THE PHYSICAL NUMBERS: A NEW FOUNDATIONAL LOGIC-NUMERICAL STRUCTURE FOR MATHEMATICS AND PHYSICS

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ABSTRACT. The boundless nature of the natural numbers imposes paradoxically a high formal bound to the use of standard artificial computer programs for solving conceptually challenged problems in number theory. In the context of the new cognitive foundations for mathematics' and physics' program immersed in the setting of artificial mathematical intelligence, we proposed a refined numerical system, called the physical numbers, preserving most of the essential intuitions of the natural numbers. Even more, this new numerical structure additionally possesses the property of being a bounded object allowing us to work with quite similar axioms like the classic Peano axioms, but in a finite environment and with an enriched physical dimension. Finally, we present several enlightening examples and we conclude that the physical numbers provide a natural formal setting for approaching classic problems in number theory from a more hybrid perspective, i.e. with a potential participation in the generation of solutions, not only of working mathematicians but also of more sophisticated artificial interactive (mathematical) intelligent agents (within the context of cognitive-computational metamathematics or artificial mathematical intelligence) and more controlled by physically-inspired principles. Finally, this paper addresses a highly new, bottom-up and paradigm-shifting approach to the foundations of physics: One should refine and improve the conceptual formal setting of the language that we use for understanding physical phenomena (e.g. the mathematical grounding concepts and theories) for being able to understand better more subtle and complex physical phenomena.

Mathematical Subject Classification (2020): 11U99, 11Y99, 11Z99

Keywords: Cognitive-Computational Metamathematics; The Physical Numbers; Number Theory; Discrete Mathematics

1. Introduction and the New Cognitive Foundations for Physics' Program

The New Cognitive Foundations for Mathematics' program (NCFM-P) is the first seminal pillar for Cognitive-Computational Metamathematics (or Artificial Mathematical Intelligence (AMI)) [6, Part I]. It encompasses, roughly speaking, the generation of new and refined grounding structures for (modern) mathematics with a stronger multidisciplinary basis and more suitable to be 'computationally simulated' by an Universal Mathematical Artificial Agent [8, 7]. In particular, one of its central principles is to transcend and to extend the mono-disciplinary and purely logical setting of the fundamental mathematical structures (e.g. the numerical systems, sets, the membership relation, first-order logic) by including cognitive, computational and physical principle, among others [7]. Now, due to the omnipresent fact that mathematics is the most natural language for physics, it is obvious that the materialization of this new (trans-, inter-, intra- and) multidisciplinary foundational program for mathematics would immediately set the initial basis of a corresponding *New Cognitive Foundations for Physics' Program* (NCFP-P). Even more, this NCFP program should be extended for seminal (physical) notions that come from an experimental and heuristic setting and that goes beyond the purely mathematical domain (e.g. physical unities, measurements, uncertainty principles, approximations, etc.).

So, the NCFP program constitutes the first pillar of *Artificial Physical Intelligence* as originally described in [10], i.e., the generation of all the necessary theoretical and computational setting needed for the implementation of a co-creative interactive artificial agent being able to help us solving (mainly theoretical and secondary experimental) open problem in physics.

The aim of this article is to present one central primary notion for both foundational programs, i.e. *the physical numbers*. In other words, a new numerical system that not only refines the most basic counting structures that we use very often in mathematics and physics, i.e. the natural numbers [21, Ch. 4]; but also that conceptually fits better into the design of a new generation of artificial (mathematical/'physical') agents immersed in the context of artificial mathematical (physical) intelligence. Along the way, we present as well, several major consequences that the integration of the physical numbers would have in the development of a more precise re-framing of central (solved and open) questions in (elementary) number theory.

Our approach here goes in the direction of sculpting the theoretical highway of mathematics and physics in such a way that the fundamental questions can be solved more naturally and straightforwardly with the building made [18], [16]. In contrast with the classical procedure of producing highly tricky solutions to the open problems based on the grounding theories at hand. In the following sections, we present the formal setting required for structuring the physical numbers, their connections with the classic Peano axioms and the standard arithmetic.

Additionally, we present a formal way to connect the physical numbers with the natural numbers via the physical natural map, and we discuss the 'physical versions' foundational problems in number theory like Fermat's last theorem, Goldbach's Conjecture, some Diophantine equations and Hilbert's tenth problem. Finally, we state the main conclusions of our work with an implicit projection toward future work.

2. Initial Foundational Intuitions about the Physical Numbers

In this section, we review the basic informal features about the physical numbers. The reader may consul [11] for the major reference on these subjects. In fact, for more examples and initial intuitions of how this construction refines a lot of aspect of the natural numbers, the reader is invited to see in more detail [11]. Indeed, the primary calculation tool for physical numbers is the formation of physical partitions of external reality, rather than the more traditional technique of 'counting' objects that served as the intuitive foundation for natural numbers [26]. Furthermore, for being able of forming such a partitions, we make two seminal assumptions: i) The existence of an external physical reality (going beyond the existence of subjective observers[1]), and ii) the assumption that there exists a finite number of physical quanta in the external reality (which may be quantitatively conditioned by a temporal parameter of consideration). With both assumptions, we establish the existence of the two most fundamental physical numbers, namely, the initial physical number α, which represents the trivial physical partition consisting of the external reality as a whole; and the final physical number ω, which represents the most refined partition consisting of each quantum of the physical realm as a constituent member.

