Mathematics
an Imagined Tool for Rational Cognition

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*"...numbers are free creations of the human mind;
they serve as a means of apprehending more easily and more sharply the difference of things."*

Richard Dedekind

Abstract. Analysing several characteristic mathematical models: natural and real numbers, Euclidean geometry, group theory, and set theory, I argue that a mathematical model in its final form is a junction of a set of axioms and an internal partial interpretation of the corresponding language. It follows from the analysis that (i) mathematical objects do not exist in the external world: they are our internally imagined objects, some of which, at least approximately, we can realize or represent; (ii) mathematical truths are not truths about the external world but specifications (formulations) of mathematical conceptions; (iii) mathematics is first and foremost our imagined tool by which, with certain assumptions about its applicability, we explore nature and synthesize our rational cognition of it.

keywords: mathematical models; mathematical objects; mathematical truths; applicability of mathematics

The basic problem of the philosophy of mathematics (not mathematics itself) is to answer the following intertwined questions:

- Are there mathematical objects, and if so, in what way?
- What is mathematical truth and how do we establish it?
- How is mathematics applied?

This paper presents a solution that can be considered an elaboration of Dedekind's quotation from the beginning of the article. The basic theses that I intend to argue in this article are the following:

- Mathematical objects do not exist in the outside world. They are our internally imagined objects, some of which, at least approximately, we can realize or represent.
- Mathematical truths are not truths about the external world but specifications (formulations) of mathematical conceptions.
- Mathematics is first and foremost our thinking tool by which, with certain assumptions about its applicability, we explore nature and synthesize our rational cognition of it.

I will try to make clear what is absolutely clear to the famous physicist Percy W. Bridgman: "It is the merest truism, evident at once to unsophisticated observation, that mathematics is a human invention." [Bri27]. Having practised mathematics all my life, by vocation and by profession, just as breathing was natural to me, so it was natural for me to consider mathematics as a human invention and a free creation of the human mind whose purpose is to be a tool of our rational cognition and rational activities in general. When I decided to clarify to myself what human invention mathematics was, it proved to me, I believe, a far more difficult task than explaining what breathing is. In this article, I have outlined what I came up with along the way. One thing is for sure. After this I understand philosophers much better.

For the purpose of illustration, I will consider classical, but still most important, mathematical models, the natural number system, the real number system, and Euclidean geometry, as well as today's standard mathematical models, group theory, and set theory. To quote the famous mathematician Saunders Mac Lane: "...a philosophy of Mathematics is not convincing unless it is founded on an examination of Mathematics itself." [ML86].

Natural numbers are the result of modelling our intuition about the size of a collection of objects. We measure the collection by process of counting, and natural numbers are objects for counting. To start counting we must have the first number, to associate it to the first chosen object in the collection. To continue counting, after each number we must have the next new number, to associate it with the next chosen object in the collection. There is no special reason to sort out some particular objects as natural numbers. Merely for the needs of calculation we sort out a particular realization, in the past collections of marbles on an abacus, and today sequences of decimal numerals on paper and of bits in a computer. It means that for counting it is not important how numbers are realized, but only the structure of the set of natural numbers which enables us to count is important. It seems that they exist in the same way as chess figures, in the sense that we can always realize them in some way. However, the structure of natural numbers, as opposed to the structure of chess, brings in itself an idealization. To be always possible to continue counting, each natural number must have the next natural number. Therefore, there are infinitely many natural numbers. So, although we can say for small natural numbers that they exist in some standard sense of that word, the existence of big natural numbers is in the best case some kind of idealized potential existence.

We form the conception of natural numbers in a particular language which, among other symbols, contains the name " 1 " for the first number and the function symbol " S " for the immediate successor operation ($n \mapsto \mathrm{S}(n)$). This allows us to name each number. What exactly the names name is not so important to us as it is important to us that what they name fulfils the required structural role of natural numbers (they can even name themselves). We further specify the structure that is important to us by certain claims of the language itself. This is necessary because, although we have the interpretation of the language, the recursive definition of the truth of sentences in the given interpretation is not a computable function due to the infinite domain of the interpretation. I will not specify these claims here, nor the language in which they were made. In that language, they must express, inter alia, (i) that 1 is the first number, (ii) that each number n has its successor $\mathrm{S}(n)$ which is a new natural number in relation to all previous natural numbers, (iii) and that any natural number can thus be obtained. I will hereinafter call these claims axioms of natural numbers. The axioms of natural numbers are neither true nor false. They are a means of specifying our ideas – we are interested in interpretations of the language (structures) for which they are true. Gödel's incompleteness theorems [Göd31] and Lowenheim-Skolem theorems [Löw15, Sko20] tell us that we cannot have a complete specification in a first-order language, even if we include sets in the specification. Although in classical set theory all structures of natural numbers are mutually isomorphic, classical set theory itself has non-isomorphic interpretations. Therefore, in addition to the axiomatic specification, it is necessary to have an interpretation of the language, too. In the case of natural numbers it is a partial interpretation in our internal world of activities. The interpretation is partial due to the idealization of the existence of extremely big numbers. The interpretation belongs to our internal world because we consider numbers to be our imagined constructions that we can partially, to the point of isomorphism, realize in the world available to us. Literally, such a structure does not exist. Probably there is no structure in the external world that satisfies the axioms of natural numbers, and if there is, I would not consider it the structure of numbers, because it does not belong to our internal world of activities. Since we can never build it, it does not exist in our internal world either. There is only the conception of natural numbers, formulated by axioms, and partially realized in our internal world. This is the final result (mathematical model) of modelling our intuition about natural numbers as objects for counting. It carries with it incompleteness and the ever-present tension in mathematics between basic intuition and the constructed model, as well as between the axiomatic specification and the constructive content of mathematical concepts. This tension is a positive source of accepting new axioms as well as improving intuition.

