

Contraction, Infinitary Quantifiers, and Omega Paradoxes

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Abstract Our main goal is to investigate whether the infinitary rules for the quantifiers endorsed by Elia Zardini in a recent paper are plausible. First, we will argue that they are problematic in several ways, especially due to their infinitary features. Secondly, we will show that even if these worries are somehow dealt with, there is another serious issue with them. They produce a truth-theoretic paradox that does not involve the structural rules of contraction.

Keywords Substructural Logic · Infinitary Quantifiers · Paradoxes · Truth

1 Introduction

The last few years have witnessed the development of a variety of new and interesting non-classical approaches to the paradoxes of self-reference. Some of these approaches question one or more of the structural properties usually attributed to the classical consequence relation, such as reflexivity, exchange, monotonicity, contraction, and transitivity. It is a well-known fact that the liar paradox, Russell's paradox, Curry's paradox, and other famous conundrums all involve, explicitly or implicitly, the rule of structural contraction. Very roughly, this rule embodies the idea that if something follows from two occurrences of a sentence, then it follows from just one occurrence of it. In this paper we will focus on a specific non-contractive account that has been advocated by Elia Zardini in a number of recent papers (e.g., in [21], [22], and [24]).

One interesting consequence of the failure of contraction is that it is possible to make a distinction between two different conjunctions and two different disjunctions¹ (which we will abbreviate to 'junctions' in what follows): the so-called 'additive' and 'multiplicative' junctions.²

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¹ A similar distinction is available if the rule of structural weakening is rejected (or if both rules are rejected).

² The terminology varies, though. On occasions, the labels 'extensional/intensional' or 'lattice-/group-theoretic' are used.

The usual quantifiers (which are typically represented as \forall and \exists) interact in the expected way with the additive junctions, in the sense that we can understand \forall as a generalized (additive) conjunction and we can understand \exists as a generalized (additive) disjunction. Unfortunately, as far as we know, there is no consensus in the literature regarding how to define multiplicative quantifiers. For instance, discussing the prospects of a set theory based on a substructural logic, Mares & Paoli point out that

A proof system (axiomatic or otherwise) for set theory would presuppose, at least, a proof system for first order logic—and in substructural logics that’s exactly what is beyond the state of the art. The current approaches to first order substructural logics superimpose to propositional logics which contain intensional and extensional connectives a first-order upper layer that only contains extensional quantifiers, simply because there are several conflicting intuitions about what intensional quantifiers look like or what rules they should obey (...). The fascinating proposal in the direction of a contraction-free theory of naive truth recently advanced by Elia Zardini (...), where intensional quantifiers of sorts are an important part of the picture, stands in need of a closer assessment, but may be inadequate exactly for specular reasons, since this system only contains intensional quantifiers (and connectives)—while it is the interplay between the two families of quantifiers that is likely to be especially difficult to unwind. However, until some more light is shed on what we consider one of the most important philosophical, as well as technical, problems about substructural logics, we have to confine ourselves to blocking those paradoxes arising at the level of propositional connexion, while being unable to follow up with a positive proposal as to the form our alternative set theory, or formal truth theory, should assume. [9, p. 463]

We will argue for two things in this paper. First, we will claim in the next section that the rules given by Zardini in [21] for the quantifiers are problematic in several ways. After that, we will show in Section 3 that, even if these worries are somehow taken care of, these quantifiers generate a paradox (due to Andrew Bacon) that does not involve the rules of structural contraction. In Section 4 we provide some closing remarks.

2 Multiplicative quantifiers and infinitary rules

The general framework we will employ relies on sequents. A sequent is an ordered pair of multisets of formulas, represented as $\Gamma \Rightarrow \Delta$.³ The distinction between additive and multiplicative connectives originates in the literature on linear logic, which was first developed by J.Y. Girard in [6]. Linear logic not only rejects the structural rules of contraction but it also rejects the structural rules of weakening.

Definition 1 (*Multiplicative-Additive Linear Logic MALL*) Let Γ , Δ , Π , and Σ be (finite) multisets of formulas, let ϕ and ψ be formulas. The system *MALL* is given by the following rules:⁴

³ Multisets are basically sets with repetitions. That is, a multiset is just like a set except for the fact that it is sensitive to the different occurrences of a member. For example, the multiset with members a , a , and b is *not* the same as the multiset with members a and b . Obviously, we need to use collections that are sensitive to the occurrences of formulas because we do not want to take for granted that contraction holds. Also, since multisets are insensitive to the order in which its members occur, the structural rules of exchange are assumed to hold.

⁴ The name *MALL* and the notation for the logical vocabulary is taken from [5].

Structural rules

$$Id \frac{}{\phi \Rightarrow \phi}$$

$$Cut \frac{\Gamma \Rightarrow \phi, \Delta \quad \Pi, \phi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

Operational rules

$$L\neg \frac{\Gamma \Rightarrow \phi, \Delta}{\Gamma, \neg\phi \Rightarrow \Delta} \quad R\neg \frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\phi, \Delta}$$

Additive operational rules

$$L\sqcap \frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, \phi \sqcap \psi \Rightarrow \Delta} \quad R\sqcap \frac{\Gamma \Rightarrow \phi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \sqcap \psi, \Delta}$$

$$L\sqcup \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \sqcap \psi \Rightarrow \Delta} \quad R\sqcup \frac{\Gamma \Rightarrow \phi, \Delta}{\Gamma \Rightarrow \phi \sqcup \psi, \Delta}$$

$$L\oplus \frac{\Gamma, \phi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \oplus \psi \Rightarrow \Delta} \quad R\oplus \frac{\Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \sqcup \psi, \Delta}$$

Multiplicative operational rules

$$L\otimes \frac{\Gamma, \phi, \psi \Rightarrow \Delta}{\Gamma, \phi \otimes \psi \Rightarrow \Delta} \quad R\otimes \frac{\Gamma \Rightarrow \phi, \Delta \quad \Pi \Rightarrow \psi, \Sigma}{\Gamma, \Pi \Rightarrow \phi \otimes \psi, \Delta, \Sigma}$$

$$L\oplus \frac{\Gamma, \phi \Rightarrow \Delta \quad \Pi, \psi \Rightarrow \Sigma}{\Gamma, \Pi, \phi \oplus \psi \Rightarrow \Delta, \Sigma} \quad R\oplus \frac{\Gamma \Rightarrow \phi, \psi, \Delta}{\Gamma \Rightarrow \phi \oplus \psi, \Delta}$$

Since they will play no role in what follows, we have left out of *MALL* the so-called exponentials and, to simplify things, we have also omitted the additive and multiplicative truth and falsity constants (see [14] for more details).

As the presentation of *MALL* makes clear, to introduce a disjunction on the right-hand side of the sequent arrow, we have two different options available: $R\sqcup$ and $R\oplus$. A dual point applies if we want to introduce a conjunction on the left-hand side of the sequent arrow. We might do so by using the rule $L\sqcap$ or the rule $L\otimes$ (we have two rules for each of $R\sqcup$ and $L\sqcap$, for we can apply these rules if *exactly* one of the disjuncts/conjuncts occurs in the place indicated). There is a similar liberty if we want to introduce a disjunction on the left-hand side of the sequent arrow. We might do so using either $L\sqcup$ or $L\oplus$. And once more, a dual point applies to conjunction regarding the rules $R\sqcap$ and $R\otimes$. Here the disparity has to do with the context (i.e., the formulas

in Γ , Δ , Π , and Σ). In the rules $L\sqcup$ and $R\sqcap$, which are additive, both premises have the same context, namely Γ and Δ . This is not necessarily so for the other rules, which are multiplicative.

