A PARTIAL MODEL THEORY AND SOME OF ITS APPLICATIONS

Rodolfo Cunha Carnier
University of São Paulo (BR)

Abstract

In this paper, we introduce the basics of what we shall call *partial model theory*, which is an extension of traditional model theory to partial structures. These are a specific kind of structure developed within the partial structures approach, which is a view constituting the semantic approach of theories. And together with other related semantical concepts, like the concept of quasi-truth, partial structures have been used in contemporary philosophy of science for several purposes. Nonetheless, those uses presuppose certain technical results, or could be improved by resorting to certain technical notions and results, that have not been put forward so far. Thus, in the present work, we intend to introduce part of this particular technical apparatus that is still lacking especially in logic, throughout the development of partial model theory. We begin by extending traditional notions of model theory for partial structures, like the notions of substructure and homomorphism, and by proving some results concerning the notion of quasi-truth. Posteriorly, we show how the content introduced can be used to improve a particular application of partial structures and quasi-truth in the philosophy of science.

Key-words: partial model theory, partial structures approach, partial structures, quasi-truth, philosophy of science

1 Introduction

Recently in the philosophy of science, emerged a proposal called *partial structures approach*[1]. One of the main aims of this proposal is to supply a conceptual frame that allows us to formally accommodate *epistemic partiality*; that is, to supply a conceptual frame that allows us to employ concepts such as the concept of structure and others, in contexts where our knowledge about a certain domain under investigation is partial and hence incomplete. There are three main concepts developed within the partial structures approach which constitute its core. They are the concepts of *partial relation*, *partial structure* and *quasi-truth*.

In short, an n-ary partial relation defined on a set A is a relation that is not defined for every n-tuple of objects of A; a partial structure, in turn, is

[1] As far as we know, the partial structures approach came out in 1986, in the paper *Pragmatic truth and approximation to truth*, from Newton da Costa, Rolando Chuaqui and Irene Mikenberg [1]. Later this proposal received contributions from several authors, notably Steven French and James Ladyman (cf. [2] and [3]), but also Otávio Bueno, whose work will be analyzed in section 3 below.

a structure whose relations defined over its universe are partial in the referred sense. The idea is that a partial structure models a given domain of investigation from an epistemic point of view, i.e. based on what is known and unknown about it. So if we are analyzing a domain of investigation and we do not know whether or not certain entities stand in a relation to each other, we can formally accommodate this situation by employing a partial structure. Partial structures also play the same role as that of usual structures, in the sense that they also provide an interpretation for the symbols of a language, and thus one can talk about *partial models for a language* (cf. [4] p. 191).

As to quasi truth, from a philosophical point of view it may be considered a *pragmatic* truth notion, since it was conceived with the aim of apprehending some of the ideas underlying the truth notions of Charles Sanders Peirce and William James (cf. [1] p. 201 and [2] pp. 12-6); accordingly, quasi-truth can also be called *pragmatic truth*. From a logical point of view, nevertheless, quasi-truth was inspired by the standard notion of truth of Alfred Tarski, which is reflected in some formal characteristics that both notions share, like the fact that a sentence can only be quasi-true in a partial structure with respect to an interpretation, the same way a sentence can only be true (in the Tarskian sense) in a structure with respect to an interpretation.

1.1 The motivations underlying our proposals

As we have already pointed out, the partial structures approach emerged in the context of the philosophy of science; in effect, it constitutes the *semantic approach of theories*, which means that it shares with the other members of this view some specific commitments, such as the commitment with the thesis according to which the identity of a scientific theory is given by the class of its models, in the model-theoretic sense. But the partial structures approach has a distinctive feature, which is the fact that it allows to incorporate in the semantical analysis of theories the kind of epistemic partiality discussed previously, and this feature has been used to defend other members of the semantic approach against criticisms. To be more specific, in his paper *Empirical Adequacy: A Partial Structures Approach*, Otávio Bueno uses the partial structures approach to defend van Fraassen's *constructive empiricism* against a criticism directed to the notion of *empirical adequacy* (cf. [5]).

However, this use is highly problematic because of how Bueno develops the formal aspects of his proposals: the way he introduces certain notions is rather vague, and he also resorts to concepts whose properties call into question his defense of constructive empiricism (cf. pp. 25-6 below). It is our contention, nonetheless, that Bueno's proposals can be preserved and indeed can be used to defend constructive empiricism against the criticism in question, if their logical features are reformulated. Now this reformulation requires changes in how some concepts have been defined within the partial structures approach, the introduction of new ones, and the proof of some technical results about them. So, motivated in part by the intention of solving the problems we see in Bueno's work, we will develop what we shall call *partial model theory*, which is an extension of traditional model theory to partial structures.

By developing partial model theory we will also reformulate the partial structures approach, and this reformulation is prompted both by a criticism that could be addressed to this view, and by a criticism that was indeed addressed to it.

According to the latter criticism, made by Sebastian Lutz (cf. [6]), the partial structures approach fails in its aim of providing a conceptual frame that allows one to formally accommodate epistemic partiality. The problems Lutz sees in the partial structures approach have to do with partial structures; more precisely, he criticizes the fact that partial structures interpret every constant symbol of a language, and claims that the interpretation of a function symbol, which is a *partial function*, is not enough to accommodate the kind of lack of knowledge about functions typically found in science.

The problem with a partial structure interpreting every constant symbol of a language is that in some contexts we may simply not know the value of a given constant, at least up to a certain instant of time (cf. [6] pp. 1356-7). As to partial functions, the problem is that we may not know the value of an n-ary (partial) function f, defined over the domain A of a (partial) structure, for an argument $(a_1, ..., a_n) \in A^n$, but we may know that some elements of A do not correspond to this value. Nevertheless, if f is an n-ary partial function defined over A such that the value of f for the argument $(a_1, ..., a_n)$ is not defined, then we cannot say whether or not a is the value assigned to $(a_1, ..., a_n)$ by f, for any $a \in A$. "The result", Lutz says, "is that for each argument of the function, one either knows the value of the function with absolute precision [in case its value is defined for the argument], or has no information whatsoever about its possible values ([6] p. 1357)."

To overcome Lutz's criticism we will define partial structures in such a way that they do not interpret constant symbols, and assign functions symbols to a new kind of partial function that shall be introduced throughout the concept of partial relation. As we shall see, those new partial functions behave precisely as they should in order to accommodate the sort of epistemic partiality regarding functions pointed out by Lutz. More specifically, they will be such that even if the value of an n-ary function f over a set A is not defined for a given argument $(a_1, ..., a_n) \in A^n$, there may be an element $a \in A$ such that a cannot be the value of f for $(a_1, ..., a_n)$. Then it will be clear that the problems posed by Lutz to partial structures are actually related to *how they have been defined so far*, and that both epistemic partiality regarding the interpretation of constants and about functions can be easily incorporated into the semantic analysis of theories supplied by the partial structures approach.

As regards the criticism that could be addressed to the partial structures approach, it has to do with partial relations. In every work where partial relations are employed they are defined as a relation over a specific set, in such a way that both the domain and the counterdomain of the relation coincide. But this prevents the application of partial relations (and partial structures) in contexts where one does not know whether a relation holds between individuals belonging to distinct sets. In its turn, this entails a further problem for our purposes, for as we have said we will define partial functions throughout partial relations, but in many reconstructions of scientific theories along the lines of the semantic approach there are functions whose respective domains and counterdomains are not the same. In Patrick Suppes' reconstruction of *classicle particle mechanics*, for instance, the notion of mass is represented by a function $m : P \to \mathbb{R}^+$ which assigns to each particle $p \in P$ the numerical value of its mass $m(p)$ (cf. [7]

p. 294)[2]. So if one wishes to use partial structures as they shall be defined - in such a way that they assign function symbols to our new notion of partial function - to reconstruct theories like classicle particle mechanics, one needs to have partial relations such that their domains and counterdomains do not coincide. Accordingly, we will introduce a new notion of partial relation, which is actually a generalization of the traditional one. To be more precise, instead of defining partial relations *over a given set*, we will define simply partial relations, allowing that their respective domains be different from their counterdomains.

We conclude this introduction with an outline of how we shall proceed in what follows. We start to develop our partial model theory in the next section by defining first our new version of partial relations and partial functions. Then we will introduce the notion of partial structure for a fixed language, as well as the notions of *expansion* - which is a relation between partial structures -, *substructure*, *homomorphism* and quasi-truth. In the third section we show how the content developed in section 2 can be employed to solve the problems of Bueno's proposals. We begin this section with a succinct account of the notion of empirical adequacy, then we present the referred criticism against this notion; afterwards we will introduce Bueno's proposals, followed by our criticisms against them. Finally, we will put forward our reformulation of Bueno's proposals using the content of section 2.

2 Partial model theory

Definition 1. A *binary partial relation* $\mathbf{R}$ is an ordered triple $(\mathbf{R}_+, \mathbf{R}_-, \mathbf{R}_0)$ such that:

1. $\mathbf{R}_+, \mathbf{R}_-, \mathbf{R}_0$ are mutually disjoint sets;
2. $\mathbf{R}_+ \cup \mathbf{R}_- \cup \mathbf{R}_0 = A \times B$, for two sets A and B.

If $A = B$, then $\mathbf{R}$ is said to be a binary partial relation *over* A. In case $\mathbf{R}_0 = \emptyset$, $\mathbf{R}$ is a usual relation that may be identified with $\mathbf{R}_+$, and is said to be a *total relation.* The set $\{a \in A :$ there exists $b \in B$ such that $(a, b) \in \mathbf{R}_+\}$ is called the *domain* of $\mathbf{R}$, whereas the set $\{b \in B :$ there exists $a \in A$ such that $(a, b) \in \mathbf{R}_+\}$ is called the *counterdomain* of $\mathbf{R}$.

The idea is that $\mathbf{R}$ is a relation which is not defined for every pair of objects of A and B, so that $\mathbf{R}_+$ is the set of the pairs that satisfy $\mathbf{R}$, $\mathbf{R}_-$ is the set of the pairs that do not satisfy $\mathbf{R}$, and $\mathbf{R}_0$ is the set of the pairs for which it is not defined whether or not they satisfy $\mathbf{R}$.

Definition 2. Let $\mathbf{R}$, $\mathbf{S}$ be binary partial relations and A, B, C be sets such that $\mathbf{R}_+ \cup \mathbf{R}_- \cup \mathbf{R}_0 = A \times B$ and $\mathbf{S}_+ \cup \mathbf{S}_- \cup \mathbf{S}_0 = B \times C$. We define the *composition* of $\mathbf{R}$ and $\mathbf{S}$ as the ordered triple $\mathbf{S} \circ \mathbf{R} = ((\mathbf{S} \circ \mathbf{R})_+, (\mathbf{S} \circ \mathbf{R})_-, (\mathbf{S} \circ \mathbf{R})_0)$ such that:

1. $(\mathbf{S} \circ \mathbf{R})_+ = \{(a, c) : (a, b) \in \mathbf{R}_+$ and $(b, c) \in \mathbf{S}_+$, for some $b \in B\}$;

[2] The belonging of Suppes' programme to the semantic approach has been called into question by some authors (cf. [8] p. 103), but is championed by others (cf. [9] pp. 5 - 20).

2. $(\mathbf{S} \circ \mathbf{R})_{\text{-}} = \{(a, c) : \text{ either } (a, b) \in \mathbf{R}_{\text{-}} \text{ or } (b, c) \in \mathbf{S}_{\text{-}}, \text{ for every } b \in B\}$;

3. $(\mathbf{S} \circ \mathbf{R})_0 = A \times C - ((\mathbf{S} \circ \mathbf{R})_+ \ \cup \ (\mathbf{S} \circ \mathbf{R})_{\text{-}})$.

Definition 3. Let $\mathbf{R}$ be a binary partial relation and A, B be sets such that $\mathbf{R}_+ \cup \mathbf{R}_{\text{-}} \cup \mathbf{R}_0 = A \times B$. Let C be a set such that $C \subseteq A$. We define the *restriction* of $\mathbf{R}$ to C as the ordered triple $\mathbf{R} \upharpoonright C = ((\mathbf{R} \upharpoonright C)_+, (\mathbf{R} \upharpoonright C)_{\text{-}}, (\mathbf{R} \upharpoonright C)_0)$ such that:

1. $(\mathbf{R} \upharpoonright C)_+ = \{(c, b) \in \mathbf{R}_+ : c \in C\}$;

2. $(\mathbf{R} \upharpoonright C)_{\text{-}} = \{(c, b) \in \mathbf{R}_{\text{-}} : c \in C\}$;

3. $(\mathbf{R} \upharpoonright C)_0 = \{(c, b) \in \mathbf{R}_0 : c \in C\}$.

Definition 4. Let $\mathbf{R}$ be a binary partial relation and A, B be sets such that $\mathbf{R}_+ \cup \mathbf{R}_{\text{-}} \cup \mathbf{R}_0 = A \times B$. We define the *inverse* of $\mathbf{R}$ as the ordered triple $\mathbf{R}^{-1} = ({\mathbf{R}^{-1}}_+, {\mathbf{R}^{-1}}_{\text{-}}, {\mathbf{R}^{-1}}_0)$ such that:

1. ${\mathbf{R}^{-1}}_+ = \{(b, a) : (a, b) \in \mathbf{R}_+\}$;

2. ${\mathbf{R}^{-1}}_{\text{-}} = \{(b, a) : (a, b) \in \mathbf{R}_{\text{-}}\}$;

3. ${\mathbf{R}^{-1}}_0 = \{(b, a) : (a, b) \in \mathbf{R}_0\}$.

Affirmation 1. *Let $\boldsymbol{R}$ be a binary partial relation and A, B be sets such that $\boldsymbol{R}_+ \cup \boldsymbol{R}_{\text{-}} \cup \boldsymbol{R}_0 = A \times B$. The following assertions are true:*

1. *If $\boldsymbol{S}$ is a binary partial relation and B, C are sets such that $\boldsymbol{S}_+ \cup \boldsymbol{S}_{\text{-}} \cup \boldsymbol{S}_0 = B \times C$, then $\boldsymbol{S} \circ \boldsymbol{R}$ is also a binary partial relation;*

2. *If D is a set such that $D \subseteq A$ then $\boldsymbol{R} \upharpoonright D$ is a binary partial relation;*

3. *$\boldsymbol{R}^{-1}$ is a binary partial relation.*

Definition 5. Let A and B be sets. An *1*-ary *partial function* f from A to B is a binary partial relation $f = (f_+, f_{\text{-}}, f_0)$, such that:

1. $f_+ \cup f_{\text{-}} \cup f_0 = A \times B$

2. For every $a \in A$ and $b \in B$:

   i. If $(a, b) \in f_+$ then $(a, b') \in f_{\text{-}}$ for every $b' \in B$ such that $b \neq b'$;

   ii. If $(a, b) \in f_{\text{-}}$ then there exists $b' \in B$ such that either $(a, b') \in f_+$ or $(a, b') \in f_0$;

   iii. If $(a, b) \in f_0$, then for every $b' \in B$ either $(a, b') \in f_{\text{-}}$ or $(a, b') \in f_0$.

If $A = B$, then f is said to be a *1*-ary partial function *over* A. In case $f_0 = \emptyset$, f is a usual *1*-ary function from A to B, and is said to be a *total function*.

