Non-deterministic semantics for cocanonical and semi-cocanonical deduction systems

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Abstract

This article aims to dualize several results concerning various types (including possibly Cut-free and Identity-free systems) of canonical multiple-conclusion sequent calculi, i.e. Gentzen-style deduction systems for sequents, equipped with well-behaved forms of left and right introduction rules for logical expressions. In this opportunity, we focus on a different kind of calculi that we dub cocanonical, that is, Gentzen-style deduction systems for sequents, equipped with well-behaved forms of left and right elimination rules for logical expressions. These systems, simply put, have rules that proceed from sequents featuring complex formulas to sequents featuring their component subformulas. Our main goals are to prove soundness and completeness results for the target systems' consequence relations in terms of their characteristic 3- or 4-valued non-deterministic matrices.

1 Introduction

This article aims to *dualize* several results proved in [3], [12] and [13] concerning various types (including possibly Cut-free and Identity-free systems) of *canonical* multiple-conclusion sequent calculi, i.e. Gentzen-style deduction systems for sequents, equipped with well-behaved forms of left and right introduction rules for logical expressions.[1]

In this opportunity, we focus on a different kind of calculi that we dub *cocanonical*, that is, Gentzen-style deduction systems for sequents, equipped with well-behaved forms of left and right elimination rules for logical expressions. These systems, simply put, have rules that proceed from sequents featuring complex formulas to sequents featuring their component subformulas. Our main goals are to prove soundness and completeness results for the target systems' consequence relations in terms of their characteristic 3- or 4-valued non-deterministic matrices.

[1] There are some other articles where Lahav and his coauthors extend these investigations to calculi designed with single-conclusions and modal logics, such as [1], [14], and [15]. We leave the generalization of our proposal to frameworks of those sorts for future work.

$$\frac{}{\varphi \Rightarrow \varphi}\ \text{Id}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ \text{LW} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\ \text{RW}$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Pi \Rightarrow \Sigma, \varphi}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ \text{Cut}$$

Figure 1: Identity, Weakening, and Cut rules

For this purpose, the article is organized as follows. In Section 2, we review the existing material on systems with canonical rules, and possibly with Identity and Cut. In Section 3, we start our dualization by introducing systems with what we call cocanonical rules, and possibly also with Identity or Cut. In Section 4, we show how to induce certain non-deterministic semantics from the previous kinds of systems. In Section 5, we provide our characterization results for canonical and cocanonical systems in terms of the aforementioned non-deterministic semantics. Finally, Section 6 closes with some remarks on extensions of the systems containing only canonical or cocanonical rules with restricted versions of the structural rules of Identity and Cut.

2 Canonical and semi-canonical systems

In what follows, we will work with (propositional) languages $\mathcal{L}$ for which the set of well-formed formulas $FOR(\mathcal{L})$ can be constructed recursively, as usual. In doing so $\varphi, \psi, \chi, ...$ will refer to formulas, and $\Gamma, \Delta, \Sigma, ...$ will refer to sets of formulas. Below, we define *sequents* of the form $\Gamma \Rightarrow \Delta$ as pairs $\langle \Gamma, \Delta \rangle$, denoting single sequents with s or enumerating them as $s_1, s_2, s_3, ...$, and sets of sequents with S or enumerating them as $S_1, S_2, S_3, ...$ and so on.

As usual, a *rule* is an expression of the form S/s, where S is a finite set of premises and s is the conclusion. When S is empty, the rule is usually called an *axiom*. A *sequent calculus* **G** is a set of rules and axioms. For future reference, in Figure 1 below we see what the structural rules of Identity, Weakening, and Cut look like. Notice that Identity constitutes, as it were, the only logical axiom.

The literature has devoted a great deal of discussion to Gentzen-style deduction systems for sequents equipped with well-behaved forms of left and right *introduction* rules for logical expressions. These systems, when their well-behavedness is taken into account, are referred to as canonical calculi, and are analyzed and studied in what nowadays are fundamental pieces in literature on such calculi, like [2] and [3]. We review a formal approach to these definitions, below.

Definition 2.1. A *canonical rule* for an n-ary connective $\diamond$ of $\mathcal{L}$ is an expression of the form S/s, where S is a finite set of premises (each one composed only by propositional letters in $p_1, ..., p_n$, if any), and the conclusion s is $\Rightarrow \diamond(p_1, ..., p_n)$ in the right rules, or $\diamond(p_1, ..., p_n) \Rightarrow$ in the left rules. An application of a canonical right rule $\{\Pi_1 \Rightarrow \Sigma_1, ..., \Pi_m \Rightarrow \Sigma_m\}/ \Rightarrow \diamond(p_1, ..., p_n)$ is any inference step of the following form:

$$\frac{\Gamma_1, \sigma(\Pi_1) \Rightarrow \Delta_1, \sigma(\Sigma_1) \qquad ... \qquad \Gamma_m, \sigma(\Pi_m) \Rightarrow \Delta_m, \sigma(\Sigma_m)}{\Gamma_1, ..., \Gamma_m \Rightarrow \Delta_1, ..., \Delta_m, \sigma(\diamond(p_1, ..., p_n))}$$

where σ is a substitution function on $FOR(\mathcal{L})$ and Γ, Δ are sets of formulas.[2] Similarly for the left canonical rules.

