Euclidean Geometry is a Priori

Boris Čulina
Department of Mathematics, University of Applied Sciences Velika Gorica,
Zagrebačka cesta 5, Velika Gorica, CROATIA
email: boris.culina@vvg.hr

"I come more and more to the conviction, that the necessity of our [Euclidean] geometry cannot be proven, at least not by human understanding nor for human understanding. Perhaps in another life we come to different insights into the essence of space, that are now impossible for us to reach. Until then, we should not put geometry on the same rank with arithmetic, which stands purely a priori, but say with mechanics."

Carl Friedrich Gauss, in a letter to Olbers from 1817

Abstract. in the article, an argument is given that Euclidean geometry is a priori in the same way that numbers are a priori, the result of modelling, not the world, but our activities in the world.

keywords: symmetry, Euclidean geometry, axioms, Weyl's axioms, philosophy of geometry

Until the appearance of non-Euclidean geometries, Euclidean geometry and numbers had an equal status in mathematics. Indeed, until then, mathematics was described as the science of numbers and space. Whether it was thought that mathematical objects belong to a special world of ideas(Plato), or that they are ultimate abstractions drawn from the real world (Aristotle),or that they are a priori forms of our rational cognition (Kant), mathematical truths were considered, because of the clarity of their subject matter, a priori objective truths that are not subject to experimental verification. Descartes in Meditations (1641) writes: "I counted as the most certain the truths which I concieved clearly as regards figures, numbers, and other matters which pertain to arithmetic and geometry, and, in general to pure and abstract mathematics.". Even Hume considered mathematics to be a non-empirical science that deals not with facts but with relations of ideas.

It seems that Euclid himself, judging by the way he formulated the fifth postulate, considered that the postulate does not have the status of obvious truth like other postulates because the postulate speaks of very distant parts of the plane about which we have no clear idea. Unsuccessful attempts to prove the fifth postulate from the remaining postulates (Saccheri, Lambert), which would establish its unquestionability, eventually resulted in the development of non-Euclidean geometries (Lobachevsky, Bolyai, Gauss) (see [Tor78]). In the 19th century, it became clear that non-Euclidean geometries were equal to Euclidean geometry both in their internal consistency and as candidates for the ”true” geometry of the world. Mathematics ceases to be a set of unquestionable truths about the world and an era of a different understanding of its nature begins. The view of Euclidean geometry is also changing - it loses its mathematical status equal to numbers. Plato's motto "God geometrizes" turns into Dedekind's motto "Man arithmetizes". The period of arithmetization of mathematics begins – the differential and integral calculus is separated from the hitherto dominant geometric intuition and is logically based on number systems and sets. In Riemann's work, geometry is also arithmetized. It is no longer seen as a priori truth but as a mathematical basis for examining real physical space. Arithmetic and set theory are becoming mathematically well-founded theories, primarily in the works of Dedekind and Cantor. The numbers themselves are beginning to be seen as human creations, and the mathematics based on sets is beginning to be considered a priori in a new sense, as a free creation of the human mind. The beginning of this transformation of mathematics as well as the view of its nature can be most strongly connected with the Göttingen circle (Gauss, Dirichlet, Riemann, Dedekind, and even Cantor - see [Fer07]). In parallel, geometry is increasingly seen as part of physics, the science of real space (Riemann, Helmholtz) in which mathematics has the same role as it does in all natural sciences – it considers various mathematical models for the theoretical description of physical phenomena. Thus Euclidean geometry loses the status of a priori mathematical theory and becomes only one of the possible models for physical space distinguished only by the fact that it is a good approximation of the space in which practical science takes place.

Contrary to the proclaimed separation of the mathematical status of Euclidean geometry and numbers, in real mathematical practice they still have equal status. Geometric intuition is still an inexhaustible source of mathematical ideas, and it is certainly not the intuition of non-Euclidean but of Euclidean geometry. Euclidean structures permeate modern mathematics as much as number structures. For example, to visualize the various abstract spaces of functions, we look for an Euclidean structure in them rather than a structure of non-Euclidean geometry. Note that this is the Euclidean structure determined by Weyl's vector axioms and not Euclidean axioms. The geometry that is present in this way in mathematical practice and which permeates the mathematical way of thinking, and which I will term mathematical geometry here, should be distinguished from physical geometry as the science of real physical space. Of course, mathematical models of physical geometry are very important, but Euclidean geometry is only one of the models. Contrary to this, Euclidean geometry forms the very core of mathematical geometry. I think that we cannot explain this by the fact that very important notions of linearity and approximation have a simple formulation in Euclidean structures, nor by Poincare's conventionalism according to which Euclidean geometry has a prominent role because it is the simplest geometry. In my opinion, such a situation can arise only if Euclidean geometry is a priori. Furthermore, in mathematical Euclidean geometry the Weyl's system of axioms is dominant to the Euclid's system. Although the Euclid's system of axioms for Euclidean geometry is taught in school, the Weyl's system of axioms is used in modern mathematics, physics and engineering.[1] Today, the Weyl's system of axioms is one of the essential synthesizing tools of modern mathematics while Euclid's system is of a secondary importance. My thesis, which I will argue below,

