(GENERAL) CONCEPTUAL SUBSTRATUM AS A NEW FOUNDATIONAL METAMATHEMATICAL COGNITIVE MECHANISM IN ARTIFICIAL MATHEMATICAL INTELLIGENCE

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ABSTRACT. We describe (essential features and an axiomatization of) a new metamathematical (cognitive) ability, i.e., functional conceptual substratum, used implicitly in the generation of several mathematical proofs and definitions, and playing a fundamental role in Artificial Mathematical Intelligence (or Cognitive-computational metamathematics). Furthermore, we present an initial (first-order) formalization of this mechanism together with its characterizing relation with classic notions like primitive positive definability and Diophantiveness. Additionally, we analyze the semantic variability of functional conceptual substratum when small syntactic modifications are done. Finally, we describe cognitively natural inference rules for (mathematical) definitions inspired by functional conceptual substratum and we show that they are sound and complete w.r.t. standard calculi.

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INTRODUCTION: INITIAL ONTOLOGICAL AND COGNITIVE MOTIVATIONS

During the last decades outstanding interdisciplinary research has emerged involving the identification and subsequently formalization of the most basic cognitive mechanisms used by the mind during mathematical invention/creation. Among these processes one can mention formal conceptual blending [Bou et al., 2015], [Fauconnier and Turner, 2003]; analogical reasoning [Gick and Holyoak, 1980], [Schwering et al., 2009]; and metaphorical thinking [Lakoff and Johnson, 2008], [Lakoff and Núñez, 2000], among others.

One of the most outstanding modern research programs in this direction is the foundation of artificial mathematical intelligence (AMI) or cognitive-computational metamathematics Gomez-Ramirez [2020]. In particular, within the second pillar of the AMI research program, a new cognitive metamathematical mechanism called conceptual substratum is proposed and used widely as a fundamental tool for the subsequent theoretical (and pragmatic) construction of universal mathematical artificial agents (UMAA-s) [Gomez-Ramirez, 2020, Ch.1-Ch.9].

More explicitly, in [Gomez-Ramirez, 2020, Ch.9] a new cognitive (metamathematical) mechanism was explicitly introduced, which encompasses the prototypical ability of the mind to use abstract morpho-syntactic configurations of symbols as valuable tokens for performing advanced formal inferences, specially in the field of mathematical research. Moreover, several examples were used for supporting the introduction and the importance of this new mechanism. Furthermore, an initial formalization of conceptual substratum was made in a quite general setting as well as in a setting of first-order logic. Specifically, a cognitive characterization of the Church-Turing Thesis was proved in terms of the former formalization and in terms of well-known arithmetical-computational results. Finally, the dual cognitive ability, called conceptual lining was introduced.

In this paper, we present an natural continuation and enrichment of the results presented in [Gomez-Ramirez, 2020, Ch.9]. Explicitly, we enhance the original motivations originating this (metamathematical) ability with a new philosophical dimension coming from a Lockean perspective (§1). In addition, we offer a plethora of new motivating examples expanding the range of usefulness of this mechanism (§2). For the sake of completeness we also present here the original formalization of conceptual substratum originally described in [Gomez-Ramirez, 2020, Ch.9], with the seminal difference that we prove the additional fundamental result regarding the invariance of the arithmetic context (involving either the natural or the integer numbers) for a concept to possess (functional) conceptual substratum (§3). And not only that, but we also prove how to generate with the help of our formalization of conceptual substratum, cognitively solid inference rules being sound and complete with respect to the standard calculi (§4). So, we present results supporting the natural evolution and coherence of conceptual substratum in the context of the three pillars of the artificial mathematical intelligence's program: 1) the new cognitive-computational foundations for mathematics' program (§1 and §2), 2) the generation of a solid and global taxonomy of (metamathematical) cognitive mechanisms supporting and structuring mathematical formal research (§3) and the generation of initial computational-feasible formal structures (e.g. proof systems) and software being able to materialize initial versions of an UMAA (Universal Mathematical Artificial Agent) (§4) Gomez-Ramirez [2020].

1. Initial Ontological and Cognitive Motivations

One of the most fundamental questions related with the metamathematical, cognitive and pragmatic aspects of mathematical research involves the description of a global taxonomy of the cognitive mechanisms used (for instance) by working mathematicians for creating/inventing new mathematical results.

So, in this paper we present an additional metamathematical (and, at some extent cognitive) ability, called *(formal) conceptual substratum*, used frequently and implicitly in the construction of mathematical arguments and definitions. From a philosophical point of view, specially assuming a Lockean approach, conceptual substratum of a (mathematical) notion X can be considered as a concrete form of taking the (morpho-syntactic) nominal essence of X [Owen, 1991]. Informally, conceptual substratum involves the essential morpho-syntactic configurations of (mathematical) concepts and structures that we (sub-)conscious- ly used when we attempt to solve a specific conjecture or problem. Here, we are basically pragmatic in our approach, i.e., we focus on the identification of the explicit symbolic configurations that allow us to capture mentally and visually the essential features of (mathematical) concepts.

Regarding the ontological nature of conceptual substrata, we implicitly assume the thesis that conceptual substrata exists at least at the nominal, mental and linguistic levels. Now, a deeper philosophical and metaphysical analysis of this notion goes beyond the scope of this presentation and has no explicit implications on the results developed in the following sections.

The name *conceptual substratum* was chosen as the simplest ways of referring to a core symbolic configuration which codes mentally as well as pragmatically the essential features of a (mathematical) notion.

Metaphorically speaking, we would describe a conceptual substratum as the most compact semantic and syntactic characterization of a concept, given in terms of a symbolic configuration and immersed in a specific conceptual environment, from which one could reconstruct explicitly the whole meaning of it.

From a cognitive perspective, conceptual substrata can be seen as metamathematical form of doing (abstract) 'line drawings' (of a specific concept), where the graphic initial input corresponds to the explicit formal (logic) description of the mathematical structure in consideration (e.g. given in a first-order logic setting) [Liu et al., 2016]. Now, due to the fact doing line drawings (or sketchs) involves processes like perception, memory and judgment, there can exist several conceptual substrata of a single (mathematical) notion, depending of the specific conceptual environment and the concrete goal involved (see further sections for explicit examples) [Liu et al., 2016, §2].