In classical logic these subtle distinctions are irrelevant. Sequent calculi presentations of classical logic with multiple conclusions contain (explicitly or implicitly) versions of the rules of structural weakening and contraction:

$$\begin{array}{ll} LK \frac{\Gamma \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta} & RK \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \phi, \Delta} \\ LW \frac{\Gamma, \phi, \phi \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta} & RW \frac{\Gamma \Rightarrow \phi, \phi, \Delta}{\Gamma \Rightarrow \phi, \Delta} \end{array}$$

With these rules in place, the additive and the multiplicative junctions are equivalent, in the sense that $\phi \sqcup \psi$ implies $\phi \oplus \psi$ and vice versa (and similarly for \sqcap and \otimes). In other words, in classical logic it does not matter if we present disjunction (conjunction) additively or multiplicatively, as the following derivations make clear (for \sqcap and \otimes , there are similar derivations):

$$\begin{array}{ll} L\oplus \frac{\phi \Rightarrow \phi \quad \psi \Rightarrow \psi}{\phi \oplus \psi \Rightarrow \phi, \psi} & RK \frac{\phi \Rightarrow \phi}{\phi \Rightarrow \phi, \psi} \quad RK \frac{\psi \Rightarrow \psi}{\psi \Rightarrow \psi, \phi} \\ R\sqcup \frac{\phi \oplus \psi \Rightarrow \phi, \psi}{\phi \oplus \psi \Rightarrow \phi \sqcup \psi, \psi} & R\oplus \frac{\psi \Rightarrow \psi, \phi}{\psi \Rightarrow \phi \oplus \psi} \\ R\sqcap \frac{\phi \oplus \psi \Rightarrow \phi \sqcup \psi, \phi \sqcup \psi}{\phi \oplus \psi \Rightarrow \phi \sqcup \psi} & L\sqcup \frac{\phi \sqcup \psi \Rightarrow \phi \oplus \psi}{\phi \sqcup \psi \Rightarrow \phi \oplus \psi} \\ RW \frac{\phi \oplus \psi \Rightarrow \phi \sqcup \psi}{\phi \oplus \psi \Rightarrow \phi \sqcup \psi} & \end{array}$$

So, putting things a bit differently, if contraction and weakening rules are available, the additive/multiplicative divide vanishes.

The absence of these structural rules changes things drastically. The additive junctions \sqcap and \sqcup are no longer equivalent to the multiplicative junctions \otimes and \oplus . If the contraction rules are rejected, we no longer have $\phi \sqcap \psi \Rightarrow \phi \otimes \psi$ or $\phi \oplus \psi \Rightarrow \phi \sqcup \psi$. And if the weakening rules are rejected, we cannot prove $\phi \otimes \psi \Rightarrow \phi \sqcap \psi$ or $\phi \sqcup \psi \Rightarrow \phi \oplus \psi$. Hence, in *MALL* and other substructural logics we need to be careful when formulating the rules for conjunction and disjunction. One conceptually interesting way of viewing the situation that is sometimes put forward (see [9] and [14]) is that logical constants such as conjunction and disjunction are ambiguous expressions in natural language, with two possible readings, an additive reading and a multiplicative reading. The problem with classical logic is that it is unable to make this distinction, and so it conflates the two readings.

Fascinating as it is, the distinction between different kinds of junctions brings technical complications with it. Since existential (universal) quantification is sometimes understood as a kind of generalized disjunction (conjunction), the presence of two disjunctions (conjunctions) seems to call for the presence of two existential (universal) quantifiers. Of course, we can define additive quantifiers quite naturally in this framework. We will use \exists^A and \forall^A for the additive existential and universal quantifiers, respectively. In the rules below, t is any term and a is a term not occurring in the conclusion-sequent:

$$\begin{array}{ll} L\exists^A \frac{\Gamma, \phi(a) \Rightarrow \Delta}{\Gamma, \exists^A x \phi \Rightarrow \Delta} & R\exists^A \frac{\Gamma \Rightarrow \phi(t), \Delta}{\Gamma \Rightarrow \exists^A x \phi, \Delta} \\ L\forall^A \frac{\Gamma, \phi(t) \Rightarrow \Delta}{\Gamma, \forall^A x \phi \Rightarrow \Delta} & R\forall^A \frac{\Gamma \Rightarrow \phi(a), \Delta}{\Gamma \Rightarrow \forall^A x \phi, \Delta} \end{array}$$

In fact, it is straightforward to add these rules to the system above, thus obtaining a first-order version of *MALL*.⁵

The issue here is how to define the multiplicative quantifiers.⁶ There have been a few proposals in the literature⁷, but we will only focus on Zardini's idea in [21]. Zardini offers a non-contractive theory of naive truth *IKT ω* (the *I* stands for the identity axioms, the *K* stands for weakening, the *T* stands for the truth predicate, and the ω stands for the infinitary quantifiers) that is based on a logic expressed in a purely multiplicative language. The logic (without the truth predicate, whose consideration will be deferred until the next section) can be specified as follows.

Definition 2 (*IK ω*) Let $\Gamma, \Gamma', \dots, \Delta, \Delta', \dots$ be (possibly infinite but at most denumerable) multisets of formulas, let ϕ be a formula, and let t_0, t_1, t_2, \dots be closed singular terms. The system *IK ω* is given by the rule of identity (*Id*), the cut rule (*Cut*), the negation rules ($L\neg$ and $R\neg$), the multiplicative fragment of *MALL* ($L\oplus, R\oplus, L\otimes$, and $R\otimes$), and the following additional rules:

Weakening rules⁸

$$LK^* \frac{\Gamma \Rightarrow \Delta}{\Gamma' \Rightarrow \Delta}$$

$$RK^* \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta'}$$

Quantificational rules

$$L\exists^Z \frac{\Gamma, \phi(t_0) \Rightarrow \Delta \quad \Gamma', \phi(t_1) \Rightarrow \Delta' \quad \Gamma'', \phi(t_2) \Rightarrow \Delta'' \quad \dots}{\Gamma, \Gamma', \Gamma'', \dots, \exists^Z x\phi \Rightarrow \Delta, \Delta', \Delta'', \dots}$$

$$R\exists^Z \frac{\Gamma \Rightarrow \phi(t_0), \phi(t_1), \phi(t_2), \dots, \Delta}{\Gamma \Rightarrow \exists^Z x\phi, \Delta}$$

$$L\forall^Z \frac{\Gamma, \phi(t_0), \phi(t_1), \phi(t_2), \dots \Rightarrow \Delta}{\Gamma, \forall^Z x\phi \Rightarrow \Delta}$$

$$R\forall^Z \frac{\Gamma \Rightarrow \phi(t_0), \Delta \quad \Gamma' \Rightarrow \phi(t_1), \Delta' \quad \Gamma'' \Rightarrow \phi(t_2), \Delta'' \quad \dots}{\Gamma, \Gamma', \Gamma'', \dots \Rightarrow \forall^Z x\phi, \Delta, \Delta', \Delta'', \dots}$$

⁵ The idea that quantifiers obeying these rules should be identified as *the* additive quantifiers (or, at least, that there is an additive—rather than a multiplicative—flavor to them) has been endorsed on several occasions (e.g., in [5, p. 509], in [15, p. 316], in [16, p. 304], and even in [21, p. 509-512]). The identification is likely grounded in the algebraic treatment of these quantifiers, where the universal quantifier is characterized as the infimum in a lattice of values and the existential quantifier as the supremum, thus mimicking additive conjunction and additive disjunction, respectively. While this can be regarded as a compelling reason to support the identification, the matter is not entirely obvious. In fact, below (see footnote 13) we will point to one way in which someone might reject the identification that strikes us at least as worth mentioning.

⁶ This, we reckon, is particularly troubling for those concerned with the concept of truth. If truth is to serve as a device that (perhaps among other things) allows us to express restricted universal and existential quantifications, then we should have the appropriate resources in our theory for truth to fulfill this requirement.

⁷ For example, [11] introduces multiplicative quantifiers for many-valued logic. And [15] and [23] provide philosophical arguments to show how interesting non-extensional quantifiers can be. In particular, [23] argues that in the case of the universal quantifier the additive/multiplicative distinction can be more or less equated with the natural language distinction between ‘anything’ and ‘everything’.

⁸ In these weakening rules, Γ (Δ) is properly included in Γ' (Δ'), where inclusion between multisets is understood in the following way: Γ is a sub-multiset of Γ' if and only if every member of Γ is a member of Γ' and occurs at least as many times in Γ' . In particular, there is nothing preventing an application of the rules where, say, Γ is a finite multiset and Γ' is an infinite multiset.

In the quantifier rules, t_0, t_1, t_2, \dots is a complete enumeration of the closed terms of the language. As Zardini points out, these rules are sound only if every object is denoted by a term in the language, and this means that the objects the quantifiers range over have to be countably many.

Despite their non-standard features, the quantifiers behave nicely in a number of respects. For instance, we can prove Universal Instantiation (i.e., $\forall^Z x\phi \Rightarrow \phi(t)$) and Particular Generalization (i.e., $\phi(t) \Rightarrow \exists^Z x\phi$), if negation behaves in the standard way, we can prove the Quantificational De Morgan rules (those are $\forall^Z x\phi \Leftrightarrow \neg\exists^Z x\neg\phi$, $\exists^Z x\phi \Leftrightarrow \neg\forall^Z x\neg\phi$, $\neg\forall^Z x\phi \Leftrightarrow \exists^Z x\neg\phi$, and $\neg\exists^Z x\phi \Leftrightarrow \forall^Z x\neg\phi$)⁹, among other things. Not only that, but also Zardini's quantifiers seem to interact very nicely with the multiplicative junctions and, at least under certain assumptions, the interplay between them and the additive quantifiers is exactly as one would expect. The first point is very easy to see. As Zardini observes in [21, p. 515], it can be proved that the principles of quantificational adjunction and abjunction are valid (i.e., $\phi(t_0), \phi(t_1), \phi(t_2), \dots \Leftrightarrow \forall^Z x\phi$ and $\exists^Z x\phi \Leftrightarrow \phi(t_0), \phi(t_1), \phi(t_2), \dots$). Recall that in this sort of sequent calculus the structural comma corresponds to the multiplicative conjunction when it occurs on the left and to the multiplicative disjunction when it occurs on the right. So, clearly, we can view these quantifiers as infinitary multiplicative junctions.