[1]It follows from this fact that an appropriate reform of school geometry needs to be made.

is that this happens because the original Euclidean system of axioms reflects a posteriori intuition (hence physical intuition) about Euclidean geometry while the Weyl's system of axioms reflects a priori intuition (hence mathematical intuition) about Euclidean geometry.

My aim is to show that Euclidean geometry is a priori in the same sense in which numbers are a priori, and that its a priori nature is expressed precisely by Weyl's axioms. This will explain why the core of mathematical geometry is precisely Euclidean geometry in Weyl's axiomatics. I will show that just as number systems are idealized conceptions derived from intuition about our internal activities of counting and measuring, so too is Euclidean geometry an idealized conception derived from intuition about our internal spatial activities. By our internal activities, I mean activities that we organize and design according to our human measure. In [Č20] I argued that our internal world of activities is the source of all mathematics and that mathematics is precisely in this sense a priori – it is the result of modelling not the world but our activities in the world.

The basic intuition about our internal spatial activities is an a priori ignorant approach to space: all places are the same to us (the homogeneity of space), all directions are the same to us (the isotropy of space) and all units of length we use for constructions in space are the same to us (the scale invariance of space). In [Č17] I developed a system of axioms which on the one hand is directly based on these three principles of symmetry and on the other hand, through the process of algebraic simplification, entails the equivalent Weyl system of axioms of Euclidean geometry. In this way I have shown that the nature of Euclidean geometry is a priori: Euclidean geometry is an idealization of our a priori ignorant approach to space, expressed in an algebraically simplified form by Weyl's axioms. This argument could satisfy Gauss who expressed in the quote from the beginning of the article his dissatisfaction with the epistemic status of Euclidean geometry I will sketch this argument below. All details can be found in [Č17].

The connection of Euclidean geometry with the three symmetry principles has a long history. In 17th century John Wallis proved, assuming other Euclid's postulates, that the scale invariance principle "For every figure there exists similar figure of arbitrary magnitude." is equivalent to the Euclid's fifth postulate [Wal99]. Wallis considered his postulate to be more convincing than Euclid's fifth postulate. Tracing back to the famous Riemann lecture Über die Hypothesen welche der Geometrie zu Grunde liegen" ([Rie67]) at Göttingen in 1854 , it is well known that among all Riemann manifolds Euclidean geometry is characterized by the three symmetry principles. However, this characterisation is not an elementary one because it presupposes the whole machinery of Riemann manifolds. As I am aware, there is no an elementary description (a description in terms of intuitive relations between points) of Euclidean geometry that is based on the three symmetry principles. In [Č17] I developed a system of axioms that provides such an elementary description.

The importance and validity of the three symmetry principles has been recognized also a long time ago. William Kingdon Clifford in [Cli73] and [Cli85] considers the three symmetry principles as the most essential geometrical assumptions. He considers that the principles are based on observations of the real space. Hermann von Helmholtz has the same opinion for the first two symmetry principles which he unifies in his principle of the free mobility of rigid bodies ([Hel68]). Henri Poincaré, in his analysis of the real space [Poi02], comes to the conclusion that the first two symmetry principles are the most essential properties of the so called geometric space which for him is not the real space but a ”conventional space” – the most convenient description of the real space. An interesting explanation of the validity of the three symmetry principles comes from Joseph Delboeuf ([Del60]). He considers what remains when we ignore all differences of things caused by their movements and mutual interactions. According to Delboeuf, in the ultimate abstraction from all diversities of real things we gain the homogeneous, isotropic, and scale invariant space - the true geometric space which is Euclidean and which is different from the real space. However, for my argument about the a priori nature of Euclidean geometry, it is crucial that my interpretation of these principles is a different one: they are not a posteriori principles, the result of analysing the real space, but they are a priori principles in a sense that they express our a priori ignorant approach to space.

