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IN-cross Entropy Based MAGDM Strategy under Interval Neutrosophic Set Environment

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Abstract. Cross entropy measure is one of the best way to calculate the divergence of any variable from the priori one variable. We define a new cross entropy measure under interval neutrosophic set (INS) environment, which we call IN-cross entropy measure and prove its basic properties. We also develop weighted IN-cross entropy measure and investigats its basic properties. Based on the weighted IN-cross entropy measure, we develop a novel strategy for multi attribute group decision making (MAGDM) strategy under interval neutrosophic environment. The proposed multi attribute group decision making strategy is compared with the existing cross entropy measure based strategy in the literature under interval neutrosophic set environment. Finally, an illustratative example of multi attribute group decision making problem is solved to show the feasibility, validity and efficiency of the proposed MAGDM strategy.

Keywords: Interval neutrosophic set, IN-cross entropy measure, MAGDM strategy.

1. Introduction

In our daily life we frequently meet with the quantitative measure to take appropriate decision for solving many problems. Entropy measure provides us a quantitative measure of two variables. In 1968, Zadeh [1] introduced fuzzy entropy measure. According to Liu [2], under fuzzy environment, entropy should meet at least three basic following requirements: the entropy of a crisp number is zero; the entropy of an equipossible fuzzy variable is maximum and the entropy is applicable not only to finite and infinite cases but also to discrete and continuous cases. Shang and Jiang [3] proposed a cross entropy measure and symmetric discrimination measure between fuzzy sets. Atanassov [4] introduced intuitionistic fuzzy set (IFS) in 1989, which is the extension of fuzzy set. Some recent applications of IFS are found in [5-11] in the literature. Vlachos and Sergiadis [12] defined cross entropy measure in IFS environment and showed a mathematical connection between the notions of entropy for fuzzy sets and IFSs in terms of fuzziness and intuitionism. In 1998, Smarandache [13] introduced the concept of neutrosophic set (NS) by introducing truth membership, falsity membership and indeterminacy membership functions as independent components and their sum lies ($^-0$, 3^+). Thereafter, Wang et al. [14] introduced single valued neutrosophic set (SVNS) as a subclass of NS. Thereafter, many researchers paid attention to apply NS and SVNS in many field of research such as conflict resolution [15], clustering analysis [16, 17], decision making [18-47], educational problem [48, 49], image processing [50, 52], medical diagnosis [53], optimization [54-59], social problem [60, 61]. Ye [62] introduced cross entropy measure in SVNS and applied it to multi criteria decision- making (MCDM) problems. Ye [63] defined an improved cross entropy measure for SVNS to overcome drawbacks in [62]. In 2005, Wang et al. [64] introduced interval neutrosophic set (INS) considering truth membership, indeterminate membership and falsity membership as interval number in [0, 1]. Broumi and Smarandache [65] defined correlation coefficient of INS and proved its basic properties. Zhang et al. [66] defined correlation coefficient for interval neutrosophic number (INN) and applied it MAGDM problems. Zhang et al. [67] presented an outranking approach for INS and applied its MCDM problems. Recently, Yu et al. [68] use VIKOR method to solve MAGDM problem with INN. Ye [69] defined similarity measure in INS environment and applied to solve MCDM problem. Pramanik and Mondal [70] extended the single valued neutrosophic grey relational analysis strategy to interval neutrosophic environment and applied it to multi-attribute decision-making (MADM) problems. Zhao et al. [71] proposed a MADM strategy based on generalized weighted aggregation operator with INS. Zhang et al. [72] proposed a MCDM strategy based on two interval neutrosophic number aggregation operators. Sahin [73] defined two cross entropy measures with INS based on fuzzy cross entropy measure and single valued neutrosophic cross entropy measure and applied for solving MCDM problem. Tian et al. [74] proposed a cross entropy measure with INS and TOPSIS for solving MCDM problems.

Sahin [73], Tian et al. [74] proposed cross entropy measures under the interval-valued neutrosophic set environment, which is suitable for single decision maker only. So multiple decision maker cannot participate in their strategies in [73, 74].

The aforementioned applications of cross entropy [63, 73, 74] can be effective in dealing with neutrosophic MADM problems. However, they also bear some limitations, which are outlined below:

i. The strategies [63, 73, 74] are capable of solving neutrosophic MADM problems.
ii. In the strategies [73, 74], interval-valued neutrosophic set are transformed to SVNS by suitable transform operators.
iii. The strategies [63, 73, 74] have a single decision-making structure, and not enough attention is paid to improving robustness when processing the assessment information.

Research gap:

MAGDM strategy based on cross entropy measure.

This study answers the following research questions:

i. Is it possible to define a new cross entropy measure under interval-valued neutrosophic set environment that is free from asymmetrical phenomena?
ii. Is it possible to define a new weighted cross entropy measure under interval-valued neutrosophic set that is free from asymmetrical phenomena?
iii. Is it possible to develop a new MAGDM strategy based on the proposed cross entropy measure under interval-valued neutrosophic set environment?
iv. Is it possible to develop a new MAGDM strategy based on the proposed weighted cross entropy measure under interval-valued neutrosophic set environment?

Motivation:

The above-mentioned analysis describes the motivation behind proposing a novel IN-cross entropy-based strategy for tackling MAGDM under the interval-valued neutrosophic environment. This study develops a novel IN-cross entropy-based MAGDM strategy that can deal with multiple decision-makers and free from the drawbacks that exist in [63, 72, 73].

The objectives of the paper are:

1. i. To define a new cross entropy measure under interval-valued neutrosophic set environment without using any transformation operator and prove its basic properties,
2. ii. To define a new weighted cross measure and prove its basic properties.
3. iii. To develop a new MAGDM strategy based on weighted cross entropy measure under interval-valued neutrosophic set environment.

To fill the research gap, we propose IN-cross entropy-based MAGDM, which is capable of dealing with multiple decision-makers.

The main contributions of this paper are summarized below:

1. i. We define a new IN-cross entropy measure and prove its basic properties. It is straightforward symmetric.
2. ii. We define a new weighted IN-cross entropy measure in the single-valued neutrosophic set environment and prove its basic properties. It is straightforward symmetric
3. iii. In this paper, we develop a new MAGDM strategy based on weighted IN cross entropy to solve MAGDM problems.
4. iv. In this paper, we solve a MAGDM problem based on the proposed MAGDM strategy.
5.

The paper unfolds as follows: In section 2, we describe the basic definitions and operations of SVNS, INS. In section 3, we present the definition of proposed IN-cross entropy measure, weighted IN-cross entropy measure and their basic properties. In section 4, we develop a MAGDM strategy with the proposed weighted IN-cross entropy measure. In section 5, we solve a MAGDM problem to show the feasibility, validity and efficiency of the proposed strategy. In section 6, we present conclusion and future direction of this study.

2. Preliminaries

2.1 Definition: Single valued neutrosophic set (SVNS) [14]

Assume that U be a space of points (objects) with generic elements $u \in U$. A SVNS H in U is characterized by a truth-membership function $T_H(u)$, an indeterminacy-membership function $I_H(u)$, and a falsity-membership function $F_H(u)$, where $T_H(u), I_H(u), F_H(u) \in [0, 1]$ for each point u in U. Therefore, a SVNS A can be expressed as H = { u, $T_H(u)$, $I_H(u)$, $F_H(u)$ | u∈ U}, whereas, the sums of $T_H(u)$, $I_H(u)$ and $F_H(u)$ satisfy the condition

$0 \le T_H(u) + I_H(u) + F_H(u) \le 3$.

2.2 Definition: Interval neutrosophic sets (INSs) [64]

Assume that U be a space of points (objects) with generic elements $u \in U$. An INSs J in U is characterized by a truth-membership measure $T_J(u)$, an indeterminacy-membership measure $I_J(u)$, and a falsity-membership measure $F_J(u)$, where,

$T_J(u) = [T_J^-(u), T_J^+(u)]$, $I_J(u) = [I_J^-(u), I_J^+(u)]$,

$F_J(u) = [F_J^-(u), F_J^+(u)]$ for each point u in U. Therefore, a INSs J can be expressed as J = {u, $[T_J^-(u), T_J^+(u)]$, $[I_J^-(u), I_J^+(u)]$, $[F_J^-(u), F_J^+(u)]$ | u∈ U}. Where, $T_J^-(u), T_J^+(u)$, $I_J^-(u), I_J^+(u)$, $F_J^-(u), F_J^+(u) \subseteq [0,1]$.

2.3 Definition: Inclusion of two INSs [64]

Let J_1 = {u, $[T_{J_1}^-(u), T_{J_1}^+(u)]$, $[I_{J_1}^-(u), I_{J_1}^+(u)]$, $[F_{J_1}^-(u), F_{J_1}^+(u)]$ | u∈ U} and J_2 = {u, $[T_{J_2}^-(u), T_{J_2}^+(u)]$, $[I_{J_2}^-(u), I_{J_2}^+(u)]$, $[F_{J_2}^-(u), F_{J_2}^+(u)]$ | u∈ U} be any two INSs in U, then $J_1 \subseteq J_2$ iff $T_{J_1}^-(u) \le T_{J_2}^-(u)$, $T_{J_1}^+(u) \le T_{J_2}^+(u)$, $I_{J_1}^-(u) \ge I_{J_2}^-(u)$, $I_{J_1}^+(u) \ge I_{J_2}^+(u)$, $F_{J_1}^-(u) \ge F_{J_2}^-(u)$, $F_{J_1}^+(u) \ge F_{J_2}^+(u)$ for all u∈ U.

2.4 Definition: Complement of an INS [64]

The complement J^c of an INS J = {u, $[T_J^-(u), T_J^+(u)]$, $[I_J^-(u), I_J^+(u)]$, $[F_J^-(u), F_J^+(u)]$ | u∈ U} is defined as follows:

J^c = {u, $[1-T_J^+(u), 1-T_J^-(u)]$, $[1-I_J^+(u), 1-I_J^-(u)]$, $[1-F_J^+(u), 1-F_J^-(u)]$ | u∈ U}.

2.5 Definition: Equality of two INSs [64]

Let J_1 = {u, $[T_{J_1}^-(u), T_{J_1}^+(u)]$, $[I_{J_1}^-(u), I_{J_1}^+(u)]$, $[F_{J_1}^-(u), F_{J_1}^+(u)]$ | u∈ U} and J_2 = {u, $[T_{J_2}^-(u), T_{J_2}^+(u)]$, $[I_{J_2}^-(u), I_{J_2}^+(u)]$, $[F_{J_2}^-(u), F_{J_2}^+(u)]$ | u∈ U} be any two INSs in U, then $J_1 = J_2$ iff $T_{J_1}^-(u) = T_{J_2}^-(u)$, $T_{J_1}^+(u) = T_{J_2}^+(u)$, $I_{J_1}^-(u) = I_{J_2}^-(u)$, $I_{J_1}^+(u) = I_{J_2}^+(u)$, $F_{J_1}^-(u) = F_{J_2}^-(u)$, $F_{J_1}^+(u) = F_{J_2}^+(u)$ for all u∈ U.

