Mature Intuition and Mathematical Understanding

Forthcoming in the *Journal of Mathematical Behavior*

William D'Alessandro
Department of Philosophy, William & Mary

Irma Stevens
Department of Mathematics and Applied Mathematical Sciences, University of Rhode Island

Abstract

Mathematicians often describe the importance of well-developed intuition to productive research and successful learning. But neither education researchers nor philosophers interested in epistemic dimensions of mathematical practice have yet given the topic the sustained attention it deserves. The trouble is partly that intuition in the relevant sense lacks a usefully clear characterization, so we begin by offering one: mature intuition, we say, is the capacity for fast, fluent, reliable and insightful inference with respect to some subject matter. We illustrate the role of mature intuition in mathematical practice with an assortment of examples, including data from a sequence of clinical interviews in which a student improves upon initially misleading covariational intuitions. Finally, we show how the study of intuition can yield insights for philosophers and education theorists. First, it contributes to a longstanding debate in epistemology by undermining *epistemicism*, the view that an agent's degree of objectual understanding is determined exclusively by their knowledge, beliefs and credences. We argue on the contrary that intuition can contribute directly and independently to understanding. Second, we identify potential pedagogical avenues towards the development of mature intuition, highlighting strategies including *adding imagery*, *developing associations*, *establishing confidence* and *generalizing concepts*.

1 Introduction

Intuition has been called "an essential part of mathematics" (Hersh 1997, 61) which plays "a basic and indispensable role in mathematical research" (Wilder, 1967, p. 605) and "a fundamental role in securing mathematical truths" (Kline, 1980, p. 319). The prevalence of such attitudes among mathematicians would seem to mark intuition as a natural target for education researchers, philosophers, and others interested in the development of mathematical thought.

Reality, alas, has yet to catch up with this hope. In a widely read 1999 paper, Burton asked "Why Is Intuition so Important to Mathematicians but Missing from Mathematics Education?" (Burton 1999). The complaint in Burton's question still rings true a quarter-century later. While the education literature contains many passing references to intuition and a handful of substantial studies, these piecemeal contributions have failed to coalesce into a stable research program. And present vital signs are worrying. Lajos suggests in a recent survey that intuition has "disappeared from the organized and collaborative platforms of special issues in journals and collective disseminations" as of late (Lajos, 2023, p. 1).

The situation in philosophy of mathematics is perhaps even more stark. In spite of a wave of increased interest in mathematical practice and its implications for epistemology—a wave which has brought new attention in the 2000s to topics like explanation, understanding, imagination and the social dimensions of mathematical knowledge[1]—philosophers have paid little attention to intuition in the relevant sense.[2]

The lack of a clearly delineated central concept has likely contributed to the slow progress in both fields. Uses of 'intuition' relevant to mathematical cognition span multiple eras, literatures, and traditions, each carrying its own understanding and associations. (Sticking only to the most prominent examples, one could name Kantian transcendental idealism, Brouwerian philosophy of mathematics, Gödelian epistemology, Fischbein's program in mathematics education, Kahneman's System 1 / System 2 paradigm, Gigerenzer's work on heuristics in psychology, and the informal discourse of present-day mathematicians.) These specialized usages coexist with various commonsense notions, and the relations between and within the two sets of concepts are tangled. Without careful attention to definitions, then, researchers may find themselves chasing chimeras, reinventing wheels or talking past one another (cf. Osbeck, 1999).

In view of these conceptual troubles, Bubp noted a decade ago that "theoretical research on developing a standard working definition of intuition in mathematics would benefit the mathematics

[1] Overviews and collections on philosophy of mathematical practice include Van Kerkhove & van Bendegem (2007), Mancosu (2008) and Carter (2019). Hamami & Morris (2020) is a "primer" aimed specifically at mathematics educators.

[2] Philosophy has historically had more to say about intuition as an alleged faculty for direct apprehension of mathematical objects or facts. Versions of this concept can be found in Kant, Husserl, Wittgenstein, Gödel and elsewhere. Some insist that this notion alone represents intuition properly so called, while typical uses of 'intuition' by contemporary mathematicians are merely metaphorical (Cellucci, 2017, 231–2). If so, the metaphor is quite dead; we see no sign that these mathematicians understand themselves to be speaking loosely or figuratively.

Elijah Chudnoff's work (e.g. 2014, 2019, 2020) is a notable exception to philosophy's neglect of expert intuition; his (2019) focuses specifically on intuition in mathematics. We discuss Chudnoff's views (and our disagreements with some of them) in §2 below.

education community” (2014, p. 243; quoted in Lajos, 2023, p. 2). The need for such a definition—or, perhaps, several alternative definitions suited for different purposes—persists today.

Our first goal here is therefore to single out a particular concept of intuition, in terms which we hope are clear enough to ground future research. Intuition in our target sense plays an important role in mathematical practice (as we show in §3). Specifically, to illustrate the ideas of naïve/mature intuition, we draw on data from clinical interviews one of the authors conducted to learn about pre-service teachers’ development of meanings of formulas via reasoning with dynamic geometric contexts (Stevens, 2019). Moreover, the study of intuition can help advance debates in epistemology and education theory (as we argue in §4 and §5, respectively).

Our focal concept is that of *mature* intuition, which we understand as the capacity for fast, fluent, reliable and insightful inference with respect to some subject matter. We take this to be the operative concept in most discussions by contemporary mathematicians of the value of intuition. §2 develops an account of mature intuition, contrasting it with naïve forms of intuition commonly cited in the education literature and other nearby notions. The main philosophical claim of the paper is that mature intuition contributes to (objectual) understanding in a way that existing accounts of understanding fail to capture.

§5 considers how existing research in mathematics education can clarify naïve and mature intuition and learners’ transitions between them. Mathematics education research on intuition relies heavily on the work of Fischbein (cf. Zagorianakos & Shvarts, 2015), who in 1987 wrote in depth about intuition in science and mathematics. Synthesizing Fischbein’s insights and conjectures, we frame the construct of intuition around more recent developments, particularly relying on the ideas of concept image/definition/image offered by Tall & Vinner (1981) and Thompson (1996).

2 An account of mature intuition

We characterized mature intuition above as the capacity for fast, fluent, reliable and insightful inference with respect to some subject matter.[3] Let us say a bit more.

We call intuition *fast*. Characterizing the timescales on which intuitive inferences unfold is a job for empirical science, and not centrally important to our purposes here; it’s enough to contrast the relative speediness of intuition with the plodding pace of step-by-step deliberative thought.

By *fluent* we mean that mature intuition produces inferences automatically and with little conscious effort on the reasoner’s part. This is the feature of intuition that distinguishes it most from ordinary deliberative reasoning, which requires cognitive control and thus places concurrent demands

[3] As this section will make clear, most of the elements of our characterization have appeared elsewhere in the various intuition literatures. Our goal isn’t to invent a new concept from scratch, but rather to draw together and systematize the observations most useful for elucidating expert intuition in mathematics.

on memory, planning, attention, inhibition and other executive functions. Recent work in cognitive psychology has often highlighted this aspect of intuitive thinking (Klein, 1998; Gigerenzer 2008, Kahneman, 2011).

The fluency of intuition should not be confused with the claim that *cultivating*, *eliciting* or *assessing* one's intuitions are always easy tasks. As discussed below, mathematicians speak of the difficulty of developing high-quality intuition and the occasional need to expend effort on "listening to one's intuition" (Thurston, 1994). The idea is rather that, when the conditions are right for intuition to operate, it does so relatively frictionlessly. Though we otherwise find much to like in Chudnoff's account of expert intuition (2014, 2019, 2020), our views part ways on this point. Chudnoff rejects fluency as a hallmark of intuition on the grounds that it seems inappropriate to describe certain "hard-won" intuitions as reflexive or fluent (e.g. 2019, p. 466). By contrast, we locate the effortfulness of hard-won intuition in the preparatory work required to develop the appropriate cognitive scaffolding rather than in the act of intuition itself.