We support our presentation by a significant amount of examples. Additionally, we show an initial (first-order) formalization of this mechanism and its relation with classic notions like primitive positive definability, recursive enumerability and Diophantiveness. In addition, we analyze how strongly the semantic range of this meta-notion varies (or not) when gradual changes are done to the language and to the formal structures in consideration. Finally, we present natural inference rules, inspired by the notion functional conceptual substratum, and prove them sound and complete w.r.t. standard calculi.

Here is worth to mention that the role of the conceptual substratum is so central regarding cognitive-computational metamathematics that one can characterize cognitively the classic Church-Turing Thesis in terms of (functional) conceptual substratum (see [Gomez-Ramirez, 2020, Thesis 9.1]).

2. Taking inspiration from Examples

Suppose that one should solve the following elementary question:

Why when we add two even (integer) numbers is the result again an even number?

This seems to be true for small pairs of numbers $2 + 6 = 8$, $12 + 18 = 30$ and $214 + 674 = 888$. Now, for getting a general proof of this fact, we should consider syntactic expressions which can allow us to represent the even numbers in a compact way. Therefore, we typically come up with a mental-symbolic representation of the form $2 \cdot n$. This means that essentially we are able to represent the collection of even numbers simultaneously with the single expression $2 \cdot n$, where we assume implicitly that n is an integer. On the other hand, if we know that a number c can be written as $2 \cdot d$, where d is an integer, then by definition c should be an even number. In conclusion, we have found a compact (morphological-syntactic) expression for representing every even number in a unified way.

Now, let us consider again the former question with the former representation in mind: First, we need to consider two (potentially different) even numbers, so we consider (or imagine) a first even number $2 \cdot a$ and a second one $2 \cdot b$, where a and b are integers. Second, we sum these numbers generically, namely, we obtain the expression $2 \cdot a + 2 \cdot b$. In addition, we check if the final syntactic expression corresponds to an even number. Thus, we try to give it the desired form $2 \cdot \#$, where $\#$ is an integer. So, we factorize the former algebraic expression and get an expression of the form $2 \cdot (a + b)$. Lastly, we realize that this number has the desired form $2 \cdot x$, where $x = a + b$ is an integer. In conclusion, we explicitly justified an affirmative answer for the former question by performing symbolic operations on morphological generic representations for even numbers.

More generally, when someone tries to solve a mathematical problem, (s)he considers, in a lot of cases, generic representations for the most standard elements living in the corresponding mathematical structures and, subsequently (s)he performs symbolic computations with these representations for solving the problem and for obtaining further insights towards a final solution.

Let us consider a second example: Let $f(x)$ be a polynomial with integer coefficients. Then the polynomial $h(x) = f(x)f''(x)$ has even degree, where $f''(x)$ denotes the second derivative of f.

A very usual way for finding a proof of this statement is by taking a syntactic formal representation for $f(x)$. Effectively, from the hypothesis we see that $f(x)$ can be explicitly written as $a_m x^m + \cdots + a_0$, where $m \in \mathbb{N}$, $a_i \in \mathbb{Z}$ and $a_m \neq 0$. So, we find a representation of $f''(x)$ as $m(m-1)a_m x^{m-2} + \cdots + 2a_2$.

In conclusion, we write $h(x)$ as

$$m(m-1)a_m^2 x^{2(m-1)} + \cdots + 2a_0a_2.$$

So, from this representation we verify that $h(x)$ has even degree.

These morphological-syntactic representations are the seminal tools which allow us to perform general logical inferences with single syntactic elements and, simultaneously prevent us from repeating the same kind of arguments for several specific instances of $f(x)$ varying on their degrees or coefficients.

A third example comes from linear algebra. Let us assume that we have two bases $A = \{u_1, \ldots, u_n\}$ and $B = \{v_1, \ldots, v_m\}$ for a vector space V. So, if we want to prove (in a standard way) that the cardinality of these two bases is the same, i.e., $m = n$, then we need to use syntactic representations of the elements of V such as $\sum_{i=1}^n \alpha_i u_i$ (or $\sum_{j=1}^m \beta_j v_j$). Effectively, one of the simplest arguments consists of replacing gradually the elements of one base with the elements of the other in such a way that the resulting finite set again builds a basis. So, one begins by writing u_1 in terms of the elements of B, i.e., $u_1 = \sum_{j=1}^m \gamma_j v_j$, and, subsequently, one chooses a coefficient $\gamma_{j_1} \neq 0$ in order to obtain a expression of the form

$$v_{j_1} = \frac{1}{\gamma_{j_1}} u_1 + \sum_{j=1, j\neq j_1}^{m} (\frac{\gamma_j}{\gamma_{j_1}}) v_j.$$

Thus, one can replace v_{j_1} by u_1 in B. Now, the next steps go essentially in the same (symbolical) way.

Fourth, the classic Euclidean proof (by contradiction) of the existence of infinitely many prime numbers uses in its core argument a kind of global syntactic description for a number $\prod_{i=1}^n p_n + 1$ bigger than one, which has no prime divisors.[1]

Finally, the classic proof of the fact that the cardinality of the real numbers between zero and one (i.e. $[0, 1]$) is uncountable uses as seminal argument

[1]Here, the assumption is that there exist finitely many prime numbers denoted by $p_1, \ldots, p_n$.

the formal existence of a real number $\lambda = \sum_{i=1}^{\infty} b_i 10^{-i}$, whose explicit decimal representation was chosen based on the corresponding decimal representations of the elements of (an hypothetical enumeration) of $[0,1]$, (i.e., $a_j = \sum_{r=1}^{\infty} a_{j,r} 10^{-r}$) such that for all $i \in \mathbb{N}$, $9 \neq b_i \neq a_{i,i}$.[2]

In conclusion, this kind of generic syntactic representation is fundamental in several mathematical areas.