To understand why the second point holds (i.e., that the interplay with the additive quantifiers is as one would expect), we need to give a slightly different presentation of the rules of structural contraction:

$$\text{LW}^* \frac{\Gamma, \phi, \phi, \phi, \dots \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta} \qquad \text{RW}^* \frac{\Gamma \Rightarrow \phi, \phi, \phi, \dots, \Delta}{\Gamma \Rightarrow \phi, \Delta}$$

The idea behind these contraction rules is that we can contract infinitely many occurrences of a formula all at once.¹⁰ It is interesting to observe that in a framework where these rules are available, the following two sequents are derivable: $\exists^Z x\phi(x) \Leftrightarrow \exists^A x\phi(x)$ (and similarly for the universal quantifiers). More specifically, if contraction is available, there is a proof that Zardini's existential quantifier implies the additive existential quantifier and if weakening is available, there is a proof of the converse:

$$\begin{array}{c} \text{R}\exists^A \frac{\phi(t_0) \Rightarrow \phi(t_0)}{\phi(t_0) \Rightarrow \exists^A x\phi(x)} \quad \text{R}\exists^A \frac{\phi(t_1) \Rightarrow \phi(t_1)}{\phi(t_1) \Rightarrow \exists^A x\phi(x)} \quad \text{R}\exists^A \frac{\phi(t_2) \Rightarrow \phi(t_2) \quad \dots}{\phi(t_2) \Rightarrow \exists^A x\phi(x) \quad \dots} \\ \text{L}\exists^Z \frac{\quad}{\exists^Z x\phi(x) \Rightarrow \exists^A x\phi(x), \dots, \exists^A x\phi(x), \dots} \\ \text{RW}^* \frac{\quad}{\exists^Z x\phi(x) \Rightarrow \exists^A x\phi(x)} \end{array}$$

$$\begin{array}{c} \text{RK}^* \frac{\phi(t_i) \Rightarrow \phi(t_i)}{\phi(t_i) \Rightarrow \phi(t_0), \phi(t_1), \dots, \phi(t_i), \phi(t_{i+1}), \dots} \\ \text{R}\exists^Z \frac{\quad}{\text{L}\exists^A \frac{\phi(t_i) \Rightarrow \exists^Z x\phi(x)}{\exists^A x\phi(x) \Rightarrow \exists^Z x\phi(x)}} \end{array}$$

What these derivations show is that if the infinitary versions of contraction and weakening are available, then the relation between the \exists^Z and \exists^A is exactly parallel to the relation between \oplus and \sqcup , which is what we should expect from the multiplicative quantifier.¹¹

⁹ We are using ' $\Gamma \Leftrightarrow \Delta$ ' as an abbreviation of ' $\Gamma \Rightarrow \Delta$ and $\Delta \Rightarrow \Gamma$ '.

¹⁰ It is worth noticing that, in virtue of the compactness property, these contraction rules, as well as the weakening rules introduced above, are equivalent over classical logic to the standard rules of contraction and weakening, which only allow us to contract or weaken on a single formula per application.

¹¹ The reader might be skeptical of the use of infinitary structural rules in the proof of the equivalence between \exists^Z and \exists^A . But if we are already on board with the idea of infinitary rules for the quantifiers, it seems unproblematic

In spite of these nice properties, we should be wary of accepting the claim that Zardini’s quantifiers are *the* multiplicative quantifiers. Here we will mostly discuss the existential quantifier, but similar considerations apply to the universal quantifier. Let’s start with $R\exists^Z$. This rule simply equates the existential quantifier to an infinite (multiplicative) disjunction and informally states that there is an object having a certain property ϕ if this object is denoted by the term t_0 or it is denoted by the term t_1 or... . $L\exists^Z$ is a generalized version of the ω -rule. Ignoring the context, it informally states that if we reject that the object denoted by t_0 has the property ϕ and we reject that the object denoted by t_1 has the property ϕ and ..., then we must reject the claim that there is an object having the property ϕ .

One feature that is hard to accept about \exists^Z is that one of its rules is classically unsound. As we mentioned, the $L\exists^Z$ rule implies the ω -rule, a rule that is not classically valid. So, in a way, the resulting logic will extend classical logic, for it accepts some inferential patterns that the classical logician would deem incorrect. Notice that nothing like this happens in *MALL* or in the propositional fragment of *IK $^\omega$* . While multiplicative conjunction and disjunction respect their classical counterparts, the infinitary quantifiers introduced by Zardini do not.¹² In addition to this, these rules treat the quantifiers—from a proof-theoretic perspective—as substitutional, requiring each object in the domain to be denoted by a term in the language. The substitutional interpretation of the quantifiers is subject to a number of problems which are very well-known, so we do not need to rehearse them here (e.g., since the language contains denumerably many terms and formulas, the correctness of the rules depends on the quantifiers ranging over only countably many objects; but if so, then a very restrictive assumption is being made about the domain of quantification).

Instead, we will turn our attention to a puzzling feature of the quantifiers that is specific to Zardini’s infinitary non-contractive framework. Although the distinction between additive and multiplicative junctions is clear, the distinction becomes problematic if we try to extend it to the quantifiers. Here is why. Zardini is operating under the assumption that the existential quantifier should be understood as a generalized disjunction. This assumption is what allows us to distinguish between the additive right rule for the existential quantifier, where only one of the substitution instances occur in the premise-sequent, and the multiplicative right rule, where all the substitution instances (for closed terms) occur in the premise-sequent. However, the assumption also tells us something about the left additive and multiplicative rules for the existential quantifier. In particular, it seems to entail that the rule $L\exists^A$ is not the additive left rule for the existential quantifier (even admitting that $R\exists^A$ is its additive right rule). The reason is that the assumption pushes us towards a multi-premise left rule (in the same way as the left rule for \vee has multiple premises), which in turn means that in this case we should distinguish the additive and the multiplicative rules in terms of shared/independent contexts. Thus, the additive left existential rule would look like this:

$$L\exists^? \frac{\Gamma, \phi(t_0) \Rightarrow \Delta \quad \Gamma, \phi(t_1) \Rightarrow \Delta \quad \Gamma, \phi(t_2) \Rightarrow \Delta \quad \dots}{\Gamma, \exists^? x\phi \Rightarrow \Delta}$$

To put the point a bit differently: treating the multiplicative existential quantifier as an infinitary multiplicative disjunction would be more reasonable if the additive existential quantifier were treated as an infinitary additive disjunction, as in the rule above.¹³ But the consensus seems

to endorse structural rules like these. In any case, we will see below that there are independent reasons to reject Zardini’s quantifiers, so we need not worry at this point about other infinitary rules.

¹² See [16] for a criticism along these lines.

¹³ To further elaborate on the idea hinted at in footnote 5, we should note that it is not entirely out of the question for someone like Zardini to endorse the claim that both the additive and the multiplicative quantifiers should obey infinitary rules. In fact, if infinitary rules are taken to be unproblematic, $L\exists^?$ seems to be a generalization

to be that the additive existential quantifier is the quantifier obeying the standard rules $L\exists^A$ and $R\exists^A$, and the rule above is not equivalent to $L\exists^A$, at least not in every kind of setting. In particular, if the theory under consideration includes arithmetic, we should not expect that disproving $\phi(\bar{n})$, for every n , is sufficient to reject $\exists^A x\phi(x)$.¹⁴ Now, because this rule contracts the context, it might be unappealing for someone like Zardini. But that is besides the point. The mere possibility of giving a context-sharing infinitary rule for the existential quantifier should cast some doubts on the idea of categorizing a quantifier as multiplicative solely because it obeys an infinitary rule that resembles one of the rules for \oplus .¹⁵

In sum, there are a number of reasons to be suspicious of Zardini’s infinitary quantifiers. And interestingly, the case against them can be made stronger. For we will see in the next section that even if none of the arguments we have given are sufficient to conclude that the rules for the quantifiers should be jettisoned, there is another problem affecting them. To this we now turn.