As far as I know, there is only one elaborate attempt to establish the a priori nature of Euclidean geometry. This is the protogeometry of Lorenzen that has its roots in Dingler's ideas. This geometry is conceived ”as a theory about the conditions under which spatial measurements are possible” [Lor87]. Protogeometry is also based on symmetry. However, this symmetry is not an expression of our basic geometric intuition, which is equal in its depth to arithmetic intuition, but a special intuition about plane, parallelism and orthogonality. Lorenzen shows how plane, parallelism, and orthogonality can be defined using symmetry, and how these concepts lead to Euclidean geometry. However, in his derivation, he uses existence axioms: ”For each plane E and each point P there exists a unique plane parallel to E through P, and similarly there exists a unique line through P and orthogonal to E.” [Lor87]. These axioms are not justified by anything, and that devalues his a priori foundation of Euclidean geometry. Furthermore, Lorenzen himself writes that ”other standards of length measurement are possible” [Lor87], ie that other (non-Euclidean) a priori constructions are also possible as preconditions for spatial measurements.The conclusion is that, even if the a priori nature of Euclidean geometry were shown in such a way, that a priori nature would be of a specialized nature for mathematics and not the core of mathematical geometry.

Now I will gradually introduce an elementary system of axioms about space of points which have an immediate support (i) in intuitive ideas about a relation between two points, (ii) in the three symmetry principles, and (iii) in the idea of continuity of space. The primitive terms of the system of axioms are: (i) equivalence of pairs of points (arrows), (ii) multiplication of a pair of points by a real number and (iii) distance between points. The multiplication could be avoided. Although, from the point of view of the foundation of the theory, it is better to define multiplication, the procedure is somewhat lengthy and I prefer to introduce the multiplication as a new primitive term. Also, it is more simple to introduce the distance function (to add an arbitrary unit of measurement) as a new primitive term than to introduce congruence between pairs of points as a new primitive term and define the distance function relative to the choice of a unit of measurement.

Geometrical space S will be modelled as a non empty set of objects termed **points**. The basic geometrical relation is the position of one point relative to another (not necessarily different) point. That the position of a point B relative to a point A is the same as the position of a point B' relative to a point A' we will denote $AB \sim A'B'$ and we will say that pairs or arrows AB and $A'B'$ are **equivalent**. This is the first primitive term of our system. It expresses a basic intuitive idea about the relative positions of points. The idea itself *to be* *in the same relative position* implies that it is a relation of equivalence. This is the content of the first axiom.

Axiom (A1). $\sim$ *is an equivalence relation.*

In more detail, it means:

Axiom (A1.1). $AB \sim AB$ *(reflexivity)*

Axiom (A1.2). $AB \sim A'B' \;\rightarrow\; A'B' \sim AB$ *(symmetry)*

Axiom (A1.3). $AB \sim A'B' \;\wedge\; A'B' \sim A''B'' \;\rightarrow\; AB \sim A''B''$ *(transitivity)*

Concerning the relative positions of points to a given point A we can easily describe the equivalence relation $\sim$: by the very idea of the relative position of points, different points have different relative positions to A:

Axiom (A2). $AB \sim AC \;\rightarrow\; B = C.$

Basic operations with arrows are to invert an arrow and to add an arrow to another arrow. The definitions follow:

inverting arrow: $AB \;\mapsto\; -AB = BA$

addition of arrows; $AB,\ BC \;\mapsto\; AB + BC = AC$

Because of axiom A2 we can extend addition of arrows:

generalized addition of arrows; $AB + CD = AB + BX$, where $BX \sim CD$, under the condition that there is such a point X.

By the homogeneity principle, the operations are invariant under the equivalence of arrows:

Axiom (A3.1). $AB \sim A'B' \;\rightarrow\; BA \sim B'A'.$

Axiom (A3.2). $AB \sim A'B' \;\wedge\; BC \sim B'C' \;\rightarrow\; AC \sim A'C'$

Until now, we know only that AB is equivalent to itself (reflexivity of $\sim$) and to no other arrow from the point A (axiom A2). All other axioms are conditional statements. It remains to describe the equivalence of arrows originating from different points. **Multiplication of an arrow by a real number** will give us a description of the equivalence of arrows originating from different points. It is a new primitive operation based on an idea of stretching arrows and of an idea of iterative addition of the same arrow (numbers will be labelled with letters from the Greek alphabet):

$$\cdot : \mathbb{R} \times S^2 \rightarrow S^2 \qquad \lambda, AB \;\mapsto\; \lambda \cdot AB$$

Sometimes, since it is a common convention, we will not write the multiplication sign at all.