By *reliable* we mean that mature intuition tends to produce correct inferences—not infallibly, but often enough to count on for many high-stakes intellectual purposes. This differentiates our account from those that focus on the untrustworthy intuitions of beginners (cf. Lajos, 2023, §3.4.2). Silvia De Toffoli's work has explored this issue in detail (De Toffoli & Giardino, 2014; De Toffoli 2021), focusing on the circumstances in which appeals to intuition can play a legitimate role in rigorous informal proof. On De Toffoli's account, "intuition can be reliable and thus acceptable in specific contexts", namely those in which "it is shared by mathematicians with the appropriate training and is systematically linked to precise mathematical concepts and operations" (De Toffoli, 2021, 1790)—as has been the case, for instance, with proofs in knot theory requiring complex acts of manipulative imagination.

By *insightful* we mean that mature intuition often points the way toward substantive, non-obvious truths. While it can also serve as a shortcut to bypass routine verifications of easy facts, this isn't its most important role: rather, the often profound, generative, inquiry-shaping nature of mature intuition is responsible for much of its interest for philosophy and education research. As Poincaré says, "analysis puts at our disposal a multitude of procedures… Who shall tell us which to choose? We need a faculty which makes us see the end from afar, and intuition is this faculty. It is necessary to the explorer for choosing his route" (1900, V, 1017–8).

We adopt a broad notion of 'inference' here, covering any putatively rational way of adjusting one's beliefs, credences or other epistemic attitudes (whether or not the grounds for the adjustment are introspectively accessible). This inclusive notion of inference reflects the heterogeneity of intuition at the levels of phenomenology, content and epistemic force. Let us briefly elaborate.

First, intuitions come with varying *phenomenological trappings*. They may be experienced as stronger or weaker in various senses (with respect to clarity, apparent conclusiveness or affective force, for instance). Some involve a feeling of insight or understanding, others a sense of inclination or pull. Yet others feel like blind guesses. (These dimensions of strength and certainty aren't, of course, infallible predictors of the accuracy of the associated intuitions.) Transparency is another axis of difference: one can sometimes see through an intuition to its underlying motivation, while in other cases any such rational basis is completely obscure. In addition to these varying kinds of "cognitive phenomenology" (Smithies, 2013), intuitions can share (or lack) experiential qualities associated with other types of mental states: sensory components, hedonic coloring, emotional texture, and so on.

Our account differs from Chudnoff's here as well. Chudnoff takes the essential feature of intuition to be its special and relatively uniform phenomenology, which he characterizes as "forceful or pushy", presentational, conferring assertoric authority and involving a feeling of rightness (2019, pp. 470–1). We think, by contrast, that intuitions vary considerably in their possession of these and other phenomenological qualities. Indeed, it seems to us *a priori* possible for intuition to occur with no accompanying phenomenology.[4]

The *contents* of intuition are also diverse. Plausibly, some intuitions have purely propositional content: many who consider the question seem to intuit that every set has a choice function, for instance, as Zermelo and other early set theorists did.[5] (For an overview of recent philosophical discussion of propositional intuition, see Pust, 2019.) Alternatively, intuitions may involve imagistic, objectual or other non-propositional content: a mathematician might intuit a visualization of a group structure, or a prospective counterexample to a conjecture.

Finally, the *epistemic force* of intuition is variable. As stereotype would have it, the function of intuition is to deliver a compelling impression of truth leading to a confident state of outright belief. Some instances fit this description well enough. But more often in practice, perhaps, intuition results in a judgment that some proposition is probable, appropriate to assume, or worth trying to verify. While these subtler epistemic adjustments are often overlooked, they play central roles in guiding attention and steering inquiry.[6]

---

[4] §4 below presents an AI thought experiment. We think the scenarios described make sense if one imagines that the systems in question possess cognitive capabilities without phenomenological consciousness (interpreting "a strong feeling that *P*" as "a strong disposition to assert that *P*", e.g.). For general arguments that AI systems might possess psychological and cognitive states in the absence of consciousness, see e.g. Goldstein & Levinstein, MS; Yildirim & Paul, 2024.

[5] In his classic paper "A New Proof of the Possibility of a Well-Ordering", Zermelo claims that the Axiom of Choice, like Peano's axioms for arithmetic, is justified in virtue of being "intuitively evident and necessary for science" (Zermelo 1908).

[6] Cf. Feferman: "The word *intuition* as used by mathematicians has a variety of meanings… One sense is the common "Ah, hah!" *Erlebnis* of a flash of insight or illumination on the road to the solution of a problem. …Less vivid than [this], but equally common, are the mathematicians' *hunches* as to what problems it would be profitable to attack, what results are to be expected, and what methods are likely to work" (2000, 317–8).

3 Mature intuition in mathematical practice and pedagogy

3.1 Intuition in mathematical practice

Our next goal is to establish that intuition plays important roles in mathematical learning and research. While the majority of our discussion will focus on contemporary mathematics, it's worth noting that positive appraisals of intuition in our sense have a long history.

Felix Klein (1911), for instance, contrasted naïve and refined uses of intuition, the latter of which is grounded in "well-formulated axioms", systematic theory, exact proofs and clear distinctions (pp. 958–9). Klein claimed that "for the purposes of research it is always necessary to combine the intuition with the axioms" (p. 961). Perhaps the most well-known classical advocate of intuition is Poincaré, who set out his epistemological views in works like "On the Nature of Mathematical Reasoning" (Poincaré, 1894) and "Intuition and Logic in Mathematics" (Poincaré, 1900). On Poincaré's account, "logic and intuition have each their necessary role... Logic, which alone can give certainty, is the instrument of demonstration; intuition is the instrument of invention" (1900, V, p. 1018).

Recent discussions add practical and psychological detail to these claims. For instance, Thurston (1994) offers an illuminating account of mathematical creativity, documenting the author's pioneering work in low-dimensional topology in the 1970s. In it, Thurston recounts how he "gradually built up over a number of years a certain intuition for hyperbolic three-manifolds, with a repertoire of constructions, examples and proofs. ...After a while, I conjectured or speculated that all three-manifolds have a certain geometric structure; this conjecture eventually became known as the geometrization conjecture," which Thurston proved for Haken manifolds a few years later (174). For Thurston, this episode is illustrative of the key role played by intuition in mathematical progress and understanding (cf. §4 below).

The drive to develop intuition is also a common theme among mathematics learners at various levels. Intuition-seeking posts like the following are common on the research question-and-answer site MathOverflow, for instance:

> I'm beginning to learn cohomology for cyclic groups... What I don't get is what the intuition is behind the definitions of these cohomology groups. I do know what cohomology is in a geometric setting... but I don't know why we take these particular kernels modulo these particular images. What is the intuition for why they are defined the way they are?... Right now, I just see theorem after theorem, I see the algebraic manipulation and diagram chasing that proves it, but I don't see a bigger picture.[7]

[7] https://mathoverflow.net/questions/10879/intuition-for-group-cohomology.

In a more systematic spirit, Tao has outlined a three-stage model of mathematics education with a prominent role for the development of intuition (Tao, 2009).[8] Tao's first, *pre-rigorous* stage, lasting until most students' early undergraduate years, is based on "examples, fuzzy notions and hand-waving". The second *rigorous* stage emphasizes formalism, proof and theoretical systematicity. In the final *post-rigorous* stage, associated with the end of graduate school and beyond,

> one has grown comfortable with all the rigorous foundations of one's chosen field, and is now ready to revisit and refine one's pre-rigorous intuition on the subject, but this time with the intuition solidly buttressed by rigorous theory. …The emphasis is now on applications, intuition, and the "big picture". …It is only with a combination of both rigorous formalism and good intuition that one can tackle complex mathematical problems; one needs the former to correctly deal with the fine details, and the latter to correctly deal with the big picture.

Tao's remarks about intuition in post-rigorous mathematics bear many similarities to our target concept. While Tao's schema focuses on the development of intuition over a career-long trajectory, however, similar dynamics can be seen to play out with respect to students' mastery of elementary mathematics. The transition from naïve to mature intuition in classroom settings is of no less importance than (and is indeed a prerequisite for) the cultivation of intuition at the research level. We expand on this point in the remainder of the section.

3.2 Cultivation of intuition in education

Mathematics educators are often interested in students' initial approaches to mathematical problems. The intuitions which students bring to such settings are often grounded in commonsense reasoning or physical understanding. For example, in the knowledge-in-pieces framework (diSessa, 1988, 2018), "p-prims" (phenomenological primitives) are intuitions based on patterns in everyday experience (that objects fall, sounds will dissipate, multiplication makes bigger, etc.). Students often apply such intuitions to novel situations to determine what is obvious, plausible or implausible.