Through real numbers we organize and make precise our intuition about the process of measuring. Real numbers are imagined as the results of such measuring. However, we can imagine (idealized) situations in which the process of measuring never stops – we generate a potentially infinite list of digits, with no consecutive repetition of the same group of digits after some step. If we want to have the results of such processes of measuring, we must introduce, in addition to rational numbers, new results of measuring – irrational numbers. As opposed to natural numbers whose existence we can understand at least as some kind of an idealized potential existence, we cannot explain the existence of irrational numbers in this way. Although we can approximate irrational numbers by rational numbers with arbitrary precision, their existence is outside our means of construction. We have just imagined irrational numbers and they exist only in our imagination.

As with natural numbers, the final mathematical model of real numbers is a junction of axiomatic specification and partial internal interpretation of the corresponding language. For example, in the mathematical model we can identify Euler number e, the irrational number to whom the sequence $\left(1+\frac{1}{n}\right)^n$ is closer, when we take bigger natural number n. Although we can approximate number e with arbitrary precision by constructions in our human world, it certainly does not exist in the same way as my dog. It exists in the same way as an idealized material point in classical mechanics, as non - existing phlogiston in a wrong theory about chemical reactions, and as Snow White in the classical fairy tale *Snow White and the Seven Dwarfs*. However, although our language usually has only a partial interpretation, the logic of using the language assumes that it is a semantically complete language. Because of this assumption, in thinking itself there is no difference whether we think of objects that really exist or we think of objects that do not really exist. That difference can be registered only in a "meeting" with reality. And for mathematics there is no such a meeting: a mathematical model creates its own reality in our internal world of imagination. However, unlike erroneous physical models, the mathematical model of real numbers can be realized approximately in our internal world in the same way that physical models are realized approximately in the external world or children's fairy tales in real theatrical performances.

Euclidean geometry models intuition about the space of our everyday activities. Because of our a priori ignorant approach to space, we imagine that space as a totally symmetric space: all places are the same to us (the homogeneity of space), all directions are the same to us (the isotropy of space) and scales of measuring are the same to us the scale invariance of space). I show in [Č17] how this intuition leads to Weyl's axioms [Wey18] of Euclidean geometry. I consider this interpretation in space of our human activities the primary interpretation of Euclidean geometry. So, Euclidean geometry is a priori, in the same way as number systems are a priori, the result of modelling, not the world, but our activities in the world. However, we can preserve the sentence part of the theory but change the interpretation. Then it does not need to be a mathematical conception any more. It depends on a new interpretation, be it an exterior or an inner one. If we ask ourselves does the physical space have such a structure we must extract from space what we consider as points (maybe enough localized parts of space), as directions (maybe directions of light rays), and the distance between two points (maybe the time needed for light to pass from one point to another). If in such an interpretation the physical space has Euclidean structure then we have got an experimentally verifiable theory. Its sentence part is the same as in our mathematical theory of the space of our human activities, so we can transfer all results to the structure of physical space. Only the interpreted part is different. It does not belong to mathematics any more, but it is the base for an experimental verification of the theory about the external world. However, we can change an interpreted part of the originally imagined Euclidean geometry in a way that it will be still a mathematical theory. And it happens in mathematics often for the following reasons. Namely, when we investigate complex mathematical objects which we cannot perceive so easily, for example a set of functions of some kind, it is useful to find Euclidean structure in it. Then we can transfer our geometric intuition to that set – think of functions as points, measure how distant two functions are, etc. In that way we can visualize them and succeed to think about them more easily and effectively.

The example of Euclidean geometry witnesses that only the axiomatic part of a theory can belong to mathematics, while interpretation does not have to. Also, mathematical interpretation does not have to be an idealized direct interpretation in our internal world, but it can also be an interpretation in some other mathematical model (theory).

Group theory, in addition to being an elementary part of more complex mathematical theories, models above all our intuition about symmetry as invariance to certain transformations. Since different situations have different symmetries, unlike previous mathematical models which have an intended interpretation, this theory does not have an intended interpretation, but has intended non isomorphic interpretations. Thus, some mathematical models are simply sets of axioms without a specific interpretation. However, if they are modelling some important inner intuition, then they are usually very important. Probably, the most famous example is Riemann conceptions about geometry as a manifold with a metric [Rie83]. These models found their application half a century after their invention with the appearance of Einstein's general theory of relativity. Today, manifolds are an essential ingredient of mathematics and physics. Although the application was realized so late, it had to happen, because manifolds model successfully the basic mathematical idea about coordinatization of investigated objects, an idea that generalize such an efficient idea of measuring.

