Teaching & Learning Guide for: Explanation in Mathematics: Proofs and Practice

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Author's Introduction

Mathematicians value and pursue proofs they consider explanatory, as opposed to those that merely verify a result without explaining it. This is philosophically interesting for at least two reasons. One is because we care about explanation: how does a proof explain a theorem, and what does this tell us about explanation in general? Another is because we care about mathematical practice: what kinds of proofs do mathematicians consider explanatory, and what does this tell us about how mathematics is done and why? The last twenty years has seen a surge of important work on explanatory proof; the article surveys some key recent developments.

Author Recommends:

Steiner, Mark. 1978. "Mathematical Explanation." Philosophical Studies 34, 135-151, DOI: 10.1007/BF00354494.
The paper that started the current wave of philosophical interest in mathematical explanation. Argues that a proof is explanatory when it exploits a "characterizing property" of an object mentioned in the theorem. Although Steiner's theory isn't currently popular, the paper has been much discussed and contains many important examples and ideas.

Mancosu, Paolo. 2008. "Mathematical Explanation: Why It Matters." In Paolo Mancosu (ed.), *The Philosophy of Mathematical Practice*, Oxford University Press: New York, 134-150.
A useful overview of some major topics, including historical and contemporary examples of explanation in mathematics, Steiner's and Kitcher's theories and their problems, and the relevance of mathematical explanation for Quine-style indispensability arguments.

Lange, Marc. 2009. "Why Proofs by Mathematical Induction Are Generally Not Explanatory." *Analysis* 69, 203-211, DOI: 10.1093/analys/anp002.
A short, engaging and ingenious paper that's produced a small literature on the explanatory value of induction proofs.

Frans, Joachim and Erik Weber. 2014. "Mechanistic Explanation and Explanatory Proofs in Mathematics." *Philosophia Mathematica* 22, 231-248, DOI: 10.1093/philmat/nku003.
Adapts "New Mechanist" ideas about interventions, mechanisms and explanation to analyze proofs of the Butterfly Theorem from Euclidean geometry. Useful for making connections with contemporary work on scientific explanation.

Lange, Marc. 2014. "Aspects of Mathematical Explanation: Symmetry, Unity, and Salience." *Philosophical Review* 123, 485-531, DOI: 10.1215/00318108-2749730.
The most ambitious and sophisticated theory of mathematical explanation since Steiner. Argues that a proof is explanatory when its theorem exhibits a salient feature and the proof makes use of the same type of feature. Full of fascinating examples.

Pincock, Christopher. 2015. "The Unsolvability of the Quintic: A Case Study in Abstract Mathematical Explanation." ***Philosophers' Imprint*** 15, 1-19.
An important attempt to connect explanatory proof with the idea that explanations require objective ontic dependence relations.

D'Alessandro, William. 2019. "Viewing-as Explanations and Ontic Dependence." ***Philosophical Studies***, DOI: 10.1007/s11098-018-1205-5.
Argues that some mathematical explanations don't involve objective dependence relations, and hence the "ontic conception of explanation" isn't adequate in general. The main examples are proofs that involve viewing one mathematical object as another.

Inglis, Matthew and Juan Pablo Mejía-Ramos. 2019. "Functional Explanation in Mathematics." ***Synthese***, DOI: 10.1007/s11229-019-02234-5.
A novel systematic theory of explanatory proof. Argues that a proof is explanatory when it produces understanding, which in turn is identified with the creation of cognitive schemas and their consolidation in long-term memory. A valuable "epistemicist" contribution to the debate.

Online Materials:

2014 Mathematical Depth Workshop at the University of California-Irvine:
https://www.youtube.com/watch?v=ityjwqFlXb0&list=PLQw7KTnzkpXfGo93vo3kQk7_jA_HgwbnC

Talks by Andrew Arana, Mario Bonk, Bob Geroch, Jeremy Gray, Marc Lange, John Stillwell, Jamie Tappenden and Alasdair Urquhart, several of which deal with the relationship between depth and mathematical explanation.

Stanford Encyclopedia of Philosophy article on explanation in mathematics, by Paolo Mancosu:
https://plato.stanford.edu/entries/mathematics-explanation/

Sample Syllabus:

Week I: Introduction
Mancosu, "Mathematical Explanation: Why It Matters"
Lange, Marc. 2010. "What Are Mathematical Coincidences (and Why Does It Matter)?" *Mind* 119, 307-340.

Week II: Some History
Detlefsen, Michael. 1988. "Fregean Hierarchies and Mathematical Explanation." *International Studies in the Philosophy of Science* 3, 97-116.
Mancosu, Paolo. 1999. "Bolzano and Cournot on Mathematical Explanation." *Revue d'Histoire des Sciences* 52, 429-455.

Week III: Skepticism about Mathematical Explanation
Zelcer, Mark. 2013. "Against Mathematical Explanation." *Journal for General Philosophy of Science* 44, 173-192.
Weber, Erik and Joachim Frans. 2017. "Is Mathematics a Domain for Philosophers of Explanation?" *Journal for General Philosophy of Science* 48,125-142.

Week IV: Steiner and His Critics
Steiner, "Mathematical Explanation"
Resnik, Michael D. and David Kushner. 1987. "Explanation, Independence and Realism in Mathematics." *British Journal for the Philosophy of Science* 38, 141-158.

Week V: Lange's Salience Theory
Lange, "Aspects of Mathematical Explanation"

Week VI: The Metaphysics of Explanatory Proof
Pincock, "The Unsolvability of the Quintic"
D'Alessandro, "Viewing-as Explanations and Ontic Dependence"

Focus Questions

1. Is there an explanatory proof for each and every mathematical fact? If not, why not, and what can we say about the exceptions?

2. Mathematicians value proofs for lots of different reasons. Some important factors are beauty, depth, simplicity, surveyability, abstractness, generalizability, transparency, transferability, computer checkability and constructiveness. Are any of these qualities related in an interesting way to explanation?

3. Many philosophers consider counterfactuals to be a useful tool for studying explanations in science. Can counterfactuals shed any light on the nature of explanatory proof?

4. Since it was shown in the early 20th century that most of mathematics can be translated into the language of set theory, questions have arisen about what a "foundation for mathematics" should look like and whether set theory is up to the task. What's the role of explanation here? Should we expect a foundational theory to yield explanatory proofs, and does set theory actually do so?

5. Does it make sense to think of some proofs as being objectively, intrinsically explanatory? Or is explanatoriness always relative to a subject's background knowledge, interests, cognitive capabilities, and other contextual factors?

Seminar/Project Idea:

If there's one pieces of mathematics that almost everyone has seen a proof of, it's the Pythagorean Theorem: the fact that $a^2 + b^2 = c^2$, where a and b are the legs of a right triangle and c is the hypotenuse. The Pythagorean Theorem has many dozens (perhaps hundreds) of proofs, drawing on a variety of interestingly different ideas. Which type of proof is the most explanatory and why? For inspiration, you might start with Eli Maor, *The Pythagorean Theorem: A 4,000-Year History* (2007, Princeton University Press).