3 An omega paradox

The main goal of this section is to show that Zardini’s quantifiers generate a truth-theoretic paradox that does not rely on the structural rules of contraction. More specifically, what we will establish is that adding Zardini’s truth theory to a background theory that contains a modicum of arithmetic results in inconsistency. In order to prove this, we draw heavily on a recent ω -inconsistency result due to Andrew Bacon. Bacon shows in [1] that:

Proposition 1 *Any transparent truth theory¹⁶ closed under the following rules is ω -inconsistent¹⁷:*

1. If $\phi \Rightarrow \psi$, then $\exists x\phi \Rightarrow \exists x\psi$
2. $\phi \rightarrow \exists x\psi \Rightarrow \exists x(\phi \rightarrow \psi)$ (where x is not free in ϕ)

To establish that Zardini’s quantifiers generate a paradox we need to do two things. First, we have to check that (the appropriate instances of) these two principles hold in (a suitable extension of) IK^ω . Secondly, we need to make sure that Bacon’s proof does not implicitly rely on contraction at some point. This will be enough to cause trouble, since the rule $L\exists^Z$ —which is basically a generalized version of the ω -rule—will turn any ω -inconsistency into a plain inconsistency.

of $L\perp$ in exactly the same way as $L\exists^Z$ is a generalization of $L\oplus$. However, neither Zardini nor, as far as we know, anyone else has taken this route, so there is no need to discuss it here.

¹⁴ We are very grateful to an anonymous referee for suggesting this way of presenting the argument.

¹⁵ In case the reader is wondering about the interaction of this quantifier with the other ones, this will ultimately depend on the right-hand side rule for $\exists^?$ (which, presumably, should be the same as that for \exists^A). But even if we leave this unspecified, it can be shown that, assuming weakening, $\exists^? x\phi \Rightarrow \exists^Z x\phi$ is derivable, and even without weakening, $\exists^? x\phi \Rightarrow \exists^A x\phi$ is derivable.

¹⁶ A *transparent truth theory* is, roughly, a theory where for any formula ϕ , ϕ and ϕ is true are everywhere intersubstitutable. In the context of a sequent calculus this amounts to the idea that if $\Gamma \Rightarrow \Delta$ is derivable and $\Gamma' \Rightarrow \Delta'$ is obtained from $\Gamma \Rightarrow \Delta$ by replacing (possibly within a formula) ϕ for ϕ is true (or vice versa), then $\Gamma' \Rightarrow \Delta'$ is derivable too.

¹⁷ Recall that a theory is *ω -inconsistent* if it proves $\Rightarrow \exists x\phi$ and it also proves $\phi(\bar{n}) \Rightarrow$ for each n . In this context, we could equivalently say—using infinitary sequents—that a theory is ω -inconsistent if it proves $\exists x\phi \Rightarrow$ and also $\Rightarrow \phi(\bar{0}), \phi(\bar{1}), \phi(\bar{2}), \dots$. Bacon makes a distinction between strongly ω -inconsistent theories and weakly ω -inconsistent theories. However, for our purposes this distinction will be unnecessary. From a semantic point of view, the models of a consistent but ω -inconsistent arithmetical theory cannot be standard: each model of the theory contains in its domain non-standard numbers. In other words, among other undesirable consequences, ω -inconsistent theories disrupt the intended ontology of the base arithmetical theory (see [2] and [8]). The issue of ω -inconsistency has not received much attention in the literature on substructural approaches to the truth-theoretic paradoxes. One exception is [4], where Andreas Fjellstad shows how to prove the ω -inconsistency of certain theories without assuming the transitivity of the corresponding consequence relations. At certain points in the proof of Theorem 1 below we rely on some of Fjellstad’s insights.

Up until now our focus has been on conjunction and disjunction. So, in the presentation of IK^ω , we omitted the (multiplicative) conditional \rightarrow . But, since Bacon's result uses a conditional, it will be useful to have \rightarrow around. It can be defined as usual, in terms of disjunction and negation: $\neg\phi \oplus \psi := \phi \rightarrow \psi$. In fact, it is easy to check that \rightarrow satisfies the following rules:

$$\text{L}\rightarrow \frac{\Gamma \Rightarrow \phi, \Delta \quad \Pi, \psi \Rightarrow \Sigma}{\Gamma, \Pi, \phi \rightarrow \psi \Rightarrow \Delta, \Sigma} \quad \text{R}\rightarrow \frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \rightarrow \psi, \Delta}$$

Another thing that was absent from IK^ω was the truth predicate $Tr(x)$. So far, we have only been discussing the logical properties of Zardini's quantifiers, so there was no need to consider truth. But now we want to suggest that these quantifiers are prone to paradoxes in a way that the rest of the logical vocabulary of IK^ω is not. In order to substantiate this claim, we need to introduce a transparent truth predicate to IK^ω . This is done using the rules below, where as usual we take $\ulcorner \phi \urcorner$ to be a term that denotes the formula ϕ .

Definition 3 (IKT^ω) The system IKT^ω is given by IK^ω plus the following rules for the truth predicate:

$$\text{LTr} \frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, Tr\ulcorner \phi \urcorner \Rightarrow \Delta} \quad \text{RTr} \frac{\Gamma \Rightarrow \phi, \Delta}{\Gamma \Rightarrow Tr\ulcorner \phi \urcorner, \Delta}$$

The paradox we will discuss below does not show that IKT^ω is trivial. In fact, in [21] Zardini proved its non-triviality via a nice cut elimination result. What the paradox is meant to show is that there is an important expressive limitation affecting IKT^ω . In particular, there are certain self-referential sentences involving the quantifiers that cannot be expressed on pain of triviality. To prove this fact we will consider a theory that extends IKT^ω with some logical and arithmetical machinery. The machinery is very modest. We will work with a first-order language that contains all the numerals $\bar{0}, \bar{1}, \bar{2}, \dots$, a two-place function symbol $+$ for addition, a logical constant \perp , and two additional function symbols \rightarrow and \perp representing these connectives.¹⁸ We shall also assume that there is some standard Gödel coding available for the expressions of the language and we will use the usual corner-quotes notation, as above. Another thing that we will require of this theory is for it to prove all true identities expressible in the language.¹⁹ More specifically, the theory will be couched in a language that contains the following function symbol f :

$$\begin{aligned} f(\bar{0}, x) &= x \rightarrow \perp \\ f(\bar{n} + \bar{1}, x) &= x \rightarrow f(\bar{n}, x) \end{aligned} \quad ^{20}$$

Intuitively, this function tracks down the number of embedded conditionals pointing to \perp . For example, $f(\bar{1}, \ulcorner \phi \urcorner)$ stands for the number coding the formula $\phi \rightarrow (\phi \rightarrow \perp)$, $f(\bar{2}, \ulcorner \phi \urcorner)$ stands for the number coding the formula $\phi \rightarrow (\phi \rightarrow (\phi \rightarrow \perp))$, and so on. The function f is of course available in Peano arithmetic and weaker theories. That is, these theories can prove for each sentence ϕ that:

¹⁸ We assume familiarity with the dot notation and with the overlining notation. The reader can look at [7] for the details.

¹⁹ Despite appearances, this is not too demanding. There are extremely weak arithmetical theories satisfying this requirement, such as Baby Arithmetic (see e.g. [19]).

²⁰ The function f has its origins in Shaw-Kwei's [18]. See also [1] and [10]. For reasons of readability, we will write f instead of f , omitting the dot notation for the function.

$$f(\bar{0}, \ulcorner \phi \urcorner) = \ulcorner \phi \urcorner \rightarrow \perp$$

and for each number n

$$f(\overline{n+1}, \ulcorner \phi \urcorner) = \ulcorner \phi \urcorner \rightarrow f(\bar{n}, \ulcorner \phi \urcorner).$$

Finally, following Bacon's [1], we will also assume that the language contains a sentence μ that is provably equivalent to $\exists^Z xTrf(x, \ulcorner \mu \urcorner)$. That is, we will assume that the following two sequents are available: $\mu \Rightarrow \exists^Z xTrf(x, \ulcorner \mu \urcorner)$ and $\exists^Z xTrf(x, \ulcorner \mu \urcorner) \Rightarrow \mu$.

So, our official theory can be specified as follows.

Definition 4 (IKT_+^ω) The system IKT_+^ω is given by IKT^ω plus the following rules:

Identity rules²¹

$$L= \frac{\Gamma, \phi(t) \Rightarrow \Delta}{\Gamma, s = t, \phi(s) \Rightarrow \Delta} \qquad R= \frac{\Gamma \Rightarrow \phi(t), \Delta}{\Gamma, s = t \Rightarrow \phi(s), \Delta}$$

\perp rule²²

$$L\perp \frac{}{\perp \Rightarrow}$$

Rules for μ

$$L\mu \frac{}{\mu \Rightarrow \exists^Z xTrf(x, \ulcorner \mu \urcorner)} \qquad R\mu \frac{}{\exists^Z xTrf(x, \ulcorner \mu \urcorner) \Rightarrow \mu}$$

Rule for terms

If t and s are terms such that $t = s$, then we have:

²¹ Typically, the identity predicate also requires the following additional rule:

$$=pop \frac{\Gamma, t = t \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

However, as an anonymous referee rightly pointed out, this rule is admissible through the application of the *Terms* rule (see below) and *Cut*.