The source of mathematical models does not have to be our inner intuition but they can be ”borrowed” from other disciplines. The nature of our thought and use of language, as well as the way how we manage a vast complexity of the world, leads to extracting a certain structure from such a domain. Our rational cognition of the world is necessarily structural. We extract from the subject certain objects, relations and operations and we describe their properties. If we have thus obtained an important model from that field then its sentence (axiomatic) part is a mathematical model important for examination. For example, we can use classical mechanics. Although particles, motion and forces do not belong to mathematics, mathematics can take the structural properties of phenomena (usually described as a set of sentences in an appropriate language) and formally investigate them: the consequences (for example, in the problem of three bodies), the equivalent formulations (for example, Lagrangian and Hamiltonian formulations of Newtonian mechanics emerged in this way), etc.

In the language of group theory, due to the existence of non-isomorphic interpretations, we sometimes think of a definite, and sometimes of an indefinite ("any") interpretation. However, the very use of language and its logic requires that when we think in the language we necessarily assume that it is a semantically complete language, no matter how we imagine the interpretation. The situation is the same as when we use variables in our thinking. Whether we attach a certain value to a variable or not, in thinking within classical mathematical language we necessarily assume that it has some value. If we think of groups in the language of set theory, then the groups themselves are the values of variables, not interpretations of the whole language, and we think of them differently. The language of set theory allows us to connect and compare groups with each other, without having to know the true nature of individual groups, but possibly their isomorphic copies in the world of sets without urelements. Thus, the process of modelling the initial intuition and the way of working with the constructed models depends on the language in which we model the intuition.

In the consideration of any objects, the consideration of the sets (collection) of those objects naturally occurs. In mathematics, this step has a deeper meaning. Namely, the foundational mathematical modelling should model just the intuition about mathematical and general thought modelling itself. In the process of thought modelling, we extract a structure from a set of objects, that is to say, we extract some distinguished objects, and some relations and functions over the set of objects. Therefore, the subject of the foundational modelling must be the structures themselves and their parts. We can reduce the description of the structures to the description of their parts. It is the standard result of mathematics that we can describe distinguished objects by functions, functions by relations, and relations by sets. In this way we can reduce the foundational modelling to the analysis of sets. From sets we can build all structures. Also, we can compare such structures using functions between them. Furthermore, the language of sets provides simple means for describing structures and constructing new structures from old ones. In this way, sets give us the universal language for mathematics. Besides this, sets are often necessary for specification. For example, the specification of natural numbers requires the axiom of induction, which, in its full formulation, needs the notion of a set. Likewise, the specification of real numbers requires the axiom of continuity and the specification of Euclidean geometry requires Hilbert's axiom of maximality, and they both need the notion of a set. However, in what way are there the set of natural numbers, the set of real numbers, and the set of space points, when they are infinite? Moreover, when we think of sets, we also consider sets of sets. If we want to have an elegant, rounded and universal set theory, infinite sets are naturally imposed on us, truly the whole infinite hierarchy of infinite sets, together with infeasible operations on them. How can we understand the existence of such sets and operations? Should we reject this theory which has proven to be very successful because its objects can be realized only when they are finite? Hilbert, who certainly knew what is good mathematics, said on Cantor's set theory of infinite sets: "This appears to me to be the most admirable flower of the mathematical intellect and in general one of the highest achievements of purely rational human activity," [Hil26].

The language of set theory presupposes an intended interpretation. In addition to the fact that we can only partially realize it, the interpretation itself is not clear to us in many ways. It is clear that the idea of a set derives from our activities of grouping, arranging and connecting objects and that set theory is an idealized mathematical model for these activities. However, in the very finite part, when we talk, for example, about the set containing three concrete objects o_1, o_2 and o_3, it is not clear what kind of object the set itself is, let's call it $s = \{o_1, o_2, o_3\}$. Formally, we can describe the situation by saying that we have added a new object to the objects, which we call the set of these objects and which has the unique property that only the objects o_1, o_2 and o_3 belong to it, and no other. Such a description would correspond to a combinatorial approach and obviously has a structured overtone - the very nature of sets is not important but their relationship to other objects is important. The description $s = \{x | x = o_1 \text{ or } x = o_2 \text{ or } x = o_3\}$ has a different connotation. Now the set is given by a predicate, so it is a kind of extensional abstraction of a one-place predicate. In this view of sets, as extensions of one-place predicates, they have no structural role but have their individual nature in our world of meaningful linguistic forms, in the same way that points and directions have their individual nature in the geometry of our world. Let us note that both the structural and individual view of sets change the initial intuition about the impossibility of realizing infinite sets. In the structural view, the set of all natural numbers is a new object equal to finite sets - the only difference is that infinitely many objects enter with it into the relation of belonging. In the individual view, it is also equal to finite sets - the only difference is that infinitely many objects satisfy the condition "to be a natural number". The idea that a set must be "made" of its elements is not present here at all, an idea according to which we can never make an infinite set. Despite all the doubts related to the notion of a set, the constructed mathematical model is very successful. Today ZFC axioms form its axiomatic part, and the model has an intended interpretation, although there are doubts as to what the interpretation is, what we can realize and how we can realize. Here we only have a more pronounced tension between the basic intuition and the final model, which can ultimately lead to model refinement, model change, or even separation into multiple models.