So, what lies behind the above examples is simply a specific and basic cognitive ability in which our minds choose *conceptual substrata* of certain mathematical notions (e.g. even numbers and polynomials in one variable with coefficient in the integers) at a suitable level of generality, and in such a way that solving the problem simultaneously for several instances of the concepts involved can be translated into formal manipulations of fixed single conceptual representations chosen in advance.

In other words, the cognitive ability of conceptual substratum can be seen as a way of identifying and effectively using the essential (e.g. proto-typical) information of a concept in order to carry out successful deductions for solving several kinds of (mathematical) problems. Implicitly, we assume morpho-syntactic mental representations for seminal mathematical concepts like concrete (natural) numbers, sets and the membership relation, geometrical figures, (graphics of mathematical) functions, among others. Subsequently, the conceptual substrata of more complex mathematical structures are built.[3]

Let us consider several additional examples which allow us to enhance our initial intuitions about what the substratum of a (mathematical) concept is, and about how we can get more elements towards a first precise formalization of it. As a matter of notation we will write conceptual substrata between brackets "$[-]$", in order to clarify that we are talking about cognitive representations of the underlying concepts and not explicitly about the concepts themselves.

So, if D denotes a mathematical concept (e.g., even numbers, polynomials, matrices, vector spaces), then we will denote by $CS(D)$ a conceptual substratum of D. It is important to clarify at this point that one single concept can have several conceptual substrata depending on the way in which we express such a concept syntactically. For instance, the concept of a (positive) prime number has the following two natural definitions:

$$\pi(p) = (\forall d \in \mathbb{N})(d|p \to (d = 1 \vee d = p)),$$

or equivalently

[2]The additional condition given by $9 \neq b_i$ can be added for avoiding difficulties involving the ambiguity of the decimal representation.

[3]The cognitive processes and metamathematical causes behind the conceptual substrata of the above elementary structures go beyond the scope of this paper.

$$\pi(p) = (\forall a, b \in \mathbb{N})(p|a \cdot b \to (p|a \vee p|b)).$$

From these notions one can obtain two conceptual substrata as follows:

$$CS(\text{Prime Numbers}) = [d \in \mathbb{N}, d|p \to (d = 1 \vee d = p)],$$

and

$$CS(\text{Prime Numbers}) = [a, b \in \mathbb{N}, p|a \cdot b \to (p|a \vee p|b)].$$

Now, if one wishes to capture the essence of the notion of a prime number through an expression given by a term instead of the former expressions given by formulas, one can use a result of Ruiz [Ruiz, 2000] (among others) in order to find a quite explicit substratum for being a prime number:

$$CS(\text{Primes}) = \left[1 + \sum_{k=1}^{2(\lfloor n \ln n \rfloor + 1)} \left(1 - \left\lfloor \frac{\sum_{j=2}^{k} 1 + \left\lfloor \frac{-\sum_{s=1}^{j} \left(\left\lfloor \frac{j}{s} \right\rfloor - \left\lfloor \frac{j-1}{s} \right\rfloor - 2 \right)}{j} \right\rfloor}{n} \right\rfloor \right) \right].$$

Most of the former conceptual substrata were expressions describing terms. Nonetheless, there are also a whole collection of concepts whose substrata are typically syntactic descriptions of relations, e.g. the number-theoretic concept of perfect number [Apostol, 1976]. Effectively, for this concept we can write

$$CS(\text{Perf. Numbers}) = \left[2 \cdot n = \sum_{(d|n),(d>0)} d, n \in \mathbb{N} \right].$$

Another enlightening example is the concept of 'representation of the natural numbers in base m ($m \in \mathbb{N}$)'. Here we get

$$CS(\text{m-ary Rep.}) = \left[\sum_{i=0}^{n} \alpha_i m^i : m \in \mathbb{N}, \alpha_i \in \mathbb{N}, 0 \leqslant \alpha_i < m \right].$$

We write the minimal amount of syntactic information that is required for recovering the essential features of this kind of representation.

Our approach has some informal similarities to the one based on (proto-)typicality presented in [Osherson and Smith, 1997]. In fact, finding the conceptual substratum of a concept can be seen as trying to present explicitly a morphological mathematical description of arbitrary instances of the corresponding concept, by starting with the typical ones. For instance, in our second example related with polynomials with coefficient into the integers, one can say that an expression of the form $\sum_{i=0}^{2} c_i x_i = c_0 + c_1 x + c_2 x^2$, is a more typical instance of a polynomial than a constant c_0, or a monomial x^m, since the first one uses the whole spectrum of potential operations which constitute a polynomial (e.g., addition, multiplication and exponentiation), and the last ones use at most one of them. Effectively, the description of the quadratic polynomial resembles the formal substratum of the ring of polynomials better than constants or monomials.

3. Towards a First Formalization

As argued above, the ability to represent an arbitrary object having a certain property in a syntactic-morphological way plays a key role from a cognitive point of view. From a logical point of view this means that we are dealing with a definition *by a term*, or, in the case of an r-ary property, by a tuple of terms. Such a conceptual substratum will be called *functional conceptual substratum*. Let us fix a first-order logic language L and an $L-$structure M. Now, taking inspiration from some of the former examples we state the following definition:

Definition 3.1. We say that a concept defined by a (r-ary) property Ω in M (i.e. $\Omega \subseteq M^r$) has a *functional conceptual substratum*, if there exist terms t_i (for $i = 1, \ldots, r$) and atomic formulas $A_1, \ldots, A_m$ whose variables are contained in $\{x_1, \ldots, x_n\}$, such that for all $a_1, \ldots, a_r \in M$, $(a_1, \ldots, a_r) \in \Omega$ if and only if

$$M \models (\exists x_1) \cdots (\exists x_n)(y_1 = t_1 \wedge \cdots \wedge y_r = t_r \wedge A_1 \wedge \cdots \wedge A_m)[y_1 \mapsto a_1, \cdots, y_r \mapsto a_r]$$

where $t_1, \ldots, t_n$ are L-terms whose variables are among $x_1, \ldots, x_n$.