3. Definition: IN-cross-entropy measure

Let J_1 and J_2 be any two INSs in U = $\{u_1, u_2, u_3, ..., u_n\}$. Then, the interval neutrosophic cross-entropy measure of J_1 and J_2 is denoted by $CE_{IN}(J_1, J_2)$ and defined as follows:

$$
\begin{aligned}
CE_{IN}(J_1, J_2) = \frac{1}{4}\sum_{i=1}^{n}\Bigg\{\Bigg\langle & \left[\frac{2\left|T_{J_1}^-(u_i) - T_{J_2}^-(u_i)\right|}{\sqrt{1+\left|T_{J_1}^-(u_i)\right|^2}+\sqrt{1+\left|T_{J_2}^-(u_i)\right|^2}} + \frac{2\left|(1-T_{J_1}^-(u_i)) - (1-T_{J_2}^-(u_i))\right|}{\sqrt{1+\left|(1-T_{J_1}^-(u_i))\right|^2}+\sqrt{1+\left|(1-T_{J_2}^-(u_i))\right|^2}}\right] + \\
& \left[\frac{2\left|T_{J_1}^+(u_i) - T_{J_2}^+(u_i)\right|}{\sqrt{1+\left|T_{J_1}^+(u_i)\right|^2}+\sqrt{1+\left|T_{J_2}^+(u_i)\right|^2}} + \frac{2\left|(1-T_{J_1}^+(u_i)) - (1-T_{J_2}^+(u_i))\right|}{\sqrt{1+\left|(1-T_{J_1}^+(u_i))\right|^2}+\sqrt{1+\left|(1-T_{J_2}^+(u_i))\right|^2}}\right] + \\
& \left[\frac{2\left|I_{J_1}^-(u_i) - I_{J_2}^-(u_i)\right|}{\sqrt{1+\left|I_{J_1}^-(u_i)\right|^2}+\sqrt{1+\left|I_{J_2}^-(u_i)\right|^2}} + \frac{2\left|(1-I_{J_1}^-(u_i)) - (1-I_{J_2}^-(u_i))\right|}{\sqrt{1+\left|(1-I_{J_1}^-(u_i))\right|^2}+\sqrt{1+\left|(1-I_{J_2}^-(u_i))\right|^2}}\right] + \\
& \left[\frac{2\left|I_{J_1}^+(u_i) - I_{J_2}^+(u_i)\right|}{\sqrt{1+\left|I_{J_1}^+(u_i)\right|^2}+\sqrt{1+\left|I_{J_2}^+(u_i)\right|^2}} + \frac{2\left|(1-I_{J_1}^+(u_i)) - (1-I_{J_2}^+(u_i))\right|}{\sqrt{1+\left|(1-I_{J_1}^+(u_i))\right|^2}+\sqrt{1+\left|(1-I_{J_2}^+(u_i))\right|^2}}\right] + \\
& \left[\frac{2\left|F_{J_1}^-(u_i) - F_{J_2}^-(u_i)\right|}{\sqrt{1+\left|F_{J_1}^-(u_i)\right|^2}+\sqrt{1+\left|F_{J_2}^-(u_i)\right|^2}} + \frac{2\left|(1-F_{J_1}^-(u_i)) - (1-F_{J_2}^-(u_i))\right|}{\sqrt{1+\left|(1-F_{J_1}^-(u_i))\right|^2}+\sqrt{1+\left|(1-F_{J_2}^-(u_i))\right|^2}}\right] + \\
& \left[\frac{2\left|F_{J_1}^+(u_i) - F_{J_2}^+(u_i)\right|}{\sqrt{1+\left|F_{J_1}^+(u_i)\right|^2}+\sqrt{1+\left|F_{J_2}^+(u_i)\right|^2}} + \frac{2\left|(1-F_{J_1}^+(u_i)) - (1-F_{J_2}^+(u_i))\right|}{\sqrt{1+\left|(1-F_{J_1}^+(u_i))\right|^2}+\sqrt{1+\left|(1-F_{J_2}^+(u_i))\right|^2}}\right]\Bigg\rangle\Bigg\}
\end{aligned}
\tag{1}
$$

Theorem 1.

Interval-valued neutrosophic cross entropy $CE_{IN}(J_1, J_2)$ for any two INSs J_1 and J_2 of U, satisfies the following properties:

i) $CE_{IN}(J_1, J_2) \ge 0$.

ii) $CE_{IN}(J_1, J_2) = 0$ if and only if $T_{J_1}^-(u_i) = T_{J_2}^-(u_i)$, $T_{J_1}^+(u_i) = T_{J_2}^+(u_i)$, $I_{J_1}^-(u_i) = I_{J_2}^-(u_i)$, $I_{J_1}^+(u_i) = I_{J_2}^+(u_i)$, $F_{J_1}^-(u_i) = F_{J_2}^-(u_i)$, $F_{J_1}^+(u_i) = F_{J_2}^+(u_i)$ for all $\forall u_i \in U$.

iii) $CE_{IN}(J_1, J_2) = CE_{IN}(J_1^c, J_2^c)$

iv) $CE_{IN}(J_1, J_2) = CE_{IN}(J_2, J_1)$

Proof: i)

For all values of $u_i \in U$, $\left|T^-_{J_1}(u_i)\right| \ge 0$, $\left|T^-_{J_2}(u_i)\right| \ge 0$, $\left|T^-_{J_1}(u_i) - T^-_{J_2}(u_i)\right| \ge 0$, $\sqrt{1+\left|T^-_{J_1}(u_i)\right|^2} \ge 0$, $\sqrt{1+\left|T^-_{J_2}(u_i)\right|^2} \ge 0$, $\left|(1-T^-_{J_1}(u_i))\right| \ge 0$, $\left|(1-T^-_{J_2}(u_i))\right| \ge 0$, $\left|(1-T^-_{J_1}(u_i))-(1-T^-_{J_2}(u_i))\right| \ge 0$, $\sqrt{1+\left|(1-T^-_{J_1}(u_i))\right|^2} \ge 0$, $\sqrt{1+\left|(1-T^-_{J_2}(u_i))\right|^2} \ge 0$

$\Rightarrow$

$$\left[\frac{2\left|T^-_{J_1}(u_i)-T^-_{J_2}(u_i)\right|}{\sqrt{1+\left|T^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^-_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^-_{J_1}(u_i))-(1-T^-_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^-_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^-_{J_2}(u_i))\right|^2}}\right] \ge 0$$

and $\left|T^+_{J_1}(u_i)\right| \ge 0$, $\left|T^+_{J_2}(u_i)\right| \ge 0$, $\left|T^+_{J_1}(u_i)-T^+_{J_2}(u_i)\right| \ge 0$, $\sqrt{1+\left|T^+_{J_1}(u_i)\right|^2} \ge 0$, $\sqrt{1+\left|T^+_{J_2}(u_i)\right|^2} \ge 0$, $\left|(1-T^+_{J_1}(u_i))\right| \ge 0$, $\left|(1-T^+_{J_2}(u_i))\right| \ge 0$, $\left|(1-T^+_{J_1}(u_i))-(1-T^+_{J_2}(u_i))\right| \ge 0$, $\sqrt{1+\left|(1-T^+_{J_1}(u_i))\right|^2} \ge 0$, $\sqrt{1+\left|(1-T^+_{J_2}(u_i))\right|^2} \ge 0$

$\Rightarrow$

$$\left[\frac{2\left|T^+_{J_1}(u_i)-T^+_{J_2}(u_i)\right|}{\sqrt{1+\left|T^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^+_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^+_{J_1}(u_i))-(1-T^+_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^+_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^+_{J_2}(u_i))\right|^2}}\right] \ge 0$$

Similarly, we can show that

$$\left[\frac{2\left|I^-_{J_1}(u_i)-I^-_{J_2}(u_i)\right|}{\sqrt{1+\left|I^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^-_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^-_{J_1}(u_i))-(1-I^-_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^-_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^-_{J_2}(u_i))\right|^2}}\right] \ge 0$$

,

$$\left[\frac{2\left|I^+_{J_1}(u_i)-I^+_{J_2}(u_i)\right|}{\sqrt{1+\left|I^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^+_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^+_{J_1}(u_i))-(1-I^+_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^+_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^+_{J_2}(u_i))\right|^2}}\right] \ge 0$$

,

$$\left[\frac{2\left|F^-_{J_1}(u_i)-F^-_{J_2}(u_i)\right|}{\sqrt{1+\left|F^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^-_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^-_{J_1}(u_i))-(1-F^-_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^-_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^-_{J_2}(u_i))\right|^2}}\right] \ge 0$$

and

$$\left[\frac{2\left|F^+_{J_1}(u_i)-F^+_{J_2}(u_i)\right|}{\sqrt{1+\left|F^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^+_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^+_{J_1}(u_i))-(1-F^+_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^+_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^+_{J_2}(u_i))\right|^2}}\right] \ge 0$$

.

Hence, we can conclude that $CE_{IN}(J_1, J_2) \ge 0$.

ii). For all values of $u_i \in U$,

$$\left[\frac{2\left|T^-_{J_1}(u_i)-T^-_{J_2}(u_i)\right|}{\sqrt{1+\left|T^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^-_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^-_{J_1}(u_i))-(1-T^-_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^-_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^-_{J_2}(u_i))\right|^2}}\right] = 0$$

$\Leftrightarrow T^-_{J_1}(u_i) = T^-_{J_2}(u_i)$

$$\left[\frac{2\left|T^+_{J_1}(u_i)-T^+_{J_2}(u_i)\right|}{\sqrt{1+\left|T^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^+_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^+_{J_1}(u_i))-(1-T^+_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^+_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^+_{J_2}(u_i))\right|^2}}\right] = 0$$

$\Leftrightarrow T^+_{J_1}(u_i) = T^+_{J_2}(u_i)$.

$$\left[\frac{2\left|I^-_{J_1}(u_i)-I^-_{J_2}(u_i)\right|}{\sqrt{1+\left|I^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^-_{J_2}(u_i)\right|^2}}+\frac{2\left\|(1-I^-_{J_1}(u_i))-(1-I^-_{J_2}(u_i))\right\|}{\sqrt{1+\left|(1-I^-_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^-_{J_2}(u_i))\right|^2}}\right] = 0$$

$\Leftrightarrow I^-_{J_1}(u_i) = I^-_{J_2}(u_i)$

$$\left[\frac{2\left|I^+_{J_1}(u_i)-I^+_{J_2}(u_i)\right|}{\sqrt{1+\left|I^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^+_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^+_{J_1}(u_i))-(1-I^+_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^+_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^+_{J_2}(u_i))\right|^2}}\right] = 0$$

$\Leftrightarrow I^+_{J_1}(u_i) = I^+_{J_2}(u_i)$

$$\left[\frac{2\left|F^-_{J_1}(u_i)-F^-_{J_2}(u_i)\right|}{\sqrt{1+\left|F^-_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^-_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^-_{J_1}(u_i))-(1-F^-_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^-_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^-_{J_2}(u_i))\right|^2}}\right] = 0$$

$\Leftrightarrow F^-_{J_1}(u_i) = F^-_{J_2}(u_i)$

$$\left[\frac{2\left|F^+_{J_1}(u_i)-F^+_{J_2}(u_i)\right|}{\sqrt{1+\left|F^+_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^+_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^+_{J_1}(u_i))-(1-F^+_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^+_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^+_{J_2}(u_i))\right|^2}}\right] = 0$$

$\Leftrightarrow F^{+}_{J_1}(u_i) = F^{+}_{J_2}(u_i)$

So, $CE_{IN}(J_1, J_2) = 0$ if and only if

$T^{-}_{J_1}(u_i) = T^{-}_{J_2}(u_i)$, $T^{+}_{J_1}(u_i) = T^{+}_{J_2}(u_i)$, $I^{-}_{J_1}(u_i) = I^{-}_{J_2}(u_i)$,

$I^{+}_{J_1}(u_i) = I^{+}_{J_2}(u_i)$, $F^{-}_{J_1}(u_i) = F^{-}_{J_2}(u_i)$, $F^{+}_{J_1}(u_i) = F^{+}_{J_2}(u_i)$ $\forall u_i \in U$.