Previously described conceptions bring all the essential characteristics of mathematical conceptions. First of all, mathematical conceptions, like other useful types of conceptions that do not pretend to be about the external world, have a goal. Fairy tales have the goal to edify. The game of chess has the goal of being an intellectual amusement. The goal of mathematical conceptions is to be successful imagined tools in the process of rational cognition, and in rational activities in general. We can use them directly like the use of numbers, through an ideal model of interaction with the world. We can use them indirectly: (i) like the use of Euclidean geometry – by changing the interpreted part of the theory into an external interpretation,

(ii) like the use of group theory – by giving just the axioms and their consequences regardless of interpretation, which could be the external one, or (iii) like the use of set theory – by organizing effectively other mathematical tools. Also, we use mathematical conceptions indirectly, (iv) as ingredients of more complex mathematical conceptions – as it is, for example, the case with Euclidean space as a tangent space on a Riemann manifold, or (v) we use them indirectly in the way described above with Euclidean geometry – to interpret them in collections of complex mathematical objects for the purpose of making them more intuitive and more manageable.

The second moral from the previous examples is that mathematics is an inner organization of rational cognition and knowledge, a thoughtful shaping of the part of the cognition that belongs to us. For example, we organize possible results of measurement in an appropriate number system. It needs to be distinguished from (but not opposed to) the outer organization of rational cognition, a real shaping of an environment that comprises construction of a physical means for cognition (for example, an instrument for measuring temperature). Mathematics is a process and result of shaping our intuitions and ideas about our internal human reality, into thoughtful models which enable us to understand and control better the whole reality. For example, we shape our sense for quantity into a system of measuring quantities by numbers. Thoughtful modelling of other intuitions about our internal human world, for example intuitions about symmetry, flatness, closeness, etc., leads to another mathematical conceptions. By our internal human world I consider our behaviour as a biological species in an environment in which we exist, a space and time of our immediate senses and activities which we organize and design by our human measure. For the sake of simplicity, an intuition about that world I shall term inner intuition.

Major mathematical models, like the previously described, arise from intuition about our internal activities and organization. It is from these concrete activities that the idea of an idealized world emerges. Due to the essential role of language in thought processes, as it is analysed in [Č20], we can only realize this idea by building an appropriate language. By choosing names, function symbols and predicate symbols, we shape the initial intuition into one structured conception. Since the conception goes beyond our real capabilities, the constructed language has only partial interpretation in our internal world. Since interpretation is partial, and because the domain of interpretation is usually infinite, we cannot determine the truth of all sentences of the language. Therefore, we must further specify the conception by appropriate choice of axioms. Thus, the final mathematical model is a junction of axioms and partial internal interpretation of an adequate language.[1] Sometimes, as we have seen on the example of the theory of groups, a mathematical model can be reduced to a set of axioms without an internal interpretation. Sometimes, the internal interpretation can be a total interpretation in another mathematical theory, as we have seen on the example of Euclidean geometry. However, we must not forget that although a mathematical model is the final product of modelling inner intuition, in real mathematical activities it is never isolated from the source from which it originated. This is especially important because the mathematical model, generally speaking, is not complete - there are multiple interpretations that are extensions of partial interpretation and that satisfy axioms; and intuition always leaves room for completion. Likewise, mathematical models are not isolated from each other. Set theory is a natural environment for formulating and comparing mathematical models. In such an approach, axioms become the definition of a certain type of structure. However, set theory analyses the described structures in a uniform way, without going into their nature, whether they are extracted from the external world or from the world of our internal activities. Thus, although it gives an elegant mathematical description, set theory can also hide the true nature of mathematical models.

Mathematics is, in a great part, an elaborated language. The ”magic” of mathematics is, in a great part, the ”magic” of language. Inferring logical consequences from axioms, we establish what is true in the mathematical model. This can be very creative and exciting work and it seems that we discover truths about some existing exotic world, but we only unfold the specification. The key difference with scientific theories is that the interpretation here is in our internal and not in the external world. The external interpretation of a scientific theory enables us to test the theory, whether it has a power of rational cognition of nature. If the theory has such power then at least some objects of the theory exist in the primary sense of the word and at least some sentences of the theory are truths about nature. If the language does not have such a part, and that is the case with mathematics, then the objects we are talking about exist only within the conception (story), although they do not exist in the external world. Equally, if the language does not have an experimentally verifiable part then sentences we