So, it is straightforward to verify that the notions of even, odd and composite numbers; perfect squares and (more generally) nth-powers have functional conceptual substrata.

In addition one can prove that this notion coincides with primitive positive definability (see for example [Bodirsky and Nešetřil, 2006]).

3.1. Classic Arithmetic Structures.

Now, let us see how the fact that having this kind of 'functional conceptual representations' materializes for several language-structure combinations.

First, it is worth noting that if we do not put any additional restriction on the atoms A_j in the former definition, then for some Ω it could happen that these atoms contain even more important information about the concept C than the terms t_i, for $i = 1, \ldots, n$. Later, we will show explicitly this phenomenon with an example.

Let us consider the language $L = \{0, 1, +, -, *, =, <\}$ and the structure $\mathbb{Z}$, the integers. Then, each A_j has the form of either $u_1(x_1, \ldots, x_n) = u_2(x_1, \ldots, x_n)$ or $u_1(x_1, \ldots, x_n) < u_2(x_1, \ldots, x_n)$, where u_1 and u_2 are the corresponding polynomials in $\mathbb{Z}[x_1, \ldots, x_n]$ representing the terms appearing in A_j.

Now, in the first case A_j can be rewritten as $h(x_1, \ldots, x_n) = 0$, where $h = u_1 - u_2$. For the second case, we can use the well-known fact that any natural number can be written as the sum of four perfect squares [Hardy and Wright, 2008] (i.e., Lagrange's theorem) in order to express the condition described by A_j in a Diophantine way, i.e.,

$$(\exists z_1 \cdots z_4)(u_1 - u_2 = z_1^2 + z_2^2 + z_3^2 + z_4^2 + 1).$$

In addition, one can also express finite conjunctions of polynomial equations through a single equation by using the fact that over the integers $\sum_{i=1}^{m} a_i^2 = 0$ if and only if each $a_i = 0$. So, combining all the former steps one can construct an explicit polynomial $H(y_1, \ldots, y_r, x_1, \ldots, x_n)$ such that for all $a_1 \ldots, a_r \in \mathbb{Z}$, $a_1, \ldots, a_r \in \Omega$ if and only if

$$\mathbb{Z} \models (\exists x_1 \cdots x_n)(H(y_1, \ldots, y_r, x_1, \ldots, x_n) = 0))[y_1 \mapsto a_1, \cdots, y_r \mapsto a_r].$$

In other words, Ω defines a Diophantine set [Matiyasevich, 1993, Ch. 1].

Furthermore, by the MRDP theorem [Matiyasevich, 1993, Ch. 2] Ω defines a recursively enumerable set. In fact, one can easily prove that a concept C over the integers, described with the former language which has a functional conceptual substratum, must be recursively enumerable. Conversely, if C is a concept defining a recursively enumerable property Θ, then by the MRDP theorem Θ is Diophantine. Thus, for all $a_1 \ldots, a_r \in \mathbb{Z}$, $a_1, \ldots, a_r \in \Theta$ if and only if

$$\mathbb{Z} \models (\exists x_1 \cdots x_m)(F(y_1, \ldots, y_r, x_1, \ldots, x_m) = 0))[y_1 \mapsto a_1, \cdots, y_r \mapsto a_r].$$

We can rewrite this formula as

$$\mathbb{Z} \models (\exists x_1 \cdots x_m)(\exists x_1' \cdots x_r')(a_1 = x_1' \wedge \cdots \wedge a_r = x_r' \wedge A_1))$$

where A_1 denotes the atom $F(x_1', \ldots, x_r', x_1, \ldots, x_m) = 0))$.[4]

In conclusion, for $\mathbb{Z}$ expressed in the language $L = \{0, 1, +, -, *, =, <\}$ a concept C describing an $n-$ary property Ω has functional conceptual substratum if and only if Ω is recursively enumerable, which is equivalent to being Diophantine.

This fact can be interpreted as a kind of extension of the Church-Turing thesis to the cognition of (elementary) arithmetical creation in the following sense. All the concepts describing recursively enumerable sets (in the classic sense of Church-Turing computation) are exactly the concepts that can be

[4] In this case, the essential information of the concept can be, at least formally, codified more in the atom A_1 rather than in the initial polynomial expressions.

explicitly characterized as a concrete functional conceptual substratum of an ideal cognitive agent.[5]

3.2. **The Notion of a Prime Number.** By the former considerations, the set of prime numbers has a functional conceptual substratum. More explicitly, one can find an explicit polynomial inequality in the integers characterizing the positive prime numbers. For example, based on the main result of [Jones et al., 1976] we can describe an (atomic) conceptual substratum of the prime numbers as follows

$$CS(\text{Prime Numbers}) = [k+2 \in \mathbb{N}, (k+2)(1-(wz+h+j-q)^2$$

$$-((gk+2g+k+1)(h+j)+h-z)^2-(2n+p+q+z-e)^2-(16(k+1)^3(k+2)(n+1)^2+1-f^2)^2$$

$$-(e^3(e+2)(a+1)^2+1-o^2)^2-((a^2-1)y^2+1-x^2)^2-(16r^2y^4(a^2-1)+1-u^2)^2$$

$$-(((a+u^2(u^2-a))^2-1)(n+4dy)^2+1-(x+cu)^2)^2-(n+l+v-y)^2-((a^2-1)l^2+1-m^2)^2$$

$$-(ai+k+1-l-i)^2-(p+l(a-n-1)+b(2an+2a-n^2-2n-2)-m)^2$$

$$-(q+y(a-p-1)+s(2ap+2a-p^2-2p-2)-x)^2-(z+pl(a-p)+t(2ap-p^2-1)-mp)^2)>0,$$

$$a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z \in \mathbb{N}]$$

This representation can only be fully comprehended and capture as a conceptual substratum of the prime numbers if one gradually studies and understands each of the arithmetical issues codified by its construction (see explicitly [Jones et al., 1976]).[6]

In addition, by Lagrange's theorem and by adding four new existentially quantified variables replacing each of the former 26 variables, one can show that there exists a polynomial $P(x_1,\ldots,x_{104})$ with integer coefficients, such that

$$CS(\text{Prime Numbers}) =$$

[5]An ideal cognitive agent is, in our context, simply an agent possessing an ideal human mind without any restriction regarding working memory and such that with enough time it can 'understand' any intellectual task that a human being can.