Unsurprisingly, reliance on these heuristics can lead to overgeneralizations. A student may be tempted to think that multiplication always yields a larger result, for instance, even for multipliers less than 1. The scope of this phenomenon is not limited to elementary arithmetic and geometry; Roh & Lee (2017) discuss students' primary intuitions about convergence in undergraduate real analysis. (Fischbein (1987) defines primary intuitions as those held prior to learning a formal definition.)

---

8 A related distinction between pre-formal, formal and post-formal theories and proofs can be found in Lakatos, 1978.

It is worth noting, however, that such naïve intuitions are often based on reasonable interpretations of repeated experiences. For instance, in the same way that a primary intuition about gravity may represent it as a downward physical force based on experiences with falling objects, primary intuitions about convergence as approaching but never reaching a value may be based on mathematical experiences with limits or asymptotes.

In the following example, we describe how Alexandria, a pre-service secondary mathematics teacher, invoked intuitive reasoning via covariational reasoning (i.e., reasoning about how two quantities change together; see Carlson et al., 2002). Although sophisticated in her assimilation of quantitative ways of approaching the problem, we classify Alexandria's initial intuitions as naive, as they lack the components of reliability and insightfulness discussed above. We conclude by noting how a later clinical interview with Alexandria pointed to signs of greater maturity.

Alexandria was engaged in a sequence of four interviews, each session lasting between 75–120 minutes (see Stevens, 2019). Throughout these clinical interviews (Clement, 2000), Alexandria reasoned about and constructed several formulas representing various quantities in various dynamic geometric environments (e.g. triangles, rectangles, parallelograms, spherical caps and cylinders). The goal of the set of interviews, also undertaken by three other students, was to gain insight into pre-service secondary mathematics students' ways of reasoning covariationally with formulas. Analysis of the interviews involved coding for covariational reasoning and the refinement of a conceptual analysis for developing students' covariational reasoning with formulas used to inform a follow-up teaching experiment (Steffe & Thompson, 2000).

Throughout the interviews, Alexandria used covariational reasoning to identify constant or varying rates of change in dynamic geometric contexts. For instance, she determined that a cylinder has equal changes in surface area for equal changes in height, and that a cone has increasing changes in surface area for equal changes in height. This reasoning relied on imagining strips of surface area being partitioned along equal changes of height for each of these shapes. For the cylinder, the strips of surface area were equal, while for the cone, they were not.

Alexandria relied primarily on her intuitive judgments in these cases, often producing explicit reasoning on paper only when prompted by the interviewer. When engaged in intuitive thinking, she generated a visualization of the relevant dynamic shape, and by mentally operating on this representation she was able to draw covariational conclusions quickly and efficiently. In this way, we consider Alexandria's reasoning *operational* (i.e., rooted in quantitative logico-mathematical operations), as defined by Beth & Piaget (1996). Alexandria's intuitions thus displayed the characteristics of speed and fluency discussed above.

But Alexandria's story does not stop there. We do not consider Alexandria's covariational reasoning as rooted in perceptual features, but we do consider her intuition for amounts of change as based on quick assimilations of perceptual features. For instance, an outward bend in a geometric shape was associated with increasing amounts of change in area. This can be seen from Alexandria's reasoning about the Spherical Cap Problem. In this problem, a student considers the changing surface area of a spherical cap as it grows from a point to a hemisphere. Alexandria initially conjectured a non-linear relationship between the height and surface area of the spherical cap. When asked why, she marked changes in surface area for equal changes in height as before, pointing to each strip, and concluded that the changes in surface areas were increasing.

We highlight Alexandria's intuition in this moment as demonstrating an assimilation of her prior experiences with covariational reasoning with dynamic geometric objects. In this case, however, Alexandria's conclusion was incorrect: the surface area of the spherical cap actually grows linearly with respect to its height. Her inferences, though rooted in covariational reasoning, were generalized in ways constrained by perceptual features. This limited the reliability and insightfulness of her intuition in a domain with unfamiliar features.

Alexandria's reasoning in the Spherical Cap Problem aligns with Azzouni's (2005) notion of *inference packages*. These packages are of assumptions "knit together" with representations of objects that can be used to explore how changes in circumstances might affect the representation. For Alexandria, this involved exploring how the surface area of the spherical cap changed as the cap increased in height. Alexandria's intuition relied on the assumptions she placed on her diagram of this phenomenon.

Azzouni notes that diagram-based reasoning enables one to operate with packages of assumptions to "quickly see what they imply" (p. 25). Of course, in doing so, the danger of using diagrams becomes apparent. It is up to the individual to recognize whether the assumptions they've placed on the diagram (in Alexandria's case, the impact of the widening of the sphere relative to the curvature of the sphere) are appropriate for the given objects.

It was not until later in the clinical interviews that Alexandria identified a way to link her imagery with formal reasoning in a way that supported better inferences about a range of geometric objects. At this later stage, Alexandria's reasoning no longer relied on perceptual features. Instead it made use of a formula for surface area, and the relationships made evident by cognitively uniting and varying specific quantities while other symbols remain constant.

This style of intuitive reasoning is more reliable, because it is rooted in the structure of the relevant formula rather than in imprecise visual assessment. It also proved more insightful, as it allowed Alexandria to move from viewing formulas as fixed entities associated with a single type of problem to understanding them as flexible vehicles for representing a variety of situations, depending on which quantities are regarded as constant or varying. Alexandria thus made concrete progress toward more mature intuition over the course of the interviews.

This discussion points to two sites for further research on the development of intuition at all levels of education. On the one hand, researchers need to better understand the nature of students' initial, untutored intuitions as developed through personal experience and informal transmission (e.g., in the form of p-prims or primary notions). On the other hand, more work is needed on the evolution of intuition alongside the introduction of formal mathematics and reasoning strategies (e.g., definitions and covariational reasoning).

This concludes our general case for the studyworthiness of skilled mathematical intuition. We turn to more concrete problems in the next sections, beginning with the relevance of intuition to an epistemological debate about the nature of understanding.

4 Intuition and the epistemology of understanding

4.1 Intuition as a factor in objectual understanding

A longstanding goal for epistemologists is to characterize the mental states which constitute or contribute to understanding. Scientific understanding in particular has been on philosophers' agenda for at least fifty years (cf. Friedman, 1974), and the new millennium has seen a significant increase in work on understanding in mathematics (Avigad, 2008, 2022; Cellucci, 2015; D'Alessandro, 2017, 2023; D'Alessandro & Lehet, 2024; Hamami & Morris, 2024; Heinzmann, 2022; Tappenden, 2005).

It is common in these debates to distinguish between two types of understanding[9], each of which plays a unique role in inquiry and is plausibly grounded in a distinctive set of mental states (Baumberger et al., 2017). So-called *objectual* understanding is the epistemic achievement we attribute to someone when we say, e.g., that they understand first-year calculus or the graphs of quadratic functions. We attribute *explanatory* understanding, on the other hand, when we say that someone understands why the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges or the reason every quadratic polynomial has exactly two complex roots. In general, objectual understanding involves familiarity with or mastery of some subject matter, topic or thing, while explanatory understanding involves grasping information relevant to some explanation-seeking question.

The relationship between objectual and explanatory understanding is a subject of controversy[10], but this debate will not be our concern here. Our primary target in this section is objectual understanding. We claim that, other things being equal, possessing better intuition with

[9] A third variety sometimes mentioned is *propositional* understanding, or understanding-that, as in "Jane understands that there's only one even prime". Since "*S* understands that *P*" plausibly expresses the same meaning as "*S* knows that *P*", propositional understanding has drawn less attention from philosophers as a distinctive type of epistemic achievement. Other notions relatively far from our core interests here include conceptual understanding (e.g. "Jane understands what a prime number is") and linguistic understanding (e.g. "Jane understands the question 'Are there infinitely many primes?'").

[10] See for instance Grimm, 2010; Khalifa, 2013; D'Alessandro & Lehet, forthcoming.

respect to some subject *S* entails objectually understanding *S* to a greater degree. Defending this claim positions us to challenge received wisdom and stake out a novel type of view, based on an expansive picture of the cognitive faculties implicated in understanding.[11]

Let us start with the relationship between intuition and objectual understanding. We present three arguments for the claim that the former contributes to the latter.