We shall use the traditional notations $f : A \rightarrow B$ or $A \xrightarrow{f} B$ to indicate that f is a *1*-ary partial function from A to B. We can also use $f(a) = b$ instead of $(a, b) \in f_+$, and $f(a) \neq b$ instead of $(a, b) \in f_-$, whenever there exists $b' \in B$ such that $(a, b') \in f_+$. Furthermore, we will say that $f(a)$ *is defined* if there exists $b \in B$ such that $f(a) = b$, and we will say that $f(a)$ *is not defined* otherwise.

As we saw, if f is a partial function from a set A to a set B according to Definition 5, then f is partial relation (f_+, f_-, f_0). Further, if $a \in A$ and $b \in B$ are any elements and $(a, b) \in f_+$, then it is defined that (a, b) satisfies f, and for every $b' \in B$ it is defined that (a, b') does not satisfy f. If $(a, b) \in f_-$ then it is defined that (a, b) does not satisfy f, and there exists $b' \in B$ such that either it is defined that (a, b') satisfies f or it is not defined whether (a, b') satisfies f. Finally, if $(a, b) \in f_0$ then it is not defined whether (a, b) satisfies f, and for each $b' \in B$, either it is not defined whether (a, b') satisfies f or it is defined that (a, b') does not satisfy f.

It is our contention that this notion of partial function is the one we need to accommodate the sort of epistemic partiality about functions that Lutz has in mind, when he criticizes the partial structures approach (cf. [6] pp. 1357-8).

Affirmation 2. *Let* $f : A \rightarrow B$. *The following holds:*

1. *If* $g : B \rightarrow C$ *then* $g \circ f : A \rightarrow C$*;*
2. *If* $D \subseteq A$ *then* $f \upharpoonright D : D \rightarrow B$.

Affirmation 3. *Let* $A \xrightarrow{f} B$. *Let* $Id_A : A \rightarrow A$ *be the identity function on* A *and* $Id_B : B \rightarrow B$ *be the identity function on* B*, such that* $Id_A(a) = a$ *for each* $a \in A$ *and* $Id_B(b) = b$ *for each* $b \in B$. *The following assertions are true:*

1. *If* $B \xrightarrow{g} C \xrightarrow{h} D$ *then* $h \circ (g \circ f) = (h \circ g) \circ f$*;*
2. $f \circ Id_A = f$ *and* $Id_B \circ f = f$.

Before proceeding, we make the following remarks. One can generalize Definitions 1 and 5 in a rather obvious way to define partial relations and partial functions of arbitrary arities. Further, partial relations and partial functions generalize the usual notions of relation and function respectively, in the sense that every relation is actually a total relation, and the same goes to functions *mutatis mutandis*. We also note that sometimes the expression *mapping* will be used to refer to total functions. Finally, given an n-ary partial function $f : A^n \rightarrow B$, we shall use the symbol $[f]$ to denote the set $\{(a_1, ..., a_n) \in A^n : f(a_1, ..., a_n) \text{ is defined}\}$ - needless to say that if f is total then $[f] = A^n$.

2.1 Partial structures

As promised, we will define the notion of partial structure for a fixed language. So before introducing this notion, we will first introduce a particular language with which we shall work in the remainder of this paper.

Definition 6. A *language* $\mathcal{L}$ is a pair $(L, (\mu, \delta))$ such that L is the union of three mutually disjoint sets $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{R}$, whilst (μ, δ) is a pair of mappings $\mu : \mathcal{F} \to \mathbb{N}^*$ and $\delta : \mathcal{R} \to \mathbb{N}^*$. The elements of $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{R}$ are called *constant symbols*, *function symbols* and *relation symbols*, respectively. We also assume that $\mathcal{L}$ is composed by a set of the usual *logical symbols* such as *variables*, *operators*, *quantifiers*, the *identity sign*, and *brackets*. The pair of mappings (μ, δ) is the *type* of $\mathcal{L}$.

In what follows, we assume a language $\mathcal{L} = (L, (\mu, \delta))$.

Definition 7. An $\mathcal{L}$-*partial structure* $\mathcal{A}$ is a pair $(A, (Z^{\mathcal{A}})_{Z \in L})$, where A is a non-empty set and the family $(Z^{\mathcal{A}})_{Z \in L}$ is such that:

1. If $Z \in \mathcal{C}$, then $Z^{\mathcal{A}} \in A$ whenever $Z^{\mathcal{A}}$ is defined;
2. If $Z \in \mathcal{F}$, then $Z^{\mathcal{A}}$ is a $\mu(Z)$-ary partial function over A;
3. If $Z \in \mathcal{R}$, then $Z^{\mathcal{A}}$ is a $\delta(Z)$-ary partial relation over A.

The set A is the *universe* of the partial structure $\mathcal{A}$ and $Z^{\mathcal{A}}$ is the *interpretation* of the symbol Z in $\mathcal{A}$. If for each $Z \in \mathcal{C}$ we have that $Z^{\mathcal{A}}$ is defined, and for each $Z \in \mathcal{F} \cup \mathcal{R}$ we have that $Z^{\mathcal{A}}$ is total, then $\mathcal{A}$ is said to be *total structure* (or simply *total*) - note that total structures are usual structures with which we work in traditional model theory, and hence the notion of partial structure generalizes the notion of structure. The cardinal of $\mathcal{A}$ is the cardinal of its universe, i.e., $Card(\mathcal{A}) = |A|$.

Two $\mathcal{L}$-partial structures $\mathcal{A} = (A, (Z^{\mathcal{A}})_{Z \in L})$ and $\mathcal{B} = (B, (Z^{\mathcal{B}})_{Z \in L})$ are equal if: **(a)** $A = B$; **(b)** For $Z \in \mathcal{C}$, it follows that $Z^{\mathcal{A}}$ is defined if and only if $Z^{\mathcal{B}}$ is defined, and if $Z^{\mathcal{A}}$ and $Z^{\mathcal{B}}$ are both defined then $Z^{\mathcal{A}} = Z^{\mathcal{B}}$; **(c)** For $Z \in \mathcal{F} \cup \mathcal{R}$, we have that $Z^{\mathcal{A}} = Z^{\mathcal{B}}$.

2.2 Expansion

Besides the notions of partial structure and quasi-truth, the next notion to be introduced is characteristic of the theory we are developing.

Definition 8. Let $\mathcal{A} = (A, (Z^{\mathcal{A}})_{Z \in L})$ and $\mathcal{B} = (B, (Z^{\mathcal{B}})_{Z \in L})$ be $\mathcal{L}$-partial structures. We say that $\mathcal{B}$ *expands* $\mathcal{A}$, in symbols $\mathcal{A} \Subset \mathcal{B}$, if:

1. $A = B$;
2. For $Z \in \mathcal{C}$, if $Z^{\mathcal{A}}$ is defined then $Z^{\mathcal{B}}$ is defined and $Z^{\mathcal{A}} = Z^{\mathcal{B}}$;
3. For $Z \in \mathcal{F}$, we have that $Z^{\mathcal{A}} \upharpoonright [Z^{\mathcal{A}}] = Z^{\mathcal{B}} \upharpoonright [Z^{\mathcal{A}}]$;

4. For $Z \in \mathcal{R}$, we have:

   i. ${Z^{\mathcal{A}}}_{+} \subseteq {Z^{\mathcal{B}}}_{+}$;
   ii. ${Z^{\mathcal{A}}}_{-} \subseteq {Z^{\mathcal{B}}}_{-}$.

Intuitively, the notion of expansion represents a possible increase of knowledge about the domain modeled by $\mathcal{A}$, so that $\mathcal{B}$ models the same domain taking this increase of knowledge into account.

Affirmation 4. *If $\mathcal{A} = (A, (Z^{\mathcal{A}})_{Z\in L})$ and $\mathcal{B} = (B, (Z^{\mathcal{B}})_{Z\in L})$ are $\mathcal{L}$-partial structures such that $\mathcal{A} \Subset \mathcal{B}$, then for $Z \in \mathcal{F}$ we have that $[Z^{\mathcal{A}}] \subseteq [Z^{\mathcal{B}}]$.*

Proposition 1. *$\Subset$ is a partial order relation in the class of $\mathcal{L}$-partial structures.*

Proof. We have to show that $\Subset$ is reflexive, antisymmetric and transitive.

1. *Reflexivity.* Straightforward.

2. *Antisymmetry.* Let $\mathcal{A} = (A, (Z^{\mathcal{A}})_{Z\in L})$ and $\mathcal{B} = (B, (Z^{\mathcal{B}})_{Z\in L})$ be $\mathcal{L}$-partial structures. Now assume that $\mathcal{A} \Subset \mathcal{B}$, $\mathcal{B} \Subset \mathcal{A}$ and let us show that $\mathcal{A} = \mathcal{B}$.

   (a) Since $A \subseteq B$ and $B \subseteq A$, it follows that $A = B$.

   (b) For $Z \in \mathcal{C}$, we have to check that $Z^{\mathcal{A}}$ is defined if and only if $Z^{\mathcal{B}}$ is defined, and if $Z^{\mathcal{A}}$ and $Z^{\mathcal{B}}$ are defined then $Z^{\mathcal{A}} = Z^{\mathcal{B}}$. We begin by the former case, noticing that if $Z^{\mathcal{A}}$ is defined, then since $\mathcal{A} \Subset \mathcal{B}$ we have that $Z^{\mathcal{B}}$ is also defined, whilst if $Z^{\mathcal{B}}$ is defined, then since $\mathcal{B} \Subset \mathcal{A}$ it follows that $Z^{\mathcal{A}}$ is defined too. Now suppose that both $Z^{\mathcal{A}}$ and $Z^{\mathcal{B}}$ are defined; then clearly $Z^{\mathcal{A}} = Z^{\mathcal{B}}$, for $\mathcal{A} \Subset \mathcal{B}$.

   (c) For $Z \in \mathcal{F}$, given that $\mathcal{A} \Subset \mathcal{B}$ and $\mathcal{B} \Subset \mathcal{A}$, by Affirmation 4 we have $[Z^{\mathcal{A}}] = [Z^{\mathcal{B}}]$, so that for every $(a_1, ..., a_{\mu(Z)}) \in [Z^{\mathcal{A}}]$, it follows that both $Z^{\mathcal{A}}(a_1, ..., a_{\mu(Z)})$ and $Z^{\mathcal{B}}(a_1, ..., a_{\mu(Z)})$ are defined, and $Z^{\mathcal{A}}(a_1, ..., a_{\mu(Z)}) = Z^{\mathcal{B}}(a_1, ..., a_{\mu(Z)})$. Note also that since $A = B$, we have $A^{\mu(Z)} = B^{\mu(Z)}$, and so for every $(a_1, ..., a_{\mu(Z)}) \in A^{\mu(Z)} - [Z^{\mathcal{A}}]$ it follows that $Z^{\mathcal{A}}(a_1, ..., a_{\mu(Z)})$ and $Z^{\mathcal{B}}(a_1, ..., a_{\mu(Z)})$ are not defined. But then $Z^{\mathcal{A}} = Z^{\mathcal{B}}$.

   (d) For $Z \in \mathcal{R}$, we have:

      i. ${Z^{\mathcal{A}}}_{+} \subseteq {Z^{\mathcal{B}}}_{+}$ and ${Z^{\mathcal{B}}}_{+} \subseteq {Z^{\mathcal{A}}}_{+}$, so that ${Z^{\mathcal{A}}}_{+} = {Z^{\mathcal{B}}}_{+}$;
      ii. ${Z^{\mathcal{A}}}_{-} \subseteq {Z^{\mathcal{B}}}_{-}$ and ${Z^{\mathcal{B}}}_{-} \subseteq {Z^{\mathcal{A}}}_{-}$, so that ${Z^{\mathcal{A}}}_{-} = {Z^{\mathcal{B}}}_{-}$;
      iii. ${Z^{\mathcal{A}}}_{0} = {Z^{\mathcal{B}}}_{0}$ (straightforward from i and ii).

   Hence, $\mathcal{A} = \mathcal{B}$.

3. *Transitivity.* Let $\mathcal{A} = (A, (Z^{\mathcal{A}})_{Z\in L})$, $\mathcal{B} = (B, (Z^{\mathcal{B}})_{Z\in L})$ and $\mathcal{D} = (D, (Z^{\mathcal{D}})_{Z\in L})$ be $\mathcal{L}$-partial structures and assume that $\mathcal{A} \Subset \mathcal{B}$ and $\mathcal{B} \Subset \mathcal{D}$. It is immediate that $A = B = D$. For $Z \in \mathcal{C}$, if $Z^{\mathcal{A}}$ is defined then both $Z^{\mathcal{B}}$ and $Z^{\mathcal{D}}$ are defined, and $Z^{\mathcal{A}} = Z^{\mathcal{B}} = Z^{\mathcal{D}}$. If, on the other hand, $Z^{\mathcal{A}}$ is not defined, clause 2 of Definition 8 is vacuously satisfied. For $Z \in \mathcal{F}$, by Affirmation 4 it follows that $[Z^{\mathcal{A}}] \subseteq [Z^{\mathcal{B}}] \subseteq [Z^{\mathcal{D}}]$. But then $Z^{\mathcal{A}} \restriction [Z^{\mathcal{A}}] = Z^{\mathcal{B}} \restriction [Z^{\mathcal{A}}] = (Z^{\mathcal{B}} \restriction [Z^{\mathcal{B}}]) \restriction [Z^{\mathcal{A}}] = (Z^{\mathcal{D}} \restriction [Z^{\mathcal{B}}]) \restriction [Z^{\mathcal{A}}] = Z^{\mathcal{D}} \restriction [Z^{\mathcal{A}}]$. For $Z \in \mathcal{R}$, we have that ${Z^{\mathcal{A}}}_{+} \subseteq {Z^{\mathcal{B}}}_{+} \subseteq {Z^{\mathcal{D}}}_{+}$ and ${Z^{\mathcal{A}}}_{-} \subseteq {Z^{\mathcal{B}}}_{-} \subseteq {Z^{\mathcal{D}}}_{-}$. Hence, $\mathcal{A} \Subset \mathcal{D}$.

Therefore, $\Subset$ is a partial order relation. $\square$

Lemma 1. *Let $\mathcal{A} = (A, (Z^{\mathcal{A}})_{Z \in L})$, $\mathcal{B} = (B, (Z^{\mathcal{B}})_{Z \in L})$ be $\mathcal{L}$-partial structures such that $\mathcal{A} \Subset \mathcal{B}$. The following holds:*

1. *For $Z \in \mathcal{C}$, if $Z^{\mathcal{A}}$ is defined then $Z^{\mathcal{A}} = Z^{\mathcal{B}}$;*
2. *For $Z \in \mathcal{F}$, if $Z^{\mathcal{A}}$ is total then $Z^{\mathcal{A}} = Z^{\mathcal{B}}$;*
3. *For $Z \in \mathcal{R}$, if $Z^{\mathcal{A}}$ is total then $Z^{\mathcal{A}} = Z^{\mathcal{B}}$.*

Proof. We prove case by case.