Hence complete the proof.

iii). Using definition (2.4), we obtain the following expression:

$$CE_{IN}(J_1^c, J_2^c) = \frac{1}{4}\left\{\sum_{i=1}^{n}\left\langle \begin{array}{l} \left[\frac{2\left|(1-T^{-}_{J_1}(u_i))-(1-T^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{-}_{J_2}(u_i))\right|^2}} + \frac{2\left|T^{-}_{J_1}(u_i)-T^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{-}_{J_2}(u_i)\right|^2}}\right] + \\ \left[\frac{2\left|(1-T^{+}_{J_1}(u_i))-(1-T^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{J_2}(u_i))\right|^2}} + \frac{2\left|T^{+}_{J_1}(u_i)-T^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{J_2}(u_i)\right|^2}}\right] + \\ \left[\frac{2\left\|(1-I^{-}_{J_1}(u_i))-(1-I^{-}_{J_2}(u_i))\right\|}{\sqrt{1+\left|(1-I^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{-}_{J_2}(u_i))\right|^2}} + \frac{2\left|I^{-}_{J_1}(u_i)-I^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{J_2}(u_i)\right|^2}}\right] + \\ \left[\frac{2\left\|(1-I^{+}_{J_1}(u_i))-(1-I^{+}_{J_2}(u_i))\right\|}{\sqrt{1+\left|(1-I^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{+}_{J_2}(u_i))\right|^2}} + \frac{2\left|I^{+}_{J_1}(u_i)-I^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{J_2}(u_i)\right|^2}}\right] + \\ \left[\frac{2\left|(1-F^{-}_{J_1}(u_i))-(1-F^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{-}_{J_2}(u_i))\right|^2}} + \frac{2\left|F^{-}_{J_1}(u_i)-F^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{J_2}(u_i)\right|^2}}\right] + \\ \left[\frac{2\left|(1-F^{+}_{J_1}(u_i))-(1-F^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{J_2}(u_i))\right|^2}} + \frac{2\left|F^{+}_{J_1}(u_i)-F^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{J_2}(u_i)\right|^2}}\right] \end{array}\right\rangle\right\}$$

$$= \frac{1}{4}\left\{\sum_{i=1}^{n}\left\langle \begin{array}{l} \left[\frac{2\left|T^{-}_{J_1}(u_i)-T^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{-}_{J_2}(u_i)\right|^2}} + \frac{2\left|(1-T^{-}_{J_1}(u_i))-(1-T^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{-}_{J_2}(u_i))\right|^2}}\right] + \\ \left[\frac{2\left|T^{+}_{J_1}(u_i)-T^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{J_2}(u_i)\right|^2}} + \frac{2\left|(1-T^{+}_{J_1}(u_i))-(1-T^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{J_2}(u_i))\right|^2}}\right] + \\ \left[\frac{2\left|I^{-}_{J_1}(u_i)-I^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{J_2}(u_i)\right|^2}} + \frac{2\left\|(1-I^{-}_{J_1}(u_i))-(1-I^{-}_{J_2}(u_i))\right\|}{\sqrt{1+\left|(1-I^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{-}_{J_2}(u_i))\right|^2}}\right] + \\ \left[\frac{2\left|I^{+}_{J_1}(u_i)-I^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{J_2}(u_i)\right|^2}} + \frac{2\left\|(1-I^{+}_{J_1}(u_i))-(1-I^{+}_{J_2}(u_i))\right\|}{\sqrt{1+\left|(1-I^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{+}_{J_2}(u_i))\right|^2}}\right] + \\ \left[\frac{2\left|F^{-}_{J_1}(u_i)-F^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{J_2}(u_i)\right|^2}} + \frac{2\left|(1-F^{-}_{J_1}(u_i))-(1-F^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{-}_{J_2}(u_i))\right|^2}}\right] + \\ \left[\frac{2\left|F^{+}_{J_1}(u_i)-F^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{J_2}(u_i)\right|^2}} + \frac{2\left|(1-F^{+}_{J_1}(u_i))-(1-F^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{J_2}(u_i))\right|^2}}\right] \end{array}\right\rangle\right\} = CE_{IN}(J_1, J_2).$$

Hence complete the proof.

iv).

$CE_{IN}(J_1, J_2) =$

$$\frac{1}{4}\left\{\sum_{i=1}^{n}\left\langle \begin{array}{l} \left[\frac{2\left|T^{-}_{J_1}(u_i)-T^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{-}_{J_2}(u_i)\right|^2}} + \frac{2\left|(1-T^{-}_{J_1}(u_i))-(1-T^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{-}_{J_2}(u_i))\right|^2}}\right] + \\ \left[\frac{2\left|T^{+}_{J_1}(u_i)-T^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{J_2}(u_i)\right|^2}} + \frac{2\left|(1-T^{+}_{J_1}(u_i))-(1-T^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{J_2}(u_i))\right|^2}}\right] + \\ \left[\frac{2\left|I^{-}_{J_1}(u_i)-I^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{J_2}(u_i)\right|^2}} + \frac{2\left|(1-I^{-}_{J_1}(u_i))-(1-I^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{-}_{J_2}(u_i))\right|^2}}\right] + \\ \left[\frac{2\left|I^{+}_{J_1}(u_i)-I^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{J_2}(u_i)\right|^2}} + \frac{2\left|(1-I^{+}_{J_1}(u_i))-(1-I^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{+}_{J_2}(u_i))\right|^2}}\right] + \\ \left[\frac{2\left|F^{-}_{J_1}(u_i)-F^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{J_2}(u_i)\right|^2}} + \frac{2\left|(1-F^{-}_{J_1}(u_i))-(1-F^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{-}_{J_2}(u_i))\right|^2}}\right] + \\ \left[\frac{2\left|F^{+}_{J_1}(u_i)-F^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{J_2}(u_i)\right|^2}} + \frac{2\left|(1-F^{+}_{J_1}(u_i))-(1-F^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{J_2}(u_i))\right|^2}}\right] \end{array}\right\rangle\right\}$$

$$= \frac{1}{4}\left\{\sum_{i=1}^{n}\left\langle \begin{array}{l} \left[\frac{2\left|T^{-}_{J_2}(u_i)-T^{-}_{J_1}(u_i)\right|}{\sqrt{1+\left|T^{-}_{J_2}(u_i)\right|^2}+\sqrt{1+\left|T^{-}_{J_1}(u_i)\right|^2}} + \frac{2\left|(1-T^{-}_{J_2}(u_i))-(1-T^{-}_{J_1}(u_i))\right|}{\sqrt{1+\left|(1-T^{-}_{J_2}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{-}_{J_1}(u_i))\right|^2}}\right] + \\ \left[\frac{2\left|T^{+}_{J_2}(u_i)-T^{+}_{J_1}(u_i)\right|}{\sqrt{1+\left|T^{+}_{J_2}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{J_1}(u_i)\right|^2}} + \frac{2\left|(1-T^{+}_{J_2}(u_i))-(1-T^{+}_{J_1}(u_i))\right|}{\sqrt{1+\left|(1-T^{+}_{J_2}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{J_1}(u_i))\right|^2}}\right] + \\ \left[\frac{2\left|I^{-}_{J_2}(u_i)-I^{-}_{J_1}(u_i)\right|}{\sqrt{1+\left|I^{-}_{J_2}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{J_1}(u_i)\right|^2}} + \frac{2\left|(1-I^{-}_{J_2}(u_i))-(1-I^{-}_{J_1}(u_i))\right|}{\sqrt{1+\left|(1-I^{-}_{J_2}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{-}_{J_1}(u_i))\right|^2}}\right] + \end{array}\right.$$

$$\left[\frac{2\left|I^{+}_{J_2}(u_i)-I^{+}_{J_1}(u_i)\right|}{\sqrt{1+\left|I^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{+}_{J_2}(u_i))-(1-I^{+}_{J_1}(u_i))\right|}{\sqrt{1+\left|(1-I^{+}_{J_2}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{+}_{J_1}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F^{-}_{J_2}(u_i)-F^{-}_{J_1}(u_i)\right|}{\sqrt{1+\left|F^{-}_{J_2}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{J_1}(u_i)\right|^2}}+\frac{2\left|(1-F^{-}_{J_2}(u_i))-(1-F^{-}_{J_1}(u_i))\right|}{\sqrt{1+\left|(1-F^{-}_{J_2}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{-}_{J_1}(u_i))\right|^2}}\right]+$$

$$\left.\left.\left[\frac{2\left|F^{+}_{J_2}(u_i)-F^{+}_{J_1}(u_i)\right|}{\sqrt{1+\left|F^{+}_{J_2}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{J_1}(u_i)\right|^2}}+\frac{2\left|(1-F^{+}_{J_2}(u_i))-(1-F^{+}_{J_1}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{J_2}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{J_1}(u_i))\right|^2}}\right]\right\}\right\rangle$$

$$=CE_{IN}(J_2,J_1).$$

Hence complete the proof.

3.1 Definition: Weighted IN-cross-entropy measure

We consider the weight w_i (i = 1, 2, 3,…, n) of u_i (i = 1, 2, 3, …, n) with $w_i\in[0,1]$ and $\sum_{i=1}^{n} w_i=1.$

Then the weighted cross entropy measure between J_1 and J_2 can be defined as follows:

$$CE^{w}_{IN}(J_1,J_2)=\frac{1}{4}\left\langle\sum_{i=1}^{n} w_i\left\{\left[\frac{2\left|T^{-}_{J_1}(u_i)-T^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^{-}_{J_1}(u_i))-(1-T^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{-}_{J_2}(u_i))\right|^2}}\right]+\right.\right.$$

$$\left[\frac{2\left|T^{+}_{J_1}(u_i)-T^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^{+}_{J_1}(u_i))-(1-T^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{J_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I^{-}_{J_1}(u_i)-I^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{-}_{J_1}(u_i))-(1-I^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{-}_{J_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I^{+}_{J_1}(u_i)-I^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{+}_{J_1}(u_i))-(1-I^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{+}_{J_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F^{-}_{J_1}(u_i)-F^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{-}_{J_1}(u_i))-(1-F^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{-}_{J_2}(u_i))\right|^2}}\right]+$$

$$\left.\left.\left[\frac{2\left|F^{+}_{J_1}(u_i)-F^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{+}_{J_1}(u_i))-(1-F^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{J_2}(u_i))\right|^2}}\right]\right\}\right\rangle \tag{2}$$

Theorem 2.