3. A Formalization of the Physical Numbers

In this article, we provide in detail the essential formal framing of physical numbers. They are a novel class of structures in which physical and abstract logic-mathematical things are combined (and subsequently blended) in a formal way.

First, we establish primal structures and relations among them, which should be understood as the improved interpretations of the concepts of set, and the membership relation in the Zermelo-Fraenkel Set Theory with the Axiom of Choice [21]. Furthermore, let us assume that we have constructed a refined logical formalism, similar to ZFC, but with features more in accordance with the new cognitive foundations for mathematics' program [7]. Thus, we will present more explicitly some of the features that such a formalism should have, without going deeper into the details, since this matter goes beyond the scope of this article, for more details on this matters, please see [7].

Second, we present our fundamental axioms in a setting similar to many-sorted logic [20]. Effectively, we quantify over entities having a purely physical nature (e.g. physical sub-spaces), as well as entities having a mixed formal/physical nature (e.g. the ph-numbers).

[1]For an initial form of this axiom see [25]. A more refined version is given in [11].

Third, let E_R be the External (Physical) Reality. Let us represent by A any well-defined sub-space of E_R (e.g. the (current) collection of all photons; planets; stars; the Milky Way; the sun, among others[2]). We use the symbol $\mathbb{F}$ to denote the ph-numbers. In this situation, we can speak about a *physical membership relation* among the *physical elements* of $\mathbb{F}$. Thus, we denote this ph-relation by $\in_p$.

Furthermore, let us fix a physical sub-space A (this can be denoted by $A \subseteq_{ss} E_R$). Generally, if A and B are physical sub-spaces, then we denote the fact that A is (physically) contained in B by $A \subseteq_{ss} B$ (i.e. each quantum of any kind of A belongs to B). $P_a(C)$ denotes the space of (all potential) finitely constructed partitions of C; i.e., 'physically-disjointed' finite gathering of sub-spaces of B, whose physical union restores B again. Note that $P_a(B)$ is a formal-physical entity, where any partition of B can be identified, but not where all the partitions of B are simultaneously found[3], [4]. If E is a fixed partition of B, then we denote this as $E \in_f P_a(B)$. Additionally, the fact that a physical sub-space e belongs to a particular partition E is indicated by $e \Subset E$. If a partition E contains exactly all of the subspaces $A_1, \cdots, A_m$, then we more clearly identify this as $E = [\![A_1, \cdots, A_m]\!]$. Let $P_1 \in_f P(A_1)$ and $P_2 \in_f P(A_2)$ denote partitions whose physical domains are disjoint (i.e. A_1 and A_2 possess no quantum in common), thus we denote the new partition generated by integrating both of them by $P_1 \uplus_{par} P_2$, with physical domain $A_1 \uplus_f A_2$, in other words, the physical union of both physical subspaces A_1 and A_2. If $e \Subset P_1$, then we symbolise by $P_1 \smallsetminus_{par} [\![e]\!]$, the partition of the (physical) subspace $A_1 \smallsetminus_f e$, consisting of all the physical subspaces of P_1 but e.

Fourth, the fundamental partial 'physical' functions are the following: The *physical cardinal*, (or *physical cardinality*) of a partition of A, $C_A : P_a(A) \to \mathbb{F}$; the *physical successor* $s_p : \mathbb{F} \nrightarrow \mathbb{F}$; the *physical addition* $+_p : \mathbb{F} \times \mathbb{F} \nrightarrow \mathbb{F}$; the

[2]It is worth ot observe that any of these physical sub-classes can be seen as things which depend on time, particularly, they depend on the time when they are being considered. For example, the collection of planets now will be different from the collection of planets a million years ago.

[3]The last requirement is due to physical considerations, due to the fact that the amount of energy required for replicating all the possible sub-spaces of B (which is required to induce all its partitions) increases exponentially, and in a lot of cases is simply not available, e.g. $B = E_R$.

[4]For a better illustration of the idea behind this concept, let us consider the entity S consisting of a person I, a board and a marker. This object can be thought of as 'the collection of all (potential) possible written (short) manifestations of thoughts of I'. In fact, one could (potentially) find in S any feasible (short) written expression of thoughts of I, just by asking I. Nonetheless, S does not consist, at any time, of the gathering of all the possible (short) written expressions of thoughts of I entirely.

physical multiplication $*_p : \mathbb{F} \times \mathbb{F} \nrightarrow \mathbb{F}$; and the *physical quotient* between ph-numbers, $\div_p : \mathbb{F} \times \mathbb{F} \nrightarrow \mathbb{F}$. The related relations are $=_p, <_p$ and $\leq_p$ contained in $\mathbb{F} \times_p \mathbb{F}$.

Fifth, the 'initial' physical number corresponding to the simplest partition consisting of just one subspace i.e. the whole corresponding subspace in consideration, will be denoted by α, i.e. $C_A([\![A]\!]) = \alpha$, for a physical subspace A. This physical number corresponds (intuitively) to the natural number zero. On the other hand, the 'last' physical number corresponding to the maximal physical partition of the whole external reality, into all its physical quanta (E_R^{max}) will be betoken by ω, i.e., $C_{E_R}(E_R^{max}) = \omega$. Analogously, for any $A \subseteq_{ss} E_R$, one can define the maximal partition relative to A, (i.e. A^{max}) in the same manner, and the corresponding maximal, or final, physical number relative to A, by $\omega_A \coloneqq C_A(A^{max})$. Therefore, $\omega = \omega_{E_R}$. Let us notice that intuitively a physical number emerging from a partition that classically possesses $n + 1$ physical subspaces ($n + 1$ understood as a natural number), will simulate the natural number n (see [11, §5.12]).