1. Straightforward.
2. Let $Z \in \mathcal{F}$ be such that $Z^{\mathcal{A}}$ is total. Then $[Z^{\mathcal{A}}] = A^{\mu(Z)}$ and hence we have $Z^{\mathcal{A}} = Z^{\mathcal{A}} \upharpoonright A^{\mu(Z)} = Z^{\mathcal{A}} \upharpoonright [Z^{\mathcal{A}}] = Z^{\mathcal{B}} \upharpoonright [Z^{\mathcal{A}}] = Z^{\mathcal{B}} \upharpoonright A^{\mu(Z)} = Z^{\mathcal{B}} \upharpoonright B^{\mu(Z)} = Z^{\mathcal{B}}$.
3. Assume that $Z \in \mathcal{R}$ is such that $Z^{\mathcal{A}}$ is total and let us check that $Z^{\mathcal{A}} = Z^{\mathcal{B}}$. We already have that ${Z^{\mathcal{A}}}_0 = \emptyset \subseteq {Z^{\mathcal{B}}}_0$. Further, since $\mathcal{A} \Subset \mathcal{B}$, we also have that ${Z^{\mathcal{A}}}_+ \subseteq {Z^{\mathcal{B}}}_+$ and ${Z^{\mathcal{A}}}_- \subseteq {Z^{\mathcal{B}}}_-$. Therefore, we just have to verify that ${Z^{\mathcal{B}}}_+ \subseteq {Z^{\mathcal{A}}}_+$, ${Z^{\mathcal{B}}}_- \subseteq {Z^{\mathcal{A}}}_-$ and ${Z^{\mathcal{B}}}_0 \subseteq {Z^{\mathcal{A}}}_0$. Thus, let $b_1, ..., b_{\delta(Z)} \in B$ be such that $(b_1, ..., b_{\delta(Z)}) \in {Z^{\mathcal{B}}}_+$. Given that ${Z^{\mathcal{A}}}_0 = \emptyset$, ${Z^{\mathcal{A}}}_- \subseteq {Z^{\mathcal{B}}}_-$ and ${Z^{\mathcal{B}}}_+ \cap {Z^{\mathcal{B}}}_- = \emptyset$, clearly $(b_1, ..., b_{\delta(Z)}) \in {Z^{\mathcal{A}}}_+$, so that ${Z^{\mathcal{A}}}_+ = {Z^{\mathcal{B}}}_+$. Using an analogous argument it follows that ${Z^{\mathcal{A}}}_- = {Z^{\mathcal{B}}}_-$. But then, it is immediate that ${Z^{\mathcal{B}}}_0 = {Z^{\mathcal{A}}}_0$ and hence $Z^{\mathcal{A}} = Z^{\mathcal{B}}$.

Therefore, the result holds. $\square$

Definition 9. Let $\mathcal{A} = (A, (Z^{\mathcal{A}})_{Z \in L})$ and $\mathcal{B} = (B, (Z^{\mathcal{B}})_{Z \in L})$ be $\mathcal{L}$-partial structures. We say that $\mathcal{B}$ is an $\mathcal{A}$-*normal* structure (or simply $\mathcal{A}$-*normal*), if $\mathcal{A} \Subset \mathcal{B}$ and $\mathcal{B}$ is total.

Proposition 2. *For each $\mathcal{L}$-partial structure $\mathcal{A}$ there exists an $\mathcal{A}$-normal structure.*

Proof. Let $\mathcal{A} = (A, (Z^{\mathcal{A}})_{Z \in L})$ be an $\mathcal{L}$-partial structure and $a \in A$ be any element. Define the pair $\mathcal{B} = (B, (Z^{\mathcal{B}})_{Z \in L})$ as follows: **(1)** $B = A$; **(2)** for $Z \in \mathcal{C}$, if $Z^{\mathcal{A}}$ is defined then $Z^{\mathcal{B}} = Z^{\mathcal{A}}$ whereas if $Z^{\mathcal{A}}$ is not defined then $Z^{\mathcal{B}} = a$; **(3)** for $Z \in \mathcal{F}$ and $b_1, ..., b_{\mu(Z)} \in B$, if $(b_1, ..., b_{\mu(Z)}) \in [Z^{\mathcal{A}}]$ then $Z^{\mathcal{B}}(b_1, ..., b_{\mu(Z)}) = Z^{\mathcal{A}}(b_1, ..., b_{\mu(Z)})$, whereas if $(b_1, ..., b_{\mu(Z)}) \in B^{\mu(Z)} - [Z^{\mathcal{A}}]$ then $Z^{\mathcal{B}}(b_1, ..., b_{\mu(Z)}) = a$; **(4)** for $Z \in \mathcal{R}$, we have that ${Z^{\mathcal{B}}}_+ = {Z^{\mathcal{A}}}_+ \cup {Z^{\mathcal{A}}}_0$, ${Z^{\mathcal{B}}}_- = {Z^{\mathcal{A}}}_-$ and ${Z^{\mathcal{B}}}_0 = \emptyset$. By construction clearly $\mathcal{B}$ is an $\mathcal{L}$-partial structure, $\mathcal{A} \Subset \mathcal{B}$, and $\mathcal{B}$ is total. Therefore $\mathcal{B}$ is $\mathcal{A}$-normal. $\square$

Proposition 3. *Let $\mathcal{A}$ be an $\mathcal{L}$-partial structure. If $\mathcal{A}$ is a total structure then $\mathcal{A}$ is $\mathcal{A}$-normal, and for every $\mathcal{L}$-partial structure $\mathcal{B}$, if $\mathcal{B}$ is $\mathcal{A}$-normal then $\mathcal{A} = \mathcal{B}$.*

Proof. Immediate by Proposition 1 and Lemma 1. □

Lemma 2. *Let $\mathcal{A} = (A, (Z^{\mathcal{A}})_{Z\in L})$ be an $\mathcal{L}$-partial structure and $\mathcal{B} = (B, (Z^{\mathcal{B}})_{Z\in L})$ be an $\mathcal{A}$-normal structure. For $Z \in \mathcal{R}$, it follows that ${Z^{\mathcal{B}}}_{+} \subseteq {Z^{\mathcal{A}}}_{+} \cup {Z^{\mathcal{A}}}_{0}$ and ${Z^{\mathcal{B}}}_{-} \subseteq {Z^{\mathcal{A}}}_{-} \cup {Z^{\mathcal{A}}}_{0}$.*

Proof. Let $b_1, ..., b_{\delta(Z)} \in B$ be such that $(b_1, ..., b_{\delta(Z)}) \in {Z^{\mathcal{B}}}_{+}$. Thus $(b_1, ..., b_{\delta(Z)}) \notin {Z^{\mathcal{B}}}_{-}$, so that since ${Z^{\mathcal{A}}}_{-} \subseteq {Z^{\mathcal{B}}}_{-}$ it follows that $(b_1, ..., b_{\delta(Z)}) \notin {Z^{\mathcal{A}}}_{-}$. But then $(b_1, ..., b_{\delta(Z)}) \in A^{\delta(Z)} - {Z^{\mathcal{A}}}_{-}$ and hence $(b_1, ..., b_{\delta(Z)}) \in {Z^{\mathcal{A}}}_{+} \cup {Z^{\mathcal{A}}}_{0}$. The argument to show that ${Z^{\mathcal{B}}}_{-} \subseteq {Z^{\mathcal{A}}}_{-} \cup {Z^{\mathcal{A}}}_{0}$ is similar. □

2.3 Substructures

In this part of the present section we define the relation of substructure between partial structures and prove that it preserves an important property of the usual relation of substructure (Proposition 4).

Definition 10. Let $\mathcal{A} = (A, (Z^{\mathcal{A}})_{Z\in L})$ and $\mathcal{B} = (B, (Z^{\mathcal{B}})_{Z\in L})$ be $\mathcal{L}$-partial structures. We say that $\mathcal{A}$ is an $\mathcal{L}$-*substructure* of $\mathcal{B}$, in symbols $\mathcal{A} \sqsubseteq \mathcal{B}$, if:

1. $A \subseteq B$;
2. For $Z \in \mathcal{C}$, we have:
   - **i.** $Z^{\mathcal{A}}$ is defined if and only if $Z^{\mathcal{B}}$ is defined;
   - **ii.** If $Z^{\mathcal{A}}$ and $Z^{\mathcal{B}}$ are both defined, $Z^{\mathcal{A}} = Z^{\mathcal{B}}$.
3. For $Z \in \mathcal{F}$, we have that $Z^{\mathcal{A}} = Z^{\mathcal{B}} \upharpoonright A^{\mu(Z)}$;
4. For $Z \in \mathcal{R}$, we have:
   - **i.** ${Z^{\mathcal{A}}}_{+} = {Z^{\mathcal{B}}}_{+} \cap A^{\delta(Z)}$;
   - **ii.** ${Z^{\mathcal{A}}}_{-} = {Z^{\mathcal{B}}}_{-} \cap A^{\delta(Z)}$;
   - **iii.** ${Z^{\mathcal{A}}}_{0} = {Z^{\mathcal{B}}}_{0} \cap A^{\delta(Z)}$.

Note that if $\mathcal{A}$ and $\mathcal{B}$ are total structures then we obtain the usual notion of substructure.

Affirmation 5. *If $\mathcal{A} = (A, (Z^{\mathcal{A}})_{Z\in L})$ and $\mathcal{B} = (B, (Z^{\mathcal{B}})_{Z\in L})$ are $\mathcal{L}$-partial structures such that $\mathcal{A} \sqsubseteq \mathcal{B}$, then for $Z \in \mathcal{F}$ we have that $[Z^{\mathcal{A}}] \subseteq [Z^{\mathcal{B}}]$.*

Lemma 3. *Let $\mathcal{A} = (A, (Z^{\mathcal{A}})_{Z\in L})$ and $\mathcal{B} = (B, (Z^{\mathcal{B}})_{Z\in L})$ be $\mathcal{L}$-partial structures such that $\mathcal{A} \sqsubseteq \mathcal{B}$. If $A = B$ then $\mathcal{A} = \mathcal{B}$.*

Proof. Assume that $A = B$ and let us check that $\mathcal{A} = \mathcal{B}$. For $Z \in \mathcal{C}$, if $Z^{\mathcal{A}}$ is defined then $Z^{\mathcal{B}}$ is defined and $Z^{\mathcal{A}} = Z^{\mathcal{B}}$, whereas if $Z^{\mathcal{A}}$ is not defined then $Z^{\mathcal{B}}$ is not defined either. For $Z \in \mathcal{F}$, it follows that $Z^{\mathcal{A}} = Z^{\mathcal{B}} \upharpoonright A^{\mu(Z)} = Z^{\mathcal{B}} \upharpoonright B^{\mu(Z)} = Z^{\mathcal{B}}$. For $Z \in \mathcal{R}$, we have that ${Z^{\mathcal{A}}}_+ = {Z^{\mathcal{B}}}_+ \cap A^{\delta(Z)} = {Z^{\mathcal{B}}}_+ \cap B^{\delta(Z)} = {Z^{\mathcal{B}}}_+$. The argument to show that ${Z^{\mathcal{A}}}_- = {Z^{\mathcal{B}}}_-$ and ${Z^{\mathcal{A}}}_0 = {Z^{\mathcal{B}}}_0$ is similar, so that $Z^{\mathcal{A}} = Z^{\mathcal{B}}$. Therefore, $\mathcal{A} = \mathcal{B}$. □

Proposition 4. *$\sqsubseteq$ is a partial order relation in the class of $\mathcal{L}$-partial structures.*

Proof. We have to show that $\sqsubseteq$ is reflexive, antisymmetric and transitive.

1. *Reflexivity.* Straightforward.

2. *Antisymmetry.* Let $\mathcal{A} = (A, (Z^{\mathcal{A}})_{Z \in L})$ and $\mathcal{B} = (B, (Z^{\mathcal{B}})_{Z \in L})$ be $\mathcal{L}$-partial structures such that $\mathcal{A} \sqsubseteq \mathcal{B}$ and $\mathcal{B} \sqsubseteq \mathcal{A}$. Thus we have that $A \subseteq B$ and $B \subseteq A$. But then $A = B$, so that $\mathcal{A} = \mathcal{B}$ by Lemma 3.

3. *Transitivity.* Let $\mathcal{A} = (A, (Z^{\mathcal{A}})_{Z \in L})$, $\mathcal{B} = (B, (Z^{\mathcal{B}})_{Z \in L})$ and $\mathcal{D} = (D, (Z^{\mathcal{D}})_{Z \in L})$ be $\mathcal{L}$-partial structures such that $\mathcal{A} \sqsubseteq \mathcal{B}$ and $\mathcal{B} \sqsubseteq \mathcal{D}$, and let us check that $\mathcal{A} \sqsubseteq \mathcal{D}$. It is immediate that $A \subseteq B \subseteq D$. For $Z \in \mathcal{C}$, if $Z^{\mathcal{A}}$ is defined then both $Z^{\mathcal{B}}$ and $Z^{\mathcal{D}}$ are defined, and $Z^{\mathcal{A}} = Z^{\mathcal{B}} = Z^{\mathcal{D}}$, since $\mathcal{A} \sqsubseteq \mathcal{B} \sqsubseteq \mathcal{D}$. On the other hand, if $Z^{\mathcal{A}}$ is not defined, then clearly neither $Z^{\mathcal{B}}$ nor $Z^{\mathcal{D}}$ is defined. For $Z \in \mathcal{F}$, it follows that $Z^{\mathcal{A}} = Z^{\mathcal{B}} \upharpoonright A^{\mu(Z)} = (Z^{\mathcal{D}} \upharpoonright B^{\mu(Z)}) \upharpoonright A^{\mu(Z)} = Z^{\mathcal{D}} \upharpoonright A^{\mu(Z)}$. For $Z \in \mathcal{R}$, we have that ${Z^{\mathcal{A}}}_+ = {Z^{\mathcal{B}}}_+ \cap A^{\delta(Z)} = ({Z^{\mathcal{D}}}_+ \cap B^{\delta(Z)}) \cap A^{\delta(Z)} = {Z^{\mathcal{D}}}_+ \cap A^{\delta(Z)}$. The argument to show that ${Z^{\mathcal{A}}}_- = {Z^{\mathcal{D}}}_- \cap A^{\delta(Z)}$ and ${Z^{\mathcal{A}}}_0 = {Z^{\mathcal{D}}}_0 \cap A^{\delta(Z)}$ is the same. Hence, $\mathcal{A} \sqsubseteq \mathcal{D}$.

Therefore, $\sqsubseteq$ is a partial order relation. □

2.4 Homomorphisms

Next we define the relation of homomorphism between partial structures, and then the relations of embedding and isomorphism (between partial structures).

Definition 11. Let $\mathcal{A} = (A, (Z^{\mathcal{A}})_{Z \in L})$ and $\mathcal{B} = (B, (Z^{\mathcal{B}})_{Z \in L})$ be $\mathcal{L}$-partial structures such that for $Z \in \mathcal{C}$, $Z^{\mathcal{A}}$ is defined if and only if $Z^{\mathcal{B}}$ is defined. An *$\mathcal{L}$-homomorphism* from $\mathcal{A}$ to $\mathcal{B}$ is a mapping $h : A \to B$ such that:

1. For $Z \in \mathcal{C}$, if $Z^{\mathcal{A}}$ and $Z^{\mathcal{B}}$ are defined then $h(Z^{\mathcal{A}}) = Z^{\mathcal{B}}$;

2. For $Z \in \mathcal{F}$, the diagram

$$\begin{array}{ccc} A^{\mu(Z)} & \xrightarrow{Z^{\mathcal{A}}} & A \\ {\scriptstyle h^{\mu(Z)}}\downarrow & & \downarrow{\scriptstyle h} \\ B^{\mu(Z)} & \xrightarrow[Z^{\mathcal{B}}]{} & B \end{array}$$

commutes, i.e., $h \circ Z^{\mathcal{A}} = Z^{\mathcal{B}} \circ h^{\mu(Z)}$;

3. For $Z \in \mathcal{R}$, we have that:

   i. If $(a_1, ..., a_{\delta(Z)}) \in Z^{\mathcal{A}}{}_+$ then $(h(a_1), ..., h(a_{\delta(Z)})) \in Z^{\mathcal{B}}{}_+$;

   ii. If $(a_1, ..., a_{\delta(Z)}) \in Z^{\mathcal{A}}{}_0$ then $(h(a_1), ..., h(a_{\delta(Z)})) \in Z^{\mathcal{B}}{}_0$.