[1] The word "model" has a different meaning here than in mathematical logic

consider true within the conception are not true in the external world. We cannot experimentally verify that $|| + || = ||||$ $(2 + 2 = 4)$, not because it is an eternal truth of numbers, but because it is the way we add tallies. Likewise, we cannot experimentally verify that $(x^2)' = 2x$ because it is the consequence of how we imagined real numbers and functions. Mathematical objects are, possibly, objects extracted from our internal activities, and mathematical truths are, possibly, truths about our internal activities. We are free to imagine any mathematical world. The real external existence of such a world is not important at all; all that matters is to be a successful thought tool in the process of rational cognition. In Cantor's words, "the essence of mathematics lies precisely in its freedom" [Can83]. The only constraint is, inside classical logic, that conceptions must not be contradictory. For Hilbert, in mathematics to exist means to be free of contradictions. In Hilbert's words: "the proof of the consistency of the axioms is at the same time the proof of the mathematical existence", [Hil00]. In Dedekind's words, "numbers are free creations of the human mind" [Ded88]. These views are in sharp contrast with historical views that mathematical truths exist really in some way and that we discover them and not create them. Historically, this change of view occurred in the 19th century with the appearance of non - Euclidean geometries. The new philosophical view of mathematics has freed the human mathematical powers and it has caused the blossom of modern mathematics. It is a nice example of how philosophical views can influence science in a positive way. According to the old views mathematical truths are a particular kind of truths about the world. An exemplar is Euclidean geometry – according to the old views, it discovers the truths about space. The appearance of non Euclidean geometries which are incompatible with Euclidean geometries but are equally logical in thinking and equally good candidates for the "true" geometry of the world has definitely separated mathematics from the truths about nature. It has become clear that mathematics does not discover the truths about the world. If it discovers the truths at all they are in the best case the truths about our own activities in that world. From my personal teaching experience, I know that looking at mathematics as a free and creative human activity is a far better basis for learning mathematics than looking at it as an eternal truth about some elusive world.

I will now analyse in more detail the question of the existence of mathematical objects. The key to the answer is in understanding the essential role of language in our thinking. In [Č20] it is shown how we synthesize our rational cognition of the world through language. An ideal language is, for example, an interpreted language of the first-order logic in which we know the semantic values of all non logical primitive symbols of the language as well as the semantic values of all descriptions and sentences in the language. In a real process of rational cognition, we use names for which we do not know completely what they name, predicate and function symbols for which we do not know completely what they symbolise, and quantified sentences for which we do not know if they are true or not. For a theory to be a scientific one, at least some names and some function and predicate symbols must have an exterior interpretation, an interpretation in the exterior world, not necessarily a complete one. This partial external interpretation enables us to perform at least part of the binary experiments described by atomic sentences. This allows nature to put its answers into our framework, so that we can test our conceptions experimentally. Without this part the theory is unusable. On the other hand, due to the partial external interpretation of the language and the impossibility to perform all binary experiments determined by atomic sentences, we necessarily complete the theory with a set of sentences that we consider true in a given situation (axioms of the theory). Thus, scientific theory is also a junction of axioms and partial external interpretations of language. Viewed at the level of the final product, scientific and mathematical theory differ in that the former has an external partial interpretation and the latter an internal one.[2] As I already mentioned when I considered how Euler's number e exists, although our language usually has only a partial interpretation, the logic of using the language assumes that it is a semantically complete language, i.e. that it is an ideal language in the sense as I described at the beginning of this paragraph. Because of this assumption, in thinking itself there is no difference whether we think of objects that really exist or we think of objects that do not really exist. That difference can be registered only in a ”meeting” with reality. Thus the question of the existence of the objects we speak and think about is completely irrelevant as long as we do not try to connect language with the outside world. Because of the “encounter” with reality, at least some of the objects that scientific theory speaks of must exist in the primary sense of the word, as objects from the outside world. What about the other objects that scientific theory talks about? Since the early 19th century, physicists and chemists have used the

[2]Here the word "theory" has a different meaning than in mathematical logic.

assumption of the existence of atoms to explain many phenomena in matter. Atoms were initially only imagined objects, and in the end it was established that they really exist. Unlike an atom, the ether was initially an imagined object, and in the end it was established that it does not exist. From this example we see that in science imagined objects can be potentially existing objects. For the next example, let us take a material particle in classical mechanics, the particle occupying a single point in space. It is imagined from the beginning as an idealized object, which does not really exist, but real objects can approximate it. Let us not forget that the basic laws of classical mechanics speak precisely of material particles. Eg. Newton's law of gravitation speaks of the gravitational force between two material particles. Only through a certain synthesis of these idealized laws do we obtain laws of the behaviour of real objects. Perhaps the notion of a material point can be avoided in the formulation of classical mechanics, but that formulation, if it exists at all, would be unnecessarily complicated. The above examples show that imagined objects together with statements that we consider true for them are very important for scientific theory. If we were to ban their use, we would literally cripple scientific theories. They are certainly a necessary linguistic tool of scientific theories and their status can change over time. However, I would point out that their significance follows exclusively from their connection with the objects of the theory that exist in the external world. The situation is analogous to mathematical theories, only here we have a partial interpretation in our inner world of activities. Just as physical models are idealizations and approximations of real processes in the external world, mathematical models are idealizations and approximations of our real activities in our internal human world. However, as with physical models, the importance of mathematical models also stems from the success of their application in rational cognition. Here the ontological situation is even better than in the case of scientific theories, because mathematical theories are completely under our control. Unlike objects from the outside world, which are under the authority of nature, we create mathematical objects, and the truths about them are in fact their specifications. Some mathematical objects are determined up to isomorphism, such as natural numbers. We realize these objects internally as part of the (partial) realization of the corresponding structure. Some can be completely realized, such as not too large natural numbers, or up to a satisfactory approximation, as irrational numbers. Some mathematical objects have a more specific nature, such as sets. We realize them through representation, for example, sets with appropriate one-place predicates (linguistic forms of a certain language). Representation is a kind of isomorphism. However, the difference is that not all isomorphic structures are equal to us, but we have a certain, perhaps not entirely clear interpretation as with sets, to which we look for "close" representatives. Some mathematical objects have a completely determined nature, such as geometric objects in the primary interpretation of Euclidean geometry. We realize them approximately, but directly (not through isomorphism or representation) in our internal world of activities.