Now, we present all the seminal axioms describing, in an implicit way, the essence of the physical numbers.

3.1. **Axioms characterizing the Initial Physical Number.**

We define the fundamental formal features of the initial physical number α.

$$(\forall A, Z \subseteq_{ss} E_R)(C_A([\![A]\!]) =_p C_Z([\![Z]\!]) =_p \alpha).$$

And the condition describing the fact that only the trivial partitions correspond to the initial number

$$(\forall B \subseteq_{ss} E_R)(\forall P \in_f P_a(B))(C_B(P) =_p \alpha \leftrightarrow P =_p [\![B]\!]).$$

3.2. **Axioms describing the Final (Global and Relative) Physical Number.**

The following axiom guarantees that ω_A is the biggest physical number relative to partitions in a physical subspace $B \subseteq_{ss} E_R$.

$$(\forall B \subseteq_{ss} E_R)(\forall P \in_f P_a(B))(C_B(P) \leq \omega_B \wedge \omega_B \leq \omega)$$

Now, we secure that the maximal relative partitions are the only ones producing the maximal relative physical numbers

$$(\forall B \subseteq_{ss} E_R)(\forall P \in_f P_a(B))(C_B(P) =_p \omega_B \leftrightarrow P = B^{max}),$$

here the symbol = among partition means that both partitions are exactly the same, i.e. they incorporate exactly the same physical subspaces.

3.3. **Axioms for describing the (Physical) Equality.** We characterize the (physical) equality among physical numbers, implicitly through a finite recursion procedure. Thus, the initial step was already defined on the former axioms structuring α by defining (physical equality) for trivial partitions.

Now, on the basis of that we define (physical) equality more generally

$$(\forall a, z \in_p \mathbb{F})((a \neq_f \alpha \wedge z \neq_p \alpha) \rightarrow ((\forall A, Z \subseteq_{ss} E_R)(\forall P_1 \in_f P_a(A))(\forall P_2 \in_f P_a(Z))$$
$$(\forall A_1 \Subset P_1)(\forall Z_1 \Subset P_2)((C_A(P_1) = a \wedge C_B(P_2) = z) \rightarrow$$
$$(C_{A \setminus_f A_1}(P_1 \setminus_{par} [\![A_1]\!]) = C_{Z \setminus_f Z_1}(P_2 \setminus_{par} [\![Z_1]\!]))))).$$

The following axiom guarantees the non-triviality of the equality:

$$(\forall D \subseteq E_R)(\forall P \in_p P_a(D))(\forall D_1 \Subset P)(C_D(P) \neq_p \alpha \rightarrow$$
$$C_D(P) \neq_p C_{D \setminus_f D_1}(P \setminus_{par} [\![D_1]\!]) \wedge C_D(P) = C_{D \setminus_f D_1}(P \setminus_{par} [\![D_1]\!]) + s(\alpha)).$$

3.4. **Partitioning Axiom.** This axiom guarantees the quantitative coherence of physical numbers when one adds (resp. suppresses) subspaces to partitions:

$$(\forall X', X'' \subseteq_{ss} E_R)(\forall P', Q' \in_f P_a(X'))(\forall P'', Q'' \in_f P_a(X''))((\exists B' \Subset P')(\exists B'' \Subset P'')$$
$$(\exists C', G', C'', G'' \subseteq_{ss} E_R)(B' = C' \uplus_f G' \wedge B'' = C'' \uplus_f G'' \wedge Q' = (P' \setminus_{par} [\![B']\!]) \uplus_{par} [\![C', G']\!]$$
$$\wedge Q'' = (P'' \setminus_{par} [\![B'']\!]) \uplus_{par} [\![C'', G'']\!]) \rightarrow (C_{X'}(P') =_p C_{X'}(Q') \leftrightarrow C_{X''}(P'') =_p C_{X''}(Q'')))$$

3.5. **Retraction-Extension Axiom.** This axiom ensures the quantitative immutability of the physical numbers with respect to the inner size of the subspaces of partitions

$$(\forall Y \subseteq_{ss} E_R)(\forall P \in_f P_Y(Y))(\forall B \Subset P)(\forall C \subseteq_{ss} E_R)(C \subseteq_{ss} B \rightarrow$$
$$C_A(P) =_p C_{(Y \setminus_f B) \uplus_f C}((P \setminus_{par} [\![B]\!]) \uplus_{par} [\![C]\!]))$$

3.6. **Axiom describing the (Physical) Successor Function.** This axiom characterizes the refinement of the classic successor function for the Peano Arithmetic (see [21, Ch.3]).

$$(\forall x, y \in_p \mathbb{F})(x =_p s(y) \leftrightarrow (\exists X', X'', X''' \subseteq_{ss} E_R)(\exists P', P'' \in_f P(E_R))(X' = X'' \uplus_f X'''$$
$$\wedge X' \Subset P' \wedge X'', X''' \Subset P'' \wedge C_{E_R}(P_1) =_p y \wedge C_{E_R}(P'') =_p x$$
$$\wedge P'' = (P' \setminus_{par} [\![X']\!]) \uplus_{par} [\![X'', X''']\!])).$$