²² Another rule that comes to mind for \perp is this:

$$\perp\text{-drop} \frac{\Gamma \Rightarrow \perp, \Delta}{\Gamma \Rightarrow \Delta}$$

In IKT_+^ω $\perp\text{-drop}$ and $L\perp$ are interderivable, so we can use either one. For definiteness, we will stick to $L\perp$.

$$\text{Terms} \frac{}{\Rightarrow t = s}$$

This last rule plays two key roles. First, it guarantees that every term t which is not a numeral is such that there is a numeral \bar{n} such that $\Rightarrow t = \bar{n}$. This is important because $\bar{0}, \bar{1}, \bar{2}, \dots$ are not all the closed terms of the language, and in applying $L\exists^Z$, we need all the closed terms of the language. For instance, $\bar{3}, \bar{2} + \bar{1}$, and $\bar{0} + \bar{3}$ are all singular terms standing for the number 3, but only the first is a numeral. However, as our theory has enough resources to prove that $\Rightarrow \bar{n} = t$, for any $t = n$, then if $\Gamma, \phi(\bar{n}) \Rightarrow \Delta$ is derivable for each n , using *Cut* and $L=$, it follows that $\Gamma, \phi(t) \Rightarrow \Delta$ has a derivation for each closed term t of the language as well.²³ More rigorously, we can prove:

Lemma 1 *The following rules for \exists^Z are derivable in IKT_+^ω :*

$$L\exists_*^Z \frac{\Gamma, \phi(\bar{0}) \Rightarrow \Delta \quad \Gamma', \phi(\bar{1}) \Rightarrow \Delta' \quad \Gamma'', \phi(\bar{2}) \Rightarrow \Delta'' \quad \dots}{\Gamma, \Gamma', \Gamma'', \dots, \exists^Z x \phi \Rightarrow \Delta, \Delta', \Delta'', \dots}$$

$$R\exists_*^Z \frac{\Gamma \Rightarrow \phi(\bar{0}), \phi(\bar{1}), \phi(\bar{2}), \dots, \Delta}{\Gamma \Rightarrow \exists^Z x \phi, \Delta}$$

Proof This is straightforward. In the case of $R\exists_*^Z$, we just apply RK^* and $R\exists^Z$. In the case of $L\exists_*^Z$, we need to use *Terms*, $L=$, *Cut*, and $L\exists^Z$. \square

Secondly, the rule *Terms* also guarantees that the function f behaves as expected. In particular, we have the following:

Lemma 2 (See [4], p. 101) *These rules for f are derivable in IKT_+^ω :*

$$f_0 \frac{\Gamma, f(\bar{0}, \ulcorner \mu \urcorner) = \ulcorner \mu \urcorner \rightarrow \perp \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad f_{n+1} \frac{\Gamma, f(\bar{n} + \bar{1}, \ulcorner \mu \urcorner) = \ulcorner \mu \urcorner \rightarrow f(\bar{n}, \ulcorner \mu \urcorner) \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \text{ For each number } n$$

Proof Immediate, by the definition of f , the rule *Terms*, and *Cut*. \square

Now we have all we need to state our main result.

Theorem 1 *IKT_+^ω is trivial.*

The rest of the section will be devoted to the proof of this. It is by no means difficult, but it is a bit long and tedious. We have added some figures at the end of the section so that the reader can get a general idea of the proof's structure.

Let's get to it. First, by $L\mu$ we have

$$\mu \Rightarrow \exists^Z x \text{Tr}f(x, \ulcorner \mu \urcorner)$$

By LTr we can infer

$$\text{Tr} \ulcorner \mu \urcorner \Rightarrow \exists^Z x \text{Tr}f(x, \ulcorner \mu \urcorner)$$

²³ And similarly for $R\forall^Z$. If $\Gamma \Rightarrow \phi(\bar{n}), \Delta$ is derivable for each n , using *Cut* and $R=$, it follows that for each closed term t , $\Gamma \Rightarrow \phi(t), \Delta$ is also derivable.

and by $R\rightarrow$

$$\Rightarrow Tr\ulcorner\mu^\neg \rightarrow \exists^Z xTrf(x, \ulcorner\mu^\neg)$$

Starting from this sequent, the strategy of the proof is as in Figure 1 (see below). Here are its main steps:

$$(1) \Rightarrow Tr\ulcorner\mu^\neg \rightarrow \exists^Z xTrf(x, \ulcorner\mu^\neg)$$

$$(2) \Rightarrow \exists^Z x(Tr\ulcorner\mu^\neg \rightarrow Trf(x, \ulcorner\mu^\neg))$$

$$(3) \Rightarrow \exists^Z xTrf(x + \bar{1}, \ulcorner\mu^\neg)$$

$$(4) \Rightarrow \exists^Z xTrf(x, \ulcorner\mu^\neg)$$

$$(5) \Rightarrow \mu$$

$$(6) Trf(\bar{n}, \ulcorner\mu^\neg) \Rightarrow, \text{ for each } n$$

$$(7) \exists^Z xTrf(x, \ulcorner\mu^\neg) \Rightarrow$$

$$(8) \Rightarrow$$

$$(9) \Gamma \Rightarrow \Delta$$

The reader familiar with [1] will notice that this is basically the structure of Bacon's proof of ω -inconsistency.

We already have the first sequent, so let's see how to obtain the second. Consider the following derivation, which we will call \mathcal{D}_0 :²⁴

$$\begin{array}{c} \vdots \\ \mu \rightarrow \perp \Rightarrow \mu \rightarrow \perp \\ \text{LTr} \frac{}{Tr\ulcorner\mu \rightarrow \perp^\neg \Rightarrow \mu \rightarrow \perp} \\ \text{L=} \frac{}{f(\bar{0}, \ulcorner\mu^\neg) = \ulcorner\mu \rightarrow \perp^\neg, Trf(\bar{0}, \ulcorner\mu^\neg) \Rightarrow \mu \rightarrow \perp} \\ \text{Subs of id and } f_0 \frac{}{Trf(\bar{0}, \ulcorner\mu^\neg) \Rightarrow \mu \rightarrow \perp} \end{array}$$

We also have the following derivation, labelled \mathcal{D}_1 :

$$\begin{array}{c} \vdots \\ \mu \rightarrow (\mu \rightarrow \perp) \Rightarrow \mu \rightarrow (\mu \rightarrow (\mu \rightarrow \perp)) \\ \text{LRT} \frac{}{Tr\ulcorner\mu \rightarrow (\mu \rightarrow \perp)^\neg \Rightarrow Tr\ulcorner\mu \rightarrow (\mu \rightarrow (\mu \rightarrow \perp))^\neg} \\ \text{Subs of id and } f_1 \frac{}{Trf(\bar{1}, \ulcorner\mu^\neg) \Rightarrow Tr(\ulcorner\mu^\neg \rightarrow f(\bar{1}, \ulcorner\mu^\neg))} \end{array}$$

Next comes \mathcal{D}_2 :

²⁴ In what follows, for the sake of simplicity, we will be sloppy with the substitution of identicals. Observe that what the rules $L=$ and $R=$ allow us to do—with the help of the rules for f , $Terms$, and Cut —is to substitute identical terms. So, instead of explicitly applying these rules, we will directly substitute identical terms in the derivations and we will mark the steps at which we do this with the label “*Subs of id*”.

$$\text{LRTr} \frac{\text{Subs of id and } f_2 \frac{\overline{\mu \rightarrow (\mu \rightarrow (\mu \rightarrow \perp))} \Rightarrow \mu \rightarrow (\mu \rightarrow (\mu \rightarrow (\mu \rightarrow \perp)))}{\text{Tr} \ulcorner \mu \rightarrow (\mu \rightarrow (\mu \rightarrow \perp)) \urcorner \Rightarrow \text{Tr} \ulcorner \mu \rightarrow (\mu \rightarrow (\mu \rightarrow (\mu \rightarrow \perp)) \urcorner}}{\text{Tr} f(\bar{2}, \ulcorner \mu \urcorner) \Rightarrow \text{Tr}(\ulcorner \mu \urcorner \rightarrow f(\bar{2}, \ulcorner \mu \urcorner))}}$$