[6]One of the reasons for that is the fact that one needs a more explicit use of working memory together with the suitable storage of former conceptual substrata materialized in the form of the single sub-polynomials (expressed as perfect squares).

$$[x_1^2 + x_2^2 + x_3^2 + x_4^2 \in \mathbb{Z}, P(x_1, \ldots, x_{108}) > 0, x_1, x_2, \ldots, x_{108} \in \mathbb{Z}]$$

So, the concept of prime numbers has a functional conceptual substratum over $\mathbb{Z}$ described in the former language.

Now, let us focus on the subsequent natural (arithmetical) question of deciding if the concept of prime numbers has a functional conceptual substratum where the atoms A_j have either the form $x_{r_j} < c_j$ or $c_j < x_{r_j}$.

So, essentially this question is equivalent to finding a polynomial $f(x_1, \ldots, x_n)$ with integer coefficients such that the set of the prime numbers is generated as the image of the domain defined by the atomic restrictions $A_1, \ldots, A_m$. Let us prove by induction on n that this cannot happen.

First, let us suppose that $f(x)$ is a polynomial in one variable with restrictions given by $A_1 \cong x < c_1$ and/or $A_2 \cong c_2 < x$. The case where the domain is either empty or finite (parametrized by two atoms) is clearly ruled out, since its image should be an infinite set. The single cases given by just one of the former atoms can be reduced to the case $x > c_1$, because the second case can be reduced to this one by means of the change of variables $y = -x$.

In conclusion, let us assume for the sake of contradiction that there exists a polynomial $f(x)$ with integer coefficients together with a constant $c \in \mathbb{Z}$ such that the image under f of the set $\mathbb{Z}_{>c}$ is the set of the prime numbers (or an infinite subset of it). Let us choose an integer $d > c$. If we denote by p the prime number $f(d)$, it is an elementary fact to see that for all $z \in \mathbb{Z}$

$$f(pz + d) \equiv f(d) \equiv 0 \pmod{p}.$$

Thus, since $f(pz + d)$ should be a prime number for all $z \geqslant 0$, then $f(pz + d) = p$. Therefore, f should be a constant polynomial, which is a contradiction.

Now, let us assume the induction's hypothesis for any $k < n$. Again, suppose by contradiction that there exists a polynomial $f(x_1, \ldots, x_n)$ with coefficients in the integers and atoms (restrictions) $A_1, \ldots, A_m$ such that the image of the domain determined by the restrictions consists of (an infinite subset of) the prime numbers. Again, by doing suitable changes and permutations of variables we can assume without loss of generality that there exists $s \in \mathbb{Z}$ with $1 \leqslant s \leqslant m$, and constants $c_i \in \mathbb{Z}$ such that $A_i \cong x_i > c_i$, for all $i = 1, \ldots, s$. Thus, since there are just finitely many potential choices for the values of the x_i's (with $i > s$) which satisfy the restrictions, we see that there are constants $e_{s+1}, \ldots, e_n \in \mathbb{Z}$ satisfying all the remaining conditions $A_{s+1}, \ldots, A_m$, such that the image of the domain described by the first s atomic restrictions under the polynomial

$$g(x_1, \ldots, x_s) = f(x_1, \ldots, x_s, e_{s+1}, \ldots, e_m)$$

is an infinite subset of the prime numbers. So, if $s < m$ we are done by the induction's hypothesis, since g has fewer variables than f.

In the second case, it is an elementary fact to see that for any non-constant polynomial $g(x_1, \ldots, x_s)$ in several variables with integer (or even real) coefficients, and for any parameters $c_1, \ldots, c_s \in \mathbb{R}$ (defining atomic restrictions as before), there exists an index i_1 and an integer (resp. a real number) $e > c_i$ such that $h = f(x_1, \ldots, x_{i-1}, e, x_{i+1}, \ldots, x_s)$ is a non-constant polynomial.

Now, using this fact, we obtain a non-constant polynomial h in $s-1$ variables, such that the image of the remaining restrictions under h is an infinite subset of the prime numbers, which is a contradiction.

Summarizing, the existence of functional conceptual representations depends strongly on the degree of freedom that we give to the corresponding atomic formulas and on the particular language used. From a cognitive perspective, this fact implicitly suggests that it is worth studying in deeper detail how to develop formal languages that permit to describe more complex concepts through (functional) conceptual substrata, and in that way to be able to find easier argumentation guidelines for open (mathematical) problems, among others.

On the other hand, let us modify the language slightly by trying to characterize the prime numbers as a kind of 'sub-concept' of the natural numbers $\mathbb{N}$ with the language $L^- = \{0, 1, +, *, =, <\}$, and with the former constraints for the atoms A_i. So, by applying basically the same method as before, we obtain again a negative answer.

However, if we do not impose any kind of restriction on the atoms, then using the same former result of Jones et al. one can find two explicit polynomial $P_1(a, b, \ldots, z)$ and $P_2(a, b, \ldots, z)$ with coefficients into the natural numbers such that

$$CS(\text{Prime Numbers}) = [k, P_1(a, b, \ldots, z) > P_2(a, b, \ldots, z), b, c, \ldots, z]$$

So, the notion of prime numbers also has a conceptual substratum over $\mathbb{N}$ with the restricted language L^-.