3.7. **(Physical) Addition Axiom.** This axiom assures the existence of the (physical) addition of two physical numbers only in the cases that the physical quantitative constrains allow such an addition to exist.[5]

$$(\forall a, h, z \in_p \mathbb{F})(a +_p h =_p z \leftrightarrow ((a =_p \alpha \to h =_p z) \wedge (a \neq_p \alpha \to (\exists a' \in_p \mathbb{F})(a =_p s(a') \wedge$$
$$(\exists A', A'', A''' \subseteq_{ss} E_R)(\exists P' \in_f P_a(A'))(\exists P'' \in_f P_a(A''))(\exists P''' \in_f P_a(A'''))$$
$$(A''' = A' \uplus_f A'' \wedge P''' = P' \uplus_{par} P'' \wedge C_{A'}(P') =_p a'$$
$$\wedge C_{A''}(P'') =_p h \wedge C_{A'''}(P''') =_p z))))).$$

3.8. **(Physical) Multiplication Axiom.** For an easier understanding, we divide this axiom in two parts by using an auxiliary multiplicative operation $*'_p$. Thus, firstly let us define this operation in an axiomatic way

$$(\forall s, t, v \in_p \mathbb{F})(s =_p t *'_p v \leftrightarrow (\exists A \subseteq_{ss} E_R)(\exists x, y, z \in_f P(E_R))$$
$$(C_{E_R}(x) = s \wedge C_{E_R}(z) = v \wedge C_{E_R}(y) = t \wedge$$
$$(\forall w \Subset y)(\exists h_w \in_f P_a(w))(C_w(h_w) = c)) \wedge \uplus_{w \Subset y}(h_w) = x),$$

where $\uplus_{w \Subset y}(h_w)$ describes the partition of E_R formed by the physical (disjoint) union of all fixed h_w. Informally, the operation $*'_p$ is the only one that is axiomatized in a manner that the multiplication of (physical) numbers corresponding to partitions with n and m elements ($n, m \in \mathbb{N}$) generates a physical number with a partition of $n * m$ elements. This minor intermediate step must be done in order to understand better the central axiom structuring the (physical) multiplication:

$$(\forall s, t, u \in_p \mathbb{F})(s =_p t *_p u \leftrightarrow ((t =_p \alpha \to s =_p \alpha) \vee$$
$$((s \neq_p \alpha \wedge t \neq_p \alpha \wedge u \neq_p \alpha) \to (\exists s', t', u' \in_p \mathbb{F})(s(s) =_p s' \wedge s(t) =_p t' \wedge s(u) =_p u'$$
$$\wedge s' =_p t' *'_p u')))).$$

3.9. **(Physical) Quotient Axiom.** This axiom defines the (physical) division in a natural way, in terms of the (physical) product:

$$(\forall f, g, h \in_p \mathbb{F})(f =_p g \div_p h \leftrightarrow (g \neq_p \alpha \wedge f =_p g *_p h)).$$

From an ontological point of view, one can say that *physical division* is a fundamental operation because it allows us to keep 'control' over the size of the ph-number, based on a physical fact (i.e., the quantitative shape of the physical partitions of sub-spaces of E_R). This type of formal regulation is harder to obtain if we consider operations which expand, quantitatively speaking, the size of computed numbers through addition and multiplication, respectively.

[5] Remember all the physical considerations done in the former sub-sections of this section.

3.10. **(Physical) Order Axioms.** The following axioms determine the meaning of the (physical) order relation for the ph-numbers.

$$(\forall a, e \in_p \mathbb{F})(a \leq_p e \leftrightarrow (\exists b \in_p \mathbb{F})(a +_p b =_p e)).$$

$$(\forall a, e \in_p \mathbb{F})(a <_p e \leftrightarrow (\exists b \in_p \mathbb{F})(b \neq_p \alpha \wedge a +_p b =_p e)).$$

All the former axioms provide a formal environment of modeling the shaping intuitions about the physical numbers, and in an extended manner some of the original intuitions that we are used to perceive about the natural numbers. Nonetheless, we enrich here the merely logical-mathematical defining principles used classically with a more multi- and trans-disciplinary perspective using implicitly more cognitive and physical precepts (see for example [12]).

4. Updating the Peano Axioms in a Bounded Discrete Context

At this point it is important to study how the classic formal axioms shaping the natural numbers (e.g. the Peano axioms) can be updated and refined from the axioms presented in the former section.

With the help of the former axioms one can see that the following axioms hold, which refine the classic Peano axioms for our multidisciplinary discrete setting. Most of them are almost identical versions of its classic counterparts, others are slightly different:

(1) $(\forall x, y, z \in_p \mathbb{F})(x =_p y \rightarrow (x =_p z \rightarrow y =_p z))$
(2) $(\forall x, y \in_p \mathbb{F})((x \neq_p \omega \wedge y \neq_p \omega) \rightarrow (x =_p y \rightarrow s(x) =_P s(y)))$
(3) $(\forall x \in_p \mathbb{F})(s(x) \neq_p \alpha)$
(4) $(\forall x \in_p \mathbb{F})(x \leq_p \omega)$
(5) $(\forall x, y \in_p \mathbb{F})(s(x) =_p s(y) \rightarrow x =_p y)$
(6) $(\forall x \in_p \mathbb{F})(x +_p \alpha =_p x)$
(7) $(\forall x, y, z \in_p \mathbb{F})(x +_p s(y) =_p z \rightarrow x +_p s(y) =_p s(x +_p y))$
(8) $(\forall x \in_p \mathbb{F})(x *_p \alpha =_p \alpha)$
(9) $(\forall x, y, z \in_p \mathbb{F})((s(\alpha) <_p y \wedge s(\alpha) \leq_p x) \rightarrow x <_p x *_p y)$
(10) $(\forall x, y, z \in_p \mathbb{F})(x *_p s(y) =_p z \rightarrow (x *_p s(y) =_p x *_p y +_p x))$
(11) (Principle of Discrete (Physical-Finite) Induction) Let $D(x)$ be a well-formed formula (in our formalism[6]) with a single free variable. Then $D(\alpha) \rightarrow ((\forall x \in_p \mathbb{F})(x \neq_p \omega \rightarrow (D(s) \rightarrow D(s(a))) \rightarrow (\forall y \in_p \mathbb{F})(D(y))))$

The former axioms represent genuine refinements of the classic Peano Axioms in the sense that all of them are formally supported by physical partitions.

[6]Here, we implicitly assume that we have constructed an enriched and expanded logical system being able to simulate some of the basic features of a first-order logic theory, and following the requirements described in [7].

5. Towards Bounded (Physical) Number Theory

The origins of the physical numbers as a modern multi- and trans-disciplinary refinement of the natural numbers, imposes the question of translating classic number theory into this new hybrid numerical system.

So, let us start with some basic considerations for a better understanding of this translation process.

Explicitly, although the physical and the natural numbers belong to two different kinds of formal worlds, we can compare them in a simple intuitive manner. Namely, let us consider the following correspondence $G : \mathbb{F} \to \mathbb{N}$, sending a physical number v to the cardinal of any partition characterizing it minus one, i.e. $G(v) = |c_v| - 1$, where $c_v \in_f P_a(A)$, for some physical sub-space $A \subseteq_{ss} E_R$, and such that $C_A(C_v) = v$. Note that G is not a surjective correspondence due to the fact that the physical numbers constitutes a finite formal structure and the natural numbers constitutes an infinite set. Furthermore, we can also see G as a correspondence from the physical numbers into the integers by composing with the natural inclusion $i : \mathbb{N} \to \mathbb{Z}$, i.e. $G : \mathbb{F} \to \mathbb{Z}$. Moreover, it is worth noting that G is also a one-to-one correspondence, which respects all the physical arithmetic operations defined in the former axioms. So, we can partially identify[7] the physical numbers with a particular subset of the natural numbers (resp. integers) respecting the corresponding arithmetic operations. Let us call the correspondence G *the physical natural map*. Furthermore, we will name the image of G, i.e. $G(\mathbb{F})$, *the physical natural numbers*. We denote the preimages of a physical natural number n, as $n_{(p)}$, in other words, $G(n_{(p)}) = n$.

From the former considerations we can deduce that for any natural number n coming from a physical number under G, we still can have a kind of physical well-founded intuition, or (physical) model, about the whole amount of elements it possesses. So, we can say that n is a *natural physical number*.

On the other hand, if n do not correspond to any physical number under G, then, per definitionem, we cannot have any kind of model of the cardinality that n represents by mean of a physical sub-space. Thus, we can talk in this case of n being a *natural meta-physical number* (or a mental natural number), i.e. a natural number whose quantitative structure goes beyond the physical realm, and is situated more in a cognitive phenomenological realm.

Let $Q(\mathbb{Z})$ be a (open) question (conjecture) in classic number theory. Then, with all the former clarifications in mind, if we want to solve $Q(\mathbb{Z})$ in the classic setting of arithmetic, we can first try to solve (prove or disprove) the corresponding (open) question (conjecture) $Q(\mathbb{F})$, which is the carefully

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[7]Here, we clarify that the identification is partial because the physical numbers as hybrid structure possesses a more robust nature, meanwhile the corresponding subset of the natural numbers (resp. integers) identified as its image under G is simply a classic mathematical (set-theoretical) structure.

translation of $Q(\mathbb{Z})$ into the physical numbers; and then try to solve the original one.

An advantage of the former procedure is that in the setting of $Q(\mathbb{F})$ we have a finite structure and an enriched gathering of intuitions coming from the physical principles framing $\mathbb{F}$, which are not present, in principle, in the setting of the natural numbers. Similarly, since we are located in a finite deductive framework working with the physical numbers, and additionally we possess an axiomatization in a many-sorted first-order-like framework, we can use computer-based specifications supported by software like the heterogeneus tool set (HETS) [22], described in the common algebraic specification language CASL [1]. More generally, one can enhance the classic human-based logical deductive methods with techniques coming from the new paradigm of conceptual computation [14].