The very idea of the multiplication as stretching arrows is formulated in the next axiom:

Axiom (A4). $\forall \lambda, A, B\ \exists C\ \ \lambda \cdot AB = AC.$

By the homogeneity principle, multiplication of an arrow by a number is invariant under the equivalence of arrows:

Axiom (**A5**). $AB \sim CD \ \rightarrow \ \lambda AB \sim \lambda CD$.

For a point C such that $AC = \lambda \cdot AB$ we will say that it is **along** AB. Also, for arrow AC we will say that it is **along** AB.

The very idea of the multiplication as addition of the same arrow leads to the next axiom:

Axiom (**A6.1**). $1 \cdot AB = AB$.

By the homogeneity principle, we can translate any arrow along AB to any point along AB. So, we can add such arrows. Specially, we can add $\lambda \cdot AB$ and $\mu \cdot AB$ and the result will be $\lambda \cdot AB + \mu \cdot AB = \nu \cdot AB$ for some number ν. Moreover, by the very idea of the multiplication as iterative addition of the same arrow, $\nu = \lambda + \mu$. This is the content of the next axiom:

Axiom (**A6.2**). $\lambda \cdot AB + \mu \cdot AB = (\lambda + \mu) \cdot AB$.

Let's note that with this equation we postulate also that the left side of the equation is defined.

If we stretch an arrow along AB the result will be an arrow along AB, too. So, $\lambda \cdot (\mu \cdot AB) = \nu \cdot AB$, for some number ν. Moreover, from the very idea of the multiplication as iterative addition of the same (stretched) arrow it follows that $\nu = \lambda \cdot \mu$. This is the content of the next axiom:

Axiom (**A6.3**). $\lambda \cdot (\mu \cdot AB) = (\lambda \cdot \mu) \cdot AB$.

Let's note that with this equation we postulate also that $\lambda \cdot (\mu \cdot AB)$ is along AB.

The last axiom (and the most important one) expresses the scale invariance principle.

Axiom (**A7**). *(the scale invariance axiom)*

If $AC = \lambda \cdot AB$ *and* $AC' = \lambda \cdot AB'$ *then* $CC' \sim \lambda \cdot BB'$. (Fig.1)

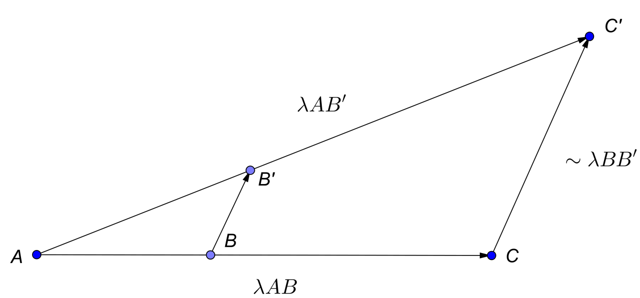

Figure 1:

Of the special interest is a somewhat modified special case of the scale invariance axiom, for $\lambda = 2$:

Theorem (A'5). *(the elementary scale invariance law)*

$AB \sim BC$ *and* $AB' \sim B'C' \ \rightarrow \ \exists\, P \ \ CP \sim PC' \sim BB'$. (Fig.2)

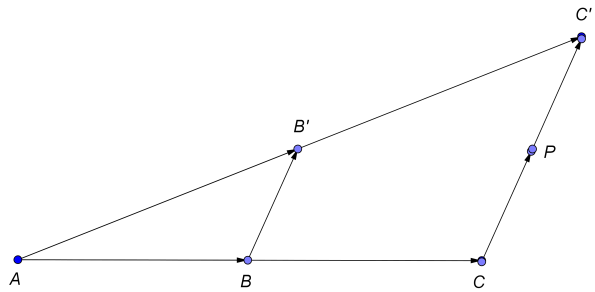

Figure 2:

This theorem has two important consequences:

Theorem (T3). *(the unique translation of arrows law)*

$\forall B, B', C \ \exists! P \ \ BB' \sim CP.$

Figure 2 gives a hint for the construction of point P.

The unique translation of arrows law enables us to add arbitrary arrows, without any condition, as we have done before.