An $\mathcal{L}$-homomorphism h from $\mathcal{A}$ to $\mathcal{B}$ is said to be *strong* if the converses of items i and ii also hold.

We use the notation $\mathcal{A} \to \mathcal{B}$ to indicate that there is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$. When the homomorphism is given by a mapping h, we write $h : \mathcal{A} \to \mathcal{B}$ or $\mathcal{A} \stackrel{h}{\to} \mathcal{B}$.

Proposition 5. *Composition of homomorphisms is homomorphism.*

Proof. Let $\mathcal{A} = (A, (Z^{\mathcal{A}})_{Z \in L})$, $\mathcal{B} = (B, (Z^{\mathcal{B}})_{Z \in L})$ and $\mathcal{D} = (D, (Z^{\mathcal{D}})_{Z \in L})$ be $\mathcal{L}$-partial structures. Suppose that $\mathcal{A} \stackrel{f}{\to} \mathcal{B} \stackrel{g}{\to} \mathcal{D}$ and let us show that $g \circ f : A \to D$ is an $\mathcal{L}$-homomorphism from $\mathcal{A}$ to $\mathcal{D}$. For $Z \in \mathcal{C}$, clearly $Z^{\mathcal{A}}$ is defined if and only if $Z^{\mathcal{D}}$ is defined. Further, if $Z^{\mathcal{A}}$ is defined then $Z^{\mathcal{B}}$ is defined and $g \circ f(Z^{\mathcal{A}}) = g(f(Z^{\mathcal{A}})) = g(Z^{\mathcal{B}}) = Z^{\mathcal{D}}$, whereas if $Z^{\mathcal{A}}$ is not defined then clause 1 of Definition 11 is vacuously satisfied. For $Z \in \mathcal{F}$, we have that $g \circ f \circ Z^{\mathcal{A}} = g \circ Z^{\mathcal{B}} \circ f^{\mu(Z)} = Z^{\mathcal{D}} \circ g^{\mu(Z)} \circ f^{\mu(Z)} = Z^{\mathcal{D}} \circ (g \circ f)^{\mu(Z)}$. For $Z \in \mathcal{R}$ and $a_1, ..., a_{\delta(Z)} \in A$, we have:

$$\begin{aligned} (a_1, ..., a_{\delta(Z)}) \in Z^{\mathcal{A}}{}_+ \quad &\Rightarrow \quad (f(a_1), ..., f(a_{\delta(Z)})) \in Z^{\mathcal{B}}{}_+ \\ &\Rightarrow \quad (g(f(a_1)), ..., g(f(a_{\delta(Z)}))) \in Z^{\mathcal{D}}{}_+ \\ &\Rightarrow \quad (g \circ f(a_1), ..., g \circ f(a_{\delta(Z)})) \in Z^{\mathcal{D}}{}_+ \end{aligned}$$

The argument to show that if $(a_1, ..., a_{\delta(Z)}) \in Z^{\mathcal{A}}{}_0$ then $(g \circ f(a_1), ..., g \circ f(a_{\delta(Z)})) \in Z^{\mathcal{D}}{}_0$ is similar. Therefore, $g \circ f$ is an $\mathcal{L}$-homomorphism from $\mathcal{A}$ to $\mathcal{D}$. □

Definition 12. Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{L}$-partial structures and h an $\mathcal{L}$-homomorphism from $\mathcal{A}$ to $\mathcal{B}$. We say that that:

1. h is an $\mathcal{L}$-*embedding* from $\mathcal{A}$ to $\mathcal{B}$ if:

   i. h is a strong $\mathcal{L}$-homomorphism from $\mathcal{A}$ to $\mathcal{B}$;

   ii. h is 1-1.

2. h is an $\mathcal{L}$-*isomorphism* from $\mathcal{A}$ to $\mathcal{B}$ if:

   i. h is an $\mathcal{L}$-embedding from $\mathcal{A}$ to $\mathcal{B}$;

   ii. h is surjective.

We use the notations $\mathcal{A} \,\tilde{\sqsubseteq}\, \mathcal{B}$ and $\mathcal{A} \cong \mathcal{B}$ respectively, to indicate that there are an embedding and an isomorphism from $\mathcal{A}$ to $\mathcal{B}$. If the embedding is given by a mapping h we write $\mathcal{A} \,\tilde{\sqsubseteq}_h\, \mathcal{B}$, and if the isomorphism is given by h we write $\mathcal{A} \cong_h \mathcal{B}$.

Notice that if $\mathcal{A}$ and $\mathcal{B}$ are total structures, we obtain the usual notions of embedding and isomorphism. (If we restrict ourselves to total structures in definition 11 as well, we also obtain the usual relation of homomorphism.)

Affirmation 6. *Let $\mathcal{A} = (A, (Z^{\mathcal{A}})_{Z \in L})$ be an $\mathcal{L}$-partial structure:*

1. *Then the identity mapping on A, Id_A, is an $\mathcal{L}$-isomorphism from $\mathcal{A}$ to $\mathcal{A}$;*
2. *If $\mathcal{B} = (B, (Z^{\mathcal{B}})_{Z \in L})$ is an $\mathcal{L}$-partial structure such that $\mathcal{A} \cong_h \mathcal{B}$, then $h^{-1} \circ h = Id_A$ and $h \circ h^{-1} = Id_B$.*

Proposition 6. *$\cong$ is an equivalence relation in the class of $\mathcal{L}$-partial structures.*

Proof. We have to show that $\cong$ is reflexive, symmetric and transitive.

1. *Reflexivity.* Immediate by item 1 of Affirmation 6.
2. *Symmetry.* Suppose that $\mathcal{A} \cong_h \mathcal{B}$ and consider h^{-1}.

   I. For $Z \in \mathcal{C}$, clearly $Z^{\mathcal{B}}$ is defined if and only if $Z^{\mathcal{A}}$ is defined. Thus, if $Z^{\mathcal{B}}$ is defined then $Z^{\mathcal{A}}$ is defined and $h(Z^{\mathcal{A}}) = Z^{\mathcal{B}}$, so that $h^{-1}(Z^{\mathcal{B}}) = Z^{\mathcal{A}}$. If $Z^{\mathcal{B}}$ is not defined, on the other hand, clause 1 of Definition 11 is vacuously satisfied.

   II. For $Z \in \mathcal{F}$ and $b_1, ..., b_{\mu(Z)} \in B$, using the fact that $\mathcal{A} \xrightarrow{h} \mathcal{B}$, item 2 of Affirmation 3 and item 2 of Affirmation 6, we have:

$$
\begin{aligned}
h^{-1} \circ Z^{\mathcal{B}}(b_1, ..., b_{\delta(Z)}) &= h^{-1} \circ Z^{\mathcal{B}}(Id_B(b_1), ..., Id_B(b_{\delta(Z)})) \\
&= h^{-1} \circ Z^{\mathcal{B}}(h \circ h^{-1}(b_1), ..., h \circ h^{-1}(b_{\delta(Z)})) \\
&= h^{-1} \circ Z^{\mathcal{B}}(h(h^{-1}(b_1)), ..., h(h^{-1}(b_{\delta(Z)}))) \\
&= h^{-1} \circ Z^{\mathcal{B}}(h^{\mu(Z)}(h^{-1}(b_1), ..., h^{-1}(b_{\delta(Z)}))) \\
&= h^{-1} \circ Z^{\mathcal{B}} \circ h^{\mu(Z)}(h^{-1}(b_1), ..., h^{-1}(b_{\delta(Z)})) \\
&= h^{-1} \circ Z^{\mathcal{B}} \circ h^{\mu(Z)}((h^{-1})^{\mu(Z)}(b_1, ..., b_{\delta(Z)})) \\
&= h^{-1} \circ Z^{\mathcal{B}} \circ h^{\mu(Z)} \circ (h^{-1})^{\mu(Z)}(b_1, ..., b_{\delta(Z)}) \\
&= h^{-1} \circ h \circ Z^{\mathcal{A}} \circ (h^{-1})^{\mu(Z)}(b_1, ..., b_{\delta(Z)}) \\
&= Id_A \circ Z^{\mathcal{A}} \circ (h^{-1})^{\mu(Z)}(b_1, ..., b_{\delta(Z)}) \\
&= Z^{\mathcal{A}} \circ (h^{-1})^{\mu(Z)}(b_1, ..., b_{\delta(Z)})
\end{aligned}
$$

   III. For $Z \in \mathcal{R}$ and $b_1, ..., b_{\delta(Z)} \in B$, using the fact that h is a strong $\mathcal{L}$-homomorphism from $\mathcal{A}$ to $\mathcal{B}$ and item 2 of Affirmation 6, we have:

$$
\begin{aligned}
(b_1, ..., b_{\delta(Z)}) \in Z^{\mathcal{B}}{}_+ &\Leftrightarrow (Id_B(b_1), ..., Id_B(b_{\delta(Z)})) \in Z^{\mathcal{B}}{}_+ \\
&\Leftrightarrow (h \circ h^{-1}(b_1), ..., h \circ h^{-1}(b_{\delta(Z)})) \in Z^{\mathcal{B}}{}_+ \\
&\Leftrightarrow (h(h^{-1}(b_1)), ..., h(h^{-1}(b_{\delta(Z)}))) \in Z^{\mathcal{B}}{}_+ \\
&\Leftrightarrow (h^{-1}(b_1), ..., h^{-1}(b_{\delta(Z)})) \in Z^{\mathcal{A}}{}_+
\end{aligned}
$$

   The argument to show that $(b_1, ..., b_{\delta(Z)}) \in Z^{\mathcal{A}}{}_0$ if and only if $(h^{-1}(b_1), ..., h^{-1}(b_{\delta(Z)})) \in Z^{\mathcal{A}}{}_0$ is similar, and hence $\mathcal{B} \cong_{h^{-1}} \mathcal{A}$.

3. *Transitivity.* Suppose that $\mathcal{A} \cong_f \mathcal{B} \cong_g \mathcal{D}$. Then the composition $g \circ f$ is a bijection from A to D as well as a an $\mathcal{L}$-homomorphism from $\mathcal{A}$ to $\mathcal{D}$ (Proposition 5), and so it only remains to check that $g \circ f$ is strong. Thus, for $Z \in \mathcal{R}$ and $a_1, ..., a_{\delta(Z)} \in A$, using the fact that both f and g are strong homomorphisms, we have:

$$\begin{aligned}(a_1, ..., a_{\delta(Z)}) \in Z^{\mathcal{A}}{}_+ &\Leftrightarrow (f(a_1), ..., f(a_{\delta(Z)})) \in Z^{\mathcal{B}}{}_+ \\ &\Leftrightarrow (g(f(a_1)), ..., g(f(a_{\delta(Z)}))) \in Z^{\mathcal{D}}{}_+ \\ &\Leftrightarrow (g \circ f(a_1), ..., g \circ f(a_{\delta(Z)})) \in Z^{\mathcal{D}}{}_+\end{aligned}$$

The argument to show that $(a_1, ..., a_{\delta(Z)}) \in Z^{\mathcal{A}}{}_0$ if and only if $(g \circ f(a_1), ..., g \circ f(a_{\delta(Z)})) \in Z^{\mathcal{D}}{}_0$ is similar, and hence $\mathcal{A} \cong_{g \circ f} \mathcal{D}$.

Therefore, $\cong$ is an equivalence relation. □

The next result is important on its own, but it will be used later to prove another result equally important, which relates the notions of isomorphism and quasi-truth.

Proposition 7. *Let $\mathcal{A} = (A, (Z^{\mathcal{A}})_{Z \in L})$, $\mathcal{B} = (B, (Z^{\mathcal{B}})_{Z \in L})$ be $\mathcal{L}$-partial structures and $h : A \to B$ be a mapping. The following holds:*

1. *If $\mathcal{A} \overset{h}{\to} \mathcal{B}$ and h is injective, then for every $\mathcal{A}$-normal structure $\mathcal{D}$, there exists a $\mathcal{B}$-normal structure $\mathcal{E}$ such that $\mathcal{D} \overset{h}{\to} \mathcal{E}$. Moreover, if in addition to be injective h is also a strong $\mathcal{L}$-homomorphism from $\mathcal{A}$ to $\mathcal{B}$, then for every $\mathcal{A}$-normal structure $\mathcal{D}$, there exists a $\mathcal{B}$-normal structure $\mathcal{E}$ such that h is a strong $\mathcal{L}$-homomorphism from $\mathcal{D}$ to $\mathcal{E}$;*

2. *If $\mathcal{A} \overset{h}{\to} \mathcal{B}$ and h is surjective, then for every $\mathcal{B}$-normal structure $\mathcal{E}$, there exists an $\mathcal{A}$-normal structure $\mathcal{D}$ such that $\mathcal{D} \overset{h}{\to} \mathcal{E}$. Moreover, if in addition to be surjective h is also a strong $\mathcal{L}$-homomorphism from $\mathcal{A}$ to $\mathcal{B}$, then for every $\mathcal{B}$-normal structure $\mathcal{E}$, there exists an $\mathcal{A}$-normal structure $\mathcal{D}$ such that h is a strong $\mathcal{L}$-homomorphism from $\mathcal{D}$ to $\mathcal{E}$;*

3. *If $\mathcal{A} \tilde{\sqsubseteq}_h \mathcal{B}$, then for every $\mathcal{A}$-normal structure $\mathcal{D}$, there exists a $\mathcal{B}$-normal structure $\mathcal{E}$ such that $\mathcal{D} \tilde{\sqsubseteq}_h \mathcal{E}$;*

4. *If $\mathcal{A} \cong_h \mathcal{B}$, then for every $\mathcal{A}$-normal structure $\mathcal{D}$ and $\mathcal{B}$-normal structure $\mathcal{E}$, there exists an $\mathcal{A}$-normal structure $\mathcal{D}'$ and a $\mathcal{B}$-normal structure $\mathcal{E}'$ such that $\mathcal{D} \cong_h \mathcal{E}'$ and $\mathcal{D}' \cong_h \mathcal{E}$.*

Proof. We prove case by case.