For example, one can easily specify the first Peano axioms for the physical numbers using a CASL language:

```
logic CASL

%%SOME PEANO AXIOMS FOR THE PHYSICAL NUMBERS

spec PeanoPhysicalAxioms =
sort F
op  alpha : F
op  omega : F
op  s__ : F -> F
op  __+__ : F * F -> F
op  __*__ : F * F -> F
preds __leq__: F * F -> F
preds __les__: F * F -> F

forall x, y : F. (x = y -> (x = z => y = z))

forall x, y : F. ((not x = omega) /\
(not y = omega) => ( x = y => s(x) = s(y)))

forall x : F. not s(x) = omega

forall x : F. x leq omega

forall x, y : F. s(x) = s(y) => x = y

forall x : F. x + alpha = x
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forall x, y, z : F. x + s(y) = z => x + s(y) = s(x + y)

forall x : F. x * alpha = x

forall x, y, z : F. (s(x) les y /\ s(alpha) leq x) => (x les x * y)

forall x, y, z : F. x * s(y) = z => x * s(y) = x * y + x

end
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Note that nowadays this hybrid approach for generating interactive proofs of theorems not only in number theory, but also in algebra, topology and many other mathematical sub-disciplines is getting more and more attention from the (multi-disciplinary) academic community (see for example [15], [13], [3], [2] and [9]).

6. Some Seminal Examples of Problems in Physical ('Observable') Number Theory

In this section we will describe how to translate some seminal problems in (elementary) classic number theory into the formal realm of the physical number.

Let us start with a well-known solved problem in number theory, the Fermat's Last Theorem.

6.1. **Fermat's Last Theorem.** This problem is one of the most famous and most astonishing 'diamonds' in modern number theory, since its solution finally founded by Andrew Wiles in 1995, involves several quite elaborated branches of mathematics, like Iwasawa theory, the theory of Elliptic curves, Galois cohomology, algebraic geometry, among others [27]. On the other hand, the statement of the problem is so simple that at the level of elementary school one could grasp the essence of the question. Explicitly, Fermat's Last Theorem states that for any natural number $n \geq 3$, the equation

$$z^n = x^n + y^n, \tag{6.1}$$

has no solution in the natural numbers with $xyz \neq 0$.

Now, note that through the physical natural map we can state the corresponding question in the context of the physical numbers, since the operations of addition, multiplication and exponentiation are defined in a partial manner as well, e.g., the binary (physical) operations defined with the physical numbers are partial (and not total) maps.

So, the corresponding statement is the following:

Physical Fermat's Last Theorem. For any physical number $n_{(p)} \geq_p 3_{(p)}$, the equation

$$z^{n_{(p)}} =_p x^{n_{(p)}} +_p y^{n_{(p)}} \tag{6.2}$$

has no solutions where $x, y, z \in_p \mathbb{P}$ and $x *_p y *_p z \neq_p \alpha =_p 0_{(p)}$.

Now, the former statement is also true for the physical numbers. In fact, let us assume by the sake of contradiction that there exist $m_{(p)}, a_{(p)}, b_{(p)}, c_{(p)} \in_p \mathbb{P}$, with $m_{(p)} \geq_p 3_{(p)}$ and $a *_p a *_p c \neq_p \alpha$ such that $c^{n_{(p)}} =_p a^{n_{(p)}} +_p b^{n_{(p)}}$. Then, via the physical natural map, we would obtain a counter-example of (classic) Fermat's last theorem given by $c^m = a^m + b^m$, where $m, a, b, c \in \mathbb{N}$ with $abc \neq 0$ and $n \geq 3$. This is a contradiction with wiles' proof. So, Physical Fermat's theorem holds as well.

6.2. **Non-solvable Diophantine equations in the Natural Numbers.** Let us consider a Diophantine equation $D(x_1, \cdots, x_s) = 0$, where $D(x_1, \cdots, x_s)$ is a polynomial with coefficients in the natural numbers. Assume that $D = 0$ has no solution over the natural numbers. Then, applying a completely similar argument as the one presented for the physical Fermat's last theorem, one can prove that the corresponding physical Diophantine equation $D_p(x_{1(p)}, \cdots, x_{s(p)}) =_p \alpha$ has no solutions over the physical numbers. More generally, if D is a polynomial with integer coefficients then, it is straightforward to see that there exists two polynomials D_1 and D_2 with coefficient in the natural numbers with s variables such that the Diophantine equation $D = 0$ has (no) solutions over the natural numbers if and only if the equation $D_1 = D_1$ has (no) solutions over the natural number, and, in fact, the solutions are exactly the same. So, again one can simulate the same argument as before for proving that the equation $D_{1p}(x_{1(p)}, \cdots, x_{s(p)}) =_p D_{2p}(x_{1(p)}, \cdots, x_{s(p)})$ has no solutions over the physical numbers.

Let us continue with with a classic old open problem in number theory known as the Goldbach's conjecture.

6.3. **Goldbach's Conjecture.** Goldbach's conjecture states that every even natural number greater than four can be written as the sum of two odd prime numbers [19]. This conjecture is around 280 years old and has been (computationally) verified for all even numbers up to 4×10^{18} [23].

Again, through the physical natural map we can state the corresponding conjecture in the setting of the physical numbers, since the notions of even, odd and prime number; addition and multiplication are defined partially as well.

Physical Goldbach Conjecture (PGC). Any even physical number bigger than $4_{(p)}$ can be written as the (physical) sum of two physical odd prime numbers.

First, note that this physical version of the Goldbach conjecture should be verified only in a finite number of cases, since the physical numbers form a finite entity.