Interval neutrosophic weighted cross-entropy measure $CE^{w}_{IN}(J_1,J_2)$ satisfies the following properties:

i). $CE^{w}_{IN}(J_1,J_2)\geq 0.$

ii). $CE^{w}_{IN}(J_1,J_2)=0$, if and only if $T^{-}_{J_1}(u_i)=T^{-}_{J_2}(u_i)$, $T^{+}_{J_1}(u_i)=T^{+}_{J_2}(u_i)$, $I^{-}_{J_1}(u_i)=I^{-}_{J_2}(u_i)$, $I^{+}_{J_1}(u_i)=I^{+}_{J_2}(u_i)$, $F^{-}_{J_1}(u_i)=F^{-}_{J_2}(u_i)$, $F^{+}_{J_1}(u_i)=F^{+}_{J_2}(u_i)$ for all $\forall u_i\in U.$

iii). $CE^{w}_{IN}(J_1,J_2)=CE^{w}_{IN}(J_1^c,J_2^c)$

iv). $CE^{w}_{IN}(J_1,J_2)=CE^{w}_{IN}(J_2,J_1)$

Proof:

i). For all values of $u_i\in U$, $\left|T^{-}_{J_1}(u_i)\right|\geq 0$, $\left|T^{-}_{J_2}(u_i)\right|\geq 0$,

$\left|T^{-}_{J_1}(u_i)-T^{-}_{J_2}(u_i)\right|\geq 0$, $\sqrt{1+\left|T^{-}_{J_1}(u_i)\right|^2}\geq 0$,

$\sqrt{1+\left|T^{-}_{J_2}(u_i)\right|^2}\geq 0$, $\left|(1-T^{-}_{J_1}(u_i))\right|\geq 0$, $\left|(1-T^{-}_{J_2}(u_i))\right|\geq 0$,

$\left|(1-T^{-}_{J_1}(u_i))-(1-T^{-}_{J_2}(u_i))\right|\geq 0$,

$\sqrt{1+\left|(1-T^{-}_{J_1}(u_i))\right|^2}\geq 0$, $\sqrt{1+\left|(1-T^{-}_{J_2}(u_i))\right|^2}\geq 0$

$$\Rightarrow\left[\frac{2\left|T^{-}_{J_1}(u_i)-T^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^{-}_{J_1}(u_i))-(1-T^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{-}_{J_2}(u_i))\right|^2}}\right]\geq 0$$

and $\left|T^{+}_{J_1}(u_i)\right|\geq 0$, $\left|T^{+}_{J_2}(u_i)\right|\geq 0$, $\left|T^{+}_{J_1}(u_i)-T^{+}_{J_2}(u_i)\right|\geq 0$,

$\sqrt{1+\left|T^{+}_{J_1}(u_i)\right|^2}\geq 0$, $\sqrt{1+\left|T^{+}_{J_2}(u_i)\right|^2}\geq 0$,

$\left|(1-T^{+}_{J_1}(u_i))\right|\geq 0$, $\left|(1-T^{+}_{J_2}(u_i))\right|\geq 0$, $\left|(1-T^{+}_{J_1}(u_i))-(1-T^{+}_{J_2}(u_i))\right|\geq 0$,

$\sqrt{1+\left|(1-T^{+}_{J_1}(u_i))\right|^2}\geq 0$, $\sqrt{1+\left|(1-T^{+}_{J_2}(u_i))\right|^2}\geq 0$

$$\Rightarrow\left[\frac{2\left|T^{+}_{J_1}(u_i)-T^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^{+}_{J_1}(u_i))-(1-T^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{J_2}(u_i))\right|^2}}\right]\geq 0$$

Similarly, we can show that

$$\left[\frac{2\left|I^{-}_{J_1}(u_i)-I^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{-}_{J_1}(u_i))-(1-I^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{-}_{J_2}(u_i))\right|^2}}\right]\geq 0$$

,

$$\left[\frac{2\left|I^{+}_{J_1}(u_i)-I^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{+}_{J_1}(u_i))-(1-I^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{+}_{J_2}(u_i))\right|^2}}\right]\geq 0$$

,

$$\left[\frac{2\left|F^{-}_{J_1}(u_i)-F^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{-}_{J_1}(u_i))-(1-F^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{-}_{J_2}(u_i))\right|^2}}\right]\geq 0$$

and

$$\left[\frac{2\left|F^{+}_{J_1}(u_i)-F^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{+}_{J_1}(u_i))-(1-F^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{J_2}(u_i))\right|^2}}\right]\geq 0$$

Since $w_i\in[0,1], \sum_{i=1}^{n}w_i=1$, we have, $CE^{w}_{IN}(J_1,J_2)\geq 0$. Hence complete the proof.

ii).

$$\left[\frac{2\left|T^{-}_{J_1}(u_i)-T^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^{-}_{J_1}(u_i))-(1-T^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{-}_{J_2}(u_i))\right|^2}}\right]=0$$

$\Leftrightarrow T^{-}_{J_1}(u_i)=T^{-}_{J_2}(u_i)$

$$\left[\frac{2\left|T^{+}_{J_1}(u_i)-T^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^{+}_{J_1}(u_i))-(1-T^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{J_2}(u_i))\right|^2}}\right]=0$$

$\Leftrightarrow T^{+}_{J_1}(u_i)=T^{+}_{J_2}(u_i)$

$$\left[\frac{2\left|I^{-}_{J_1}(u_i)-I^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{-}_{J_1}(u_i))-(1-I^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{-}_{J_2}(u_i))\right|^2}}\right]=0$$

$\Leftrightarrow I^{-}_{J_1}(u_i)=I^{-}_{J_2}(u_i)$

$$\left[\frac{2\left|I^{+}_{J_1}(u_i)-I^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{+}_{J_1}(u_i))-(1-I^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{+}_{J_2}(u_i))\right|^2}}\right]=0$$

$\Leftrightarrow I^{+}_{J_1}(u_i)=I^{+}_{J_2}(u_i)$

$$\left[\frac{2\left|F^{-}_{J_1}(u_i)-F^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{-}_{J_1}(u_i))-(1-F^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{-}_{J_2}(u_i))\right|^2}}\right]=0$$

$\Leftrightarrow F^{-}_{J_1}(u_i)=F^{-}_{J_2}(u_i)$

$$\left[\frac{2\left|F^{+}_{J_1}(u_i)-F^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{+}_{J_1}(u_i))-(1-F^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{J_2}(u_i))\right|^2}}\right]=0$$

$\Leftrightarrow F^{+}_{J_1}(u_i)=F^{+}_{J_2}(u_i)$, For all values of $u_i\in U$.

Since, $w_i\in[0,1], \sum_{i=1}^{n}w_i=1$, $w_i\geq 0$, we can show that

$CE^{w}_{IN}(J_1,J_2)=0$ iff $T^{-}_{J_1}(u_i)=T^{-}_{J_2}(u_i)$, $T^{+}_{J_1}(u_i)=T^{+}_{J_2}(u_i)$,

$I^{-}_{J_1}(u_i)=I^{-}_{J_2}(u_i)$, $I^{+}_{J_1}(u_i)=I^{+}_{J_2}(u_i)$,

$F^{-}_{J_1}(u_i)=F^{-}_{J_2}(u_i)$, $F^{+}_{J_1}(u_i)=F^{+}_{J_2}(u_i)$ and

$T_{J_1}(u_i)=T_{J_2}(u_i), I_{J_1}(u_i)=I_{J_2}(u_i), F_{J_1}(u_i)=F_{J_2}(u_i)$ for all $u_i\in U$.

iii).

Using definition (2.4), we obtain the following expression:

$$CE^{w}_{IN}(J^{c}_1,J^{c}_2)=\frac{1}{4}\left\langle\sum_{i=1}^{n}w_i\left\{\left[\frac{2\left|T^{-}_{J^c_1}(u_i)-T^{-}_{J^c_2}(u_i)\right|}{\sqrt{1+\left|T^{-}_{J^c_1}(u_i)\right|^2}+\sqrt{1+\left|T^{-}_{J^c_2}(u_i)\right|^2}}+\frac{2\left|(1-T^{-}_{J^c_1}(u_i))-(1-T^{-}_{J^c_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{-}_{J^c_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{-}_{J^c_2}(u_i))\right|^2}}\right]+\right.\right.$$

$$\left[\frac{2\left|T^{+}_{J^c_1}(u_i)-T^{+}_{J^c_2}(u_i)\right|}{\sqrt{1+\left|T^{+}_{J^c_1}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{J^c_2}(u_i)\right|^2}}+\frac{2\left|(1-T^{+}_{J^c_1}(u_i))-(1-T^{+}_{J^c_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{+}_{J^c_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{J^c_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I^{-}_{J^c_1}(u_i)-I^{-}_{J^c_2}(u_i)\right|}{\sqrt{1+\left|I^{-}_{J^c_1}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{J^c_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{-}_{J^c_1}(u_i))-(1-I^{-}_{J^c_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{-}_{J^c_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{-}_{J^c_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I^{+}_{J^c_1}(u_i)-I^{+}_{J^c_2}(u_i)\right|}{\sqrt{1+\left|I^{+}_{J^c_1}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{J^c_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{+}_{J^c_1}(u_i))-(1-I^{+}_{J^c_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{+}_{J^c_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{+}_{J^c_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F^{-}_{J^c_1}(u_i)-F^{-}_{J^c_2}(u_i)\right|}{\sqrt{1+\left|F^{-}_{J^c_1}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{J^c_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{-}_{J^c_1}(u_i))-(1-F^{-}_{J^c_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{-}_{J^c_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{-}_{J^c_2}(u_i))\right|^2}}\right]+$$

$$\left.\left.\left[\frac{2\left|F^{+}_{J^c_1}(u_i)-F^{+}_{J^c_2}(u_i)\right|}{\sqrt{1+\left|F^{+}_{J^c_1}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{J^c_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{+}_{J^c_1}(u_i))-(1-F^{+}_{J^c_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{J^c_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{J^c_2}(u_i))\right|^2}}\right]\right\}\right\rangle$$

$$=\frac{1}{4}\left\langle\sum_{i=1}^{n}w_i\left\{\left[\frac{2\left|(1-T^{-}_{J_1}(u_i))-(1-T^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{-}_{J_2}(u_i))\right|^2}}+\frac{2\left|T^{-}_{J_1}(u_i)-T^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{-}_{J_2}(u_i)\right|^2}}\right]+\right.\right.$$

$$\left[\frac{2\left|(1-T^{+}_{J_1}(u_i))-(1-T^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{J_2}(u_i))\right|^2}}+\frac{2\left|T^{+}_{J_1}(u_i)-T^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{J_2}(u_i)\right|^2}}\right]+$$

$$\left[\frac{2\left|(1-I^{-}_{J_1}(u_i))-(1-I^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{-}_{J_2}(u_i))\right|^2}}+\frac{2\left|I^{-}_{J_1}(u_i)-I^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{J_2}(u_i)\right|^2}}\right]+$$

$$\left[\frac{2\left|(1-I^{+}_{J_1}(u_i))-(1-I^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{+}_{J_2}(u_i))\right|^2}}+\frac{2\left|I^{+}_{J_1}(u_i)-I^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{J_2}(u_i)\right|^2}}\right]+$$