First, an argument from mathematical practice. Reflective mathematicians often emphasize the centrality of understanding to the epistemic aims of mathematics. Intuition features prominently in many of these accounts. Solomon Feferman, for instance, writes with characteristic clarity that "no less than the absorption of the techniques of systematic, rigorous, logically developed mathematics, *intuition is necessary for the understanding of mathematics*" (2000, p. 319; emphasis in original). Feferman is explicitly speaking here of mature intuition, "cultivated through training and practice", as opposed to "innate, 'raw', untutored intuition" (p. 318).

Likewise, §3 above discussed Thurston's remarks about the role of intuition in his pioneering topological work. Thurston situates these claims in a general epistemological framework which identifies mathematical progress with "advanc[ing] human understanding of mathematics" (162). The growth of understanding, in turn, depends on the successful cultivation of intuition (alongside linguistic, visual and deductive cognitive skills).

Mathematicians' discussions of intuition and understanding often adopt an explicitly pedagogical framing. Poincaré "insist[ed] on the place intuition should hold in the teaching of the mathematical sciences", claiming that "[w]ithout it young minds could not make a beginning in the understanding of mathematics" (1900, IV, p. 1017). Gosztonyi attributes to the Karácsony circle[12] the view that "[w]ithout [intuition and experience], neither mathematical creation, nor real understanding can be achieved, so it is important to develop intuition with the help of a handful of experiences in every level of education" (2016, p. 87). Similarly, Barnett writes that "students can be expected to gain intuition, and thereby understanding, through the acquisition of experience, a view affirmed by research on the learning process" (2000, pp. 82–83).

As experts deeply acquainted with the acquisition of understanding, these mathematicians are important sources of evidence. So those who deny intuition a role in understanding will have to explain why the motivations for their views are more credible than the picture suggested by mathematical practice.

---

[11] Connections between intuition and understanding have received some prior attention. For example, Bengson, 2015 describes a role for "sense-making" acts of intuition in the process of coming to understand (i.e., transitioning from the absence to the presence of understanding). As far as we know we're the first to address the relationship between intuition in our sense and objectual understanding.

[12] An influential group of 20th-century Hungarian thinkers with views on pedagogy and education reform, centering on the philosopher-psychologist Sándor Karácsony and including prominent mathematicians such as Kálmar and Péter.

Compelling though such testimonial evidence may be, however, its indirectness leaves something to be desired for epistemological purposes, since it does little to explain precisely *how* intuition contributes to objectual understanding. A direct argument would help make our case more vivid and theoretically informative. We offer two.

First, it's widely agreed that the possession of objectual understanding is associated with certain characteristic abilities, so that displaying many of the latter is a good indicator of the former. We find this idea plausible. (Some authors make the stronger claim that these abilities actually constitute understanding; see de Regt & Dieks, 2005; Avigad 2008; Delarivière & Van Kerkhove, 2021. We see no need to go this far, but for present purposes there is no need to decide the issue.)

Some skills linked with objectual understanding include the ability to recognize characteristic consequences of a relevant theory, to make correct counterfactual inferences, to relate one's knowledge on the topic in question to other relevant facts, to respond successfully to challenges to one's beliefs, to identify key features of a situation or problem, and to search effectively through a large space of possibilities (Delarivière & Van Kerkhove, 2021, p. 644). Mature intuition as we have characterized it is intimately related to performance on inferential and recognitional tasks like these. So it should be clear that, other things being equal, agents with superior intuition will possess such abilities to greater degrees.

Second and more conceptually, we take understanding a subject *S* to be closely related to notions like *mastery* of *S* and *cognitive proficiency* with respect to *S*. These notions, in turn, indicate a role not only for the contents of one's mental states at a particular time, nor even just one's dispositions to make correct inferences or learn new facts, but additionally for the fluency, assurance, insight and flexibility with which these contents can be accessed and these dispositions activated. This latter condition suggests something like mature intuition as an ingredient of objectual understanding.

4.2 Against epistemicism

Let *epistemicism* be the view that the possession of objectual understanding at a time is completely determined by one's epistemic state at that time, where an agent's epistemic state consists of her knowledge, beliefs, credences and related truth-directed attitudes.[13] "Epistemic" in this sense contrasts with "cognitive-psychological": whether an agent believes that $\sqrt{2}$ is irrational is part of her epistemic state, but not, for instance, whether she has a vivid imaginative representation of $\sqrt{2}$ as the diagonal

---

[13] In epistemology, a *belief* is usually understood as a state of outright commitment to the truth of a proposition (which one can model with the binary values 0 = disbelief and 1 = belief), while a *credence* is a state assigning a likelihood to the truth of a proposition (which one can model with values in the interval [0,1]; I might have credence 0.5 that a flipped coin will land on heads, for example).

length of the unit square, which definition of 'irrational' she finds easiest to remember, or whether her on intuition suggests that $\sqrt{n}$ should be irrational for other values of n.

Epistemicism has long been a popular view. Indeed, a number of philosophers endorse a strong form of epistemicism according to which understanding reduces to knowledge alone. On Christoph Kelp's view, for instance, maximal objectual understanding of a phenomenon *P* is "fully comprehensive and maximally well-connected knowledge" about *P*, with intermediate degrees of understanding corresponding to better or worse approximations to this limit (2017, p. 252). Kareem Khalifa suggests in a similar vein that "ideal understanding is maximally scientific knowledge of a complete explanatory nexus" (2017, p. 15).[14]

We think epistemicism in any of these forms fails to tell the whole story. On our view, objectual understanding is a kind of multidimensional cognitive proficiency, which can be advanced not only by gains in knowledge but by "broad-spectrum improvement[s] to a variety of epistemic states (belief, credence, expectation, attention, inquiry) and cognitive functions (reasoning, intuition, similarity-detection, problem-solving)" (D'Alessandro, 2023, p. 16).

The previous section made the case that mature intuition is a component of mathematical understanding. But this point alone is not enough to settle the epistemicism issue. Those who endorse a non-psychologistic picture can, after all, insist that intuition itself is reducible to (or a mere byproduct of) knowledge or other epistemic states. We haven't convincingly refuted epistemicism until we have thwarted this maneuver.

To this end, consider the following scenarios:

[Comparison] Yann and Zev have taken a first-year calculus course in which they have both learned precisely the same facts. (Improbable, yes; but let's suppose and see what follows, as philosophers are wont to do.) For one reason or other—because Yann has grasped these facts more deeply or internalized them more thoroughly, or perhaps because this sort of thing comes more easily to him for idiosyncratic neuropsychological reasons—Yann now possesses good intuitions about elementary calculus. (He can, for instance, quickly identify promising strategies for evaluating unfamiliar integrals, appropriate methods for modeling real-world problems with calculus techniques, and so on.) Meanwhile, Zev has little intuitive sense for these things.

[14] Khalifa's account of objectual understanding involves knowledge of explanatory relations because, on his view, objectual understanding is a species of explanatory understanding.

[Extremes] Alpha is a sentient AI system with an unusual cognitive architecture: it has no stored belief-like, knowledge-like or credence-like states at all[15], but if prompted with a mathematical question, it finds itself with a strong feeling about the correct answer. These answers turn out to be mostly correct. Alpha has no introspective access to the source of its intuitions. In fact, they're generated by a powerful cognitive module trained on a large mathematical corpus, which Alpha can access only in this way.

Meanwhile, Omega is an AI with a different architecture. It knows all the mathematical facts, and all the logical and explanatory relationships between these facts. But its beliefs enjoy no intuitive backing: on reflection, Omega does not find even the simplest mathematical truths cognitively compelling, even in light of all the other facts it knows.

Our claims: In [Comparison], Yann understands calculus to a greater degree than Zev even though their epistemic states are identical. In [Extremes], Alpha's level of objectual understanding is nonzero in spite of its lack of beliefs and knowledge, while Omega's is nonmaximal in spite of its impeccable epistemic state.

Prima facie, we find these claims quite appealing. But epistemicists could dispute them in a couple of ways.