There is another way we can understand the existence of mathematical objects. If a theory is consistent, from Henkin's [Hen49] proof of the completeness of a deductive system follows the existence of a canonical model whose objects are classes of equivalence of the corresponding terms of the language. Since this model is homomorphically embedded in any interpretation that satisfies axioms, this means that we can always represent a finite portion of imagined mathematical objects through their proper names in the language in which we speak of those objects.

To summarize, let us start from the fact that the question of the existence of the objects of language is irrelevant to the formation and use of language in the process of thinking, until the moment when we apply language to the outside world. For a scientific theory, at a given stage of its development, the existence of parts of language that speak of objects whose existence in the outside world has not been established and about which the theory makes certain claims can only be evaluated through the way these parts relate to the experimentally verifiable part of the theory. Through this interaction with the experimentally verifiable part of the theory, it can be shown that such objects exist or do not exist, and thus their ontological status is revised. However, for some objects it can be shown that they are imagined objects that enable an efficient, perhaps necessary, linguistic synthesis of rational cognition. In the latter case we may regard them as mathematical objects, our imagined tools of rational cognition. Thus we can equate them with the parts of language that belong to mathematics and reduce the problem of their existence to the problem of the existence of mathematical objects. Since mathematical language, i.e. mathematical models, do not have an experimentally verifiable part, from that point of view, the question of the existence of mathematical objects is irrelevant, and mathematical truths are only the specifications of imagined objects. The only thing that matters is how we apply mathematical language in rational cognition. Thus, in the broadest view, a view that does not restrict mathematical freedom, a mathematical model is a set of sentences of a language that we declare to be true and that has possibly a partial internal interpretation. Thereby, it is important that mathematical model is intended to be a successful tool of rational cognition or rational activities in general. It is the success of a mathematical model that determines its quality. And for a mathematical model to be successful, its language must somehow be related to the language we use to describe the outside world. For example, when we count, basically, with the words " one ", " two ", " three ", ..., we associate the objects we are counting. We don't need to know at all what those words name and whether they name something at all. Of course, we imagine the language of numbers so that these words name objects in some structure of numbers (we can assume that these words themselves form that structure). But the point is that we can count without knowing what those words name. Let us imagine another situation, that someone gave us axioms of real numbers without us ever hearing about real numbers. By deriving statements from these axioms and defining new words using existing ones, we may understand that this language allows us to describe the measurement process and express the measurement results in it. In this way, that language (that theory) itself becomes a successful tool of rational cognition, regardless of what the objects of the language are and whether they exist at all. I have given these extreme cases to show that in the broadest view of a mathematical model, as a consistent set of axioms in a language, the question of what that theory is talking about need not be important at all for its use. The set of axioms can even have multiple interpretations. And if you just want to have some interpretation, we can build a canonical model from the very strings of the language of the theory. But real and successful mathematical models do not arise in such a way. As I described in this article, they are the results of modelling intuition about activities from our human world and in that world they finally have their own partial interpretation which significantly reduces the incompleteness of the model. So the best I can say is that mathematical objects do not exist in the outside world – they are our internally imagined objects, some of which, at least approximately, we can realize or represent.

It remains to explain in more detail, how is it possible that something imagined can contribute to rational cognition of the world? I consider that the indirect contribution, which I described above, is enough clear. So, I will analyse the direct contribution on the example of numbers. Real numbers are imagined objects which can be, generally speaking, only approximately realized in our internal world. However, in the process of measuring, we connect them with the external world, enabling nature to select one of the offered numbers as its answer. The number itself is not real (in the sense that it does not belong to the external world) but nature's selection is real. Numbers belong to our experimental framework but nature's selection is a truth about the external world. For example, in the process of measuring the speed of light, between all numbers nature selects the number c. The selected number c possibly exists as our internal construction. Whether it is a rational or irrational number depends on the choice of units of measurement. However, that c is the speed of light is the idealized truth about the external world which is synthesized in the process of measuring. Idealized, because we assume that c is the result of an idealized process of measurement to which the actual measurement is only an approximation. In the same way, the simple assertion about natural numbers, that $2 + 2 = 4$, is a true sentence about the imagined world of natural numbers, and not a truth about the external world. However, through the real process of counting, we can use assertions about numbers to obtain synthesizing assertions about the external world. For example, when we put two apples in a basket which already contains two apples, we predict that there will be $2 + 2 = 4$ apples in the basket. This is the prediction about reality deduced from the mathematical assertion that $2 + 2 = 4$ and the assumption that the mathematical model of counting and adding one set of things to another set is applicable in this situation. However, we must distinguish the mathematical assertion that $2 + 2 = 4$ from the assertion about reality that when we add 2 apples to 2 apples there will be 4 apples. The best way to see the difference is to imagine a situation where we add 2 apples to 2 apples and get 5 apples. It would mean that it is not always true, as we have thought, that adding 2 apples to 2 apples gives 4 apples, but in some situations, according to as yet unknown physical laws, an additional apple emerges. However, this situation would not have any influence on the world of numbers. In that world it is still true that $2 + 2 = 4$. It only means that in some real situations we cannot apply the mathematical model of counting and addition. The natural numbers model of counting as well as the real numbers model of measuring or any other mathematical model have their assumptions of applicability. For the natural numbers model we assume that we can associate the number of elements to a collection of objects by the process of counting. For the real number model we assume that we can always continue to measure with ten times smaller unit, if it is necessary. A real process of measuring, for example, of the distance of point B to point A, must stop in some step, because the passage to a ten times smaller unit would not be possible with an existing measuring instrument or that passage revises our understanding of what we are measuring at all. For example, in measuring the distance between points A and B marked by pencil on a paper, the passage to the one hundredth part of millimetre requires a microscope. Looked at under a microscope A and B are not points any more, but diffused flecks. And what are we measuring now? The distance between the closest points of the two flecks? If we continue to magnify, we will see molecules which constitute flecks and which are in constant vibrations. And if we went to more tiny parts we would come to the world of quantum mechanics in which classical notions, on which our conception of measuring distance is based, are not valid any more. However, the question whether it is possible to apply the mathematical model of measuring in reality is not a mathematical question at all. Only when the assumptions of the model of measuring are (at least approximately) fulfilled can we employ the mathematics of real numbers to the real world. Likewise, only when the assumptions of the model of counting are fulfilled can we employ the mathematics of natural numbers to the real world. Only then we have at our disposal the whole mathematical world that can help us in asserting truths about the real world. We have at our disposal an elaborate non-verifiable language which we can connect with a verifiable language, mathematical truths which we can synthesize into truths about the external world. If contradictions occurs in an interaction with the verifiable part of the language, it does not mean that the mathematical model is false (the concept of the real truth and falsehood does not make any sense for the model), but that the assumptions about its applicability in that situation are false.