1. Suppose that $\mathcal{A} \overset{h}{\to} \mathcal{B}$, h is 1-1 and let $\mathcal{D} = (D, (Z^{\mathcal{D}})_{Z \in L})$ be an $\mathcal{A}$-normal structure. Now let $b \in B$ be any element and define the pair $\mathcal{E} = (E, (Z^{\mathcal{E}})_{Z \in L})$ as follows: **(1)** $E = B$; **(2)** for $Z \in \mathcal{C}$, if $Z^{\mathcal{B}}$ is defined then $Z^{\mathcal{E}} = Z^{\mathcal{B}}$, whilst if $Z^{\mathcal{B}}$ is not defined then $Z^{\mathcal{E}} = h(Z^{\mathcal{D}})$; **(3)** for $Z \in \mathcal{F}$ and $e_1, ..., e_{\mu(Z)} \in E$, we have:

i. If $(e_1, ..., e_{\mu(Z)}) \in h(D)^{\mu(Z)}$ then $Z^{\mathcal{E}}(e_1, ..., e_{\mu(Z)}) = h \circ Z^{\mathcal{D}}(d_1, ..., d_{\mu(Z)})$, where $e_1 = h(d_1), ..., e_{\mu(Z)} = h(d_{\mu(Z)})$;

ii. If $(e_1, ..., e_{\mu(Z)}) \in [Z^{\mathcal{B}}]$ then $Z^{\mathcal{E}}(e_1, ..., e_{\mu(Z)}) = Z^{\mathcal{B}}(e_1, ..., e_{\mu(Z)})$;

iii. If $(e_1, ..., e_{\mu(Z)}) \in E^{\mu(Z)} - (h(D)^{\mu(Z)} \cup [Z^{\mathcal{B}}])$ then $Z^{\mathcal{E}}(e_1, ..., e_{\mu(Z)}) = b$.

(4) For $Z \in \mathcal{R}$, we have that ${Z^{\mathcal{E}}}_+ = {Z^{\mathcal{B}}}_+ \cup ({Z^{\mathcal{B}}}_0 \cap h^{\delta(Z)}({Z^{\mathcal{D}}}_+))$, ${Z^{\mathcal{E}}}_- = E^{\delta(Z)} - {Z^{\mathcal{E}}}_+$ and ${Z^{\mathcal{E}}}_0 = \emptyset$. We leave to the reader the verification that $\mathcal{E}$ is an $\mathcal{L}$-partial structure and $\mathcal{E}$ is $\mathcal{B}$-normal. Next we check that $\mathcal{D} \xrightarrow{h} \mathcal{E}$, and if h is a strong $\mathcal{L}$-homomorphism from $\mathcal{A}$ to $\mathcal{B}$ then h is also a strong $\mathcal{L}$-homomorphism from $\mathcal{D}$ to $\mathcal{E}$.

I. We begin by noticing that since $\mathcal{D}$ and $\mathcal{E}$ are total structures, for $Z \in \mathcal{C}$ we have that both $Z^{\mathcal{D}}$ and $Z^{\mathcal{E}}$ are defined. Further, if $Z^{\mathcal{A}}$ is defined then $Z^{\mathcal{B}}$ is defined as well, and $h(Z^{\mathcal{D}}) = h(Z^{\mathcal{A}}) = Z^{\mathcal{B}} = Z^{\mathcal{E}}$. Now if $Z^{\mathcal{A}}$ is not defined, $Z^{\mathcal{B}}$ is not defined either and therefore $Z^{\mathcal{E}} = h(Z^{\mathcal{D}})$. For $Z \in \mathcal{F}$ and $d_1, ..., d_{\mu(Z)} \in D^{\mu(Z)}$, we have that $h \circ Z^{\mathcal{D}}(d_1, ..., d_{\mu(Z)}) = Z^{\mathcal{E}}(h(d_1), ..., h(d_{\mu(Z)})) = Z^{\mathcal{E}} \circ h^{\mu(Z)}(d_1, ..., d_{\mu(Z)})$, and hence $h \circ Z^{\mathcal{D}} = Z^{\mathcal{E}} \circ h^{\mu(Z)}$. For $Z \in \mathcal{R}$ and $d_1, ..., d_{\delta(Z)} \in D$, we have that if $(d_1, ..., d_{\delta(Z)}) \in {Z^{\mathcal{D}}}_+$ then $(d_1, ..., d_{\delta(Z)}) \in {Z^{\mathcal{A}}}_+ \cup {Z^{\mathcal{A}}}_0$ by Lemma 2. Now if $(d_1, ..., d_{\delta(Z)}) \in {Z^{\mathcal{A}}}_+$ then $(h(d_1), ..., h(d_{\delta(Z)})) \in {Z^{\mathcal{B}}}_+$, for $\mathcal{A} \xrightarrow{h} \mathcal{B}$, and thus $(h(d_1), ..., h(d_{\delta(Z)})) \in {Z^{\mathcal{E}}}_+$. If $(d_1, ..., d_{\delta(Z)}) \in {Z^{\mathcal{A}}}_0$, using again the fact that h is an $\mathcal{L}$-homomorphism from $\mathcal{A}$ to $\mathcal{B}$ we have that $(h(d_1), ..., h(d_{\delta(Z)})) \in {Z^{\mathcal{B}}}_0$. But then $(h(d_1), ..., h(d_{\delta(Z)})) \in {Z^{\mathcal{B}}}_0 \cap h^{\delta(Z)}({Z^{\mathcal{D}}}_+)$, so that $(h(d_1), ..., h(d_{\delta(Z)})) \in {Z^{\mathcal{E}}}_+$. Hence, h is an $\mathcal{L}$-homomorphism from $\mathcal{D}$ to $\mathcal{E}$.

II. Finally, assume that h is a strong $\mathcal{L}$-homomorphism from $\mathcal{A}$ to $\mathcal{B}$. Now let $Z \in \mathcal{R}$ and $d_1, ..., d_{\delta(Z)} \in D$ be such that $(h(d_1), ..., h(d_{\delta(Z)})) \in {Z^{\mathcal{E}}}_+$. Thus either $(h(d_1), ..., h(d_{\delta(Z)})) \in {Z^{\mathcal{B}}}_+$ or $(h(d_1), ..., h(d_{\delta(Z)})) \in {Z^{\mathcal{B}}}_0 \cap h^{\delta(Z)}({Z^{\mathcal{D}}}_+)$. If $(h(d_1), ..., h(d_{\delta(Z)})) \in {Z^{\mathcal{B}}}_+$ then $(d_1, ..., d_{\delta(Z)}) \in {Z^{\mathcal{A}}}_+$ and so $(d_1, ..., d_{\delta(Z)}) \in {Z^{\mathcal{D}}}_+$. If $(h(d_1), ..., h(d_{\delta(Z)})) \in {Z^{\mathcal{B}}}_0 \cap h^{\delta(Z)}({Z^{\mathcal{D}}}_+)$ then $(h(d_1), ..., h(d_{\delta(Z)})) \in h^{\delta(Z)}({Z^{\mathcal{D}}}_+)$, so that there exists $d'_1, ..., d'_{\delta(Z)} \in D$ such that $h(d_1) = h(d'_1), ..., h(d_{\delta(Z)}) = h(d'_{\delta(Z)})$ and $(d'_1, ..., d'_{\delta(Z)}) \in {Z^{\mathcal{D}}}_+$. But given that h is 1-1, it follows that $d_1 = d'_1, ..., d_{\delta(Z)} = d'_{\delta(Z)}$. Hence, $(d_1, ..., d_{\delta(Z)}) \in {Z^{\mathcal{D}}}_+$. Therefore, h is a strong $\mathcal{L}$-homomorphism from $\mathcal{D}$ to $\mathcal{E}$.

2. Suppose that $\mathcal{A} \xrightarrow{h} \mathcal{B}$, h is surjective and let $\mathcal{E} = (E, (Z^{\mathcal{E}})_{Z \in L})$ be a $\mathcal{B}$-normal structure. Let also $(a_b)_{b \in B}$ be a family of A's elements such that

$$h(a_b) = b.$$

Now, define the pair $\mathcal{D} = (D, (Z^{\mathcal{D}})_{Z \in L})$ in the following manner: **(1)** $D = A$; **(2)** for $Z \in \mathcal{C}$, if $Z^{\mathcal{A}}$ is defined then $Z^{\mathcal{D}} = Z^{\mathcal{A}}$, whilst if $Z^{\mathcal{A}}$ is not defined then $Z^{\mathcal{D}} = a \in A$ such that $h(a) = Z^{\mathcal{E}}$; **(3)** for $Z \in \mathcal{F}$ and $d_1, ..., d_{\mu(Z)} \in D$, we have:

i. If $(d_1, ..., d_{\mu(Z)}) \in [Z^{\mathcal{A}}]$, then $Z^{\mathcal{D}}(d_1, ..., d_{\mu(Z)}) = Z^{\mathcal{A}}(d_1, ..., d_{\mu(Z)})$;

ii. If $(d_1, ..., d_{\mu(Z)}) \in D^{\mu(Z)} - [Z^{\mathcal{A}}]$, then $Z^{\mathcal{D}}(d_1, ..., d_{\mu(Z)}) = a_b$, such that $b = Z^{\mathcal{E}}(h(d_1), ..., h(d_{\mu(Z)}))$.

(4) for $Z \in \mathcal{R}$, we have that $Z^{\mathcal{D}}{}_+ = Z^{\mathcal{A}}{}_+ \cup \{(d_1, ..., d_{\delta(Z)}) \in Z^{\mathcal{A}}{}_0 : (h(d_1), ..., h(d_{\delta(Z)})) \in Z^{\mathcal{E}}{}_+\}$, $Z^{\mathcal{D}}{}_- = D^{\delta(Z)} - Z^{\mathcal{D}}{}_+$ and $Z^{\mathcal{D}}{}_0 = \emptyset$. Again, we leave to the reader the verification that $\mathcal{D}$ is an $\mathcal{L}$-partial structure and $\mathcal{D}$ is $\mathcal{A}$-normal. Next we check that $\mathcal{D} \xrightarrow{h} \mathcal{E}$, and if h is a strong $\mathcal{L}$-homomorphism from $\mathcal{A}$ to $\mathcal{B}$ then h is also a strong $\mathcal{L}$-homomorphism from $\mathcal{D}$ to $\mathcal{E}$.

I. The arguments to show that for $Z \in \mathcal{C}$, $Z^{\mathcal{D}}$ is defined if and only if $Z^{\mathcal{E}}$ is defined, and that h satisfies clause 1 of Definition 11 are the same as that of item 1. For $Z \in \mathcal{F}$ and $d_1, ..., d_{\mu(Z)} \in D$, there are two cases to consider:

i. If $(d_1, ..., d_{\mu(Z)}) \in [Z^{\mathcal{A}}]$, using the fact that $\mathcal{A} \xrightarrow{h} \mathcal{B}$ we have:

$$\begin{aligned}
h \circ Z^{\mathcal{D}}(d_1, ..., d_{\mu(Z)}) &= h(Z^{\mathcal{D}}(d_1, ..., d_{\mu(Z)})) \\
&= h(Z^{\mathcal{A}}(d_1, ..., d_{\mu(Z)})) \\
&= h \circ Z^{\mathcal{A}}(d_1, ..., d_{\mu(Z)}) \\
&= Z^{\mathcal{B}} \circ h^{\mu(Z)}(d_1, ..., d_{\mu(Z)}) \\
&= Z^{\mathcal{B}}(h^{\mu(Z)}(d_1, ..., d_{\mu(Z)})) \\
&= Z^{\mathcal{E}}(h^{\mu(Z)}(d_1, ..., d_{\mu(Z)})) \\
&= Z^{\mathcal{E}} \circ h^{\mu(Z)}(d_1, ..., d_{\mu(Z)})
\end{aligned}$$

ii. If $(d_1, ..., d_{\mu(Z)}) \in D^{\mu(Z)} - [Z^{\mathcal{A}}]$, then assuming that $Z^{\mathcal{D}}(d_1, ..., d_{\mu(Z)}) = a_b$ for some $b \in B$ such that $b = Z^{\mathcal{E}}(h(d_1), ..., h(d_{\mu(Z)}))$, we have:

$$\begin{aligned}
h \circ Z^{\mathcal{D}}(d_1, ..., d_{\mu(Z)}) &= h(Z^{\mathcal{D}}(d_1, ..., d_{\mu(Z)})) \\
&= h(a_b) \\
&= b \\
&= Z^{\mathcal{E}}(h(d_1), ..., h(d_{\mu(Z)})) \\
&= Z^{\mathcal{E}}(h^{\mu(Z)}(d_1, ..., d_{\mu(Z)})) \\
&= Z^{\mathcal{E}} \circ h^{\mu(Z)}(d_1, ..., d_{\mu(Z)})
\end{aligned}$$

Hence, $h \circ Z^{\mathcal{D}} = Z^{\mathcal{E}} \circ h^{\mu(Z)}$. For $Z \in \mathcal{R}$ and $d_1, ..., d_{\delta(Z)} \in D$, we have that if $(d_1, ..., d_{\delta(Z)}) \in Z^{\mathcal{D}}{}_+$ then $(d_1, ..., d_{\delta(Z)}) \in Z^{\mathcal{A}}{}_+ \cup Z^{\mathcal{A}}{}_0$ by Lemma 2. Now if $(d_1, ..., d_{\delta(Z)}) \in Z^{\mathcal{A}}{}_+$ then $(h(d_1), ..., h(d_{\delta(Z)})) \in Z^{\mathcal{B}}{}_+$, for $\mathcal{A} \xrightarrow{h} \mathcal{B}$, and thus $(h(d_1), ..., h(d_{\delta(Z)})) \in Z^{\mathcal{E}}{}_+$. If $(a_1, ..., a_{\delta(Z)}) \in Z^{\mathcal{A}}{}_0$, then it is immediate that $(h(d_1), ..., h(d_{\delta(Z)})) \in Z^{\mathcal{E}}{}_+$. Hence, h is an $\mathcal{L}$-homomorphism from $\mathcal{D}$ to $\mathcal{E}$.

II. Suppose that h is a strong $\mathcal{L}$-homomorphism from $\mathcal{A}$ to $\mathcal{B}$. Now let $Z \in \mathcal{R}$ and $a_1, ..., a_{\delta(Z)} \in A$ be such that $(h(a_1), ..., h(a_{\delta(Z)})) \in Z^{\mathcal{E}}{}_+$. Thus, by Lemma 2, either $(h(a_1), ..., h(a_{\delta(Z)})) \in Z^{\mathcal{B}}{}_+$ or $(h(a_1), ..., h(a_{\delta(Z)})) \in Z^{\mathcal{B}}{}_0$. If $(h(a_1), ..., h(a_{\delta(Z)})) \in Z^{\mathcal{B}}{}_+$, then since h is a strong $\mathcal{L}$-homomorphism from $\mathcal{A}$ to $\mathcal{B}$, it follows that $(a_1, ..., a_{\delta(Z)}) \in$

$Z^{\mathcal{A}}{}_+$ and hence $(a_1, ..., a_{\delta(Z)}) \in Z^{\mathcal{D}}{}_+$. If $(h(a_1), ..., h(a_{\delta(Z)})) \in Z^{\mathcal{B}}{}_0$, then using again the fact that h is a strong $\mathcal{L}$-homomorphism from $\mathcal{A}$ to $\mathcal{B}$ we have $(a_1, ..., a_{\delta(Z)}) \in Z^{\mathcal{A}}{}_0$, so that $(a_1, ..., a_{\delta(Z)}) \in Z^{\mathcal{D}}{}_+$. Therefore, h is a strong $\mathcal{L}$-homomorphism from $\mathcal{D}$ to $\mathcal{E}$.