A first potential objection: contrary to what we have claimed, the cases as described do (and must) involve belief and knowledge states consistent with epistemicism. For Yann and Zev to differ with respect to the quality of their intuition just *is* for Yann to know more, and for Alpha to reliably intuit truths just *is* for it to have many accurate mathematical beliefs. Perhaps the most promising way to defend such a position is by appealing to *dispositionalism* about belief: roughly, the view that an agent believes that *P* just in case they are disposed to act as though *P* were true, even if they have no occurrent mental state whose content is *P* (Ryle, 1949; Marcus, 1990; Schwitzgebel, 2002).

We find this an unpromising analysis of the cases, however. Following Audi, we might say that Yann has a *disposition to believe* rather than a non-occurrent belief in the dispositionalist's sense. "[W]hereas a belief is—at least in good part—a (state of) readiness to act in certain ways appropriate to its content, at least by affirming the proposition believed, a disposition to believe is a readiness to form a belief" (Audi, 1994, pp. 423-4). Armed with well-trained intuition, Yann is poised to reliably find the answers to various calculus problems; prior to actually doing so, however, he's in no way ready to act as though he already knows the answers. So ascribing Yann knowledge or beliefs beyond Zev's looks

[15] Or, if you like, Alpha has the minimal set of such states necessary to be sentient and engage in the activity described here.

implausible. Similarly, Alpha's mathematical intuitions guide its behavior only in the limited context of responding to others' questions. This is surely too slender a dispositional basis to hang a belief ascription on.

An alternative strategy for the epistemicist is to grant our verdicts about [Comparison] and [Extremes], while maintaining that these verdicts—based as they are on the fine points of highly unusual artificial cases—tells us little about the relationship between intuition and understanding in real-world scenarios. On this view, while intuition may have independent epistemic value in principle, actual gains in understanding are almost always mediated by gains in knowledge and the like. So teachers and education researchers can safely act as though epistemicism were true.

On the contrary, we think the falsity of epistemicism has important practical consequences. The case of chess makes this point particularly vivid. Chess commentators note a distinction between intuitive and analytical play at all levels of competition, classifying many players as characteristically favoring one style or the other. The current world champion Magnus Carlsen is a celebrated intuitive player, to such a degree that chess observers "wonder how a player like Carlsen is so good at finding good moves, often without much calculation, while weaker grandmasters cannot find them despite considerable thinking" (Gobet, 2019, p. 70). Comparing Carlsen with then-second-ranked player Levon Aronian, for instance, former champion Viswanathan Anand credited Carlsen with "an incredible innate sense... The majority of ideas occur to him absolutely naturally. He's also very flexible... I think Aronian's a much more tactical player than Carlsen. He's always looking for various little tricks to solve technical tasks" (Anand, 2012).

We find several claims plausible. First, Carlsen's success is indicative of an extraordinary level of understanding. Second, Carlsen's understanding is in part constituted by highly developed intuition. Finally, Carlsen's unique intuitive strength is not plausibly attributable to a uniquely large amount of knowledge or other purely epistemic advantages.

We think the first claim needs little defense[16], and we can assume the second for dialectical purposes (since our epistemicist opponent concedes that intuition contributes to understanding, disagreeing with us only about whether it does so independently under realistic conditions). This leaves only the third claim. Is Carlsen's intuition better than other grandmasters' merely because he knows more, or is otherwise in a better epistemic state?

The biographical evidence suggests not. Carlsen picked up chess late compared to many top players, beginning serious play only at age eight. He is also known to spend relatively little time studying and preparing for games ("he prefers watching or playing football" (Gobet, 2019, p. 48)). Gobet & Ereku (2014) estimated the amount of practice time accumulated by then-top players,

[16] Carlsen has been the top-ranked player since 2010, is the youngest player ever to reach the top rank, and holds the highest peak Elo rating in history and the longest unbeaten streak in high-level play, among other achievements.

finding that Carlsen had likely trained least or second-least of the 11 highest-ranked (with 18 years of practice compared, for instance, to third-ranked Grischuk's 26 years or sixth-ranked Kramnik's 33).

It is conceptually possible that Carlsen managed to acquire much more knowledge than other top players in spite of having trained less intensively for many fewer years. Considering, however, that players at this level all possess extraordinary memories, strong general cognitive abilities, access to the best coaches, partners and study materials, and so on, this epistemicism-friendly hypothesis seems doubtful. A more plausible explanation is that Carlsen has comparable levels of knowledge to other elite grandmasters but more efficient, reliable and insightful intuitive faculties.

Chess is a useful anchor for claims about intuition on account of its relative straightforwardness and clear metric of success. But we expect these points to generalize to mathematics, since learning and expert performance in the two domains involve similar uses of a largely shared set of cognitive mechanisms.

Even in interesting real-world scenarios, then, the quality of one's intuition need not be fully determined by the amount of knowledge one has. So this second defense of epistemicism also fails.

Our discussion here has illustrated one way in which the study of intuition can have important philosophical consequences. We think this is far from the only example. The so-called *zetetic turn* in recent epistemology, focusing on the nature, norms and goals of the process of inquiry, has opened an expansive arena in which cognitive-psychological considerations can inform philosophical work (Friedman, 2020; Thorstad, 2021). And of course philosophers of mathematics will have much to do in characterizing the roles of mature intuition in mathematical practice.

5 Intuition in mathematics education

In this final section, we turn to implications of the study of intuition for mathematics education research.

Previously, we noted how Alexandria's intuitive reasoning was rooted in quick assimilations of perceptual features. However, in that example, we also identified Alexandria's covariational reasoning as productive in most situations. Here, we expand on this idea and illustrate (i) what types of developmental intuitions exist and (ii) how we might support students' reliable mathematical intuitions, such as those we saw from Alexandria by the end of her clinical interviews.

5.1 Image and intuition

The notions of *concept image* and *concept definition* which Tall & Vinner (1987) introduced to the field of mathematics education point to the idea that students' accumulations of formal definitions are insufficient for characterizing students' conceptions. In their discussion of students' conceptions of limits, for instance, we see that understanding the identity $\frac{1}{3} = .3333...$ may rely on a formal definition of limits, but may also rely on more informal notions of repetition, or on visually imagining magnitudes approaching a single value but never reaching it. This characterization of knowledge supports the argument we are making that objectual understanding can involve various explanatory understandings.

Similarly, Thompson (1996) described a student's image as containing not only the diagrams and other visual/sensory materials at their disposal, but also the mental abstractions they make from these materials. Such abstractions are typically influenced by sensorimotor experiences involving both conscious and subconscious understandings of their environments.

These authors are just a few of many education researchers who recognize that students' conceptions consist of more than a collection of stored facts, and that learning involves more than the accumulation of more such facts. The role of intuition in this process has remained ambiguous, though education researchers have certainly described examples of students providing intuitive explanations. For instance, Weber and Alcock (2004) describe a student's instantiation of group isomorphisms from a re-ordering of multiplication tables as an "intuitive instantiation", in contrast to students who could report a formal definition of an isomorphism. Roh & Lee (2017) describe students' initial intuitions about convergence as, for instance, values approaching but never reaching a single value. They support students in developing more refined and formal definitions of convergence that rely more heavily on the formal definition of convergence (e.g., considering intervals of an arbitrary size ε). Lastly, Zagoianakos & Shvarts (2015) describe a student physically creating mathematical representations (e.g., perpendicular lines with arms). In doing so, the authors intend to establish the importance of embodiment in students' reasoning to support intuitive generalizations, building on the relevance of students' gestures and the kinesthetic experiences associated with mathematics (see deFreitas & Sinclair, 2012) . A prevalent theme in these education researchers' descriptions of intuitive reasoning and others' (including in educational psychology) is the existence and usefulness of metaphors in students' intuitive reasoning (see also Dawkins, 2012; Raidl & Lubart, 2001).

Overall, researchers have looked favorably on the previous examples of intuitive reasoning, with some claiming that this style of thinking shows a more developed or conceptual understanding of the target ideas. Indeed, education research generally views intuition in a positive light, though it also recognizes potential risks in students overgeneralizing or relying on unsupported intuitive guesses. (P-prims, in particular, are often flagged with these kinds of warnings.)

Efforts to support the growth of students' intuition would be simplified if researchers had access to validated intuition-measuring instruments. Since such instruments do not exist at present, the path forward is less obvious. In the following section, we draw on existing findings on intuition in mathematics education to offer some preliminary suggestions about the development of intuitive reasoning.