$AB + CD = AB + BX$, where $BX \sim CD$

Theorem (T4). *(the parallelogram law)*

$AB \sim A'B' \ \rightarrow \ AA' \sim BB'$. (Fig.3)

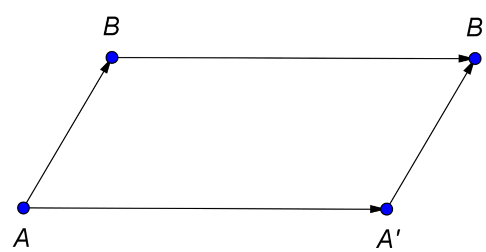

Figure 3:

Remark 1. *Assuming only axioms A1 to A3 and that* $\forall A, B \ \exists D \ \ AB \sim BD$*, without any axiom on the multiplication, we can prove that the elementary scale invariance law is equivalent to the conjunction of the unique translation of arrows law and parallelogram law.*

Remark 2. *Until now we haven't used axiom A1.1 (reflexivity). Indeed we can prove it now.*

Remark 3. *The parallelogram law is algebraically very efficient in proofs. For example, assuming only the parallelogram law, we can prove the equivalence of: (i) A1.1 (reflexivity of* $\sim$*) and that* $\forall A, B \ \ AA \sim BB$ *(all arrows of the type AA are equivalent), (ii) A1.2 (symmetry of* $\sim$*) and A3.1 (invariance under* $\sim$ *of inverting arrows) , (iii) A1.3 (transitivity of* $\sim$*) and A3.2 (invariance under* $\sim$ *of adding arrows).*

Axiom A1 enables us to define vectors. Since, by A1, $\sim$ is an equivalence relation, it classifies arrows (pairs of points) into classes of mutually equivalent arrows. We define **vectors** as these equivalence classes. The set of all vectors will be denoted $\overrightarrow{S}$. To every pair of points AB we will associate the vector $\overrightarrow{AB}$, the equivalence class to which AB belongs:

$$\overrightarrow{AB} = \{CD | CD \sim AB\}$$

So, $\rightarrow$ maps pairs of points to vectors: $\overrightarrow{\ \ } : S^2 \to \overrightarrow{S}$.

For every pair of points (arrows) belonging to a vector (an equivalence class) we say that it **represents** the vector. Thus, for example, a pair AB represents the vector $\overrightarrow{AB}$.

Introduced axioms allow to transfer operations with arrows into operations with corresponding vectors, in a way invariant under relation $\sim$:

null vector: $\overrightarrow{0} = \overrightarrow{AA}$

inverse vector: $-\overrightarrow{AB} = \overrightarrow{BA}$

Addition of vectors: $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

Multiplication of a vector by a number: $\lambda \cdot \overrightarrow{AB} = \overrightarrow{\lambda \cdot AB}$

In [Č17] is proved that the space of vectors $\overrightarrow{S}$ together with operations of addition of vectors and multiplication of a vector by a number has the structure of vector space, and that the space of points S together with the space of vectors $\overrightarrow{S}$ and the mapping $AB \mapsto \overrightarrow{AB}$, that is to say, the structure

$$(S, \overrightarrow{S}, \overrightarrow{\ \ }, +, \cdot), \text{ where } \overrightarrow{\ \ } : S^2 \to \overrightarrow{S}, \ + : \overrightarrow{S}^2 \to \overrightarrow{S}, \ \cdot : \mathbb{R} \times \overrightarrow{S} \to \overrightarrow{S},$$

is Weyl's structure of affine space. Conversely, starting from Weyl's structure of affine space we could define in a standard way the equivalence of arrows, addition of arrows and multiplication of an arrow by a number, and prove that for such a defined structure

$$(S, \sim, +, \cdot), \text{ where } S \neq \emptyset, \ \sim \subseteq S^2 \times S^2, \ + : S^2 \times S^2 \to S^2, \ \cdot : \mathbb{R} \times S^2 \to S^2$$

all A propositions are valid. Hence,

A axioms are equivalent to Weyl's axioms in this affine layer of Euclidean geometry.

Let's note that in the structure $(S, \sim, +, \cdot)$ addition $+$ is defined and multiplication $\cdot$ can be defined if we choose to introduce multiplication of a vector gradually, first by natural numbers, then by integers and rational numbers, and finally by real numbers. It means that there is essentially only one primitive term, the relation of equivalence $\sim$ between ordered pairs of points.

The basic geometric measure is a measure of the distance between points, the function $| \ | : S^2 \to \mathbb{R}$. This is the next and the final primitive term. The real number $|AB|$ will be termed the **length** of the arrow AB or **distance** from the point A to the point B.