Quine [Qui51] in his naturalized epistemology considers that every part of the web of knowledge is liable to experiment, including logic and mathematics. That is true, but there are qualitative differences between science on one side and logic and mathematics on other. Experimental evidence can affect the truth values of scientific sentences but not the truth values of mathematical and logical sentences. It can only question the applicability or adequacy of mathematical models and language frameworks in some parts of science. Scientific theories are true or false of something while mathematical models are good or bad of something.

Various mathematical models are not mutually disconnected but they are interwoven. Moreover, we express these connections also by corresponding mathematical models. First of all, there is a not so big collection of primitive mathematical models ("mother structures" in Bourbaki's terminology [Bou50]) that model the basic intuitions about our human world, intuition about near and remote (topological and metric structures), about measuring (spaces with measure), about straight and flat (linear spaces), about symmetry (groups), about order (ordered structures) etc. We use them as ingredients of more complex mathematical models. The complex mathematical models enable us to realize some simple and important mathematical ideas (for example, we use normed linear spaces to realize an idea of the velocity of change) or they have important applications (like Hilbert spaces which, among other things, describe the states of quantum systems). Therefore, the world of mathematics is built of some primitive models which model our inner ideas and of various ways of comparing and combining these models into more complex models. Corresponding mathematical theories are interpreted mutually or have internal interpretations. If a theory has an external interpretation then only its sentence part belongs to mathematics. For example, classical mechanics is a theory about the world. However, if we ignore its externally interpreted part we get a set of sentences which we can investigate mathematically. Hence, we can say that mathematics is concerned with the internal models and internal properties of external models, or more simply, it is concerned with that part of rational cognition that belongs to us. Furthermore, such diverse mathematical models are the basis for secondary mathematical models that model how to compare structures (set theory and category theory) and in what language to describe them (mathematical logic). However, regardless of the complexity of the world of modern mathematics, mathematics is an inner organization of rational cognition based on the modelling of our inner intuition. In constructing mathematical worlds the criteria of real truth and falsehood have no meaning, although every such world has its inner truths and falsehoods which shape and express the underlying conception. However, in the process of rational cognition we synthesize mathematical objects and truths into truths about the real world (for example, into Newton's law of universal gravitation). In the construction of mathematical worlds the criteria of simplicity and beauty, almost artistic criteria, are of real importance as well as how well the constructed theories model the original intuition and ideas. The fulfilment of these criteria is a good indicator, as experience shows, that the main criteria will also be fulfilled, the criteria of the direct or indirect usefulness of those worlds as our thought tools in the process of rational cognition. If the ideas are good and if they are well modelled mathematically, sooner or later they will certainly find a successful application, as we have seen in the example of Riemann's manifolds.

In the rest of the article, my intention is to position my view of mathematics in relation to some standard topics and two seemingly close views, structuralism and fictionalism, in modern philosophy of mathematics.

I consider that my view on mathematics satisfies both concerns of Benacerraf dilemma [Ben73]: "(1) the concern for having a homogeneous semantical theory in which semantics for the propositions of mathematics parallel the semantics for the rest of the language, and (2) the concern that the account of mathematical truth mesh with a reasonable epistemology.". I have shown that the only difference between mathematical and scientific language is whether it is a partial interpretation of language in the internal or external world. Since we are the creators of the mathematical worlds, their epistemological status is unquestionable.