Of course, we could continue with \mathcal{D}_3 , \mathcal{D}_4 , and so on. If we put the end-sequents of all these derivations together, we can proceed as in Figure 2 to obtain the sequent

$$\text{Tr} \ulcorner \mu \urcorner \rightarrow \exists^Z x \text{Tr} f(x, \ulcorner \mu \urcorner) \Rightarrow \exists^Z x \text{Tr}(\ulcorner \mu \urcorner \rightarrow f(x, \ulcorner \mu \urcorner)).$$

Now, we already have

$$\Rightarrow \text{Tr} \ulcorner \mu \urcorner \rightarrow \exists^Z x \text{Tr} f(x, \ulcorner \mu \urcorner)$$

So, by an application of *Cut* we can obtain

$$\Rightarrow \exists^Z x \text{Tr}(\ulcorner \mu \urcorner \rightarrow f(x, \ulcorner \mu \urcorner))$$

The next step is to obtain

$$\exists^Z x \text{Tr}(\ulcorner \mu \urcorner \rightarrow f(x, \ulcorner \mu \urcorner)) \Rightarrow \exists^Z x \text{Tr} f(x + \bar{1}, \ulcorner \mu \urcorner)$$

This can be done using the derivation in Figure 3. Then another application of *Cut* is enough to infer

$$\Rightarrow \exists^Z x \text{Tr} f(x + \bar{1}, \ulcorner \mu \urcorner)$$

Our next goal is to show that

$$\exists^Z x \text{Tr} f(x + \bar{1}, \ulcorner \mu \urcorner) \Rightarrow \exists^Z x \text{Tr} f(x, \ulcorner \mu \urcorner)$$

This is not hard to do, as the derivation in Figure 4 shows. So, by another application of *Cut*

$$\Rightarrow \exists^Z x \text{Tr} f(x, \ulcorner \mu \urcorner)$$

But $\exists^Z x \text{Tr} f(x, \ulcorner \mu \urcorner) \Rightarrow \mu$ by $R\mu$. So cutting on $\exists^Z x \text{Tr} f(x, \ulcorner \mu \urcorner)$, we have

$$\Rightarrow \mu$$

And of course, given that $L\perp$ proves $\perp \Rightarrow$, by $L\rightarrow$ we infer

$$\mu \rightarrow \perp \Rightarrow$$

which by another application of $L\rightarrow$ delivers

$$\mu \rightarrow (\mu \rightarrow \perp) \Rightarrow$$

and then

$$\mu \rightarrow (\mu \rightarrow (\mu \rightarrow \perp)) \Rightarrow$$

and so on. By the truth rules, the rules for f , $Terms$, and $L=$, we then have for each number n :

$$Trf(\bar{n}, \ulcorner \mu \urcorner) \Rightarrow$$

This means that the resulting theory is ω -inconsistent. But, since the rule $L\exists_*^Z$ is available, we have

$$\exists^Z x Trf(x, \ulcorner \mu \urcorner) \Rightarrow$$

And we can turn this ω -inconsistency into an inconsistency *simpliciter*, because we end up with both

$$\Rightarrow \exists^Z x Trf(x, \ulcorner \mu \urcorner)$$

and

$$\exists^Z x Trf(x, \ulcorner \mu \urcorner) \Rightarrow$$

By a final application of Cut we can obtain the empty sequent:

$$\Rightarrow$$

And, by weakening,

$$\Gamma \Rightarrow \Delta$$

In other words, IKT_+^ω is trivial. This is surprising since *contraction was not applied at any point in the derivation!*

Fig. 1 Strategy of the proof

$$\begin{array}{c}
 \vdots \\
 \hline
 \Rightarrow Tr\Gamma\mu^\neg \rightarrow \exists^Z xTrf(x, \Gamma\mu^\neg) \\
 \hline
 \text{Cut} \\
 \hline
 \begin{array}{ccc}
 \vdots & & \vdots \\
 \hline
 Tr\Gamma\mu^\neg \rightarrow \exists^Z xTrf(x, \Gamma\mu^\neg) \Rightarrow \exists^Z xTr(\Gamma\mu^\neg \rightarrow f(x, \Gamma\mu^\neg)) & & \vdots \\
 \hline
 \Rightarrow \exists^Z xTr(\Gamma\mu^\neg \rightarrow f(x, \Gamma\mu^\neg)) & & \exists^Z xTr(\Gamma\mu^\neg \rightarrow f(x, \Gamma\mu^\neg)) \Rightarrow \exists^Z xTrf(x + \bar{1}, \Gamma\mu^\neg) \\
 \hline
 \Rightarrow \exists^Z xTrf(x + \bar{1}, \Gamma\mu^\neg)
 \end{array}
 \end{array}$$

We call the previous derivation D_α

$$\begin{array}{c}
 \vdots \\
 \hline
 \exists^Z xTrf(x + \bar{1}, \Gamma\mu^\neg) \Rightarrow \exists^Z xTrf(x, \Gamma\mu^\neg) \\
 \hline
 \text{Cut} \\
 \hline
 \begin{array}{ccc}
 \vdots & & \vdots \quad D_\alpha \\
 \hline
 \exists^Z xTrf(x + \bar{1}, \Gamma\mu^\neg) \Rightarrow \exists^Z xTrf(x, \Gamma\mu^\neg) & & \Rightarrow \exists^Z xTrf(x + \bar{1}, \Gamma\mu^\neg) \\
 \hline
 \Rightarrow \exists^Z xTrf(x, \Gamma\mu^\neg) & & \exists^Z xTrf(x, \Gamma\mu^\neg) \Rightarrow \mu \\
 \hline
 \Rightarrow \mu
 \end{array}
 \end{array}$$

We call the previous derivation D_β

$$\begin{array}{c}
 \vdots \\
 \hline
 \Rightarrow \mu \\
 \hline
 \mu \rightarrow \perp \Rightarrow \\
 \hline
 \begin{array}{ccc}
 \vdots & & \vdots \\
 \hline
 \mu \rightarrow \perp \Rightarrow & & \Rightarrow \mu \\
 \hline
 \mu \rightarrow (\mu \rightarrow \perp) \Rightarrow & & \Rightarrow \mu \\
 \hline
 \mu \rightarrow (\mu \rightarrow (\mu \rightarrow \perp)) \Rightarrow & & \Rightarrow \mu \\
 \hline
 \dots \\
 \vdots
 \end{array}
 \end{array}$$

Call this last derivation D_δ . By D_δ we have for each n that $Trf(\bar{n}, \Gamma\mu^\neg) \Rightarrow$. So we can reason as follows:

$$\begin{array}{c}
 \vdots \quad D_\delta \quad \vdots \quad D_\delta \quad \vdots \quad D_\delta \quad \dots \quad \vdots \quad D_\beta \\
 \hline
 Trf(\bar{0}, \Gamma\mu^\neg) \Rightarrow \quad Trf(\bar{1}, \Gamma\mu^\neg) \Rightarrow \quad Trf(\bar{2}, \Gamma\mu^\neg) \Rightarrow \quad \dots \quad \Rightarrow \exists^Z xTrf(x, \Gamma\mu^\neg) \\
 \hline
 \text{Cut} \\
 \hline
 \exists^Z xTrf(x, \Gamma\mu^\neg) \Rightarrow \\
 \hline
 LRK^* \frac{\Rightarrow}{\Gamma \Rightarrow \Delta}
 \end{array}$$