5.2 Limitations in existing classifications of intuition

We have already established that mathematicians consider intuition an important part of their reasoning. We have also established that mathematicians consider mature intuition as developing only with time and effort. But "how?" is a question for education researchers. Does mature intuition develop from naïve intuition? Is naïve intuition a necessary prerequisite for mature intuition? Is it possible to achieve well-developed intuition without an awareness of formal definitions and proofs? In an effort to engage with existing literature, we briefly remark on how others have addressed the development of intuitions.

Certainly education researchers have identified students who have exhibited intuitive reasoning prior to systematic instruction. These are classified as *primary intuitions* (Fischbein, 1987). However, Keene et al. (2014) are only willing to conclude that primary intuitions may serve to help students construct more formal conceptions, of say a limit. Thus, though these primary intuitions may support the construction of additional objectual knowledge, it does not seem the authors are willing to claim that primary intuitions may develop into mature intuitions. On the other hand, Roh & Lee (2017) showed some promise in this realm by enacting an instructional sequence to support students in building primary intuitions via activities that intentionally ask the students to perturb and engage with their meanings for convergence while introducing a formal definition of convergence. Though not necessarily evidence of what we consider to be mature intuition, the study provides initial insights into supporting students in transforming their initial intuitions. This approach lies in contrast to recommendations made to ask students to ignore or doubt their primary intuitions (or p-prims, etc.).

This section has indicated the distinction in knowledge of a formal definition and the images surrounding the concept as useful to characterize objectual knowledge with surrounding explanatory knowledge. It has illustrated several examples of how education researchers view the role of intuition in learning formal definitions. In doing so, we have also identified that these education researchers either (i) do not mention the role of intuition in knowledge or (ii) use intuition to develop more formal conceptions but not explicitly to develop intuition. In the following section, we describe what mature intuition, as we have defined it, entails. We then use these characteristics to inform how intuition might develop.

5.3 Developing intuition

As noted above, though characterizations of intuition have been operationalized in the literature, less research has focused on the development of intuition, and there is no validated instrument to assess the maturity of intuition. (It is also important to note that the development of intuitions, particularly of p-prims, may be influenced by one's culture, geography and other contextual factors.) However, there are researchers who have analyzed reasoning they deem intuitive and who offer recommendations for improving intuitive reasoning. We synthesize and expand on four of these strategies: *adding imagery*, *building associations*, *developing confidence* and *generalizing concepts*.

Add imagery. When mathematicians reflect on their own intuitions, there is often a description of "seeing" something mentally. Whether it is viewing a chessboard (or other physical object), imagining curves of a graph (or other mathematical representation), or looking at a group of symbols (or other formal representation), much intuitive reasoning is associated with an image. Hogarth (2001) identifies images as important tools for educating intuition because they provide a source of feedback for the learner.

Moreover, the ability to gain new insights into a concept often requires manipulating images. With Alexandria's example with the spherical cap, we recognize the limitations of relying on representations, even when quantitative reasoning is used. Azzouni (2005) admits this as well, stating: "mathematicians (rightfully) were and are suspicious of the use of intuition" but also "successful mathematical work *can't* proceed without employing such intuitions" (p. 22). This manipulation may involve mentally manipulating objects themselves (e.g., moving a point on a line, or pieces on a chess board) or cognitively manipulating the role of an object (e.g., identifying a function that will select elements of a set).

The former idea, mentally manipulating objects, is akin to De Toffoli and Giardino's (2014) notion of manipulative imagination; they argue that novice students need to train their imagination to engage in manipulations that are both possible and effective. Watson and Mason (2008) echo this idea, noting that students' imagery (whether symbols, objects, or concepts) can either be *limited* by an overreliance on prototypical examples or else *supported* by careful attention to the possible dimensions of variation. Even mathematicians can identify limitations in intuitive reasoning. "We have here an excellent example of the value and danger of intuitional reasoning. On the credit side is the fact that it led Green to a series of important discoveries since well established. On the debit side is its unreliability, for there are, in fact, regions for which Green's function does not exist" (Kellogg, 1929, pp. 237-238). Nevertheless, as Azzouni argued, progress would not have been made if Green and others did not follow his intuitions.

The second idea—that of cognitively manipulating the role of objects—aligns with *interiorization*, in which individuals can represent and manipulate structures abstractly without the need for perceptual material (von Glasersfeld, 1982). Fischbein (1987) notes that "later stages" of intuition, which more closely resemble our definition of mature intuition, "are more and more 'abstracted' from material action and are more and more independent from any form of representation" (p. 67).

Build associations. We previously discussed Thompson's definition of image and Tall & Vinner's concept image, which collectively involve the set of mental associations, pictures, experiences, reference examples and the like which an individual associates with a given concept. Intuition draws on these networks, and mature intuition requires a particularly detailed and accurate complex of associations to fuel its far-ranging insights. Arming oneself only with formal definitions, by contrast, leaves intuition with little to work with: "unless we introduce some concept to talk about—it is difficult to find anything to say at all" (Wilder, 1952, p. 19).

Another important notion in this vicinity is that of a *schema* (Piaget, 1952): roughly speaking, a structured mental representation of a topic that facilitates remembering, recognizing, predicting and reasoning about information related to that subject. Inglis & Mejía Ramos (2021) have argued that the creation and consolidation of high-quality schemas is essential to mathematical understanding, offering a psychological model of the sorts of proofs likely to make good use of one's schemas without imposing burdensome cognitive costs. We believe the possession of many rich and interconnected schemas is at least extremely useful for improving intuition.

Develop confidence. Fischbein describes how individuals engaging in intuitive reasoning often present a highly assured disposition. As he writes, "intuitions are specifically those cognitions in which overconfidence plays an essential role" (28).

The establishment of (at least temporary) subjective certitude extends beyond establishing the givens in a situation: Fischbein is describing a cognitive state in which an individual chooses no longer to question these givens. Hogarth describes this idea as "accepting the conflict in choice" (211). We suggest that mature intuition also requires exceeding the bounds of the givens to draw insightful inferences. Moreover, it requires allowing oneself to reason without stopping to question, at least within the regimes where intuition can be expected to operate successfully. Fischbein (1987) calls this the "double game"—intuitive reasoners must "know *in principle* that they may be wrong but they go on reasoning as if they were convinced that they are correct at every step" (pp. 37–38). Excess self-monitoring, self-questioning or overcontrolled cognition may limit one's capacity to reason intuitively.

Generalize concepts. In the previous section, we noted that mature intuition provides insights going beyond the bounds of one's explicit knowledge. If an individual is to gain a reliable insight, the imagery that an individual has around a concept must be generalizable in a way that affords reliable conclusions beyond specific instantiations. That is, rather than merely gathering more facts about a situation, an individual must acquire an image that expresses, as Fischbein says, "a general, necessary relation" (Fischbein, 1987, p. 18).

These images (in Thompson's sense of the word) are associated with both naïve and mature intuitions. If the meanings one constructs rely heavily on explicitly identified patterns, then the corresponding intuitions are likely to be more limited and less insightful (cf. Fischbein's "assimilatory intuitions"). Mature intuition, by contrast, is capable of deeper syntheses producing more genuinely novel conclusions. Bills and Rowland (1999) might consider at least some of these deeper generalizations as structural in nature, given their account of mathematicians' ability to isolate structural invariants to construct generic proofs from specific examples.

Fischbein perhaps offers an insight into how assimilatory intuitions may be developed to form novel insights (though we know of no empirical evidence for this). He states that considering non-intuitive situations may allow individuals to accommodate their existing structures to better support more reliable images. The idea is that by repeatedly pushing the bounds of generalizability, the relevant mental operations become more reliable and more automated. These mental operations would perhaps be the "lines of force" which Fischbein (1987, p. 64) states are necessary to understand how individuals engage in intuitive reasoning. The idea is unconscious mindfulness, similar to the notion of developing "circuit breakers" that Hogarth (1987) describes as necessary for educating intuition.

Whether or not this act of strengthening the generalizability of concepts is what mathematicians refer to when they indicate the length of time needed to construct more expert intuitions is still an open question. However, it does seem to align with some of what researchers who have studied generalization posit—that generalizing skills are linked to students' ability to anticipate how to approach novel problems (e.g., Ellis et al., 2021).