$$\left[\frac{2\left|(1-F^{-}_{J_1}(u_i))-(1-F^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{-}_{J_2}(u_i))\right|^2}}+\frac{2\left|F^{-}_{J_1}(u_i)-F^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{J_2}(u_i)\right|^2}}\right]+$$

$$\left.\left.\left[\frac{2\left|(1-F^{+}_{Q_1}(u_i))-(1-F^{+}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{Q_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{Q_2}(u_i))\right|^2}}+\frac{2\left|F^{+}_{Q_1}(u_i)-F^{+}_{Q_2}(u_i)\right|}{\sqrt{1+\left|F^{+}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{Q_2}(u_i)\right|^2}}\right]\right\}\right\rangle$$

$$=\frac{1}{4}\left\langle\sum_{i=1}^{n}w_i\left\{\left[\frac{2\left|T^{-}_{J_1}(u_i)-T^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^{-}_{J_1}(u_i))-(1-T^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{-}_{J_2}(u_i))\right|^2}}\right]+\right.\right.$$

$$\left[\frac{2\left|T^{+}_{J_1}(u_i)-T^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|T^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-T^{+}_{J_1}(u_i))-(1-T^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-T^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T^{+}_{J_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I^{-}_{J_1}(u_i)-I^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{-}_{J_1}(u_i))-(1-I^{-}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{-}_{J_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|I^{+}_{J_1}(u_i)-I^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|I^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-I^{+}_{J_1}(u_i))-(1-I^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-I^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I^{+}_{J_2}(u_i))\right|^2}}\right]+$$

$$\left[\frac{2\left|F^{-}_{J_1}(u_i)-F^{-}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{-}_{J_1}(u_i))-(1-F^{-}_{Q_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{-}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{-}_{J_2}(u_i))\right|^2}}\right]+$$

$$\left.\left.\left[\frac{2\left|F^{+}_{J_1}(u_i)-F^{+}_{J_2}(u_i)\right|}{\sqrt{1+\left|F^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{J_2}(u_i)\right|^2}}+\frac{2\left|(1-F^{+}_{J_1}(u_i))-(1-F^{+}_{J_2}(u_i))\right|}{\sqrt{1+\left|(1-F^{+}_{J_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F^{+}_{J_2}(u_i))\right|^2}}\right]\right\}\right\rangle$$

$$=CE^{w}_{IN}(J_1,J_2), \forall u_i\in U.$$

Hence complete the proof.

iv).

Since,

$\left|T^{-}_{J_1}(u_i)-T^{-}_{J_2}(u_i)\right|=\left|T^{-}_{J_2}(u_i)-T^{-}_{J_1}(u_i)\right|$,

$\left|I^{-}_{J_1}(u_i)-I^{-}_{J_2}(u_i)\right|=\left|I^{-}_{J_2}(u_i)-I^{-}_{J_1}(u_i)\right|$,

$\left|F^{-}_{J_1}(u_i)-F^{-}_{J_2}(u_i)\right|=\left|F^{-}_{J_2}(u_i)-F^{-}_{J_1}(u_i)\right|$,

$\left|(1-T^{-}_{J_1}(u_i))-(1-T^{-}_{J_2}(u_i))\right|=\left|(1-T^{-}_{J_2}(u_i))-(1-T^{-}_{J_1}(u_i))\right|$,

$\left|(1-I^{-}_{J_1}(u_i))-(1-I^{-}_{J_2}(u_i))\right|=\left|(1-I^{-}_{J_2}(u_i))-(1-I^{-}_{J_1}(u_i))\right|$,

$\left|(1-F^{-}_{J_1}(u_i))-(1-F^{-}_{J_2}(u_i))\right|=\left|(1-F^{-}_{J_2}(u_i))-(1-F^{-}_{J_1}(u_i))\right|$.

Then, we obtain

$\sqrt{1+\left|T^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|T^{-}_{J_2}(u_i)\right|^2}=\sqrt{1+\left|T^{-}_{J_2}(u_i)\right|^2}+\sqrt{1+\left|T^{-}_{J_1}(u_i)\right|^2}$,

$\sqrt{1+\left|I^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{J_2}(u_i)\right|^2}=\sqrt{1+\left|I^{-}_{J_2}(u_i)\right|^2}+\sqrt{1+\left|I^{-}_{J_1}(u_i)\right|^2}$,

$\sqrt{1+\left|F^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{J_2}(u_i)\right|^2}=\sqrt{1+\left|F^{-}_{J_2}(u_i)\right|^2}+\sqrt{1+\left|F^{-}_{J_1}(u_i)\right|^2}$,

$\sqrt{1+\left|(1-T^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|(1-T^{-}_{J_2}(u_i)\right|^2}=\sqrt{1+\left|(1-T^{-}_{J_2}(u_i)\right|^2}+\sqrt{1+\left|(1-T^{-}_{J_1}(u_i)\right|^2}$,

, $\sqrt{1+\left|(1-I^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|(1-I^{-}_{J_2}(u_i)\right|^2}=\sqrt{1+\left|(1-I^{-}_{J_2}(u_i)\right|^2}+\sqrt{1+\left|(1-I^{-}_{J_1}(u_i)\right|^2}$,

$\sqrt{1+\left|(1-F^{-}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|(1-F^{-}_{J_2}(u_i)\right|^2}=\sqrt{1+\left|(1-F^{-}_{J_2}(u_i)\right|^2}+\sqrt{1+\left|(1-F^{-}_{J_1}(u_i)\right|^2}$,

$\forall u_i\in U$.

Similarly, $\left|T^{+}_{J_1}(u_i)-T^{+}_{J_2}(u_i)\right|=\left|T^{+}_{J_2}(u_i)-T^{+}_{J_1}(u_i)\right|$,

$\left|I^{+}_{J_1}(u_i)-I^{+}_{J_2}(u_i)\right|=\left|I^{+}_{J_2}(u_i)-I^{+}_{J_1}(u_i)\right|$,

$\left|F^{+}_{J_1}(u_i)-F^{+}_{J_2}(u_i)\right|=\left|F^{+}_{J_2}(u_i)-F^{+}_{J_1}(u_i)\right|$,

$\left|(1-T^{+}_{J_1}(u_i))-(1-T^{+}_{J_2}(u_i))\right|=\left|(1-T^{+}_{J_2}(u_i))-(1-T^{+}_{J_1}(u_i))\right|$,

$\left|(1-I^{+}_{J_1}(u_i))-(1-I^{+}_{J_2}(u_i))\right|=\left|(1-I^{+}_{J_2}(u_i))-(1-I^{+}_{J_1}(u_i))\right|$,

$\left|(1-F^{+}_{J_1}(u_i))-(1-F^{+}_{J_2}(u_i))\right|=\left|(1-F^{+}_{J_2}(u_i))-(1-F^{+}_{J_1}(u_i))\right|$,

then

$\sqrt{1+\left|T^{+}_{Q_1}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{Q_2}(u_i)\right|^2}=\sqrt{1+\left|T^{+}_{Q_2}(u_i)\right|^2}+\sqrt{1+\left|T^{+}_{Q_1}(u_i)\right|^2}$,

$\sqrt{1+\left|I^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{J_2}(u_i)\right|^2}=\sqrt{1+\left|I^{+}_{J_2}(u_i)\right|^2}+\sqrt{1+\left|I^{+}_{J_1}(u_i)\right|^2}$,

$\sqrt{1+\left|F^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{J_2}(u_i)\right|^2}=\sqrt{1+\left|F^{+}_{J_2}(u_i)\right|^2}+\sqrt{1+\left|F^{+}_{J_1}(u_i)\right|^2}$,

$\sqrt{1+\left|(1-T^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|(1-T^{+}_{J_2}(u_i)\right|^2}=\sqrt{1+\left|(1-T^{+}_{J_2}(u_i)\right|^2}+\sqrt{1+\left|(1-T^{+}_{J_1}(u_i)\right|^2}$,

$\sqrt{1+\left|(1-I^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|(1-I^{+}_{J_2}(u_i)\right|^2}=\sqrt{1+\left|(1-I^{+}_{J_2}(u_i)\right|^2}+\sqrt{1+\left|(1-I^{+}_{J_1}(u_i)\right|^2}$,

$\sqrt{1+\left|(1-F^{+}_{J_1}(u_i)\right|^2}+\sqrt{1+\left|(1-F^{+}_{J_2}(u_i)\right|^2}=\sqrt{1+\left|(1-F^{+}_{J_2}(u_i)\right|^2}+\sqrt{1+\left|(1-F^{+}_{J_1}(u_i)\right|^2}$,

$\forall u_i\in U$.

And $w_i\in[0,1]$, $\sum_{i=1}^{n}w_i=1$, $w_i\geq 0$.

So, $CE^{w}_{IN}(J_1,J_2)=CE^{w}_{IN}(J_2,J_1)$. Hence complete the proof.

4. Multi attribute group decision making strategy using IN-cross entropy measure in interval neutrosophic set environment

In this section we develop a novel MAGDM strategy based on proposed IN- cross entropy measure.

The MAGDM problem can be consider as follows:
Let $A=\{A_1,A_2,A_3,...,A_m\}$ and $G=\{G_1,G_2,G_3,...,G_n\}$ be the discrete set of alternatives and attribute respectively. Let $W=\{w_1,w_2,w_3,...,w_n\}$ be the weight vector of attributes G_j (j = 1, 2, 3, …, n), where $w_j\geq 0$ and $\sum_{j=1}^{n}w_j=1$. Let $E=\{E_1,E_2,E_3,...,E_\rho\}$ be the set of decision makers who are employ to evaluate the alternative. The weight vector of the decision makers $E_k\ (k=1,2,3,...,\rho)$ is $\lambda=\{\lambda_1,\lambda_2,\lambda_3,...,\lambda_\rho\}$ (where, $\lambda\geq 0$ and $\sum_{k=1}^{\rho}\lambda_k=1$), which can be determined according to the decision makers expertise, judgment quality and decision making knowledge.

Now, we describe the steps of the proposed MAGDM strategy (See Figure 1.) using weighted IN-cross entropy measure.

MAGDM strategy using IN-cross entropy measure

Step: 1. Formulate the decision matrices

For MAGDM with INSs information, the rating values of the alternatives A_i $(i=1,2,3,...,m)$ on the basis of critera G_j $(j=1,2,3,...,n)$ by the k-th decision maker can be expressed in INN as $a_{ij}^k = <[^-T_{ij}^k, {}^+T_{ij}^k], [^-I_{ij}^k, {}^+I_{ij}^k], [^-F_{ij}^k, {}^+F_{ij}^k]>$ (i = 1, 2, 3, …, m; j = 1, 2, 3, …, n; k = 1, 2, 3, …, ρ) . We arrange these rating values of alternatives provided by the decision makers in matrix form as follows:

$$M^k = \begin{pmatrix} & G_1 & G_2 & \dots & G_n \\ A_1 & a_{11}^k & a_{12}^k & \cdots & a_{1n}^k \\ A_2 & a_{21}^k & a_{22}^k & & a_{2n}^k \\ . & . & & \dots & . \\ A_m & a_{m1}^k & a_{m2}^k & \cdots & a_{mn}^k \end{pmatrix} \quad \dots\dots\dots(3)$$

Step: 2. Formulate the weighted aggregated decision matrix

For obtaining one group decision, we aggregate all individual decision matrices (M^k) to an aggregated decision matrix (M) using interval-valued neutrosophic weighted averaging (INNWA) operator ([72]) as follows:

$$a_{ij} = INNWA_\lambda(a_{ij}^1, a_{ij}^2, a_{ij}^3, \dots, a_{ij}^\rho) =$$
$$(\lambda_1 a_{ij}^1 \oplus \lambda_2 a_{ij}^2 \oplus \lambda_3 a_{ij}^3 \oplus \dots \oplus \lambda_\rho a_{ij}^\rho) =$$
$$<[1-\prod_{k=1}^{\rho}(1-{}^-T_{ij}^k)^{\lambda_k}, 1-\prod_{k=1}^{\rho}(1-{}^+T_{ij}^k)^{\lambda_k}], [\prod_{k=1}^{\rho}({}^-I_{ij}^k)^{\lambda_k}, \prod_{k=1}^{\rho}({}^+I_{ij}^k)^{\lambda_k}],$$
$$[\prod_{k=1}^{\rho}({}^-F_{ij}^k)^{\lambda_k}, \prod_{k=1}^{\rho}({}^+F_{ij}^k)^{\lambda_k}]> \quad (4)$$

(i = 1, 2, 3, …, m; j = 1, 2, 3, …, n; k = 1, 2, 3, …, ρ).