Even more, if we want to have more concrete upper bounds for verifying this conjecture, we can restrict ourselves a little bit more to physical numbers into the observable universe. In other words, we say that a physical number $n_{(p)}$ is *an observable physical number*, if there exists at least one partition Q, and a subspace A completely contained in the observable universe such that $C_A(Q) = n_{(p)}$, i.e., one can codify $n_{(p)}$ with a partition formed with only observable subspaces.

So, we can consider the more restricted form of the former conjecture:

Observable Physical Goldbach Conjecture (OPGC). Any even observable physical number bigger than $4_{(p)}$ can be written as the (physical) sum of two observable physical odd prime numbers.

Now, due to the fact that we can compute rough estimates of the total amount of elementary particles in the observable universe using basic results in cosmology and particle physics, we can estimate a more concrete upper bound for the OPGC. Effectively, one general accepted estimate for the number of elementary particles in the observable universe is the one given by the British physicist Antonio Padilla of 3.28×10^{80} [24].[8]

Thus, roughly speaking, we can say that for proving the OPGC, it is enough to check for physical natural numbers up to $3,28 \times 10^{80}$.

So, combining this with the results of Oliveira e Silta et al we conclude that the interval of interest for proving the OPGC is $[4 \times 10^{18}, 3.28 \times 10^{80}]$.

Therefore, in order to solve the mono-disciplinary (i.e. entirely mathematical problem) given in the Goldbach conjecture, it seems very natural to solve firstly the OPGC, which is a more multidisciplinary problem, where one can use techniques not only from pure mathematics, but also one could integrate on its potential solution estimates and techniques of high energy physics and cosmology, among others; as well as the whole cognitive-meta-mathematical setting of tools developed in artificial mathematical intelligence.

7. Hilbert's Tenth Problem

One of the most fundamental questions in the elementary theory of Diophantine equations from the computational perspective is the one asking if there is an algorithm (at least theoretically speaking) being able to decide if an arbitrary Diophantine equation $D(x_1, \cdots, x_s) = 0$ has a solution in $\mathbb{N}^s$. This question is generally known as the Hilbert's tenth problem due to historical reasons [17]. Now, the query was negatively solved in 1970 in a common effort by Martin Davis, Julia Robinson, Hilary Putnam and Jury

[8] In more refined calculations one could include massless particles like photons and gluons, but for initial simplicity in this article we only include particles with mass.

Matiyasevich and it is known as the MRDP theorem (or Matiyasevich's theorem) [4], [5]. As we have done in the former section, given a Diophantine equation $D = 0$, one can find two polynomials D_1 and D_2 with coefficients in $\mathbb{N}$ such such $D(x_1, \cdots, x_s) = 0$ has a solution in $\mathbb{N}^s$, if and only if $D_1(x_1, \cdots, x_s) = D_2(x_1, \cdots, x_s)$ has a solution in $\mathbb{N}^s$. So, in this setting we can state the counterpart of Hilbert's tenth problem for the physical number as follows:

Physical Hilbert's Tenth Problem (PHTP). There is a general algorithm which is able to decide if an arbitrary physical Diophantine equation of the form $D_{1p}(x_{1(p)}, \cdots, x_{s(p)}) =_p D_{2p}(x_{1(p)}, \cdots, x_{s(p)})$ has at least a solution in $\mathbb{P}^s$, where D_1 and D_2 are polynomials in s variables with coefficients in $\mathbb{P}$.

In comparison with the classic Hilbert's tenth problem, the former physical version of the question can be solved positively, at least theoretically. In fact, since the physical numbers form a finite structure, an algorithm for solving PHTP should simply verify if the corresponding equation $D_1(x_1, \cdots, x_s) = D_2(x_1, \cdots, x_n)$ has (or not) a solution for $s-$tuples of physical natural numbers, which form a finite set, since the physical natural numbers so are. In other words, this algorithm must check tuple by tuple the Diophantine equation tuple by tuple over all the physical natural numbers and finally decide if a solution was (or not) found. Note that the algorithm in this bare form would require an immense amount of time for computing the answer about the solvability of any Diophantine equation, however this technical fact, in principle, is not directly relevant for answering PHTP.

8. Main Logic-Epistemological Conclusions

From all the formal framework described before we can obtain general pragmatics conclusions in the following manner.

Firstly, the classic numerical system used in several ways for our understanding of 'counting', 'gathering' and arithmetic operations should be refined in a way that allows us to introduce principles and intuitions from near scientific disciplines like physics and cognitive sciences. This should be done with the purpose of founding more robust and solid foundational grounds to the study of number theory (from a multi- and trans-disciplinary perspective).

Secondly, the fast development of multifaceted methods in artificial intelligence, and in particular in artificial mathematical intelligence compel us to discover/invent enriched numerical structures for number theory that we can use in a mixed manner (i.e. a man-machine approach) to decipher the seminal open questions in modern (elementary) number theory from a bottom-up perspective, in comparison with the top-down classic point of view.