5.5 Concluding supports for developing intuitions

To conclude this section, we highlight two potential supports for developing mature intuition based on the recommendations Fischbein offered (1987) about developing intuition generally.

First, Fischbein describes the importance of allowing students to engage in practical, behaviorally meaningful situations. We consider these situations to fit this description if they allow for adding imagery, building associations, and providing students with opportunities to generalize concepts. The quantitative situations with dynamic geometry that Alexandria engaged in do so because (i) the students can construct an image of the dynamic situation and manipulate that imagery (*adding imagery*) to answer a question of interest (*building associations*). As Alexandria continued to engage in tasks that asked her to engage in similar reasoning (i.e., covariational reasoning) across new contexts, she began to create productive generalizations (*generalize concepts*) of formula structure to quickly, fluently, and reliably respond to novel tasks.

Second, Fischbein describes the importance of developing "alarm devices" which serve as "self-control schemas". These alarm devices are developed from conscious activity, and they ideally develop into automated ways of triggering the mind to avoid certain inferences or assumptions. For Alexandria, the spherical cap problem was her attempt to learn the viability of relying on perceptual features of the geometric context at hand to construct area formulas. Thus, potentially, when describing her formula, she may have had an alarm device telling her not to return to perceptual features when describing her formula as she moved to new contexts. This may have also supported her in exploring a new avenue of formula structure in which she could *build confidence* and *generalize concepts*.

Overall, this section has characterized the development of mature intuition as involving (i) adding imagery, (ii) building associations, (iii) developing confidence and (iv) generalizing concepts. We have also demonstrated how these actions could be supported via engaging in practical, behaviorally meaningful situations and developing "alarm devices", using Alexandria's engagement in the clinical interviews to demonstrate how this might be operationalized.

Conclusion

This paper has attempted to renew the discussion of intuition in mathematics. The myriad of classifications of intuition in use continues to highlight the importance philosophers, psychologists, and education researchers have for the construct of intuition. Nevertheless, this collective body of research has not provided either sufficient empirical insights into its development nor clear ideas about its epistemological significance.

We have attempted to reignite interest in this research agenda by offering an account of expert mathematical intuition and demonstrating its connections with several problems of interest. Epistemologists, for instance, have reason to care about intuition on account of its relevance for debates about understanding. Education researchers should note its role in the development of mathematical maturity and its connections with key theoretical frameworks. And scholars of mathematical practice from all disciplines ought to take notice of its well-attested roles in research and pedagogy.

Our efforts here have, of course, only revealed the tip of a large iceberg. We trust it cannot remain submerged forever. There's much more to be learned on all sides from further work on intuition in mathematics (and elsewhere).

Acknowledgements

We are grateful to Juan Pablo Mejía-Ramos and Keith Weber for organizing the Understanding Mathematical Explanation conference at Rutgers University in May 2023, which led to our meeting and this collaboration. We would also like to thank Jessi Lajos for her support and assistance with the intuition literature, as well as Silvia De Toffoli and two anonymous reviewers for their excellent feedback. Finally, we thank the NSF under Grant No. (DRL-1350342) for enabling the research described with Alexandria. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF.

Bibliography

Anand, V. (2012). "Anand's WhyChess interview." Interview by Vladislav Tkachiev. URL = https://www.chessintranslation.com/2012/05/anands-whychess-interview/. Accessed May 22, 2024.

Avigad, J. (2008). Understanding proofs. In Paolo Mancosu (ed.), *The Philosophy of Mathematical Practice*. New York: Oxford University Press.

Avigad, J. (2022). Varieties of mathematical understanding. *Bulletin (New Series) of the American Mathematical Society*, *59*, 99–117.

Azzouni, J. (2005). Is there still a sense in which mathematics can have foundations? In G. Sica (ed.) *Essays on the Foundations of Mathematics and Logic*. Monza, Italy: Polimetrica International Scientific Publisher, 9–47.

Barnett, J. H. (2000). Anomalies and the development of mathematical understanding. In Victor J. Katz (ed.), *Using History to Teach Mathematics: An International Perspective*, Washington, D.C.: Mathematical Association of America, 77–88.

Baumberger, C., Beisbart, C. & Brun, G. (2017). What is understanding? An overview of recent debates in epistemology and philosophy of science. In S. Grimm, C. Baumberger and S. Ammon (Eds.), *Explaining Understanding: New Perspectives from Epistemolgy and Philosophy of Science*, New York: Routledge, 1–34.

Bengson, J. (2015). A noetic theory of understanding and intuition as sense-maker. *Inquiry: An Interdisciplinary Journal of Philosophy*, 58 (7-8), 633-668.

Beth, E. W., and Piaget, J. (1966). *Mathematical Epistemology and Psychology*. Dordrecht: Reidel.

Bills, L., & Rowland, T. (1999). Examples, generalisation and proof. *Advances in Mathematics Education*, 1(1), 103–116. https://doi-org.uri.idm.oclc.org/10.1080/14794809909461549

Bubp, K. (2014). To prove or disprove: The use of intuition and analysis by undergraduate students to decide on the truth value of mathematical statements and construct proofs and counterexamples. [Doctoral dissertation, Ohio State University]. OhioLINK. URL = <http://rave.ohiolink.edu/etdc/view?acc_num=ohiou1417178872>.

Burton, L. (1999). Why is intuition so important to mathematicians but missing from mathematics education?. *For the Learning of Mathematics*, *19*, 27–32.

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352–378.

Carter, J. (2019). Philosophy of mathematical practice—motivations, themes and prospects. *Philosophia Mathematica*, *27*, 1–32.

Cellucci, C. (2015). Mathematical beauty, understanding, and discovery. *Foundations of Science*, *20*, 339–355.

Chudnoff, E. (2014). Intuition in mathematics. In L. Osbeck and B. Held (eds.), *Rational Intuition*, Cambridge: Cambridge University Press.

Chudnoff, E. (2019). In search of intuition. *Australasian Journal of Philosophy*, *98*, 465–480.

Chudnoff, E. (2020). *Forming Impressions: Expertise in Perception and Intuition*. Oxford: Oxford University Press.

Clement, J. 2000. Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 547–589). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Dawkins, P. C. (2012). Metaphor as a possible pathway to more formal understanding of the definition of sequence convergence. *The Journal of Mathematical Behavior*, *31*(3), 331–343.

D'Alessandro, W. (2017). Proving quadratic reciprocity: Explanation, disagreement, transparency and depth. *Synthese* 198, 8621–8664.

D'Alessandro, W. (2023). Unrealistic models in mathematics. *Philosophers' Imprint* 23 (article 27), DOI: 10.3998/phimp.1712.

D'Alessandro, W. & Lehet, E. (2024). A noetic account of explanation in mathematics. *Philosophical Quarterly*, DOI: 10.1093/pq/pqae137.

deFreitas, E. & Sinclair, N. (2012). Diagram, gesture, agency: Theorizing embodiment in the mathematics classroom. *Educational Studies in Mathematics*, 80(1–2), 133–152.

Delarivière, S. & Van Kerkhove, B. (2021). The mark of understanding: In defense of an ability account. *Axiomathes*, 31, 619–648.

De Toffoli, S. (2021). Reconciling *rigor and intuition*. *Erkenntnis*, 86(6), 1783-1802.

De Toffoli, S. & Giardino, V. (2014). Forms and roles of diagrams in knot theory. *Erkenntnis*, 79(4), 829–842.

de Regt, Henk, & Dieks, D. (2005). A contextual approach to scientific understanding. *Synthese*, *144*, 137–170.

diSessa, A. A. (1988). Knowledge in pieces. In G. Forman & P. Pufall (Eds.), *Constructivism in the computer age* (pp. 49–70). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

diSessa, A. A. (2018). A Friendly Introduction to 'Knowledge in Pieces': Modeling Types of Knowledge and Their Roles in Learning." In Kaiser, G., Forgasz, H., Graven, M., Kuzniak, A., Simmt, E., Xu, B. (Eds.) *Invited Lectures from the 13th International Congress on*

Mathematical Education. ICME-13 Monographs. Springer, Cham. https://doi.org/10.1007/978-3-319-72170-5_5 10.1007/978-3-319-72170-5_5.