Therefore, the aggregated decision matrix is defined as follows:

$$M = \begin{pmatrix} & G_1 & G_2 & \dots & G_n \\ A_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ A_2 & a_{21} & a_{22} & & a_{2n} \\ . & . & & \dots & . \\ A_m & a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \dots\dots\dots (5)$$

Step: 3. Formulate priori/ ideal decision matrix

In the MAGDM processes, the priori decision matrix is used to select the best alternatives among the set of collected feasible alternatives. In this decision making processes we use the following decision matrix as priori decision matrix.

$$P = \begin{pmatrix} & G_1 & G_2 & \dots & G_n \\ A_1 & a_{11}^* & a_{12}^* & \cdots & a_{1n}^* \\ A_2 & a_{21}^* & a_{22}^* & & a_{2n}^* \\ . & . & & \dots & . \\ A_m & a_{m1}^* & a_{m2}^* & \cdots & a_{mn}^* \end{pmatrix} \quad \dots\dots\dots\dots\dots(6)$$

Where, $a_{ij}^* = <[1,1],[0,0],[0,0]>$ for benefit type attributes and $a_{ij}^* = <[0,0],[1,1],[1,1]>$ for cost type attributes, (i = 1, 2, 3, …, m; j = 1, 2, 3, …, n).

Step: 4. Formulate the weighted IN-cross entropy matrix

Using equation (2), we calculate weighted cross entropy value between aggregate matrix and priori matrix. The cross entropy value can be present in matrix form as follows:

$$^{INS}M_{CE}^w = \begin{pmatrix} CE_{IN}^w(A_1) \\ CE_{IN}^w(A_2) \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ CE_{IN}^w(A_m) \end{pmatrix} \quad \dots\dots\dots\dots (7)$$

Step: 5. Rank the priority

Smaller value of the cross entropy reflect that an alternative is closer to the ideal alternative. Therefore, the priority order of all the alternatives can be determined according to the increasing order of the cross entropy values $CE_{IN}^w(A_i)$ (i = 1, 2, 3, …, m). Smallest cross entropy value indicates the best alternative and greatest cross entropy value indicates the worst alternative.

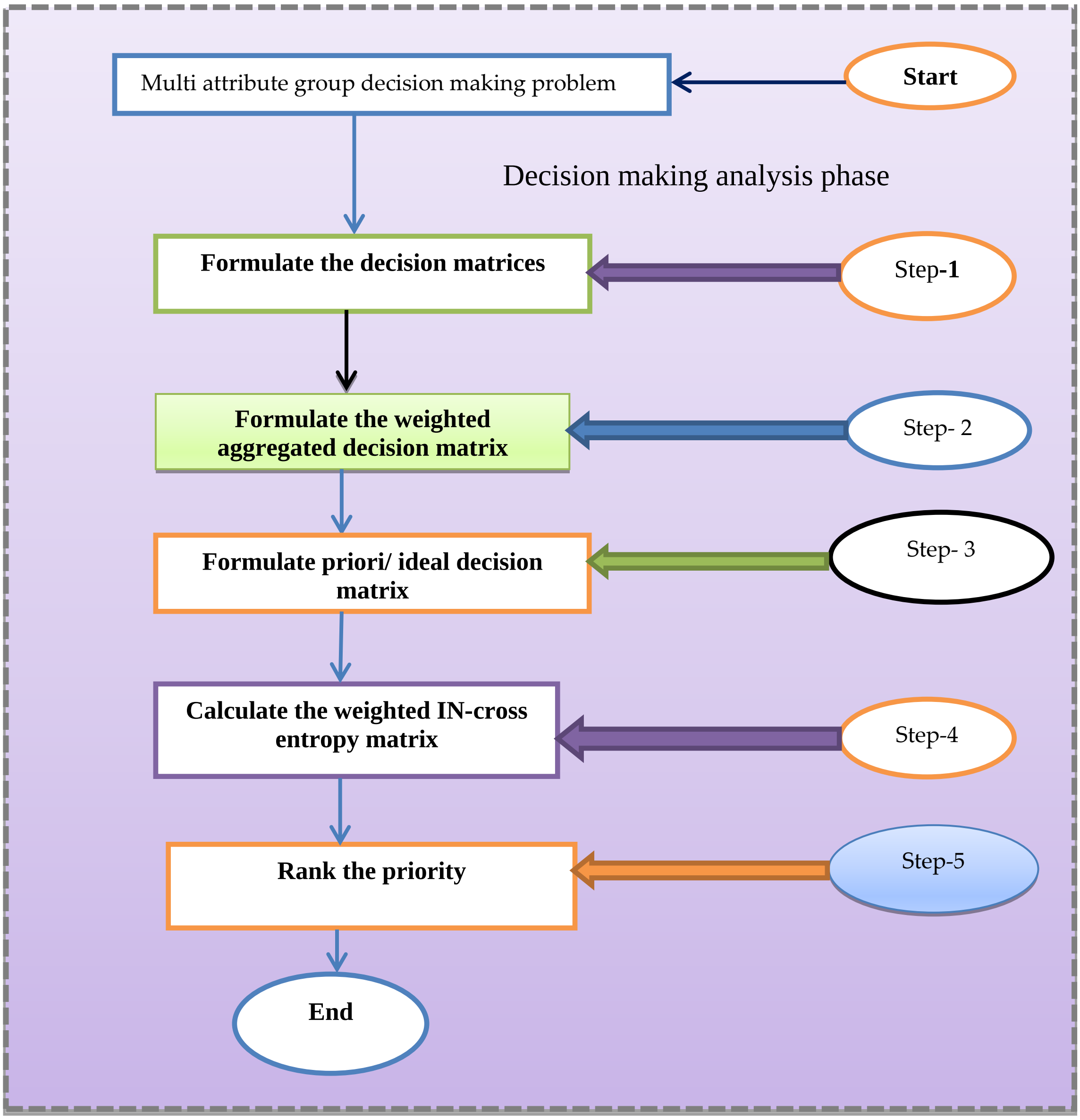

Figure.1 Decision making procedure of proposed MAGDM method

5. Illustrative example

In this section, we provide an illustrative example of MAGDM problems to reflect the validity and efficiency of our proposed strategy under INSs environment.

Now, we solve an illustrative example adapted from [9] for cultivation and analysis. A venture capital firm intends to make evaluation and selection to five enterprises with the investment potential:

1) Automobile company (A_1)
2) Military manufacturing enterprise (A_2)
3) TV media company (A_3)
4) Food enterprises (A_4)
5) Computer software company (A_5)

On the basis of four attributes namely:

1) Social and political factor (G_1)
2) The environmental factor (G_2)
3) Investment risk factor (G_3)
4) The enterprise growth factor (G_4).

The investment firm makes a panel of three decision makers $E = \{E_1, E_2, E_3\}$ having their weights vector

λ $\{0.42, 0.28, 0.30\}$ and weight vector of attributes is $W = \{0.24,\ 0.25,\ 0.23,\ 0.28\}$.

The steps of decision making strategy to rank alternatives are presented below:

Step: 1. Formulate the decision matrices

We represent the rating values of alternatives A_i (i = 1, 2, 3, 4, 5) with respects to the attributes G_j (j = 1, 2, 3, 4) provided by the decision-makers E_k (k = 1, 2, 3) in matrix form as follows:

Decision matrix for E_1 decision maker

$$M^1 = \begin{pmatrix} & G_1 & G_2 & G_3 & G_4 \\ A_1 & \langle[.7,.9],[.3,.4],[.3,.4]\rangle & \langle[.6,.7],[.3,.4],[.4,.5]\rangle & \langle[.6,.7],[.2,.3],[.2,.4]\rangle & \langle[.4,.5],[.3,.4],[.7,.8]\rangle \\ A_2 & \langle[.6,.7],[.1,.2],[.2,.3]\rangle & \langle[.7,.8],[.2,.4],[.2,.3]\rangle & \langle[.7,.9],[.5,.6],[.4,.5]\rangle & \langle[.7,.9],[.1,.2],[.1,.3]\rangle \\ A_3 & \langle[.6,.8],[.2,.4],[.3,.4]\rangle & \langle[.5,.7],[.3,.4],[.1,.2]\rangle & \langle[.8,.9],[.5,.7],[.3,.6]\rangle & \langle[.6,.7],[.1,.3],[.2,.3]\rangle \\ A_4 & \langle[.4,.5],[.7,.8],[.6,.7]\rangle & \langle[.3,.6],[.2,.3],[.3,.4]\rangle & \langle[.6,.7],[.1,.2],[.4,.5]\rangle & \langle[.4,.5],[.3,.4],[.6,7]\rangle \\ A_5 & \langle[.7,.8],[.3,.4],[.2,.3]\rangle & \langle[.4,.5],[.2,.4],[.3,.5]\rangle & \langle[.5,.6],[.2,.4],[.3,.4]\rangle & \langle[.7,.9],[.6,.7],[.4,.5]\rangle \end{pmatrix} \quad \ldots\ldots\ldots(8)$$

Decision matrix for E_2 decision maker

$$M^2 = \begin{pmatrix} & G_1 & G_2 & G_3 & G_4 \\ A_1 & \langle[.6,.7],[.1,.2],[.2,.3]\rangle & \langle[.3,.5],[.2,.4],[.4,.5]\rangle & \langle[.7,.9],[.3,.4],[.3,.5]\rangle & \langle[.4,.6],[.4,.5],[.2,.3]\rangle \\ A_2 & \langle[.4,.7],[.2,.4],[.3,.4]\rangle & \langle[.6,.7],[.2,.3],[.3,.4]\rangle & \langle[.5,.7],[.1,.3],[.3,.4]\rangle & \langle[.4,.6],[.3,.4],[.2,.3]\rangle \\ A_3 & \langle[.3,.6],[.2,.4],[.3,.4]\rangle & \langle[.4,.5],[.2,.3],[.3,.5]\rangle & \langle[.8,.9],[.2,.5],[.3,.4]\rangle & \langle[.5,.6],[.3,.5],[.3,.6]\rangle \\ A_4 & \langle[.5,.7],[.3,.5],[.1,.3]\rangle & \langle[.5,.6],[.1,.3],[.4,.6]\rangle & \langle[.4,.7],[.1,.4],[.3,.4]\rangle & \langle[.6,.8],[.3,.5],[.3,4]\rangle \\ A_5 & \langle[.6,.9],[.3,.4],[.2,.3]\rangle & \langle[.3,.6],[.3,.4],[.2,.5]\rangle & \langle[.6,.8],[.3,.5],[.,4.6]\rangle & \langle[.3,.5],[.3,.4],[.4,.5]\rangle \end{pmatrix} \quad \ldots\ldots\ldots(9)$$