3.3. **The Arithmetical Invariance of Functional Conceptual Substrata.** More generally, if we restrict ourselves to a concept C described by a $r-$ relation in $\mathbb{N}$, then the fact that C has a functional conceptual substratum does not change if we expand the language involved (resp. the corresponding structure) by adding the operation of subtraction $-(*)$. Specifically, the following general fact holds:

Proposition 3.2. *Let C be a concept described by a $r-$ary relation Ω in $\mathbb{N}$. Then C has a functional conceptual substratum in L^-, if and only if C (seen as a concept described by the corresponding $r-$ary relation $\Omega \subseteq \mathbb{Z}$) has a functional conceptual substratum in L.*

Proof. Without loss of generality, we can assume that $r = 1$ (the general argument is essentially the same). First, let us suppose that there is an $L-$functional conceptual substratum for C involving the polynomial $f(x_1, \ldots, x_n)$ and atoms $A_1, \ldots, A_r$. Now, we will add an extra variable z in order to be able to codify the fact that $a_1 = f(x_1, \ldots, x_n)$ through the atoms $a_1 = z$ and $A_{r+1} \equiv z = f(x_1, \ldots, x_n)$. This allows us to update f by a polynomial with positive coefficients.

By Lagrange's theorem and by adding (eventually) new existentially quantified variables, we can assume that all the atoms involve only the equality relation. Effectively, this follows from the relations

$$(\forall a, b \in \mathbb{Z})(a < b \leftrightarrow a + 1 \leqslant b),$$

$$(\forall c, d \in \mathbb{Z})(c \leqslant d \leftrightarrow (\exists y_1, y_2, y_3, y_4 \in \mathbb{Z})(d - c = \sum_{i=1}^{4} y_i^2)).$$

An additional simplification consists in reducing the number of atoms to one, by using the fact that

$$(\forall e, g \in \mathbb{Z}((e = 0 \wedge g = 0) \leftrightarrow e^2 + g^2 = 0)).$$

So, let us assume the we have just one atom A.

Furthermore, the fact that there are existential conditions for A involving variables z and $x_1, \ldots, x_n$ varying over $\mathbb{Z}$, can be re-written as new atom A' involving variables z' and $x'_1, \ldots, x'_n$ varying now over $\mathbb{N}$.

In fact, if $A \equiv h_1(z, x_1, \ldots, x_n) = h_2(z, x_1, \ldots, x_n)$, then the fact that there exists $z, x_1, \ldots, x_n \in \mathbb{Z}$ such that $A(z, x_1, \ldots, x_n)$ is equivalent to saying that there exist $z', x'_1, \ldots, x'_n \in \mathbb{N}$ such that

$$\bigvee (h_1(\pm z', \pm x'_1, \ldots, \pm x'_n) = h_2(\pm z', \pm x'_1, \ldots, \pm x'_n)),$$

where the former expression involves 2^{n+1} atoms corresponding to all the possible combinations of signs. Now, by writing each of the former equalities as $\varphi_j(z', \underline{x'}) = 0$, for $j = 1, \ldots, 2^{n+1}$, we can re-write the former expression as the single atomic condition

$$\Phi(z', \underline{x'}) = \prod_{j=1}^{2^{n+1}} \varphi_j(z', \underline{x'}) = 0.$$

Finally, we can re-write this condition as a polynomial equality of the form $\gamma_1(z', \underline{x'}) = \gamma_2(z', \underline{x'})$ involving only positive coefficients.

So, for all $a \in \mathbb{N}$, $a \in \Omega$ if and only if

$$(\exists w_1 \cdots w_{n+1})(a = w_1 \wedge \gamma_1(w_1, \ldots, w_{n+1}) = \gamma_2(w_1, \ldots, w_{n+1})).$$

This means that C has $L^{-}-$functional conceptual substratum.

Conversely, we replace in a $L^{-}-$functional conceptual subtratum, any variable x_j by four variables $y_{j,1}, y_{j,2}, y_{j,3}$ and $y_{j,4}$; and we replace each occurrence of x_j by $\sum_{i=1}^{4} y_i^2$. So, by Lagrange's theorem, we obtain an $L-$functional conceptual substratum for C. □

Remark 3.3. If we replace in the former proposition functional conceptual substratum by Diophantine, then the answer is quite different. Effectively, by the MRDP theorem we know that the 'Diophantine' $L-$concepts are exactly the recursively enumerable. However, the set of Diophantive $L^{-}-$concepts corresponds to a strictly smaller sub-collection of them. Specifically, if $r = 1$, then it is an elementary exercise to prove that the only two Diophantine $L^{-}-$subsets of $\mathbb{N}$ (i.e. subsets described as projections over $\mathbb{N}$ of a polynomial with non-negative coefficients) are $\{0\}$ and $\mathbb{N}$. In general, one can verify by induction over r that a subset $\Omega \subseteq \mathbb{N}^r$ is $L^{-}-$Diophantine if it has the form

$$\bigcup_{\underline{i}=(i_1,\ldots,i_k)}^{\text{finite}} \prod_{r=1}^{k} \mathbb{N}^{(i_r)},$$

where $i_r \in \{0, 1\}$ and we define $\mathbb{N}^0 = \{0\}$ and $\mathbb{N}^1 = \mathbb{N}$.

4. Natural and Complete Definition Rules for Functional Conceptual Substratum

Let us denote by $\mathrm{LK_e}$ the sequent calculus for first-order predicate logic with equality (over a language L) with the standard inference rules (see for instance Buss [1998], Takeuti [2013]). Let us enlarge the language L with a new $r-$ary predicate symbol D which we will define in terms of a functional conceptual substratum in the language L, i.e., by a definition of the form

$$D(a_1, \ldots, a_r) \Leftrightarrow (\exists x_1 \cdots x_n)(a_1 = t_1 \wedge \cdots \wedge a_r = t_r \wedge A_1 \wedge \cdots \wedge A_m)$$

where $t_1, \ldots, t_n$ are $L-$terms and $A_1, \ldots, A_m$ are L-atoms whose variables are (both) among $x_1, \ldots, x_n$.