Ellis, A. B., Lockwood, E., Tillema, E., & Moore, K. (2021). Generalization Across Multiple Mathematical Domains: Relating, Forming, and Extending. *Cognition and Instruction*, *40*(3), 351–384. <https://doi.org/10.1080/07370008.2021.2000989>

Feferman, S. (2000). Mathematical intuition vs. mathematical monsters. *Synthese*, *125*, 317–332.

Fischbein, E. (1987). *Intuition in Science and Mathematics: An Educational Approach*. Dordrecht, Holland: D. Reidel. Publishing Company.

Friedman, J. (2020). The epistemic and the zetetic. *Philosophical Review*, *129*, 501–536.

Friedman, M. (1974). Explanation and scientific understanding. *Journal of Philosophy*, *71*, 5–19.

Gigerenzer, G. (2008). *Gut Feelings: The Intelligence of the Unconscious*. New York: Penguin.

Gobet, F. (2019). *The Psychology of Chess*. New York: Routledge.

Goldenberg, P., & Mason, J. (2008). Shedding Light on and with Example Spaces. *Educational Studies in Mathematics*, *69*(2), 183–194. https://doi.org/10.1007/s10649-008-9143-3.

Goldstein, S. & Levinstein, B. (MS). Does ChatGPT have a mind? *arXiv*: https://arxiv.org/abs/2407.11015v1.

Gosztonyi, K. (2016). Mathematical culture and mathematics education in Hungary in the XXth century. In Brendan Larvor (Ed..), *Mathematical Cultures: The London Meetings 2012–2014*, Basel: Birkhäuser.

Grimm, S. R. (2010). The goal of explanation. *Studies in History and Philosophy of Science Part A*, *41*, 337–344.

Hamami, Y. & Morris, R. L. (2020). Philosophy of mathematical practice: A primer for mathematics educators. *ZDM—Mathematics Education*, *52*, 1113–1126.

Hamami, Y. & Morris, R. L. (2024). Understanding in mathematics: The case of mathematical proofs. *Noûs*, DOI: 10.1111/nous.12489.

Heinzmann, G. (2022). Mathematical understanding by thought experiments. *Axiomathes* 32, 871–886.

Hersh, R. (1997). *What Is Mathematics, Really?* Oxford: Oxford University Press.

Hogarth, R. (2001). *Educating Intuition*. Chicago: Chicago Press.

Inglis, M. and Mejía Ramos, J. P. (2021). Functional explanation in mathematics. *Synthese* 198 (Suppl. 26), S6369–S6392.

Lajos, J. (2023). An updated conceptualization of the intuition construct for mathematics education research. *Journal of Mathematical Behavior*, *71*, 1–19.

Kahneman, D. (2011). *Thinking, Fast and Slow*. New York: Farrar, Straus and Giroux.

Keene, K. A., Hall, W., & Duca, A. (2014). Sequence limits in calculus: Using design research and building on intuition to support instruction. *Zentralblatt für Didaktik der Mathematik (ZDM)*, *46*(4), 561–574.

Kellogg, O.D. (1929). Electric Images; Green's Function. In *Foundations of Potential Theory. Die Grundlehren der Mathematischen Wissenschaften*, Vol. 31. Springer, Berlin, Heidelberg. https://doi.org/10.1007/978-3-642-90850-7_9

Kelp, C. (2017). Towards a knowledge-based account of understanding. In Stephen Grimm, Christoph Baumberger and Sabine Ammon (Eds.), *Explaining Understanding: New Perspectives from Epistemology and Philosophy of Science*, New York: Routledge, 251–271.

Khalifa, K. (2013). Is understanding explanatory or objectual? *Synthese*, *190*, 1153–1171.

Klein, F. (1911). On the mathematical character of space-intuition and the relation of pure mathematics to the applied sciences. In William Ewald (Ed.), 1996, *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*, Volume II, Oxford: Clarendon Press.

Klein, G. (1998). *Sources of Power: How People Make Decisions.* Cambridge, MA: MIT Press.

Kline, M. (1980). *Mathematics: The Loss of Certainty*. Oxford: Oxford University Press.

Lakatos, I. (1978). What does a mathematical proof prove? In Worrall, J. & Currie, G. (eds.), *Mathematics, Science and Epistemology*. Cambridge University Press, 61-69.

Mancosu, P. (Ed.). (2008). *The Philosophy of Mathematical Practice*. Oxford: Oxford University Press.

Moore, K. C., Stevens, I. E., Paoletti, T., Hobson, N. L. F., & Liang, B. (2019). Pre-service teachers' figurative and operative graphing actions. *The Journal of Mathematical Behavior*. <https://doi.org/10.1016/j.jmathb.2019.01.008>

Osbeck, L. M. (1999). Conceptual problems in the development of a psychological notion of 'intuition'. *Journal for the Theory of Social Behaviour*, 29, 229–250.

Poincaré, J. H. (1894). On the nature of mathematical reasoning. Translated by Halsted, G.B., in Ewald, W. (ed.), 1996, *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*, Volume II, Oxford: Clarendon Press.

Poincaré, J. H. (1900). Intuition in logic and mathematics. Translated by Halsted, G.B., in Ewald, W. (ed.), 1996, *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*, Volume II, Oxford: Clarendon Press.

Pust, J. (2019). Intuition. In Zalta, E.N. (ed.), *The Stanford Encyclopedia of Philosophy* (Summer 2019 Edition), URL = <https://plato.stanford.edu/archives/sum2019/entries/intuition/>.

Raidl, M. H., & Lubart, T. I. (2001). An empirical study of intuition and creativity. *Imagination, Cognition and Personality,* 20(3), 217–230.

Roh, K. H., & Lee, Y. H. (2017). Designing tasks of introductory real analysis to bridge a gap between students' intuition and mathematical rigor: The case of the convergence of a sequence. *International Journal of Research in Undergraduate Mathematics Education*, 3, 34–68.

Smithies, Declan. (2013). The nature of cognitive phenomenology. *Philosophy Compass*, *8*, 744–754.

Stevens, I. E. (2019). "Pre-service teachers' constructions of formulas through covariational reasoning with dynamic objects" [Unpublished Doctoral Dissertation]. University of Georgia.

Tall, D. O., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151–169.

Tappenden, J. (2005). Proof style and understanding in mathematics I: Visualization, unification and axiom choice. In P. Mancosu, K. Jørgensen and S. Pedersen (Eds.), *Visualization, Explanation and Reasoning Styles in Mathematics*, Berlin: Springer.

Tao, T. 2009. There's more to mathematics than rigour and proofs. *What's New*, URL = <https://terrytao.wordpress.com/career-advice/theres-more-to-mathematics-than-rigour-and-proofs/.>

Thompson, P. W. (1996). Imagery and the development of mathematical reasoning. In L. P. Steffe, P. Nesher, P. Cobb, G. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 267–283). Hillsdale, NJ: Erlbaum.

Thorstad, D. (2021). Inquiry and the epistemic. *Philosophical Studies* 178, 2913–2928.

Thurston, W. P. (1994). On proof and progress in mathematics. *Bulletin (New Series) of the American Mathematical Society*, *30*, 161–177.

Van Kerkhove, B. & van Bendegem, J. P. (Eds). 2007. *Perspectives on Mathematical Practices: Bringing Together Philosophy of Mathematics, Sociology of Mathematics, and Mathematics Education*. Dordrecht: Springer.

von Glasersfeld, E. (1982). Subitizing: The role of figural patterns in the development of numerical concepts. <http://www.vonglasersfeld.com/074>

Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational studies in mathematics, 56*, 209-234.

Wilder, R. L. (1967). The role of intuition. *Science*, *156*, 605–610.

Yildirim, I. & Paul, L.A. (2024). From task structures to world models: what do LLMs know? *Trends in Cognitive Sciences*, 28, 404–415.

Zagorianakos, A., & Shvarts, A. (2015). The role of intuition in the process of objectification of mathematical phenomena from a Husserlian perspective: a case study. *Educational Studies in Mathematics*, *88*, 137–157, < https://doi.org/10.1007/s10649-014-9576-9>

Zermelo, E. (1908). A new proof of the possibility of a well-ordering. Reprinted in Jean van Heijenoort (ed.), 1967, *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*, translated by Stefan Bauer-Mengelberg, Cambridge: Harvard University Press, 183–198.