By the homogeneity of space the length of an arrow must be invariant under equivalence relation $\sim$:

Axiom (A8). *$AB \sim CD \ \to \ |AB| = |CD|$.*

By the very idea of measuring distance:

Axiom (A9.1). *$|AA| = 0$.*

Every point $B \neq A$ determines a direction in which we can go from A. Because of the isotropy of space, the algebraic sign of distance must be always the same – distance must be always negative or always positive or always zero. The zero case gives a trivial measure which does not make any difference between arrows, so, it is a useless measure. Thus, the two other possibilities remain. Technically speaking they are mutually equivalent choices, but by the very idea of measuring it is natural to choose a positive algebraic sign:

Axiom (A9.2). *$B \neq A \ \to \ |AB| > 0$. (positive definiteness)*

By the isotropy of space we also have:

Axiom (A9.3). *$|AB| = |BA|$.*

For every direction from a point A determined with a point $B \neq A$ we already have a measure of distance. If we take AB as a unit of measure, than we can take the number $\lambda > 0$ as a measure of distance of AC where $AC = \lambda AB$. Note that such a choice of measure along every direction need not be isotropic. However, along every direction the measure of distance $A, B \mapsto |AB|$ must be in accordance with this λ measuring (although it must be more than this):

Axiom (A10). *$|\lambda AB| = \lambda |AB|$, for $\lambda > 0$,*

We can express axioms A9.1, A9.3 and A10 in a uniform way by the next equivalent proposition:

Theorem (7). *(compatibility of distance with multiplication)*
$|\lambda AB| = |\lambda||AB|$, for every real number λ.

The description of distance function we have achieved until now enables us to compare distances in a given direction with distances in the opposite direction and with distances in parallel directions. What remains is to solve the main problem: how to compare distances along arbitrary directions in an isotropic way. Let's take, in a given plane, along every direction from a point S, a point at a fixed distance $r > 0$ from S. The set of such points is the **circle** with center S and radius r, $C(S, r) = \{T : |ST| = r\}$. Let's choose two points A and B on the circle and consider the unique line $p(A, B)$ through these points (Fig.4):

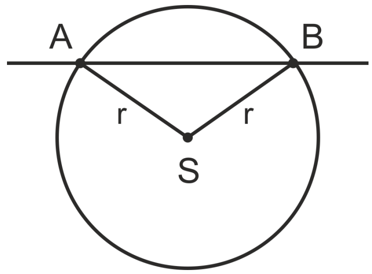

Figure 4:

Let's take an arbitrary point T on the line $p(A, B)$ and consider how the distance $d(T)$ from T to the center S of the circle varies with the choice of T. Thereby, we will use the idea of continuity of space and of continuity of function $d(T)$. Because of the isotropy of space, the function $d(T)$ must be symmetrical with respect to the relative position of the point T to the points A and B (directions SA and SB). For example, the values of the function in the points A and B are the same (equal to r). Also, the function must have the same value in a point we reach when we move a certain distance from A to B as well as in the point we reach when we move the same distance from B to A ($d(A + \lambda\overrightarrow{AB}) = d(B + \lambda\overrightarrow{BA})$) (Fig.5):

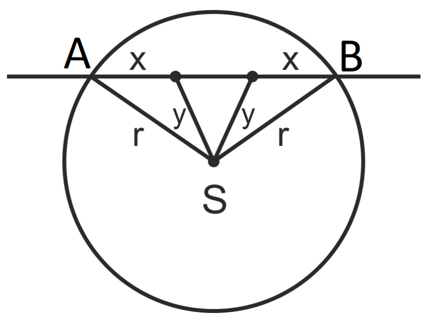

Figure 5:

Because of this symmetry, the function $d(T)$ must have a local extreme value in the midpoint of AB. To determine more precisely the character of the extreme point we will exploit knowledge of a special case, when the points A and B are diametrically opposite on the circle, that is to say, when the center S of the circle lies on $p(A, B)$. In that case, if we "move" a point T from A to B (or from B to A), the distance $d(T)$ from the center S of the circle decreases and it is smallest in the midpoint (S). Furthermore, if we move T from A in the direction opposite to the direction to B (or from B in the direction opposite to the direction to A), the distance increases. Therefore, the midpoint S is a unique point of the global minimum of the function $d(T)$. If we drag the point B slightly along the circle into the point B', the center S of the circle will no longer be on the line $p(AB')$, but, because of continuity, the behaviour of the function $d(T)$ will remain the same. That is to say, the midpoint P of AB' will remain a unique global minimum of the function on the line (Fig.6):