Thirdly, from an epistemological point of view it seems to be more accurate to face (open) questions in number theory firstly in a bounded discrete manner, i.e. in the setting of the physical numbers, which is a framework based essentially on finite structures. The main reason for this lies in the fact that the working mathematicians may be (dangerously) get used to the endless and meta-physical character of the natural numbers, without noticing that they can encode easily meaningless, or kind of semantically empty, (open) questions covered in well-established logic-mathematical formalism (e.g. first-order logic). In this sense, the first titanic work of Ludwig Wittgenstein is still very relevant and inspiring [28].

Finally, the wide and huge sophistication that lots of open questions in number theory possess should inspire us to follow a modern bottom-up integrative approach. In other words, we should re-build the very seminal structures and methods used for describing formally number theory with the help not only of disciplines beyond mathematics and (classic) logic, but also with (a stronger use of present and coming) artificial devices emerging, for example, from multidisciplinary scientific disciplines like artificial mathematical intelligence.

Acknowledgements

The author wants to thank sincerely to Maria Valeria for all the (in)visible support and love beyond words. Moreover, he thanks to Juan Carlos Díaz and Fabian Suarez for all his sincere friendship and support.

References

1. Michel Bidoit and Peter D. Mosses, *Casl user manual*, Lecture Note in Computer Science 2900, Springer-Verlag, Berlin, Heidelberg, 2004.
2. F. Bou, J. Corneli, Danny A. J. Gómez-Ramírez, E. Maclean, A. Peace, M. Schorlemmer, and A. Smaill, *The role of blending in mathematical invention*, Proceedings of the Sixth International Conference on Computational Creativity (ICCC). S. Colton et. al., eds. Park City, Utah, June 29-July 2, 2015. Publisher: Brigham Young University, Provo, Utah. (2015), 55–62.
3. Roberto Confalonieri, Alison Pease, Marco Schorlemmer, Tarek R Besold, Oliver Kutz, Ewen Maclean, and Maximos Kaliakatsos-Papakostas, *Concept invention*, Springer, 2018.
4. Martin Davis, *Hilbert's tenth problem is unsolvable*, The American Mathematical Monthly **80** (1973), no. 3, 233–269.
5. Danny A. J. Gómez-Ramírez, *Décimo problema de hilbert, curvas elipticas y algunas extensiones*, Master Thesis, Universidad Nacional de Colombia Sede Medellín (2007), 69 pages.
6. ———, *Artificial mathematical intelligence: Cognitive, (meta)mathemticial, physical and philosophical foundations*, Springer International Publishing. ISBN 978-3-030-50272-0, 2020.
7. ———, *General considerations for the new cognitive foundations' program*, Artificial Mathematical Intelligence, Springer, Cham, 2020, pp. 41–52.
8. ———, *Global introduction to the artificial mathematical intelligence general program*, Artificial Mathematical Intelligence, Springer, Cham, 2020, pp. 1–17.
9. ———, *Meta-modeling of classic and modern proofs and concepts*, Artificial Mathematical Intelligence, Springer, Cham, 2020, pp. 201–249.
10. ———, *The most outstanding (future) challenges towards global ami and its plausible extensions*, Artificial Mathematical Intelligence, Springer, Cham, 2020, pp. 251–259.

11. ______, *The physical numbers*, Artificial Mathematical Intelligence, Springer, Cham, 2020, pp. 67–89.
12. ______, *Towards the (cognitive) reality of mathematics and the mathematics of the (cognitive) reality*, Artificial Mathematical Intelligence, Springer, Cham, 2020, pp. 53–66.
13. Danny A. J. Gómez-Ramírez, Marlon Fulla, Ismael Rivera, Juan D Velez, and Edisson Gallego, *Category-based co-genration of seminal concepts and results in algebra and number theory: containment-division rings and goldbach rings*, JP Journal of Algebra, Number Theory and Applications **40** (2018), 887–901.
14. Danny A. J. Gómez-Ramírez and Yoe A. Herrera-Jaramillo, *Conceptual computation in artificial mathematical intelligence as a paradigm-shifting technique in physics and mathematics*, Journal of Physics: Conference Series, vol. 1672, IOP Publishing, 2020, p. 012015.
15. Danny A. J. Gómez-Ramírez and Alan Smaill, *Formal conceptual blending in the (co-) invention of (pure) mathematics*, Concept Invention, Springer, 2018, pp. 221–239.
16. Alexandre Grothendieck, *Récoltes et semailles*, éditeur non identifié, 1992.
17. Michiel Hazewinkel, *Hilbert's 1990 icm lecture. the 23 problems.*, Journal of Computational and Applied Mathematics (1990), 1–11.
18. Allyn Jackson, *Comme appelé du néant—as if summoned from the void: the life of alexandre grothendieck*, Notices of the AMS **51** (2004), no. 4, 1196–1212.
19. Emmanuel Markakis, Christopher Provatidis, and Nikiforos Markakis, *Some issues on goldbach conjecture*, Number Theory **29** (2012), 1–30.
20. K. Meinke and J. V. Tucker, *Many-sorted logic and its applications*, John Wiley & Sons, Inc., New York, NY, USA, 1993.
21. Elliot Mendelson, *Introduction to mathematical logic.(5th edn., new york*, 2010.
22. T. Mossakowski, C. Maeder, and M. Codescu, *Hets user guide (version 0,99)*, 2014.
23. Tomás Oliveira e Silva, Siegfried Herzog, and Silvio Pardi, *Empirical verification of the even goldbach conjecture and computation of prime gaps up to* 4×10^{18}, Mathematics of Computation **83** (2014), no. 288, 2033–2060.
24. Antonio Padilla, *How many particles in the universe*, https://www.youtube.com/watch?v=lpj0E0a0mlU (2018), –.
25. Max Tegmark, *The mathematical universe*, Foundations of physics **38** (2008), no. 2, 101–150.
26. Hao Wang, *The axiomatization of arithmetic*, The Journal of Symbolic Logic **22** (1957), no. 2, 145–158.
27. A. Wiles, *Modular elliptic curves and fermat last theorem*, Annals of Mathematics **142** (1995), 443–551.
28. Ludwig Wittgenstein, *Tractatus logico-philosophicus*, Routledge, 2013.

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