Item 3 is immediate by item 1, and item 4 is immediate by items 2 and 3. □

2.5 Quasi-Truth

This last part of section 2 will be dedicated to the notion of quasi-truth. We will also define other derived notions followed by a presentaion of some properties of quasi-truth and some characteristics of its logic.

Definition 13. Let α be an $\mathcal{L}$-sentence and $\mathcal{A}$ an $\mathcal{L}$-partial structure. We say that α is *quasi-true in* $\mathcal{A}$, in symbols $\mathcal{A} \mid\models \alpha$, if there exists an $\mathcal{A}$-normal structure $\mathcal{B}$ such that $\mathcal{B} \models \alpha$, i.e., if α is true in $\mathcal{B}$ in the Tarskian sense. Otherwise, α is said to be *quasi-false* in $\mathcal{A}$.

Definition 14. An $\mathcal{L}$-sentence α is said *quasi-valid*, in symbols $\mid\models \alpha$, if for every $\mathcal{L}$-partial structure $\mathcal{A}$ it follows that $\mathcal{A} \mid\models \alpha$.

Definition 15. If $\mathcal{A} \mid\models \alpha$, then $\mathcal{A}$ is said to be a *partial model* of α. Given a set Γ of $\mathcal{L}$-sentences, we say that $\mathcal{A}$ is a *partial model* of Γ, in symbols $\mathcal{A} \mid\models \Gamma$, if for every $\gamma \in \Gamma$ we have that $\mathcal{A} \mid\models \gamma$.

Definition 16. An $\mathcal{L}$-sentence α is a *(logical) quasi-consequence* of an $\mathcal{L}$-sentence γ, in symbols $\gamma \mid\models \alpha$, if every partial model of γ is a partial model of α. An $\mathcal{L}$-sentence α is a *(logical) quasi-consequence* of a set Γ of $\mathcal{L}$-sentences, in symbols $\Gamma \mid\models \alpha$, if every partial model of Γ is a partial model of α.

Definition 17. Two $\mathcal{L}$-sentences α and β are *(logically) quasi-equivalent* if $\alpha \mid\models \beta$ and $\beta \mid\models \alpha$.

In what follows, we prove a series of results showing the properties of $\mid\models$. According to the first result, quasi-truth is equivalent to Tarskian truth when restricted to total structures.

Proposition 8. *Let $\mathcal{A}$ be a total $\mathcal{L}$-structure and φ an $\mathcal{L}$-sentence. Then $\mathcal{A} \mid\models \varphi$ if and only if $\mathcal{A} \models \varphi$.*

Proof. (⇒) Suppose that $\mathcal{A} \mathrel{|}\!\models \varphi$. Then, there exists an $\mathcal{A}$-normal structure $\mathcal{B}$ such that $\mathcal{B} \models \varphi$. But $\mathcal{A} = \mathcal{B}$ according to Proposition 3. Hence, $\mathcal{A} \models \varphi$.

(⇐) Now suppose that $\mathcal{A} \models \varphi$. Using again Proposition 3, we have that $\mathcal{A}$ is $\mathcal{A}$-normal, and therefore $\mathcal{A} \mathrel{|}\!\models \varphi$. □

Proposition 9. *Let φ be an $\mathcal{L}$-sentence. Then $\mathrel{|}\!\models \varphi$ if and only if $\models \varphi$.*

Proof. (⇒) Suppose that $\mathrel{|}\!\models \varphi$ and let $\mathcal{A}$ be a total $\mathcal{L}$-structure. Since total structures are particular cases of partial structures, we have that $\mathcal{A} \mathrel{|}\!\models \varphi$ and so $\mathcal{A} \models \varphi$ by Proposition 8. Therefore, $\models \varphi$.

(⇐) Now suppose that $\models \varphi$ and let $\mathcal{A}$ be an $\mathcal{L}$-partial structure. by Proposition 2 above, there exists an $\mathcal{A}$-normal structure $\mathcal{B}$, so that since $\models \varphi$ we have $\mathcal{B} \models \varphi$ and thus $\mathcal{A} \mathrel{|}\!\models \varphi$. Therefore, $\mathrel{|}\!\models \varphi$. □

Proposition 10. *Two $\mathcal{L}$-sentences φ and ψ are quasi-equivalent if and only if they are equivalent.*

Proof. (⇒) Assume that φ and ψ are quasi-equivalent and let $\mathcal{A}$ be a total $\mathcal{L}$-structure. Then we have:

$$\begin{aligned} \mathcal{A} \models \varphi \quad &\Leftrightarrow \quad \mathcal{A} \mathrel{|}\!\models \varphi \qquad \text{(Proposition 8)} \\ &\Leftrightarrow \quad \mathcal{A} \mathrel{|}\!\models \psi \\ &\Leftrightarrow \quad \mathcal{A} \models \psi \end{aligned}$$

Hence, φ and ψ are equivalent.

(⇐) Suppose that φ and ψ are equivalent and let $\mathcal{A}$ be an $\mathcal{L}$-partial structure. Then we have:

$$\begin{aligned} \mathcal{A} \mathrel{|}\!\models \varphi \quad &\Leftrightarrow \quad \mathcal{B} \models \varphi \qquad \text{(for some } \mathcal{A}\text{-normal structure } \mathcal{B}) \\ &\Leftrightarrow \quad \mathcal{B} \models \psi \\ &\Leftrightarrow \quad \mathcal{A} \mathrel{|}\!\models \psi \end{aligned}$$

Therefore, φ and ψ are quasi-equivalent. □

Propositions 9 and 10 show that the notions of quasi-validity and quasi-equivalence coincide with the notions of validity and equivalence respectively; but we shall soon see that the notion of quasi-consequence does not coincide with the notion of consequence.

Proposition 11. *Let $\mathcal{A}$ be an $\mathcal{L}$-partial structure and φ an $\mathcal{L}$-sentence. The following assertions are true:*

1. *If $\mathcal{A} \mathrel{|}\!\not\models \varphi$, then $\mathcal{A} \mathrel{|}\!\models \neg\varphi$;*
2. *If $\varphi := \psi \wedge \chi$ and $\mathcal{A} \mathrel{|}\!\models \psi \wedge \chi$, then $\mathcal{A} \mathrel{|}\!\models \psi$ and $\mathcal{A} \mathrel{|}\!\models \chi$;*

3. If $\varphi := \psi \vee \chi$, then $\mathcal{A} \mid\models \psi \vee \chi$ if and only if $\mathcal{A} \mid\models \psi$ or $\mathcal{A} \mid\models \chi$;

4. If $\varphi := \psi \to \chi$, then $\mathcal{A} \mid\models \psi \to \chi$ if and only if $\mathcal{A} \mid\models \neg\psi$, or $\mathcal{A} \not\mid\models \psi$, or $\mathcal{A} \mid\models \chi$;

Proof. We prove case by case.

1. Suppose that $\mathcal{A} \not\mid\models \varphi$ and let $\mathcal{B}$ be an $\mathcal{A}$-normal structure. Then, clearly $\mathcal{B} \not\models \varphi$, so that $\mathcal{B} \models \neg\varphi$ and hence $\mathcal{A} \mid\models \neg\varphi$.

2. Let $\varphi := \psi \wedge \chi$ and assume that $\mathcal{A} \mid\models \psi \wedge \chi$. Thus there exists an $\mathcal{A}$-normal structure $\mathcal{B}$ such that $\mathcal{B} \models \psi \wedge \chi$, so that $\mathcal{B} \models \psi$ and $\mathcal{B} \models \chi$. But then, $\mathcal{A} \mid\models \psi$ and $\mathcal{A} \mid\models \chi$.

3. Let $\varphi := \psi \vee \chi$.

   ($\Rightarrow$) Suppose that $\mathcal{A} \mid\models \psi \vee \chi$. Thus there exists an $\mathcal{A}$-normal structure $\mathcal{B}$ such that $\mathcal{B} \models \psi \vee \chi$, so that $\mathcal{B} \models \psi$ or $\mathcal{B} \models \chi$. But then, $\mathcal{A} \mid\models \psi$ or $\mathcal{A} \mid\models \chi$.

   ($\Leftarrow$) Suppose now that $\mathcal{A} \mid\models \psi$ or $\mathcal{A} \mid\models \chi$. If $\mathcal{A} \mid\models \psi$ then there exists an $\mathcal{A}$-normal structure $\mathcal{B}$ such that $\mathcal{B} \models \psi$, and so $\mathcal{B} \models \psi \vee \chi$. Thus $\mathcal{A} \mid\models \psi \vee \chi$. If $\mathcal{A} \mid\models \chi$, by means of the same argument it follows that $\mathcal{A} \mid\models \psi \vee \chi$. Now if $\mathcal{A} \mid\models \psi$ and $\mathcal{A} \mid\models \chi$, there exist two $\mathcal{A}$-normal structures $\mathcal{B}$ and $\mathcal{D}$ (possibly the same), such that $\mathcal{B} \models \psi$ and $\mathcal{D} \models \chi$, so that $\mathcal{B} \models \psi \vee \chi$ and $\mathcal{D} \models \psi \vee \chi$. Hence, $\mathcal{A} \mid\models \psi \vee \chi$.

4. Let $\varphi := \psi \to \chi$.

   ($\Rightarrow$) Assume that $\mathcal{A} \mid\models \psi \to \chi$, $\mathcal{A} \not\mid\models \neg\psi$ and $\mathcal{A} \mid\models \psi$. Since $\mathcal{A} \mid\models \psi \to \chi$, there exists an $\mathcal{A}$-normal structure $\mathcal{B}$ such that $\mathcal{B} \models \psi \to \chi$. Since $\mathcal{A} \not\mid\models \neg\psi$, we have $\mathcal{B} \not\models \neg\psi$, so that $\mathcal{B} \models \psi$. But then $\mathcal{B} \models \chi$, and hence $\mathcal{A} \mid\models \chi$.

   ($\Leftarrow$) There are three cases to consider:

   I. Suppose that $\mathcal{A} \mid\models \neg\psi$. Thus $\mathcal{B} \models \neg\psi$, for some $\mathcal{A}$-normal structure $\mathcal{B}$, so that $\mathcal{B} \not\models \psi$ and therefore $\mathcal{B} \models \psi \to \chi$. But then, $\mathcal{A} \mid\models \psi \to \chi$.

   II. Now assume that $\mathcal{A} \not\mid\models \psi$ and let $\mathcal{B}$ be an $\mathcal{A}$-normal structure. Then $\mathcal{B} \not\models \psi$, so that $\mathcal{B} \models \psi \to \chi$ and hence $\mathcal{A} \mid\models \psi \to \chi$.

   III. Finally, suppose that $\mathcal{A} \mid\models \chi$. Then $\mathcal{B} \models \chi$, for some $\mathcal{A}$-normal structure $\mathcal{B}$, so that $\mathcal{B} \models \psi \to \chi$ and thus $\mathcal{A} \mid\models \psi \to \chi$.

Therefore, the result holds. □

Note that according to item 1, for every $\mathcal{L}$-partial structure $\mathcal{A}$ and $\mathcal{L}$-sentence φ, it never happens that $\mathcal{A} \not\mid\models \varphi$ and $\mathcal{A} \not\mid\models \neg\varphi$. As we shall see, however, it might happen that $\mathcal{A} \mid\models \varphi$ and $\mathcal{A} \mid\models \neg\varphi$. In other words, it never happens that both a sentence and its negation are quasi-false in a partial structure, but it may happen that they are both quasi-true.

Now we will present an example which shows what we have just said, why the converse of items 1 and 2 of Proposition 11 does not hold, and also that the notion of quasi-consequence does not coincide with the notion of consequence.

Suppose that $\mathcal{L} = (L, (\mu, \delta))$ is such that: $\mathcal{C} = \{c\}$, $\mathcal{R} = \{R\}$ and $\delta(R) = 1$. Now assume that $\mathcal{A} = (A, (Z^{\mathcal{A}})_{Z \in L})$ is an $\mathcal{L}$-partial structure where: $A = \{a\}$, $c^{\mathcal{A}}$ is not defined, $R^{\mathcal{A}}{}_0 = \{a\}$ and $R^{\mathcal{A}}{}_+ = R^{\mathcal{A}}{}_- = \emptyset$. So, it is easy to verify that there are only two $\mathcal{A}$-normal structures $\mathcal{B} = (B, (Z^{\mathcal{B}})_{Z \in L})$ and $\mathcal{D} = (D, (Z^{\mathcal{D}})_{Z \in L})$, such that: $B = D = A$, $c^{\mathcal{B}} = c^{\mathcal{D}} = a$, $R^{\mathcal{B}}{}_+ = R^{\mathcal{D}}{}_- = \{a\}$ and $R^{\mathcal{B}}{}_- = R^{\mathcal{B}}{}_0 = R^{\mathcal{D}}{}_+ = R^{\mathcal{D}}{}_0 = \emptyset$. But then, we have that $\mathcal{B} \models Rc$ and $\mathcal{D} \models \neg Rc$, so that $\mathcal{A} \mid\models Rc$ and $\mathcal{A} \mid\models \neg Rc$. On the other hand, $\mathcal{A} \mid\not\models Rc \wedge \neg Rc$, because clearly $\mathcal{B} \not\models Rc \wedge \neg Rc$ and $\mathcal{D} \not\models Rc \wedge \neg Rc$. Hence, assuming that φ is Rc and ψ is $Rc \wedge \neg Rc$, it follows that

$$\{\varphi, \neg\varphi\} \mid\not\models \psi.$$

The next result shows how quasi-truth behaves with respect to isomorphisms, but before stating it we will introduce a further concept.

Definition 18. Let $\mathcal{A} = (A, (Z^{\mathcal{A}})_{Z \in L})$ and $\mathcal{B} = (B, (Z^{\mathcal{B}})_{Z \in L})$ be $\mathcal{L}$-partial structures. We say that $\mathcal{A}$ and $\mathcal{B}$ are *elementarily equivalent*, if for each $\mathcal{L}$-sentence φ, it follows that $\mathcal{A} \mid\models \varphi$ if and only if $\mathcal{B} \mid\models \varphi$.

Proposition 12. *If $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{L}$-partial structures such that $\mathcal{A} \cong \mathcal{B}$, then $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{L}$-partial structures such that $\mathcal{A} \cong \mathcal{B}$, and let φ be an $\mathcal{L}$-sentence. Using item 4 of Proposition 7, as well as the fact that the result holds with respect to $\models$ and the relation $\cong$ restricted to total structures, we have:

$$\begin{aligned} \mathcal{A} \mid\models \varphi \quad &\Leftrightarrow \quad \mathcal{D} \models \varphi \qquad \text{(for some } \mathcal{A}\text{-normal structure } \mathcal{D}) \\ &\Leftrightarrow \quad \mathcal{E} \models \varphi \qquad \text{(for some } \mathcal{B}\text{-normal structure } \mathcal{E}) \\ &\Leftrightarrow \quad \mathcal{B} \mid\models \varphi \end{aligned}$$

Therefore, $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent. □

Thus, according to Proposition 12, isomorphic partial structures cannot be distinguished by $\mathcal{L}$-sentences, and with this result we close this section.