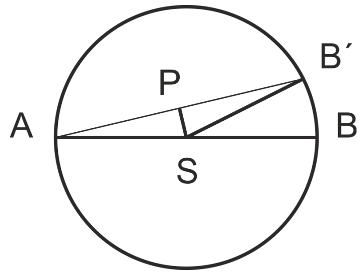

Figure 6:

Because of continuity, for every two points A and B' on the circle the function $d(T)$ will have a unique global minimum on line $p(AB')$ exactly in the midpoint of AB'. Thus, by the isotropy principle and the idea of continuity of space it follows:

Axiom (A11). *If a line has two common points with a circle, points A and B, then the midpoint P of AB is the point on the line nearest to the center of the circle.* (Fig.7)

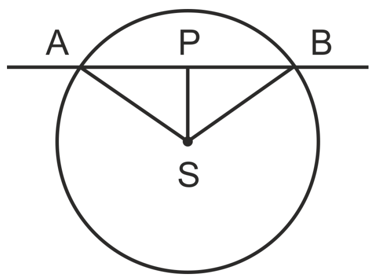

Figure 7:

From the axiom it follows immediately that a line can not have more than two common points with a circle.

Let a line p have exactly one common point with a circle, a point A. If we drag the point A slightly along the circle in one direction onto a point Al, and in another direction onto a point Ad, then the line p is dragged onto the line $p(Al, Ad)$. By axiom A11 the midpoint P of AB is the point on $p(Al, Ad)$ nearest to the center of the circle. By continuity of space, the point A must be the point on p nearest to the center of the circle (Fig.8). Thus, by the isotropy principle and the idea of continuity of space it follows:

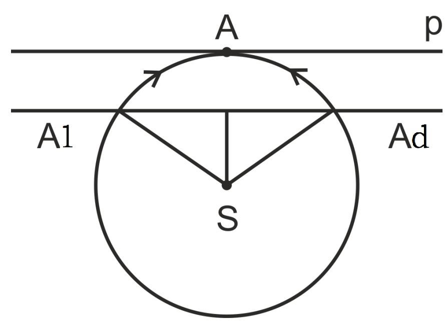

Figure 8:

Axiom (A12). *If a line has exactly one common point with a circle, then the common point is the point on the line nearest to the center of the circle.* (Fig.9)

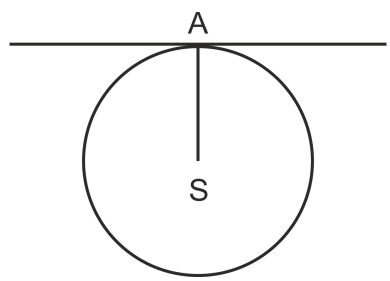

Figure 9:

Theorem (7'). *For every point S not on a line p there is a unique point P on p which is the point on p nearest to S.*

The point on the line p nearest to the point S we term **orthogonal projection** of the point S on the line p. Orthogonal projection enables us to define the scalar orthogonal projection of an arrow onto another arrow. Let $C \neq D$, and let points A and B be orthogonally projected on line $p(CD)$ into points A' and B' (Fig.10). Then $A'B' \sim \alpha CD$ for some real number α.

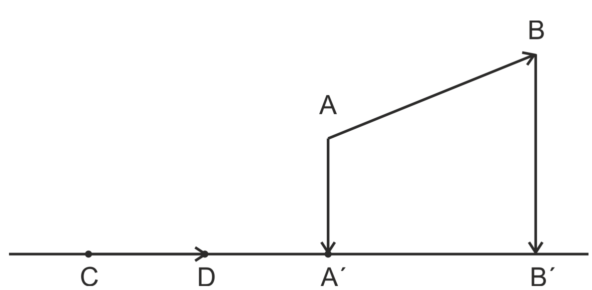

Figure 10:

We define the **scalar orthogonal projection** of the arrow AB onto the arrow CD to be the number $\alpha|CD|$. In simpler terms, it is just the $\pm$ length of the orthogonal projection of the arrow AB onto the line $p(CD)$, where the sign is $+$ if the projection is in the direction of CD, $-$ otherwise. In the extreme case of null arrow CC it is convenient to take zero for the value of the scalar projection on CC. We will denote AB_{CD} as the scalar projection of AB onto CD.