Decision matrix for E_3 decision maker

$$M^3 = \begin{pmatrix} & G_1 & G_2 & G_3 & G_4 \\ A_1 & \langle[.4,.7],[.1,.2],[.3,.5]\rangle & \langle[.3,.6],[.2,.4],[.3,.4]\rangle & \langle[.6,.7],[.2,.4],[.3,.5]\rangle & \langle[.8,.9],[.2,.4],[.1,.3]\rangle \\ A_2 & \langle[.3,.6],[.4,.5],[.4,.5]\rangle & \langle[.7,.9],[.1,.3],[.3,.4]\rangle & \langle[.5,.7],[.2,.4],[.2,.3]\rangle & \langle[.6,.8],[.2,.4],[.3,.5]\rangle \\ A_3 & \langle[.7,.8],[.1,.3],[.4,.5]\rangle & \langle[.8,.9],[.1,.3],[.3,.4]\rangle & \langle[.6,.8],[.2,.3],[.3,.4]\rangle & \langle[.6,.7],[.2,.3],[.3,.4]\rangle \\ A_4 & \langle[.6,.9],[.2,.3],[.2,.4]\rangle & \langle[.5,.6],[.1,.3],[.2,.4]\rangle & \langle[.3,.5],[.1,.2],[.2,.4]\rangle & \langle[.5,.7],[.2,.3],[.3,5]\rangle \\ A_5 & \langle[.7,.8],[.1,.3],[.2,.3]\rangle & \langle[.5,.6],[.2,.4],[.1,.3]\rangle & \langle[.4,.6],[.1,.3],[.2,.4]\rangle & \langle[.5,.7],[.2,.3],[.3,.5]\rangle \end{pmatrix} \quad \ldots\ldots\ldots\ldots(10)$$

Step: 2. Formulate the weighted aggregated decision matrix

Using equation (4), the aggregated decision matrix is presented below:

Aggregated decision matrix

$$M = \begin{pmatrix} & G_1 & G_2 & G_3 & G_4 \\ A_1 & \langle[.6,.8],[.2,.3],[.3,.4]\rangle & \langle[.5,.6],[.2,.4],[.4,.4]\rangle & \langle[.6,.8],[.2,.3],[.2,.4]\rangle & \langle[.6,.7],[.3,.4],[.3,.4]\rangle \\ A_2 & \langle[.5,.7],[.2,.3],[.3,.4]\rangle & \langle[.7,.8],[.2,.3],[.2,.4]\rangle & \langle[.6,.8],[.2,.4],[.3,.4]\rangle & \langle[.6,.8],[.2,.3],[.2,.3]\rangle \\ A_3 & \langle[.6,.8],[.2,.4],[.3,.4]\rangle & \langle[.6,.8],[.2,.3],[.2,.3]\rangle & \langle[.8,.9],[.3,.5],[.3,.5]\rangle & \langle[.6,.7],[.2,.3],[.2,.4]\rangle \\ A_4 & \langle[.5,.7],[.4,.5],[.3,.5]\rangle & \langle[.4,.6],[.1,.3],[.3,.4]\rangle & \langle[.5,.6],[.1,.2],[.3,.4]\rangle & \langle[.5,.7],[.3,.4],[.4,.5]\rangle \\ A_5 & \langle[.7,.8],[.2,.4],[.2,.3]\rangle & \langle[.4,.6],[.2,.4],[.2,.4]\rangle & \langle[.5,.7],[.2,.4],[.3,.4]\rangle & \langle[.6,.8],[.4,.5],[.4,.5]\rangle \end{pmatrix} \quad \ldots\ldots\ldots(11)$$

Step: 3. Formulate priori/ ideal decision matrix

Priori/ ideal decision matrix

$$M^1 = \begin{pmatrix} & G_1 & G_2 & G_3 & G_4 \\ A_1 & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle \\ A_2 & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle \\ A_3 & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle \\ A_4 & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle \\ A_5 & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle & \langle[1,1],[0,0],[0,0]\rangle \end{pmatrix} \quad \ldots\ldots\ldots\ldots(12)$$

Step: 4. Calculate the weighted IN-cross entropy matrix

Using equation (2), we calculate the interval neutrosophic weighted cross entropy values between ideal matrixes (12) and weighted aggregated decision matrix (11).

$$^{IN}M_{CE}^{w} = \begin{pmatrix} 0.86 \\ 0.77 \\ 0.78 \\ 0.95 \\ 0.90 \end{pmatrix} \quad \ldots\ldots\ldots\ldots(13)$$

Step: 5. Rank the priority

The position of cross entropy values of alternatives arranging in increasing order is

0.77 < 0.78 < 0.86 < 0.90 < 0.95. Since, smallest values of cross entropy indicate the alternative is closer to the ideal alternative. Thus the ranking priority of alternatives is $A_2 > A_3 > A_1 > A_5 > A_4$. Hence, military manufacturing enterprise (A_2) is the best alternative for investment.

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In Figure 2, we draw a bar diagram to represent the cross entropy values of alternatives which shows that A_2 is the

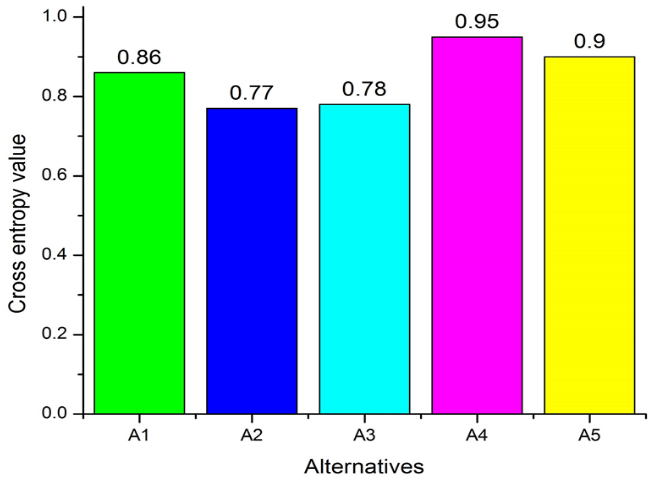

Figure.2. Bar diagram of alternatives versus cross entropy values of alternatives

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2. Conclusion

In this paper we have defined IN-cross entropy measure in INS environment which is free from all the drawback of existence cross entropy measures under interval neutrosophic set environment. We have proved the basic properties of the cross entropy measures. We have also defined weighted IN- cross entropy measure and proved its basic properties. Based on the weighted IN-cross entropy measure, we have proposed a novel MAGDM strategy. Finally, we solve a MAGDM problem to show the feasibility and efficiency of the proposed MAGDM making strategy. The proposed IN-cross entropy based MAGDM strategy can be employed to solve a variety of problems such as logistics center selection, teacher selection, renewable energy selection, fault diagnosis, etc.

References

[1] L. A. Zadeh. Probability measures of fuzzy events. Journal of Mathematical Analysis and Applications, 23 (1968), 421–427.

[2] B. Liu. A survey of entropy of fuzzy variables. Journal of Uncertain Systems, **1**(1) (2007), 4–13.

[3] X. G. Shang, and W. S. Jiang. A note on fuzzy information measures. Pattern Recognition Letters, 18 (1997), 425-432.

[4] K. T. Atanassov. Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 20 (1986), 87-96.

[5] M. D. Wade, A. Telik, D. S. Malik, and J. N. Mordeson. Political stability: analysis using TOPSIS and intuitionistic fuzzy sets. New Mathematics and Natural Computation, 13 (1) (2017), 1-11. doi: 10.1142/s1793005717500016.

[6] S. Pramanik, and D. Mukhopadhyaya. Grey relational analysis based intuitionistic fuzzy multi-criteria group decision-making approach for teacher selection in higher education. International Journal of Computer Applications, 34 (10) (2011), 21-29. doi: 10.5120/4138-5985.

[7] K. Mondal, and S. Pramanik. Intuitionistic fuzzy multi criteria group decision making approach to quality-brick selection problem. Journal of Applied Quantitative Methods, 9 (2) (2014), 35–50.

[8] P. P. Dey, S. Pramanik, and B. C. Giri. Multi-criteria group decision making in intuitionistic fuzzy environment based on grey relational analysis for weaver selection in Khadi institution. Journal of Applied Quantitative Methods, 10 (4) (2015**)**, 1–14.

[9] X. He, and W. F. Liu. An intuitionistic fuzzy multi-attribute decision-making method with preference on alternatives. Operations Research & Management Science, 22 (2013), 36–40.

[10] K. Mondal, S. Pramanik. Intuitionistic fuzzy similarity measure based on tangent function and its application to multi-attribute decision making. Global Journal of Advanced Research, 2 (2), (2015), 464-471.

[11] P. Biswas, S. Pramanik, and B. C. Giri. A study on information technology professionals' health problem based on intuitionistic fuzzy cosine similarity measure. Swiss Journal of Statistical and Applied Mathematics, 2 (1) (2014), 44-50.

[12] I. K. Vlachos, and G. D. Sergiadis. Intuitionistic fuzzy information applications to pattern recognition. Pattern Recognition Letters, 28 (2007), 197-206.

[13] F. Smarandache. A unifying field of logics. Neutrosophy: neutrosophic probability, set and logic, American Research Press, Rehoboth, (1998).

[14] H. Wang, F. Smarandache, Y. Zhang, and R. Sunderraman. Single valued neutrosophic sets. Multi-space and Multi-structure, 4 (2010), 410-413.

[15] S. Pramanik, and T. K. Roy. Neutrosophic game theoretic approach to Indo-Pak conflict over Jammu-Kashmir. Neutrosophic Sets and Systems, 2 (2014) 82-101.

[16] J. Ye. Single valued neutrosophic minimum spanning tree and its clustering method. Journal of Intelligent Systems, 23(2014), 311–324.

[17] J. Ye. Clustering methods using distance-based similarity measures of single-valued neutrosophic sets. Journal of Intelligent Systems, 23 (2014), 379–389.

[18] P. Biswas, S. Pramanik, and B. C. Giri. Entropy based grey relational analysis method for multi-attribute decision making under single valued neutrosophic assessments. Neutrosophic Sets and Systems, 2 (2014), 102–110.

[19] P. Biswas, S. Pramanik, and B. C. Giri. A new methodology for neutrosophic multi-attribute decision making with unknown weight information. Neutrosophic Sets and Systems, 3 (2014), 42–52.

[20] P. Biswas, S. Pramanik, and B. C. Giri. TOPSIS method for multi-attribute group decision-making under single valued neutrosophic environment. Neural Computing and Applications, (2015), doi: 10.1007/s00521-015-1891-2.