Fig. 2

$$\begin{array}{c}
\begin{array}{c} \mathcal{D}_0 \quad \mathcal{D}_1 \quad \mathcal{D}_2 \quad \dots \end{array} \\
\frac{\text{L}\exists^Z \frac{\text{Trf}(\bar{0}, \ulcorner \mu \urcorner) \Rightarrow \mu \rightarrow \perp}{\text{L}\rightarrow \frac{\exists^Z x \text{Trf}(x, \ulcorner \mu \urcorner) \Rightarrow \mu \rightarrow \perp, \text{Tr}(\ulcorner \mu \urcorner \rightarrow f(\bar{1}, \ulcorner \mu \urcorner)), \text{Tr}(\ulcorner \mu \urcorner \rightarrow f(\bar{2}, \ulcorner \mu \urcorner)), \dots}{\text{R}\rightarrow \frac{\mu, \text{Tr}\ulcorner \mu \urcorner \rightarrow \exists^Z x \text{Trf}(x, \ulcorner \mu \urcorner) \Rightarrow \mu \rightarrow \perp, \text{Tr}(\ulcorner \mu \urcorner \rightarrow f(\bar{1}, \ulcorner \mu \urcorner)), \text{Tr}(\ulcorner \mu \urcorner \rightarrow f(\bar{2}, \ulcorner \mu \urcorner)), \dots}{\text{RTr} \frac{\text{Tr}\ulcorner \mu \urcorner \rightarrow \exists^Z x \text{Trf}(x, \ulcorner \mu \urcorner) \Rightarrow \mu \rightarrow (\mu \rightarrow \perp), \text{Tr}(\ulcorner \mu \urcorner \rightarrow f(\bar{1}, \ulcorner \mu \urcorner)), \text{Tr}(\ulcorner \mu \urcorner \rightarrow f(\bar{2}, \ulcorner \mu \urcorner)), \dots}{\text{Subs of id} \frac{\text{Tr}\ulcorner \mu \urcorner \rightarrow \exists^Z x \text{Trf}(x, \ulcorner \mu \urcorner) \Rightarrow \text{Tr}\ulcorner \mu \urcorner \rightarrow (\mu \rightarrow \perp)^\neg, \text{Tr}(\ulcorner \mu \urcorner \rightarrow f(\bar{1}, \ulcorner \mu \urcorner)), \text{Tr}(\ulcorner \mu \urcorner \rightarrow f(\bar{2}, \ulcorner \mu \urcorner)), \dots}{\text{R}\exists^Z_* \frac{\text{Tr}\ulcorner \mu \urcorner \rightarrow \exists^Z x \text{Trf}(x, \ulcorner \mu \urcorner) \Rightarrow \text{Tr}(\ulcorner \mu \urcorner \rightarrow f(\bar{0}, \ulcorner \mu \urcorner)), \text{Tr}(\ulcorner \mu \urcorner \rightarrow f(\bar{1}, \ulcorner \mu \urcorner)), \text{Tr}(\ulcorner \mu \urcorner \rightarrow f(\bar{2}, \ulcorner \mu \urcorner)), \dots}{\text{Tr}\ulcorner \mu \urcorner \rightarrow \exists^Z x \text{Trf}(x, \ulcorner \mu \urcorner) \Rightarrow \exists^Z x \text{Trf}(\ulcorner \mu \urcorner \rightarrow f(x, \ulcorner \mu \urcorner))}}}}}}}{\text{RTTr} \frac{\text{Id} \frac{\mu \Rightarrow \mu}{\mu \Rightarrow \text{Tr}\ulcorner \mu \urcorner}}}}
\end{array}$$

Fig. 3

$$\begin{array}{c}
\begin{array}{c} \vdots \\ \mu \rightarrow (\mu \rightarrow \perp) \Rightarrow \mu \rightarrow (\mu \rightarrow \perp) \\ \vdots \end{array} \quad \begin{array}{c} \vdots \\ \dots \\ \mu \rightarrow (\mu \rightarrow (\mu \rightarrow \perp)) \Rightarrow \mu \rightarrow (\mu \rightarrow (\mu \rightarrow \perp)) \dots \\ \vdots \end{array} \\
\text{LRTTr} \frac{\mu \rightarrow (\mu \rightarrow \perp) \Rightarrow \mu \rightarrow (\mu \rightarrow \perp)}{\text{Subs of id} \frac{\text{Tr}\ulcorner \mu \urcorner \rightarrow (\mu \rightarrow \perp)^\neg \Rightarrow \text{Tr}\ulcorner \mu \urcorner \rightarrow (\mu \rightarrow \perp)^\neg}{\text{L}\exists^Z_* \frac{\text{Tr}(\ulcorner \mu \urcorner \rightarrow f(\bar{0}, \ulcorner \mu \urcorner)) \Rightarrow \text{Trf}(\bar{0} + \bar{1}, \ulcorner \mu \urcorner)}{\text{R}\exists^Z_* \frac{\exists^Z x \text{Tr}(\ulcorner \mu \urcorner \rightarrow f(x, \ulcorner \mu \urcorner)) \Rightarrow \text{Trf}(\bar{0} + \bar{1}, \ulcorner \mu \urcorner), \text{Trf}(\bar{1} + \bar{1}, \ulcorner \mu \urcorner), \text{Trf}(\bar{2} + \bar{1}, \ulcorner \mu \urcorner), \dots}{\exists^Z x \text{Tr}(\ulcorner \mu \urcorner \rightarrow f(x, \ulcorner \mu \urcorner)) \Rightarrow \exists^Z x \text{Trf}(x + \bar{1}, \ulcorner \mu \urcorner)}}}}}} \quad \begin{array}{c} \text{LRTTr} \frac{\mu \rightarrow (\mu \rightarrow (\mu \rightarrow \perp)) \Rightarrow \mu \rightarrow (\mu \rightarrow (\mu \rightarrow \perp)) \dots}{\text{Subs of id} \frac{\text{Tr}\ulcorner \mu \urcorner \rightarrow (\mu \rightarrow (\mu \rightarrow \perp))^\neg \Rightarrow \text{Tr}\ulcorner \mu \urcorner \rightarrow (\mu \rightarrow (\mu \rightarrow \perp))^\neg \dots}{\text{Tr}(\ulcorner \mu \urcorner \rightarrow f(\bar{1}, \ulcorner \mu \urcorner)) \Rightarrow \text{Trf}(\bar{1} + \bar{1}, \ulcorner \mu \urcorner)} \dots} \\
\text{R}\exists^Z_* \frac{\exists^Z x \text{Tr}(\ulcorner \mu \urcorner \rightarrow f(x, \ulcorner \mu \urcorner)) \Rightarrow \text{Trf}(\bar{0} + \bar{1}, \ulcorner \mu \urcorner), \text{Trf}(\bar{1} + \bar{1}, \ulcorner \mu \urcorner), \text{Trf}(\bar{2} + \bar{1}, \ulcorner \mu \urcorner), \dots}{\exists^Z x \text{Tr}(\ulcorner \mu \urcorner \rightarrow f(x, \ulcorner \mu \urcorner)) \Rightarrow \exists^Z x \text{Trf}(x + \bar{1}, \ulcorner \mu \urcorner)}
\end{array}
\end{array}$$

Fig. 4

$$\begin{array}{c}
\begin{array}{c} \vdots \\ \text{Trf}(\bar{0} + \bar{1}, \ulcorner \mu \urcorner) \Rightarrow \text{Trf}(\bar{1}, \ulcorner \mu \urcorner) \\ \vdots \end{array} \quad \begin{array}{c} \vdots \\ \dots \\ \text{Trf}(\bar{1} + \bar{1}, \ulcorner \mu \urcorner) \Rightarrow \text{Trf}(\bar{2}, \ulcorner \mu \urcorner) \\ \vdots \end{array} \quad \begin{array}{c} \vdots \\ \dots \\ \text{Trf}(\bar{2} + \bar{1}, \ulcorner \mu \urcorner) \Rightarrow \text{Trf}(\bar{3}, \ulcorner \mu \urcorner) \dots \\ \vdots \end{array} \\
\text{L}\exists^Z_* \frac{\text{Trf}(\bar{0} + \bar{1}, \ulcorner \mu \urcorner) \Rightarrow \text{Trf}(\bar{1}, \ulcorner \mu \urcorner)}{\text{RK}^* \frac{\exists^Z x \text{Trf}(x + \bar{1}, \ulcorner \mu \urcorner) \Rightarrow \text{Trf}(\bar{1}, \ulcorner \mu \urcorner), \text{Trf}(\bar{2}, \ulcorner \mu \urcorner), \text{Trf}(\bar{3}, \ulcorner \mu \urcorner), \dots}{\text{R}\exists^Z_* \frac{\exists^Z x \text{Trf}(x + \bar{1}, \ulcorner \mu \urcorner) \Rightarrow \text{Trf}(\bar{0}, \ulcorner \mu \urcorner), \text{Trf}(\bar{1}, \ulcorner \mu \urcorner), \text{Trf}(\bar{2}, \ulcorner \mu \urcorner), \text{Trf}(\bar{3}, \ulcorner \mu \urcorner), \dots}{\exists^Z x \text{Trf}(x + \bar{1}, \ulcorner \mu \urcorner) \Rightarrow \exists^Z x \text{Trf}(x, \ulcorner \mu \urcorner)}}}}}}
\end{array}$$