For two equally long arrows with the same initial point, because of the isotropy of space, the scalar projection of the first arrow on the second arrow must be the same as the scalar projection of the second arrow on the first arrow. This is the content of the last axiom:

Axiom (A13). $|AB| = |AC| \;\rightarrow\; AB_{AC} = AC_{AB}$. (Fig.11)

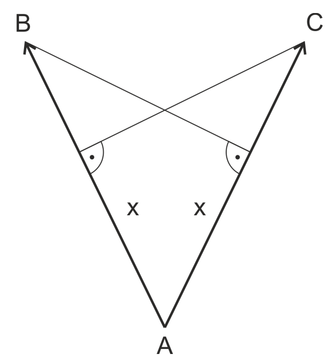

Figure 11:

Now, we can define the **scalar product of arrows** AB and CD. It is the product of the scalar projection of the arrow AB onto CD and the length of the arrow CD. More formally:

$AB \cdot CD = AB_{CD} \cdot |CD|$

This operation is invariant under $\sim$ relation.

The **scalar product of vectors** is defined by arrows which represent vectors:

$\overrightarrow{AB} \cdot \overrightarrow{CD} = AB \cdot CD$

Since scalar product of arrows is invariant under $\sim$ the definition is correct, that is to say, it doesn't depend on the choice of arrows representing vectors.

In [Č17] is shown that the scalar product of vectors, together with the affine structure of space, satisfy Weyl's axioms of Euclidean geometry. Conversely, by Weyl's axioms we can define in a standard way the length of an arrow and deduce all A axioms about length. This means that A axioms of the structure of set of points S with relation $\sim$ between pairs of points, addition of pairs, multiplication of a pair by a number and distance function of a pair:

$(S, \sim, +, \cdot, |\;\;|)$, where $S \neq \emptyset$,

$\sim \subseteq S^2 \times S^2,\; + : S^2 \times S^2 \rightarrow S^2,\; \cdot : \mathbb{R} \times S^2 \rightarrow S^2,\; |\;\;| : S^2 \rightarrow \mathbb{R},$

are equivalent to Weyl's axioms of the corresponding structure of the set of points and the set of vectors together with the operation from pairs of points to vectors, addition of vectors, multiplication of a vector by a number and the scalar product of vectors:

$(S, \overrightarrow{S}, \ \vec{}\ , +, \cdot, \cdot)$, where $S \neq \emptyset$,

$\vec{}\ : S^2 \to \overrightarrow{S}$, $+, \cdot : \overrightarrow{S}^2 \to \overrightarrow{S}$, $\cdot : \mathbb{R} \times \overrightarrow{S} \to \overrightarrow{S}$

Thus, showing that (i) the a priori ignorant approach to space gives three principles of symmetry, (ii) that the three principles of symmetry immediately support the introduced axioms, and that (iii) the introduced axioms, through algebraic simplification, entail Weyl's axioms of Euclidean geometry, I argued that Euclidean geometry is a priori.

References

[Cli73] William Kingdon Clifford. *The Postulates of the Science of Space*. 1873. the third in a series of lectures Clifford delivered at the Royal Institution in London.

[Cli85] William Kingdon Clifford. *The Common Sense of the Exact Sciences*. Kegan, Paul, Trench, 1885.

[Del60] Joseph Rémi Léopold Delboeuf. *Proiegomenes Philosophiques de la Geometrie et Solution des Postulats*. J. Desoer, 1860.

[Fer07] José Ferreirós. *Labyrinth of Thought: A History of Set Theory and Its Role in Modern Mathematics*. Birkhäuser, 2 edition, 2007.

[Hel68] Hermann Ludwig Ferdinand Helmholtz. Über die tatsachen, die der geometrie zu grunde liegen. *Nachrichten der Königlichen Gesellschaft der Wissenschaften und der Georg - Augustus - Universität*, (9):193 – 221, 1868.

[Lor87] Paul Lorenzen. *Constructive Philosophy*. University of Massachusetts Press, 1987.

[Poi02] Jules Henri Poincaré. *La Science et l'Hypothèse*. Flammarion, 1902.

[Rie67] Georg Friedrich Bernhard Riemann. *Über die Hypothesen, welche der Geometrie zu Grunde liegen*, volume 13 of *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*. 1867.

[Tor78] Roberto Torretti. *Philosophy of geometry from Riemann to Poincaré*. Springer, 1978.

[Č17] Boris Čulina. An elementary system of axioms for euclidean geometry based on symmetry principle. *Axiomathes*, 2017. DOI 10.1007/s10516-017-9358-y.

[Č20] Boris Čulina. Mathematics as an imagined tool for rational cognition. unpublished, 2020.

[Wal99] John Wallis. *Opera Mathematica*. E Theatro Sheldoniano, 1695-1699. 3 Vols.