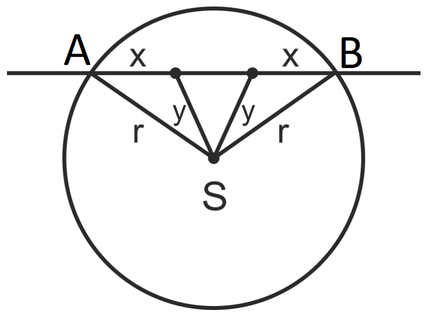

Figure 5:

Because of this symmetry, the function $d(T)$ must have a local extreme value in the midpoint of AB. To determine more precisely the character of the extreme point we will exploit knowledge of a special case, when the points A and B are diametrically opposite on the circle, that is to say, when the centre S of the circle lies on $p(A, B)$. In that case, if we "move" a point T from A to B (or from B to A), the distance $d(T)$ from the centre S of the circle decreases and it is smallest in the midpoint (S). Furthermore, if we move T from A in the direction opposite to the direction to B (or from B in the direction opposite to the direction to A), the distance increases. Therefore, the midpoint S is a unique point of the global minimum of the function $d(T)$. If we drag the point B slightly along the circle into the point B', the centre S of the circle will no longer be on the line $p(AB')$, but, because of continuity, the behaviour of the function $d(T)$ will remain the same. That is to say, the midpoint P of AB' will remain a unique global minimum of the function on the line (Fig.6):

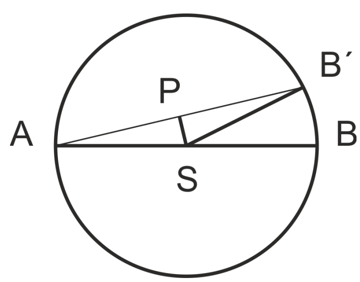

Figure 6:

Because of continuity, for every two points A and B' on the circle the function $d(T)$ will have a unique global minimum on line $p(AB')$ exactly in the midpoint of AB'. Thus, by the isotropy principle and the idea of continuity of space it follows:

Axiom (A11). *If a line has two common points with a circle, points A and B, then the midpoint P of AB is the point on the line nearest to the centre of the circle.* (Fig.7)

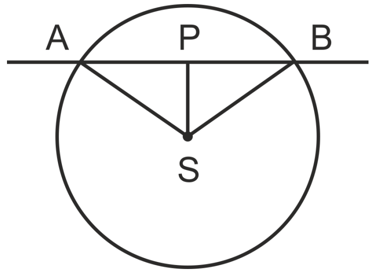

Figure 7:

From the axiom it follows immediately that a line can not have more than two common points with a circle.

Let a line p have exactly one common point with a circle, a point A. If we drag the point A slightly along the circle in one direction onto a point Al, and in another direction onto a point Ad, then the line p is dragged onto the line $p(Al, Ad)$. By axiom A11 the midpoint P of AB is the point on $p(Al, Ad)$ nearest to the circle. By continuity of space, the point A must be the point on p nearest to the centre of the circle (Fig.8). Thus, by the isotropy principle and the idea of continuity of space it follows:

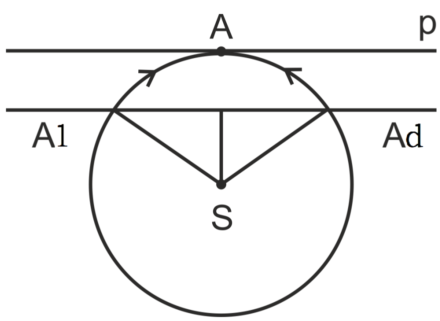

Figure 8:

Axiom (A12). *If a line has exactly one common point with a circle, then the common point is the point on the line nearest to the centre of the circle.* (Fig.9)

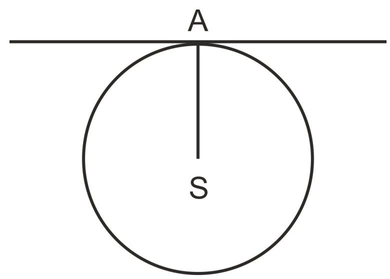

Figure 9:

Theorem (7'). *For every point S not on a line p there is a unique point P on p which is the point on p nearest to S.*

The point on the line p nearest to the point S we term **orthogonal projection** of the point S on the line p. Orthogonal projection enables us to define the scalar orthogonal projection of an arrow onto another arrow. Let $C \neq D$, and let points A and B be orthogonally projected on line $p(CD)$ into points A' and B' (Fig.10). Then $A'B' \sim \alpha CD$ for some real number α.

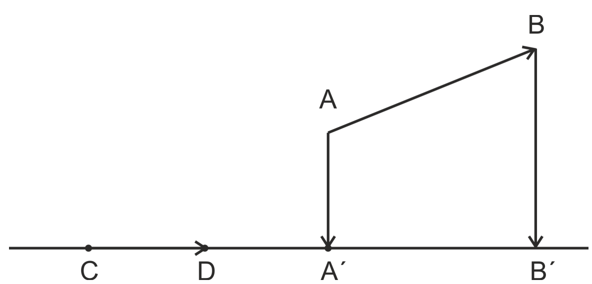

Figure 10:

We define the **scalar orthogonal projection** of the arrow AB onto the arrow CD to be the number $\alpha|CD|$. In simpler terms, it is just the $\pm$ length of the orthogonal projection of the arrow AB onto the line $p(CD)$, where the sign is $+$ if the projection is in the direction of CD, $-$ otherwise. In the extreme case of null arrow CC it is convenient to take zero for the value of the scalar projection on CC. We will denote AB_{CD} as the scalar projection of AB onto CD.

For two equally long arrows with the same initial point, because of the isotropy of space, the scalar projection of the first arrow on the second arrow must be the same as the scalar projection of the second arrow on the first arrow. This is the content of the last axiom:

Axiom (A13). $|AB| = |AC| \ \rightarrow \ AB_{AC} = AC_{AB}$. (Fig.11)

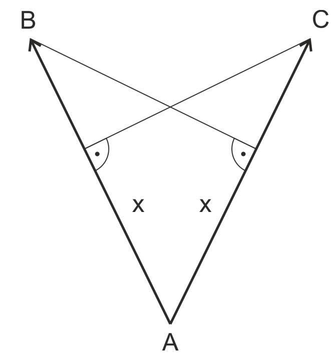

Figure 11:

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