3 An application of partial model theory

As we have said before (p. 2 above), Bueno uses the partial structures approach to defend van Fraassen's constructive empiricism against a criticism directed to the notion of empirical adequacy. Constructive empiricism is a conception according to which science has to do with the *observable*, so the only phenomena a scientific theory has to account for or "save" are those we observe. And a scientific theory $\mathcal{T}$ "saves" this kind of phenomenon if the theory is empirically adequate, that is, if there is a model $\mathcal{M}$ of $\mathcal{T}$, such that every model of a phenomenon in the intended scope of $\mathcal{T}$ is *isomorphic* to certain substructures of $\mathcal{M}$ - the models of the phenomena in the *intended scope* of $\mathcal{T}$ are called *appearances*, whilst the empirical substructures of $\mathcal{M}$ that must be isomorphic to the appearances (for the theory to be empirically adequate) are called *empirical substructures* (cf. [10] p. 64). Notice that this characterization is tantamount to require the existence of an *embedding* between every appearance and $\mathcal{M}$, which means that a theory is empirically adequate if and only if there exists an embedding between every appearance and one of its models. This point is important because the referred criticism to the notion of empirical adequacy, made by Mauricio Suárez, is actually directed against the notion of embedding.

3.1 Suárez's criticism to the notion of empirical adequacy

For Suárez, embeddings are too strong to characterize the relationship between theory and phenomenon. He also contends that as conceived within constructive empiricism, empirical adequacy does not coincide with what means for a theory to "save" the phenomena, according to scientists. To exemplify, Suárez presents the case of the *phenomenon of superconductivity*, that was considered "saved" by *classical electromagnetic* theory - according to the scientific community -, but over time it was shown that the required embedding between the appearances and a model of the theory could not exist.

In order to have a better understand of Suárez's criticism and to which extent his example contributes to it, let us first see what Bueno says about the phenomenon of superconductivity:

> Roughly speaking, there are two main factors that characterize superconducting behaviour: (1) a conductivity with almost no resistance, and (2) the Meissner effect, that is, the sudden expulsion of magnetic flux from the inside of the material when the material is taken into the superconducting domain. As Suarez points out, the first of these was accounted for by a certain 'acceleration equation' deduced from classical electromagnetic theory. However, (...) such an equation unfortunately did not describe the Meissner effect - in fact, it contradicted it. A new model was then called for, and when such a model was constructed both features of superconductivity were accounted for. ([5] pp. 590-1)

Summarizing, there are three relevant structures (cf. [11] p. 96): a model $\mathcal{M}_{\mathcal{E}}$ of the electromagnetic theory, an empirical substructure $\mathcal{ES}$ of $\mathcal{M}_{\mathcal{E}}$, and the new phenomenological model $\mathcal{LM}$, proposed by the brothers Fritz and Heinz London, called *London model*.

The empirical substructure $\mathcal{ES}$ is also called *'acceleration' model* by Suárez, since it satisfies the 'acceleration equation'

$$\Lambda d\mathbf{j}/dt = \mathbf{E},$$

where $\Lambda = m/ne^2$, m is the mass of an electron and e its charge, n is the number of electrons per cubic centimetre, $\mathbf{j}$ is the current density, and $\mathbf{E}$ is the electric field (cf. [11] p. 40). The London model, in turn, accounts for the Meissner effect because it satisfies the equation

$$\Lambda c^2 \nabla^2 \mathbf{H} = \mathbf{H},$$

called by Suárez *London equation*, which properly describes the Meissner effect; as we can see, this equation does that so as to relate the terms Λ, c and $\mathbf{H}$, where Λ is the same constant of the 'acceleration equation', c is the velocity of light, and $\mathbf{H}$ is the magnetic field (cf. [11] p. 96).

The Meissner effect was discovered later, and until then the electromagnetic theory seemed to be properly related to the phenomenon of superconductivity, since it accounted for the first characteristic of this phenomenon (the fact that when an object is in the superconducting state it presents a conductivity with almost no resistance)[3], which is reflected in the fact that when we restrict ourselves to this characteristic, the London model and the acceleration model might be considered equivalent. As Suárez claims:

> The London model is close to being isomorphic to the 'acceleration' model. First, the two domains are isomorphic: for every physical entity ($\mathbf{j}$, $\mathbf{H}$, etc.) in the domain over which the 'acceleration' model is defined, there is a corresponding entity in the domain of the London model. Second, at least one relation over the domain is isomorphic, namely the relation that accounts for a constant current (...), expressed in the equation:
>
> $$\nabla\Lambda \frac{d\mathbf{j}}{dt} = -\frac{1}{c}\frac{d\mathbf{H}}{dt} \text{ ([11] pp. 96-7).}$$

But as Bueno notes, the Meissner effect presented inconsistencies with the electromagnetic theory, which implies that it is not possible for the London model to be embedded in some model of the theory. On this point, Suárez says:

> The construction of the London model was clearly not theory-driven. (...) The *'fundamental law'* of superconductivity cannot be derived from theory. As a matter of fact the equation that can be derived from theory (...) yields results that are inconsistent with the London model. The description of the phenomena involves predicates that are not interpretable by any of the relations available in the theoretical structure. If we were to formalize the London model in set theory we would find a relation R_1 between the magnetic field before and after the phase transition that is lacking in electromagnetic theory. The theory contains a putative relation between the field inside the material before the phase transition and the field inside the material after the transition. This relation cannot account for the Meissner effect. But the relation in the London model does (*apud* [5] p. 591; the italics in [5] were suppressed).

Hence, if we take *any* bijection f from the domain of $\mathcal{LM}$ to the domain of $\mathcal{ES}$, which relates each phenomenological entity to its theoretical correspondent, and if we denote the relation between the magnetic field before and after the phase transition in $\mathcal{LM}$ and $\mathcal{ES}$, respectively, by $R^{\mathcal{LM}}$ and $R^{\mathcal{ES}}$, then what prevents f from being an isomorphism is the fact that the acceleration equation negates the Meissner effect, whereas the London equation "affirms it". Formally, then, we have that

[3] This first characteristic was discovered in 1911 by Kamerlingh Onnes, whilst the Meissner effect was discovered in 1933, by Walther Meissner and Robert Ochsenfeld (cf. [11] pp. 36 and 38).

$$(\Lambda, c, \mathbf{H}) \in R^{\mathcal{LM}}$$
and
$$(f(\Lambda), f(c), f(\mathbf{H})) \notin R^{\mathcal{ES}}.$$

Therefore, $\mathcal{LM}$ is not isomorphic to $\mathcal{ES}$.

3.2 Bueno's proposals

Before introducing Bueno's proposals, we observe that the application of the conceptual frame supplied the partial strucutres approach to constructive empiricism results in a new version of the latter, called *structural empiricism* (cf. [5] p. 607). Moreover, within structural empiricism both the models of the theories and their empirical substructures are conceived as partial structures, just like the appearances[4]. Now the idea of Bueno to circumvent Suárez's criticism is to use structural empiricism in order to develop a new notion of empirical adequacy weaker than the original one, but which preserves its main features and could be applied to Suárez's case.

Nonetheless, Bueno introduces two new notions of empirical adequacy. The first of them is called *partial empirical adequacy*, and it amounts to an isomorphism between the appearances and the (partial) empirical substructures of a model of the theory:

> A theory T (thought of, in conformity to the semantic view, as a family of partial structures) is partially empirically adequate if for some of its models there is a partial isomorphism holding between all the models of phenomena (conceived as partial structures) and the partial empirical substructures of the model (...) ([5] p. 596; both in this and in the subsequent quotations we will use our notation).

Partial empirical adequacy could be applied to Suárez's case in the following manner. Consider the partial structures $\mathcal{LM}'$ and $\mathcal{ES}'$, differing from the structures $\mathcal{LM}$ and $\mathcal{ES}$ above only with respect to the relations $R^{\mathcal{LM}}$ and $R^{\mathcal{ES}}$. Let f be a bijection from the domain of $\mathcal{LM}$ to the domain of $\mathcal{ES}$, like the one described on this page and on page 22. Assume also that the relations $R^{\mathcal{LM}'}$ and $R^{\mathcal{ES}'}$, corresponding to the relations $R^{\mathcal{LM}}$ and $R^{\mathcal{ES}}$ respectively, are such that:

- **i.** ${R^{\mathcal{LM}'}}_+ = {R^{\mathcal{LM}}}_+ - \{(\Lambda, c, \mathbf{H})\}$ and ${R^{\mathcal{ES}'}}_+ = {R^{\mathcal{ES}}}_+$;
- **ii.** ${R^{\mathcal{LM}'}}_- = {R^{\mathcal{LM}}}_-$ and ${R^{\mathcal{ES}'}}_- = {R^{\mathcal{ES}}}_- - \{(f(\Lambda), f(c), f(\mathbf{H}))\}$;
- **iii.** ${R^{\mathcal{LM}'}}_0 = \{(\Lambda, c, \mathbf{H})\}$ and ${R^{\mathcal{ES}'}}_0 = \{(f(\Lambda), f(c), f(\mathbf{H}))\}$.

Putting another way, $\mathcal{LM}'$ and $\mathcal{ES}'$ are the partial structures obtained by "removing" only the triples $(\Lambda, c, \mathbf{H})$ of ${R^{\mathcal{LM}}}_+$ and $(f(\Lambda), f(c), f(\mathbf{H}))$ of ${R^{\mathcal{ES}}}_-$, and by "passing" them to ${R^{\mathcal{LM}'}}_0$ and ${R^{\mathcal{ES}'}}_0$ respectively. The structure $\mathcal{LM}'$

[4]An empirical substructure of a model of a theory, conceived as a partial structure, is called *partial empirical substructure* by Bueno (cf. [5] p. 596). He also uses the expression *partial isomorphism* in order to refer to isomorphisms between partial structures, as defined on page 12 above (cf. [5] pp. 595-6).

could be considered as modeling the phenomenon of superconductivity before the finding of the Meissner effect, whilst $\mathcal{ES}'$ corresponds to its theoretical counterpart. Notice also that f is an isomorphism between $\mathcal{LM}'$ and $\mathcal{ES}'$, and so classical electromagnetic theory can be considered partially empirically adequate[5].

However, partial empirical adequacy presents some difficulties, such as its impossibility of holding if we have an increase of knowledge that involves the discovery of new entities. This entails that, depending on the case, the cardinality of the domain of the empirical substructures will be different from that of the domain of phenomena (cf. [5] p. 597). It is in this context that Bueno introduces his second proposal, which consists in an *identification of empirical adequacy with quasi-truth.*

In addition to the concepts of the partial strucutres approach and those of constructive empiricism, this identification is undertaken resorting also to the notion of *hierarchy of partial models of phenomena*, which is a finite set $\{\mathcal{AP}_1, \mathcal{AP}_2, ..., \mathcal{AP}_{k-1}, \mathcal{AP}_k\}$ of partial structures, where $\mathcal{AP}_k$ is a total structure, and for each $1 \leq m < k$ and each partial relation $R^{\mathcal{A}_m}$ defined over the universe of $\mathcal{A}_m$, it follows that $Card(R^{\mathcal{A}_m}{}_0) > Card(R^{\mathcal{A}_{m+1}}{}_0)$ (cf. [5] p. 601). The idea is that this hierarchy models the progress of our knowledge about the phenomenon until the point where our knowledge becomes complete, which is represented by the structure $\mathcal{AP}_k$.

With this notion at hand, Bueno identifies empirical adequacy with quasi-truth as follows:

> (...) we (...) say that [a theory] $\mathcal{T}$ is *empirically adequate* if it is *pragmatically true* in the (partial) empirical substructure $\mathcal{ES}$ according to a structure $\mathcal{A}$, where $\mathcal{A}$ is the last level of the hierarchy of models of phenomena (being thus a total structure) ([5] pp. 602-3; emphases in original).

It is easy to see how this identification supposedly circumvents the issue of increasy cardinality in the domain of the phenomena, for it simply eliminates the use of partial isomorphisms (more especifically, it eliminates the use of functions). So, it is enough that the last level $\mathcal{A}$ of the hierachy of partial models of phenomena be $\mathcal{ES}$-normal, in order for $\mathcal{T}$ to be empirically adequate[6].

[5]Note that van Fraassen characterizes empirical adequacy as an isomorphism between the empirical substructures of some model of the theory and the appearances, but Bueno characterizes partial empirical adequacy as an isomorphism between the appearances and the partial empirical substructures of some partial model of the theory. As we have shown in Proposition 6, nevertheless, the relation of isomorphism between partial structures is symmetric, so if an appearance is isomorphic to a partial empirical substructure (of a partial model of the theory), then the partial empirical substructure is isomorphic to the appearance as well. Thus, despite the fact that Bueno defines partial empirical adequacy differently from how van Fraassen defines empirical adequacy, the first notion coincides with the second when extended to partial structures.

[6]That $\mathcal{A}$ has to be $\mathcal{ES}$-normal for the theory to be empirically adequate, is made clear in the following passage:

> The basic point of such a proposal [of identification of empirical adequacy with quasi-truth] consists in the fact that, intuitively, a theory is empirically adequate if that part of it which is concerned with the observable phenomena (its empirical substructures) can be extended to a total structure that represents the information provided by the observational side of "experience" (the last level in the hierarchy of partial models of phenomena) ([5] p. 603).

It is our contention, nonetheless, that both Bueno's identification of empirical adequacy with quasi-truth and his inspiration to put forward this identification have problems, to which we now address.

3.3 The problems of Bueno's proposals

We would like to start with the difficulties we see in Bueno's argument against partial empirical adequacy. The problem that he poses to this notion can be illustrated as follows. Assume we have a theory $\mathcal{T}$ that might be considered partially empirically adequate. Thus everything indicates that there exists a model $\mathcal{M}$ of $\mathcal{T}$, such that for every appearance $\mathcal{AP}$ there is a partial empirical substructure $\mathcal{ES}$ of $\mathcal{M}$, partially isomorphic to $\mathcal{AP}$. But if the universe of $\mathcal{AP}$ is finite - and hence the same goes to the universe of $\mathcal{ES}$ -, we discover a new object o in the domain of the phenomenon modeled by $\mathcal{AP}$, and we construct a new appearance $\mathcal{AP}'$ whose universe contains this new object, then obviously $\mathcal{ES}$ and $\mathcal{AP}'$ will not be partially isomorphic.

However, it is not clear why there is no corresponding partial empirical substructure $\mathcal{ES}'$ on the theoretical level as well. Note that $\mathcal{AP}$ is a substructure of $\mathcal{AP}'$, so the only difference between both structures is that the latter has one more element in its universe. Thus what prevents the existence of another partial empirical substructure $\mathcal{ES}'$ of $\mathcal{M}$, which in turn possess $\mathcal{ES}$ as a partial substructure and is partially isomorphic to $\mathcal{AP}'$?

One possible and rather trivial reason is if the universe of $\mathcal{M}$ is a singleton set. In this case the only partial (empirical) substructure of $\mathcal{M}$ is $\mathcal{M}$ itself. But then we could look for another model $\mathcal{M}'$ of $\mathcal{T}$, whose cardinal is bigger than the cardinal of $\mathcal{M}$, $\mathcal{M}$ is one of its partial substructures, and the universe M' of $\mathcal{M}'$ contains a theoretical entity o' correponding to o. In this case, $\mathcal{AP}'$ could be isomorphic to $\mathcal{M}'$, or to some empirical substructure of $\mathcal{M}'$.