Now, a standard approach to incorporate definitions into a sequent calculus is to add definition rules which allow unfolding the defined predicate symbol. In our setting this gives rise to the rules

$$\frac{\phi(a_1, \ldots, a_r), \Gamma \to \Delta}{D(a_1, \ldots, a_r), \Gamma \to \Delta}\ D_{\mathrm{L}} \qquad \text{and} \qquad \frac{\Gamma \to \Delta, \phi(a_1, \ldots, a_r)}{\Gamma \to \Delta, D(a_1, \ldots, a_r)}\ D_{\mathrm{R}}$$

where $\phi(a_1, \ldots, a_r)$ abbreviates the formula defining $D(a_1, \ldots, a_r)$ as above. We denote the sequent calculus obtained from adding these rules to $\mathrm{LK_e}$ as $\mathrm{LK_e}(D)$. These rules correspond to inferences that syntactically replace into a proof the former definition of the new relational symbol within the left and right part of a sequent, respectively.

Lemma 4.1. *For any formula ψ, $\mathrm{LK_e}(D) \vdash \psi \leftrightarrow \psi[D\backslash\phi]$, where $\psi[D\backslash\phi]$ denotes the formula obtained after replacing D by ϕ in ψ.*

Proof. This fact can be straightforwardly proved by induction on the (syntactic) complexity of ψ, decomposing the equivalence into two implications and using the new pair of rules. □

The calculus $\mathrm{LK_e}(D)$ is a conservative extension of $\mathrm{LK_e}$ in the following sense:

Theorem 4.2. *For any formula ψ, $\mathrm{LK_e}(D) \vdash \psi$ if and only if $\mathrm{LK_e} \vdash \psi[D\backslash\phi]$.*

Proof. ($\Rightarrow$) Let P be a proof of ψ in $\mathrm{LK_e}(D)$. Then, by replacing D in P by ϕ and removing D_{L}- and D_{R}-inferences, we obtain a proof P' of $\psi[D\backslash\phi]$ in $\mathrm{LK_e}$.

($\Leftarrow$) Let P be an $\mathrm{LK_e}$-proof of $\psi[D\backslash\phi]$. Obtain an $\mathrm{LK_e}(D)$-proof Q of $\psi[D\backslash\phi] \to \psi$ from Lemma 4.1. Then a cut on P and Q gives an $\mathrm{LK_e}(D)$-proof of ψ. □

The above definition rules treat definitions in general. However, a definition of a concept that has a functional conceptual substratum is typically used in a more specific way in mathematical proofs. For example, when showing that the sum of n and m is even if m and n are, one may start the proof by a phrase like "Since n is even, $n = 2a$ (for some $a \in \mathbb{N}$)". For the general case, this is formalized by the rule

$$\frac{a_1 = t_1[\underline{x}\backslash\underline{\zeta}], \ldots, a_r = t_r[\underline{x}\backslash\underline{\zeta}], A_1[\underline{x}\backslash\underline{\zeta}], \ldots, A_m[\underline{x}\backslash\underline{\zeta}], \Gamma \to \Delta}{D(a_1, \ldots, a_r), \Gamma \to \Delta}\ D_{\mathrm{L}}^{\mathrm{fcs}}$$

Similarly, one may end the proof with a phrase like "$2 \cdot (a + b)$ is even". For the general case, this is formalized by the rule

$$\frac{\Gamma \to \Delta, A_1[\underline{x}\backslash\underline{u}] \quad \cdots \quad \Gamma \to \Delta, A_m[\underline{x}\backslash\underline{u}]}{\Gamma \to \Delta, D(t_1[\underline{x}\backslash\underline{u}], \ldots, t_r[\underline{x}\backslash\underline{u}])}\ D_{\mathrm{R}}^{\mathrm{fcs}}$$

We write $\mathrm{LK_e^{fcs}}$ for the calculus obtained from $\mathrm{LK_e}$ by adding these two rules. We will now verify that $\mathrm{LK_e^{fcs}}(D)$ is sound and complete w.r.t. $\mathrm{LK_e}(D)$. To that aim, we first relate it to $\mathrm{LK_e}$.

Lemma 4.3. *For any formula ψ, $LK_{\mathrm{e}}^{\mathrm{fcs}}(D) \vdash \psi \leftrightarrow \psi[D\backslash\phi]$.*

Proof. We proceed by induction on the syntactic complexity of ψ. The only non-trivial case is when ψ is $D(v_1, \ldots, v_r)$.

We obtain an $\mathrm{LK_e^{fcs}}(D)$-proof of $D(v_1, \ldots, v_r) \to \phi(v_1, \ldots, v_r)$ by applying a $D_{\mathrm{L}}^{\mathrm{fcs}}$-inference, n $\exists_{\mathrm{r}}$-inferences, and $r + m - 1$ $\wedge_{\mathrm{r}}$-inferences.

In the other direction, we obtain an $\mathrm{LK}_{\mathrm{e}}^{\mathrm{fcs}}(D)$-proof of $\phi(v_1,\ldots,v_r) \to D(v_1,\ldots,v_r)$ as follows:

$$\dfrac{\dfrac{\dfrac{A_1[\underline{x}\backslash\underline{\zeta}] \to A_1[\underline{x}\backslash\underline{\zeta}] \quad \cdots \quad A_m[\underline{x}\backslash\underline{\zeta}] \to A_m[\underline{x}\backslash\underline{\zeta}]}{A_1[\underline{x}\backslash\underline{\zeta}],\ldots,A_m[\underline{x}\backslash\underline{\zeta}] \to D(t_1[\underline{x}\backslash\underline{\zeta}],\ldots,t_r[\underline{x}\backslash\underline{\zeta}])}\,D_{\mathrm{R}}^{\mathrm{fcs}}}{v_1 = t_1[\underline{x}\backslash\underline{\zeta}],\ldots,v_r = t_r[\underline{x}\backslash\underline{\zeta}], A_1[\underline{x}\backslash\underline{\zeta}],\ldots,A_m[\underline{x}\backslash\underline{\zeta}] \to D(v_1,\ldots,v_r)}\,\text{eq.}}{\dfrac{v_1 = t_1[\underline{x}\backslash\underline{\zeta}] \wedge \cdots \wedge v_r = t_r[\underline{x}\backslash\underline{\zeta}] \wedge A_1[\underline{x}\backslash\underline{\zeta}] \wedge \cdots \wedge A_m[\underline{x}\backslash\underline{\zeta}] \to D(v_1,\ldots,v_r)}{\phi(v_1,\ldots,v_r) \to D(v_1,\ldots,v_r)}\,\exists_{\mathrm{l}}^{n}}\,\wedge_{\mathrm{l}}^{r+m-1}$$