4 Final remarks

There are a number of interesting issues arising from the previous two sections. The first thing we want to point out is that the proof above does not seem to go through with the additive quantifier \exists^A . The reason is simple. The most obvious way of obtaining $\exists^A x Tr(\ulcorner \mu \urcorner \rightarrow f(x, \ulcorner \mu \urcorner))$ from $Tr \ulcorner \mu \urcorner \rightarrow \exists^A x Tr f(x, \ulcorner \mu \urcorner)$ involves an application of right contraction. This step is an instance of the classically valid rule that allows us to go from $\phi \rightarrow \exists^A x \psi$ to $\exists^A x(\phi \rightarrow \psi)$:

$$\begin{array}{c}
 \vdots \\
 \hline
 \text{R}\rightarrow \frac{\phi \Rightarrow \phi, \psi}{\Rightarrow \phi, \phi \rightarrow \psi} \\
 \text{R}\exists^A \frac{\Rightarrow \phi, \phi \rightarrow \psi}{\Rightarrow \phi, \exists^A x(\phi \rightarrow \psi)} \\
 \text{L}\rightarrow \frac{\Rightarrow \phi, \exists^A x(\phi \rightarrow \psi)}{\phi \rightarrow \exists^A x \psi \Rightarrow \exists^A x(\phi \rightarrow \psi), \exists^A x(\phi \rightarrow \psi)} \\
 \text{RW} \frac{\phi \rightarrow \exists^A x \psi \Rightarrow \exists^A x(\phi \rightarrow \psi), \exists^A x(\phi \rightarrow \psi)}{\phi \rightarrow \exists^A x \psi \Rightarrow \exists^A x(\phi \rightarrow \psi)} \\
 \hline
 \vdots \\
 \hline
 \text{R}\rightarrow \frac{\psi, \phi \Rightarrow \psi}{\psi \Rightarrow \phi \rightarrow \psi} \\
 \text{R}\exists^A \frac{\psi \Rightarrow \phi \rightarrow \psi}{\psi \Rightarrow \exists^A x(\phi \rightarrow \psi)} \\
 \text{L}\exists^A \frac{\psi \Rightarrow \exists^A x(\phi \rightarrow \psi)}{\exists^A x \psi \Rightarrow \exists^A x(\phi \rightarrow \psi)}
 \end{array}$$

This rule does hold for Zardini's quantifier \exists^Z . So, even though the proof offered in the previous section does not involve any explicit application of contraction, it might be suggested that the rule $\text{R}\exists^Z$ absorbs contraction, at least for existentially quantified formulas on the right-hand side of the sequent arrow.²⁵ While the premise-sequent of that rule is $\Gamma \Rightarrow \phi(t_1), \phi(t_2), \phi(t_3), \dots, \Delta$, its conclusion-sequent is $\Gamma \Rightarrow \exists^Z x \phi, \Delta$, but we can see this last sequent as the result of contracting on the existentially quantified formula infinitely many times, as in Section 2. As one anonymous referee puts it, the aforementioned surprise in the proof of theorem 1 might be said to vanish once we realize that $\text{R}\exists^Z$ is a contraction-absorbing rule.²⁶

This, we reckon, is too hasty. The rule $\text{R}\exists^Z$ might be said to be contraction-absorbing in the same way that, say, the rule $\text{R}\oplus$ is contraction-absorbing. Of course, to fully absorb contraction we need to suitably combine these rules with context-sharing left introduction rules. However, the whole point of Zardini's approach is to have a purely multiplicative vocabulary. And although it is this choice of vocabulary that allows him to retain a number of classically valid principles (such as *modus ponens*, the law of excluded middle, and so on) that would otherwise require contraction, there is no paradox affecting the quantifier-free fragment of Zardini's theory. Having one of the contraction-absorbing rules is not enough to wreak havoc in the case of the connectives, so we do find it a bit surprising that this is enough in the case of the quantifiers.²⁷

²⁵ In this respect, $\text{R}\exists^Z$ is similar to the more well-known (see [20], p. 77):

$$\text{R}\exists^* \frac{\Gamma \Rightarrow \exists^* x A(x), A(t), \Delta}{\Gamma \Rightarrow \exists^* x A(x), \Delta}$$

in that they both can be used to provide a sequent calculus where contraction is absorbed (pending the appropriate choice of the other rule for the existential quantifier and the rest of the logical rules). Moreover, in the proof of ω -inconsistency developed by Fjellstad in [4], it is precisely this kind rule that is used (actually, because Fjellstad's proof is given in terms of the universal quantifier, what is needed is a dual rule for introducing the universal quantifier on the left-hand side of the sequent arrow, but this is not important).

²⁶ Moreover, the referee also points out that a quick inspection of Fjellstad's proof will reveal that for it to go through it is only required that the right-hand side rule of the quantifier employed admits contraction.

²⁷ Another interesting question that might be posed is if the proof holds for the purely additive fragment of the language (i.e., the fragment with the additive quantifier *and the additive conditional*). The rules for the additive conditional (\sqsupset) are as follows:

Another feature worth noting has to do with the notion of ω -inconsistency. For Zardini's quantifiers there is no interesting notion of ω -inconsistency. Given the rules for the quantifiers (along with the rules of cut and weakening), both notions are indistinguishable from the notion of plain inconsistency or triviality. If we have a derivation of $\Rightarrow \phi(\bar{0}), \phi(\bar{1}), \phi(\bar{2}), \dots$, by $R\exists_*^Z$ we also have a derivation of $\Rightarrow \exists^Z x\phi(x)$ and if we have a derivation of $\phi(\bar{n}) \Rightarrow$ for every n , by $L\exists_*^Z$ we also have a derivation of $\exists^Z x\phi(x) \Rightarrow$. This is why the proof we used, which is in essence an ω -inconsistency proof, turns out to be a full triviality proof.²⁸

Before finishing, let's consider one more thing about the proof in the last section that might strike the reader as doubtful. The function f used to construct the self-referential sentence μ is usually taken to require an arithmetical theory, but non-contractive accounts of truth (like the ones in [9], [17] and [21]) handle self-reference without introducing arithmetic. This is so for a reason. Simply adding the axioms of Peano or Robinson's arithmetic is not enough. We cannot expect all arithmetical theorems to follow from the axioms in the absence of contraction. It won't work either to use Negri and von Plato's method from [12] and [13] to introduce a version of Robinson's arithmetic with contraction built-in. This is because in order to have contraction admissible over the logical part of the theory, the method requires single-premise multiplicative rules and multi-premise additive (context-sharing) rules for the logical connectives, and we have seen that this is not available in Zardini's approach.

Nevertheless, we think that this is unproblematic. For instance, there is no difficulty, as far as we can tell, with restricting contraction to arithmetical formulas. This will simply give us the machinery to construct μ in the usual way using a predicate capturing the diagonalisation function. Another legitimate path that we could have followed—but didn't—is simply to *assume* that all the theorems of arithmetic are at our disposal. So, for example, the diagonal lemma would hold in full generality and it would apply to extensions of our basic language that contain \exists^Z in the usual way, thus allowing us to construct self-referential sentences with this quantifier in the same way that we can construct self-referential sentences with new predicates or operators that we might like to add to the basic language of arithmetic.²⁹

Although we think that there is nothing wrong with these approaches, we have taken a different route in the previous section. In fact, IKT_+^ω is not strong enough to prove all the theorems of Robinson's or Peano arithmetic. We have not required IKT_+^ω to represent all recursive functions, but only to prove the relevant facts about f . Nor have we required IKT_+^ω to prove the diagonal lemma in full generality, but only the instance involving μ . In other words, the triviality proof for IKT_+^ω requires much less than a proper arithmetical theory or a full theory of syntax. This is not to say that it could not be extended to such a theory. Our point here is that much less is needed to cause trouble.

$$L\Box \frac{\Gamma \Rightarrow \phi, \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \Box \psi \Rightarrow \Delta} \qquad R\Box \frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma \Rightarrow \phi \Box \psi, \Delta} \quad R\Box \frac{\Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \Box \psi, \Delta}$$

With these rules, it is easy to check that the step from $\Rightarrow \phi \Box \exists^A x\psi$ to $\Rightarrow \exists^A x(\phi \Box \psi)$ is indeed valid (because $L\Box$ contracts the context). However, the interest of this is moot. As far as we know, no one has advocated a truth theory based on the additive conditional in the literature. Therefore, we will not explore this alternative.

²⁸ As expected, for the additive quantifier the notion of ω -inconsistency is indeed different from the notion of plain inconsistency or triviality.

²⁹ The situation is the same as that for non-classical theories of truth that include a non-standard negation. When studying paradoxes in a logic that is, say, paraconsistent or paraconsistent, we assume that there is a sentence λ which is equivalent to (in the sense that it has the same value as) $\neg Tr^{\ulcorner \lambda \urcorner}$, where \neg is the non-classical negation of the logic. In doing so, we are assuming a background syntax theory which proves that equivalence.

Lastly, it is worth mentioning that we are not the firsts to wonder about the relation between Zardini’s approach and arithmetic. In [3, p.862] the authors point out that:

(...) while IKT^ω is known to be non-trivial, its relation to models of PA has not yet been explored.

The proof in the previous section goes some way towards an answer. Of course, Zardini proves the consistency of his truth theory—which includes the infinitary quantifiers. However, what the result of this paper shows is that there are self-referential sentences that are simply not expressible in the language of Zardini’s theory. If they were, the resulting theory would be trivial.

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