Now we turn to the problems of Bueno's identification of empirical adequacy with quasi-truth. Some of the main problems of this identification lies in Bueno's reliance on the notion of "normal" structures, for as we saw, a theory is empirically adequate if it has a model and this model has an empirical substructure $\mathcal{ES}$, such that the last level in the hierarchy of partial models of phenomena is $\mathcal{ES}$-normal. Notice, nevertheless, that the domain of phenomena does not need to be, and as a rule it is not, the same as that of the empirical substructures. On the other hand, one of the conditions for a structure $\mathcal{B}$ to be $\mathcal{A}$-normal - given a partial structure $\mathcal{A}$ - is that $\mathcal{A}$ and $\mathcal{B}$ have the same domain (cf. [5] p. 592). Hence, for a theory to be empirically adequate in the sense of the account at work, it is necessary that the domain of the last level of the hierarchy of partial models of phenomena be the same as that of some empirical substructure. But then we have two problems: firstly, this notion of empirical adequacy does not have much scope and hence does not fulfill very well its function of circumventing Suárez's criticism; recall that this criticism was about the notion of embedding, which Suárez considers too restrictive to represent the relationship between theory and phenomenon, but the requirement that the domains of empirical substructures and phenomena be the same is even more restrictive. Secondly, this notion of empirical adequacy does not serve either to deal with the question of discovering new entities at the phenomenological level, for if we have two partial structures $\mathcal{A} = (A, (Z^{\mathcal{A}})_{Z \in L})$ and $\mathcal{B} = (B, (Z^{\mathcal{B}})_{Z \in L})$ such that $\mathcal{B}$ is $\mathcal{A}$-normal, and throughout an addition of information we obtain a structure $\mathcal{D} = (D, (Z^{\mathcal{D}})_{Z \in L})$ such that $B \subset D$, then $\mathcal{D}$ is not $\mathcal{A}$-normal.

Another problem of Bueno's second proposal is in the way he introduces the hierarchy of partial models of phenomena. Given a hierarchy $\{\mathcal{AP}_1, \mathcal{AP}_2, ..., \mathcal{AP}_{k-1}, \mathcal{AP}_k\}$, few things are said beyond the fact that for each $1 \leq m < k$ and each partial relation $R^{\mathcal{A}_m}$ defined over the universe of $\mathcal{A}_m$, it follows that $Card(R^{\mathcal{A}_m}{}_0) > Card(R^{\mathcal{A}_{m+1}}{}_0)$. In the end, we are not informed about some important things, such as what is the relationship between the universes of the structures $\mathcal{A}_m$ and $\mathcal{A}_{m+1}$; We do not know if $A_m \subset A_{m+1}$, $A_m \subseteq A_{m+1}$, $A_m = A_{m+1}$, or even if A_m and A_{m+1} are disjoint.

The motivations behind this hierarchy can be found in Matthias Kaiser's paper *From rocks to graphs - the shaping of phenomena*. From a distinction between *data* and *phenomenon*[7], in this paper the author introduces the hierarchy used by Bueno in his proposal to show how the data contributes to the construction of the empirical phenomenon (cf. [14] pp. 121-3), which is the proper object of explanation of scientific theories (contrarily to data). Kaiser makes a case study, which he takes as the paradigm of the way in which the referred process of phenomena construction must occur. In short, he shows how a set of objects (rocks colected from certain sites of the earth), throughout a series of procedures (analysis, mensurations, among others), have become evidence for the phenomenon of *continental drift* (cf. [14] pp. 113-21). By analyzing this whole process, Kaiser identifies some fundamental features, such as the fact that the data can (and should) be represented in terms of structures, so that as the investigations advance the description of the objects worked initially may change, requiring the construction of new structures, with the set of all them consisting in the hierarchy. Kaiser also notes that every structure is related to the objects of immediate experience, placed at the first level. Now in order for this to happen, he identifies certain conditions, and one of them has as consequence the fact that the domains of the structures which constitute the hierarchy cannot be disjoint (cf. [14] pp. 125-9)[8].

By the way Bueno characterizes the notion of hierarchy of partial models of phenomena he seems to suggest that the relation between the partial structures which constitute a given hierarchy is the relation of expansion, in which case the universe of all those structures would be the same. But this is not clear.

3.4 A solution to the problems of Bueno's proposals

In this part of our work we put forward a solution to the problems recently pointed out. Our main proposal in this regard is the development of a new identification of empirical adequacy with quasi-truth. This new identification will be made by resorting to some features of Bueno's partial empirical adequacy

[7] A distinction established by James Woodward and James Boogen in [12], by Woodward in [13], and assumed by Bueno in [5] pp. 599-600.

[8] More precisely, if we consider a hierarchy of models of phenomena as a set $\mathbb{H} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, ..., \mathcal{A}_{k-1}, \mathcal{A}_k\}$, and let $U = \{A_1, A_2, A_3, ..., A_{k-1}, A_k\}$ be the set of the universes of $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, ..., $\mathcal{A}_{k-1}$, $\mathcal{A}_k$, then the condition in question demands that there be a *filter* F (over a set G) such that $U \subseteq F$. As for G, Kaiser suggests it be identified with $\bigcup_{A_n \in U} A_n$.

and some features of his identification of empirical adequacy with quasi-truth, but we will also employ new concepts which are extensions of some concepts of section 2[9].

Definition 19. Let $\mathcal{A} = (A, (Z^{\mathcal{A}})_{Z\in L})$, $\mathcal{B} = (B, (Z^{\mathcal{B}})_{Z\in L})$ be $\mathcal{L}$-partial structures and $h : A \to B$ be a mapping. We say that:

1. $\mathcal{B}$ *expands* $\mathcal{A}$ with respect to h, in symbols $\mathcal{A} \Subset_h \mathcal{B}$, if there exists an $\mathcal{L}$-partial structure $\mathcal{D}$ such that:

   i. $\mathcal{A} \cong_h \mathcal{D}$;

   ii. $\mathcal{D} \Subset \mathcal{B}$.

2. $\mathcal{B}$ is a $\langle h, \mathcal{A}\rangle$-*normal* structure (or $\mathcal{A}$-*normal* with respect to h) if:

   i. $\mathcal{A} \Subset_h \mathcal{B}$;

   ii. $\mathcal{B}$ is total.

The idea behind the notion of expansion with respect to a function is allowing for a partial structure $\mathcal{B}$ to expand a partial structure $\mathcal{A}$, without the need for their universes to be the same. This can be done throughout the notion of isomorphism, since two isomorphic partial structures are practically the same (as the results of section 2 show). Thus if h is an isomorphism between a partial structure $\mathcal{A}$ and a partial structure $\mathcal{D}$, and a partial structure $\mathcal{B}$ expands $\mathcal{D}$, then $\mathcal{B}$ might be seen as expanding $\mathcal{A}$ as well (with respect to h). Now if in addition $\mathcal{B}$ is total, then $\mathcal{B}$ might also be seen as $\mathcal{A}$-normal (with respect to h, or yet $\langle h, \mathcal{A}\rangle$-normal).

Affirmation 7. *Let $\mathcal{A}$ be an $\mathcal{L}$-partial structure. Then:*

1. $\mathcal{A} \Subset_{Id_A} \mathcal{A}$;

2. *For every $\mathcal{L}$-partial structure $\mathcal{B}$, it follows that $\mathcal{B}$ is $\mathcal{A}$-normal if and only if $\mathcal{B}$ is $\langle Id_A, \mathcal{A}\rangle$-normal.*

Definition 20. Let α be an $\mathcal{L}$-sentence and $\mathcal{A}$ an $\mathcal{L}$-partial structure. We say that:

1. α is *quasi-true* in $\mathcal{A}$ according to an $\mathcal{A}$-normal structure $\mathcal{B}$, if α is true in $\mathcal{B}$ in the Tarskian sense. Otherwise, α is said to be *quasi-false* in $\mathcal{A}$ with respect to $\mathcal{B}$;

2. α is *quasi-true* in $\mathcal{A}$ according to a $\langle h, \mathcal{A}\rangle$-normal structure $\mathcal{D}$, in symbols $\mathcal{A} \mid\models_{\mathcal{D}} \alpha$, if α is true in $\mathcal{D}$ in the Tarskian sense. Otherwise, α is said to be *quasi-false* in $\mathcal{A}$ with respect to $\mathcal{D}$.

[9]Those new concepts will be introduced just to undertake this new identification of empirical adequacy with quasi-truth, which is why we decided to introduce them here and not in section 2. For this section was dedicated to what is strictly fundamental to develop our partial model theory.

Proposition 13. *Let $\mathcal{A} = (A, (Z^{\mathcal{A}})_{Z \in L})$ be an $\mathcal{L}$-partial structure and φ an $\mathcal{L}$-sentence. Let $\mathcal{B} = (B, (Z^{\mathcal{B}})_{Z \in L})$ be an $\mathcal{L}$-partial structure and $h : A \to B$ be a mapping such that $\mathcal{B}$ is $\langle h, \mathcal{A} \rangle$-normal. If $\mathcal{A} \mid\models_{\mathcal{B}} \varphi$ then $\mathcal{A} \mid\models \varphi$.*

Proof. Assume that $\mathcal{A} \mid\models_{\mathcal{B}} \varphi$. It is immediate that $\mathcal{B}$ is total and $\mathcal{B} \models \varphi$. Further, since $\mathcal{B}$ is $\langle h, \mathcal{A} \rangle$-normal there exists an $\mathcal{L}$-partial structure $\mathcal{D}$ such that $\mathcal{D} \Subset \mathcal{B}$ and $\mathcal{A} \cong_h \mathcal{D}$. So $\mathcal{B}$ is $\mathcal{D}$-normal and hence $\mathcal{D} \mid\models \varphi$. But then, using the fact that $\mathcal{A} \cong_h \mathcal{D}$, it follows that $\mathcal{A} \mid\models \varphi$ by Proposition 12. □

Definition 21. Let $\mathcal{OP}$ be an observable phenomenon. A *hierarchy of partial models of* $\mathcal{OP}$ is a set $\mathbb{H}_{\mathcal{OP}} = \{\mathcal{AP}_1, \mathcal{AP}_2, ..., \mathcal{AP}_{k-1}, \mathcal{AP}_k\}$ of $\mathcal{L}$-partial structures, such that:

1. $\mathcal{AP}_i$ is a partial model of $\mathcal{OP}$ for every i $(1 \leq i \leq k)$;
2. For every i, j $(1 \leq i, j \leq k)$, if $i \leq j$ then $\mathcal{AP}_i \Subset \mathcal{AP}_j$;
3. $\mathcal{AP}_k$ is total.

The partial structures in $\mathbb{H}_{\mathcal{OP}}$ are called *partial appearances*, and the structure $\mathcal{AP}_k$ belonging to the last level of $\mathbb{H}_{\mathcal{OP}}$ is also called an *appearance*. It goes without saying that $\mathcal{AP}_k$ is $\mathcal{AP}_i$-normal for evey i $(1 \leq i \leq k)$. Furthermore, clause 2 implies that $\mathbb{H}_{\mathcal{OP}}$ is totally ordered by $\Subset$, and since $\Subset$ is a partial order relation (Proposition 1), it follows that $\mathbb{H}_{\mathcal{OP}}$ is a *chain*.

Definition 22. Let $\mathcal{T}$ be a theory, Γ the set of $\mathcal{T}'$s axioms formulated in $\mathcal{L}$, and $\mathcal{A}$ an $\mathcal{L}$-partial structure. We say that $\mathcal{A}$ is a *partial model* of $\mathcal{T}$, if $\mathcal{A} \mid\models \Gamma$.

Definition 23. Let $\mathcal{T}$ be a theory, $\mathcal{M}$ a partial model of $\mathcal{T}$, $\mathcal{ES}$ an empirical substructure of $\mathcal{M}$, and $\mathcal{I}_{(\mathcal{T})}$ the intended scope of $\mathcal{T}$. Let $\mathbb{H}_{\mathcal{OP}}$ be the hierarchy of partial models of an observable phenomenon $\mathcal{OP} \in \mathcal{I}_{(\mathcal{T})}$, $\mathcal{AP}_k$ the last level of $\mathbb{H}_{\mathcal{OP}}$, and $\mathcal{AP}_j$ $(1 \leq j \leq k)$ an arbitrary level of $\mathbb{H}_{\mathcal{OP}}$. Lastly, let h_j be a mapping from the universe of $\mathcal{SE}$ into the universe of $\mathcal{AP}_j$. We say that:

1. $\mathcal{T}$ is *quasi-true* in $\mathcal{ES}$ with respect to $\mathcal{AP}_k$, if $\mathcal{AP}_k$ is $\langle h_j, \mathcal{ES} \rangle$-normal;
2. $\mathcal{T}$ is *empirically adequate*, if $\mathcal{T}$ has a partial model $\mathcal{M}'$ such that for every appearance $\mathcal{AP}$ in the last level of the hierarchy of partial models of a phenomenon $\mathcal{OP}' \in \mathcal{I}_{(\mathcal{T})}$, there exists an empirical substructure $\mathcal{ES}'$ of $\mathcal{M}'$ such that $\mathcal{T}$ is quasi-true in $\mathcal{ES}'$ with respect to $\mathcal{AP}$.

In other terms, the empirical adequacy of a theory $\mathcal{T}$ can be identified with its quasi-truth in some empirical substructure(s) of $\mathcal{M}'$, with respect to the last level of the hierarchies of partial models of every phenomenon in $\mathcal{I}_{(\mathcal{T})}$.

It is easy to see how this identification of empirical adequacy with quasi-truth circumvents the problems of Bueno's identification. In particular, it is easy to see that a theory $\mathcal{T}$ can be empirically adequate according to Definition 23, even if the universes of the empirical substructures of $\mathcal{T}'$s partial models are different from the universe of the last level of a hierarchy $\mathbb{H}_{\mathcal{OP}}$, where $\mathcal{OP} \in \mathcal{I}_{(\mathcal{T})}$. One can also easily see how Definition 21 solves the problem of vagueness which affects the way Bueno characterizes the notion of hierarchy of partial models of a phenomenon.

4 Conclusion

The present paper introduces the basics to the development of our partial model theory, and despite the fact that this theory has a technical content, it has also applications in the philosophy of science (especially in the semantic approach); what was presented in section 2 above could be used to reconstruct another views constituting the semantic approach such as *quasi-realism* and *structuralism*[10], the same way Bueno did with constructive empiricism. Epistemological issues can also be addressed with the content of section 2; we recall that the main motivation for defining partial structures as we did, i.e. in such a way that they may not interpret constant symbols, and assign function symbols to a specific kind of partial function, was Sebastian Lutz's criticism to partial structures (see p. 3 above), which touches upon the issue of how one can properly accommodate epistemic partiality in a formal - or model-theoretic - setting.

But certainly the content of section 2 has applications in more complex branches of model theory than those we dealt with, and it has applications in other logical theories as well. An example is *category theory*; note that Affirmation 3, Proposition 5 and item 1 of Affirmation 6 imply that $\mathcal{L}$-partial structures and $\mathcal{L}$-homomorphisms between $\mathcal{L}$-partial structures form a category (cf. [16] pp. 24-5), and this fact naturaly raises the question *what are the properties of this category?*

We shall answer this question and proceed in the development of our partial model theory in other works, but this can only be done with the content of the present paper. So, this paper can be seen as the starting point of an interdisciplinary research programme centered on partial structures and to a lesser extent on partial relations and quasi-truth, with applications and implications in all those research areas mentioned, to wit logic, philosophy of science, and epistemology.

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