$\square$

Theorem 4.4. *For any formula ψ, $\mathrm{LK}_{\mathrm{e}}^{\mathrm{fcs}}(D) \vdash \psi$ if and only if $\mathrm{LK}_{\mathrm{e}} \vdash \psi[D\backslash\phi]$.*

Proof. ($\Rightarrow$) Let P be a proof of ψ in $\mathrm{LK}_{\mathrm{e}}^{\mathrm{fcs}}(D)$. We replace D in P by ϕ, simulating a $D_{\mathrm{L}}^{\mathrm{fcs}}$-inferece by n $\exists_{\mathrm{l}}$-inferences, and $m+r-1$ $\wedge_{\mathrm{l}}$-inferences and a $D_{\mathrm{R}}^{\mathrm{fcs}}$-inference by

$$\dfrac{\dfrac{\dfrac{\Gamma \to \Delta, A_1[\underline{x}\backslash\underline{u}] \quad \cdots \quad \Gamma \to \Delta, A_m[\underline{x}\backslash\underline{u}]}{\Gamma \to \Delta, A_1[\underline{x}\backslash\underline{u}] \wedge \cdots \wedge A_m[\underline{x}\backslash\underline{u}]}\,\wedge_{\mathrm{r}}^{m-1}}{\Gamma \to \Delta, t_1[\underline{x}\backslash\underline{u}] = t_1[\underline{x}\backslash\underline{u}] \wedge \cdots \wedge t_r[\underline{x}\backslash\underline{u}] = t_r[\underline{x}\backslash\underline{u}] \wedge A_1[\underline{x}\backslash\underline{u}] \wedge \cdots \wedge A_m[\underline{x}\backslash\underline{u}]}\,\text{eq.}}{\Gamma \to \Delta, \phi(t_1[\underline{x}\backslash\underline{u}],\ldots,t_r[\underline{x}\backslash\underline{u}])}\,\exists_{\mathrm{r}}^{n}$$

Thus we obtain a proof P' of $\psi[D\backslash\phi]$ in LK_{e}

($\Leftarrow$) Let P be an LK_{e}-proof of $\psi[D\backslash\phi]$. Obtain an $\mathrm{LK}_{\mathrm{e}}(D)$-proof Q of $\psi[D\backslash\phi] \to \psi$ from Lemma 4.3. Then a cut on P and Q gives an $\mathrm{LK}_{\mathrm{e}}^{\mathrm{fcs}}(D)$-proof of ψ. $\square$

Corollary 4.5. *For any formula ψ, $\mathrm{LK}_{\mathrm{e}}^{\mathrm{fcs}}(D) \vdash \psi$ iff $\mathrm{LK}_{\mathrm{e}}(D) \vdash \psi$.*

Thus one does not loose power by using these specialized definition rules for defined predicate symbols with functional conceptual substratum. On the other hand, one gains a mathematically more natural use of these defined symbols.

5. Conclusions

The general meta-notion of conceptual substratum (and its particular form as functional conceptual substratum) serves as a new kind of meta-mathematical mechanism of outstanding importance with a strong cognitive dimension used (implicitly) in mathematical creation/invention.

Moreover, the initial first-order formalization of this meta-concept turns out to be equivalent to central notions in theoretical computer sciences and elementary number theory. In addition, (functional) conceptual substratum suggests an additional way of developing proof-theoretical frameworks with a stronger human-style structure. One of the main reasons supporting this thesis is the fact that the new deduction rules and definitions simply mirror specific instances of the human mathematical practice in an explicit morpho-syntactic way, and, simultaneously preserving exactly the same deductive power.

So, subsequent formalizations of conceptual substrata (and of the notion of conceptual substratum) in higher-order frameworks could bring new light in our quest for understanding how mathematical creation/invention works and for developing software able to solve mathematical problems at higher levels of abstraction.

More explicitly, the results presented here, together with the general formalizations of pragmatic and purely formal conceptual substratum presented in [Gomez-Ramirez, 2020, Def. 10.18,Ch.10] suggests a research program, where the local conceptual substrata for each relevant mathematical sub-discipline should be characterized and used for computing the corresponding pseudo-precode aiming to obtain general algorithmic commonalities, which can served as the basis for generating the corresponding computer programs serving as (local) co-creative assistants in several mathematical sub-areas.

Even more, a suitable analysis for conceptual substrata in related field like physics, chemistry and biology, is the initial key point for being able to materialize the corresponding programs of of artificial physical/chemical/biological intelligence from the perspective of artificial mathematical intelligence [Gomez-Ramirez, 2020, Ch.12]. Moreover, the same strategy can be potentially used for developing more concrete pragmatic and morpho-syntactic prevention guidelines for viruses like COVID-19, as well as similar ones, following the lines along Gómez-Ramírez et al. [2021] and Herrera-Jaramillo et al. [2021].

The relation with existing theorem provers is more implicit and at the meta-level. In other words, any theorem prover explicitly based on versions of the sequent calculus can be potentially updated with specific exemplifications of the new functional calculus, that preserve the theoretical deductive scope that increase the cognitive plausibility.

The importance of conceptual substratum for mathematical creation/invention refers to the fact that obtaining more useful and explicit conceptual substrata for mathematical structures has the potential of facilitating and speeding up the creativity in mathematical reasoning.

Extensions of the formalization of (functional) conceptual substratum to second order logic could also be used for helping us to understand even more deeply the technical deductive aspects of our daily (natural) language [Mostowski and Szymanic, 2012].

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