Also, I consider that the analysis conducted in this article has shown that my approach fully meets Bueno's [Bue09] "five desiderata that an account of mathematics should meet to make sense of mathematical practice: (1) The view explains the possibility of mathematical knowledge. (2) It explains how reference to mathematical entities is achieved. (3) It accommodates the application of mathematics to science. (4) It provides a uniform semantics for mathematics and science. (5) It takes mathematical discourse literally."

Concerning Quine-Putnam indispensability argument for existence of mathematical objects as it is spelled in [Col19], my view fully supports the second premise "Mathematical entities are indispensable to our best scientific theories." and rejects the first premise "We ought to have ontological commitment to all and only the entities that are indispensable to our best scientific theories.". I have previously shown in the simplest example of counting, which is certainly indispensable to our best scientific theories, that one can use the language of numbers to count in (scientific) application without even knowing what numbers are and whether they exist at all.

I consider that my view on the nature of mathematics belongs in large part to the tradition which has begun with Dedekind, Cantor and Hilbert. It has spread into the mathematics community through works of Emmy Noether, van der Waerden and Bourbaki. The subject of discussion is whether and in what form psychologism is present in Dedekind's work, also in the quotation from the beginning of the article (see [Rec03] for a detailed analysis). If Dedekind, under the free creation of natural numbers (as well as other numbers), considered their creation as a creation of abstract thoughts then that is different from my view. In my view, mathematical objects are always imagined as concrete objects, because that is how our language works, regardless of whether we can realize them in our internal world or not. We already do abstraction with language itself, which ensures that we consider only those properties of mathematical objects that interest us. Thus, language is the bearer of the required abstraction and not the objects of language.

The structuralism that is part of my view of mathematics is structuralism that is present in mathematical practice. He is deprived of all philosophical additions on the nature of mathematical structures and mathematical objects by which various structuralist views in the philosophy of mathematics are distinguished, as nicely dissected in [RP00]. In what follows, by structuralism I will mean precisely this structuralism. The approach in mathematics according to which only the structure of objects is important is named in [RP00] structuralist methodology and is described as follows: "Mathematicians with a structuralist methodology stress the following two principles in connection with them: (i) What we usually do in mathematics (or, in any case, what we should do) is to study the structural features of such entities. In other words, we study them as structures, or insofar as they are structures. (ii) At the same time, it is (or should be) of no real concern in mathematics what the intrinsic nature of these entities is, beyond their structural features.". I consider that structuralism is only one, although very important, aspect of mathematical modelling but we cannot reduce all mathematics to it. As I previously stated, the nature of our thought and use of language, as well as the way how we manage a vast complexity of the world, leads to extracting a certain structure from such a domain. Thus, in the study of a phenomenon, we limit ourselves to the study of a certain structure of that phenomenon. However, structuralism is an approach that studies structures formally and does not enter into their nature. It thus naturally falls under mathematics, and the language of set theory is a natural language for the study of structures. However, not only do the structures we extract from the outside world (and thus do not belong to mathematics) have their content, but mathematical structures can also have their content. We have seen that only structural properties are important for natural numbers, and here the whole mathematics is covered by the structural approach. However, what about sets, for example? Although we do not have a completely clear interpretation of what sets are, we do not think that only structure is important for sets, but that they have additional content. The situation is even more pronounced with Euclidean geometry which I consider to be a mathematical model of space of our human activities and which is on an equal footing to number systems, and not part of physics. Its objects are completely determined in our internal world and are not just "places" in a certain structure. Although we can study various structures of the same type as the structure of Euclidean geometry, or connect it with isomorphic structures, thus we cannot exhaust its content.

The main difference between fictionalism and my view of mathematics is that fictionalism considers that mathematical objects do not exist (or at least that it does not matter whether they exist at all) and that mathematical claims are not true, while I consider that mathematical objects are imagined objects that we can at least partially realize in the world of our internal activities and that mathematical truths are specifications of mathematical ideas, idealized truths about the internal world of our activities. This connection between mathematics (mathematical objects and mathematical truths) and our internal world of activities is crucial - it is the source from which mathematics arises and the environment in which it is applied. Furthermore, I believe that fiction is always linguistic fiction. And I do not think that when we use mathematical language, we pretend that there are mathematical objects and pretend that what we say about them is true. The very nature of language use requires us to assume the existence of the objects we are talking about. Only when we step out from that language can we speak of the existence of these objects (in this new language), as Carnap explained long ago in [Car50]. In my opinion, fictionalism has two different versions. In one version, fictionalism does not take mathematical language literally but figuratively, according to Field [Fie80] as a conservative extension of a content language, while Yablo [Yab02] considers it as representational aid. The second version (Balaguer [Bal98], Leng [Len10], Bueno [Bue09] takes the mathematical language literally. Since I, too, take mathematical language literally, my view of mathematics may have common points only with the second version of fictionalism, and I will here refer to this version. I have previously shown by the example of natural and real numbers how a mathematical model can be applied purely linguistically without even knowing what its objects are and whether they exist at all. This way of using mathematics is in line with fictionalism and it shows that, if we look at the final product, fictionalism is very close to formalism. However, such an approach to mathematics is artificial and limiting - neither mathematics is practiced in this way nor can all mathematical activities be reduced to such an approach. It is a partial internal interpretation that gives and sustains life to mathematical models.

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