[21] P. Biswas, S. Pramanik, and B. C. Giri. Aggregation of triangular fuzzy neutrosophic set information and its application to multi-attribute decision making. Neutrosophic Sets and Systems, 12 (2016), 20-40.

[22] P. Biswas, S. Pramanik, and B. C. Giri. Value and ambiguity index based ranking method of single-valued trapezoidal neutrosophic numbers and its application to multi-attribute decision making. Neutrosophic Sets and Systems 12 (2016), 127-138.

[23] P. Biswas, S. Pramanik, and B. C. Giri. Multi-attribute group decision making based on expected value of neutrosophic trapezoidal numbers. New Trends in Neutrosophic Theory and Applications-Vol-II. Pons Editions, Brussels (2017). In Press.

[24] P. Biswas, S. Pramanik, and B. C. Giri. Non-linear programming approach for single-valued neutrosophic TOPSIS method. New Mathematics and Natural Computation, (2017). In Press.

[25] I. Deli, and Y. Subas. A ranking method of single valued neutrosophic numbers and its applications to multi-attribute decision making problems. International Journal of Machine Learning and Cybernetics, (2016), doi:10.1007/s13042016-0505-3.

[26] P. Ji , J. Q. Wang, and H. Y. Zhang. Frank prioritized Bonferroni mean operator with single-valued neutrosophic sets and its application in selecting third-party logistics providers. Neural Computing and Applications, (2016). doi:10.1007/s00521-016-2660-6.

[27] A. Kharal. A neutrosophic multi-criteria decision making method. New Mathematics and Natural Computation, 10 (2014), 143–162.

[28] R. X. Liang, J. Q. Wang, and L. Li. Multi-criteria group decision making method based on interdependent inputs of single valued trapezoidal neutrosophic information. Neural Computing and Applications, (2016), doi:10.1007/s00521-016-2672-2.

[29] R. X. Liang, J. Q. Wang, and H. Y. Zhang. A multi-criteria decision-making method based on single-valued trapezoidal neutrosophic preference relations with complete weight information. Neural Computing and Applications, (2017). Doi: 10.1007/s00521-017-2925-8.

[30] P. Liu, Y. Chu, Y. Li, and Y. Chen. Some generalized neutrosophic number Hamacher aggregation operators and their application to group decision making. International Journal of Fuzzy System, 16(2) (2014), 242–255.

[31] P. D. Liu, and H. G. Li. Multiple attribute decision-making method based on some normal neutrosophic Bonferroni mean operators. Neural Computing and Applications, 28 (2017), 179–194.

[32] P. Liu, and Y. Wang. Multiple attribute decision-making method based on single-valued neutrosophic normalized weighted Bonferroni mean. Neural Computing and Applications, 25(7) (2014), 2001–2010.

[33] J. J. Peng, J. Q. Wang, J. Wang, H. Y. Zhang, and X. H. Chen. Simplified neutrosophic sets and their applications in multi-criteria group decision-making problems. International Journal of Systems Science, 47 (10) (2016), 2342-2358.

[34] J. Peng, J. Wang, H. Zhang, and X. Chen. An outranking approach for multi-criteria decision-making problems with simplified neutrosophic sets. Applied Soft Computing, 25:336–346.
[35] S. Pramanik, D. Banerjee, and B. C. Giri. Multi – criteria group decision making model in neutrosophic refined set and its application. Global Journal of Engineering Science and Research Management, 3(6) (2016), 12-18.
[36] S. Pramanik, S. Dalapati, and T. K. Roy. Logistics center location selection approach based on neutrosophic multi-criteria decision making. New Trends in Neutrosophic Theories and Applications. Pons-Editions, Brussels, 2016, 161-174.
[37] R. Sahin, and M. Karabacak. A multi attribute decision making method based on inclusion measure for interval neutrosophic sets**.** International Journal of Engineering and Applied Sciences**,** 2(2) (2014), 13–15.
[38] R. Sahin, and A. Kucuk. Subsethood measure for single valued neutrosophic sets. Journal of Intelligent and Fuzzy System, (2014), doi:10.3233/IFS-141304.
[39] R. Sahin, and P. Liu. Maximizing deviation method for neutrosophic multiple attribute decision making with incomplete weight information. Neural Computing and Applications, (2015), doi: 10.1007/s00521-015-1995-8.
[40] M. Sodenkamp. Models, methods and applications of group multiple-criteria decision analysis in complex and uncertain systems. *Dissertation*, University of Paderborn, (2013), Germany.
[41] J. Ye. Multi criteria decision-making method using the correlation coefficient under single-valued neutrosophic environment. International Journal of General Systems,42 (2013), 386–394.
[42] E. K. Zavadskas, R. Baušys, and M. Lazauskas. Sustainable assessment of alternative sites for the construction of a waste incineration plant by applying WASPAS method with single-valued neutrosophic set. Sustainability, 7 (2015)**,** 15923-15936.
[43] J. Ye. A multi criteria decision-making method using aggregation operators for simplified neutrosophic sets. Journal of Intelligent and Fuzzy Systems, 26 (2014), 2459–2466.
[44] J. Ye. Trapezoidal neutrosophic set and its application to multiple attribute decision-making. Neural Computing and Applications, 26 (2015),1157–1166.
[45] J. Ye. Bidirectional projection method for multiple attribute group decision making with neutrosophic number. Neural Computing and Applications, (2015), doi: 10.1007/s00521-015-2123-5.
[46] J. Ye. Projection and bidirectional projection measures of single valued neutrosophic sets and their decision – making method for mechanical design scheme. Journal of Experimental and Theoretical Artificial Intelligence, (2016), doi:10.1080/0952813X.2016.1259263.
[47] S. Pramanik, P. Biswas, and B. C. Giri. Hybrid vector similarity measures and their applications to multi-attribute decision making under neutrosophic environment. Neural Computing and Applications, 28 (2017), 1163–1176. doi:10.1007/s00521-015-2125-3.
[48] K. Mondal, and S. Pramanik. Multi-criteria group decision making approach for teacher recruitment in higher education under simplified Neutrosophic environment. Neutrosophic Sets and Systems, 6 (2014), 28-34.
[49] K. Mondal, and S. Pramanik. Neutrosophic decision making model of school choice. Neutrosophic Sets and Systems, 7 (2015), 62-68.
[50] H. D. Cheng, and Y. Guo. A new neutrosophic approach to image thresholding. New Mathematics and Natural Computation, 4 (2008), 291–308.
[51] Y. Guo, and H. D. Cheng. New neutrosophic approach to image segmentation. Pattern Recognition, 42 (2009), 587–595.
[52] Y. Guo, A. Sengur, and J. Ye. A novel image thresholding algorithm based on neutrosophic similarity score. Measurement, 58 (2014), 175–186.
[53] J. Ye. Improved cosine similarity measures of simplified neutrosophic sets for medical diagnoses. Artificial Intelligence in Medicine, 63 (2015b), 171–179.
[54] M. Abdel-Baset, I.M. Hezam, and F. Smarandache. Neutrosophic goal programming, Neutrosophic Sets and Systems, 11 (2016), 112-118.
[55] P. Das, and T. K. Roy. Multi-objective non-linear programming problem based on neutrosophic optimization technique and its application in riser design problem. Neutrosophic Sets and Systems, 9 (2015), 88-95.
[56] I.M. Hezam, M. Abdel-Baset, and F. Smarandache. Taylor series approximation to solve neutrosophic multiobjective programming problem. Neutrosophic Sets and Systems, 10 (2015), 39-45.
[57] S. Pramanik. Neutrosophic multi-objective linear programming. Global Journal of Engineering Science and Research Management, 3(8) (2016), 36-46.
[58] S. Pramanik. Neutrosophic linear goal programming, Global Journal of Engineering Science and Research Management, 3(7) (2016), 01-11.
[59] R. Roy, and P. Das. A multi-objective production planning roblem based on neutrosophic linear rogramming approach. Internal Journal of Fuzzy Mathematical Archive, 8(2) (2015), 81-91.

[60] K. Mondal, and S. Pramanik. A study on problems of Hijras in West Bengal based on neutrosophic cognitive maps. Neutrosophic Sets and Systems, 5(2014), 21-26.

[61] S. Pramanik, and S. Chakrabarti. A study on problems of construction workers in West Bengal based on neutrosophic cognitive maps. International Journal of Innovative Research in Science. Engineering and Technology, 2(11) (2013), 6387-6394.

[62] J. Ye. Single valued neutrosophic cross-entropy for multi criteria decision making problems. Applied Mathematical Modelling, 38 (3) (2014), 1170 – 1175.

[63] J. Ye. Improved cross entropy measures of single valued neutrosophic sets and interval neutrosophic sets and their multi criteria decision making methods. Cybernetics and Information Technologies, 15 (2015), 13-26. doi: 10.1515/cait-2015-0051.

[64] H. Wang, F. Smarandache, Y. Q. Zhang, and R. Sunderraman. Interval neutrosophic sets and logic: theory and applications in computing. Hexis; Neutrosophic book series, No. 5 (2005).

[65] S. Broumi, and F. Smarandache. Correlation coefficient of interval neutrosophic set. Applied Mechanics and Materials, 436 (2013), 511–517.

[66] H.Y. Zhang, P. Ji, J. Wang, and X.H. Chen. An improved weighted correlation coefficient based on integrated weight for interval neutrosophic sets and its application in multi-criteria decision-making problems. International Journal of Computational Intelligence Systems, 8 (2015), 1027–1043.

[67] H.Y. Zhang, J.Q. Wang, and X.H. Chen. An outranking approach for multi-criteria decision-making problems with interval-valued neutrosophic sets. Neural Computing and Applications, 17 (2016), 615–627.

[68] Y. Huang, G. Wei, and C. Wei. VIKOR method for interval neutrosophic multiple attribute group decision-making. Information 8 (2017), 144; doi:10.3390/info8040144.

[69] J. Ye. Similarity measures between interval neutrosophic sets and their applications in multi criteria decision-making. Journal of Intelligent and Fuzzy Systems, 26 (2014), 165–172.

[70] S. Pramanik and K. Mondal. Interval neutrosophic multi-attribute decision-making based on grey relational analysis. Neutrosophic Sets and Systems, 9 (2015), 13-22.

[71] A. W. Zhao, J. G. Du, and H. J. Guan. Interval valued neutrosophic sets and multi-attribute decision-making based on generalized weighted aggregation operator. Journal of Intelligent and Fuzzy Systems, 29 (2015), 2697–2706.

[72] H.Y. Zhang, J.Q. Wang, and X. H. Chen. Interval neutrosophic sets and their application in multi-criteria decision making problems. The Scientific World Journal, 2014 (2014), Article ID 645953. http://dx.doi.org/10.1155/2014/645953.

[73] R. Sahin. Cross-entropy measure on interval neutrosophic sets and its applications in multi criteria decision making. Neural Computing and Applications, 2015, doi 10.1007/s00521-015-2131-5.

[74] Z. P. Tian, H. Y. Zhang, J. Wang, J. Q. Wang, and X. H. Chen. Multi-criteria decision-making method based on a cross-entropy with interval neutrosophic sets. International Journal of Systems Science, 2015. doi:10.1080/00207721.2015.1102359.

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