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma, \neg\varphi \Rightarrow \Delta}\text{L}\neg \qquad \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}\text{R}\neg$$

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\text{L}\wedge \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Pi \Rightarrow \Sigma, \psi}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \varphi \wedge \psi}\text{R}\wedge$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Pi, \psi \Rightarrow \Sigma}{\Gamma, \Pi, \varphi \vee \psi \Rightarrow \Delta, \Sigma}\text{L}\vee \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}\text{R}\vee$$

Figure 2: Some canonical rules for $\neg$, $\wedge$, $\vee$

For future reference, we follow the literature, as in e.g. [17], saying that in a canonical rule for an n-ary connective $\diamond$ of $\mathcal{L}$ we refer to the $\diamond$-formula being introduced as the *principal* formula and to the distinguished formulas appearing in the premise of the rules as the *auxiliary* formulas. Thus, the usual introduction rules for the classical connectives—as depicted, for instance, in Gentzen's famous [9]—are canonical. For instance, in Figure 2 we can see applications of canonical rules for $\neg, \wedge$ and $\vee$.

Definition 2.2. A sequent calculus containing all the structural rules, and canonical rules for the logical expressions is called *canonical*. If it lacks the Cut rule but has the Identity axiom it's (`-C+A`) *semi-canonical*, if it's the other way around it's (`+C-A`) *semi-canonical*, and if it has neither it's (`-C-A`) *semi-canonical.*

Having presented what is already known in the literature, we now move on to our contribution for broadening the scope of study of Gentzen-style deduction systems for sequents.

3 Cocanonical and semi-cocanonical systems

In the introduction we mentioned our intention to study deduction systems for sequents equipped with well-behaved forms of left and right *elimination* rules for logical expressions. These systems, simply put, have rules that proceed from sequents featuring complex formulas to sequents featuring their component subformulas. We refer to rules of this form as cocanonical, and we define them precisely as follows.

[2]In another context, sequent-calculi and rules can be defined for different collections of formulas, e.g. sequences, multisets, etc. Here we are using sets, since this makes our proofs below easier to carry out, but nothing substantial hinges on this choice. Also, notice that rules with multiple premises are always context-independent or multiplicative. As is clarified in [14], since Weakening and Contraction are always assumed, they are interderivable with their context-sharing or additive counterparts.

Definition 3.1. A *cocanonical rule* for an n-ary connective $\diamond$ of $\mathcal{L}$ is an expression of the form S/s, where S is a single sequent of the form $\Rightarrow \diamond(p_1, ..., p_n)$ in the right rules, or $\diamond(p_1, ..., p_n) \Rightarrow$ in the left rules and the conclusion s is a single sequent (composed only by propositional letters in $p_1, ..., p_n$, if any). An application of a cocanonical right rule $\{\Rightarrow \diamond(p_1, ..., p_n)\}/\Pi \Rightarrow \Sigma$ is any inference step of the following form:

$$\frac{\Gamma \Rightarrow \Delta, \sigma(\diamond(p_1, ..., p_n))}{\Gamma, \sigma(\Pi) \Rightarrow \Delta, \sigma(\Sigma)}$$

where σ is a substitution function on $FOR(\mathcal{L})$ and Γ, Δ are sets of formulas. Similarly for the left cocanonical rules.

$$\frac{\Gamma, \neg\varphi \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta}\ \text{EL}\neg \qquad \frac{\Gamma \Rightarrow \Delta, \neg\varphi}{\Gamma, \varphi \Rightarrow \Delta}\ \text{ER}\neg$$

$$\frac{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}{\Gamma, \varphi, \psi \Rightarrow \Delta}\ \text{EL}\wedge \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}{\Gamma \Rightarrow \Delta, \varphi(\psi)}\ \text{ER}\wedge$$

$$\frac{\Gamma, \varphi \vee \psi \Rightarrow \Delta}{\Gamma, \varphi(\psi) \Rightarrow \Delta}\ \text{EL}\vee \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \vee \psi}{\Gamma \Rightarrow \Delta, \varphi, \psi}\ \text{ER}\vee$$

Figure 3: Some cocanonical rules for $\neg$, $\wedge$, $\vee$

Once more, for future reference, in a cocanonical rule for an n-ary connective $\diamond$ of $\mathcal{L}$, we call the $\diamond$-formula being eliminated the *coprincipal* formula and its subformulas appearing in the conclusion of the rules the *coauxiliary* formulas. Therefore, the inverse of the usual (invertible) introduction rules for the classical connectives—as defined, for instance, in Negri and von Plato [17]—are cocanonical rules. For example, those in Figure 3 are applications of cocanonical rules for $\neg, \wedge$ and $\vee$, where in $ER\wedge$ and in $EL\vee$, $\varphi(\psi)$ means that the connective can be eliminated in any of the two formulas.

Definition 3.2. A sequent calculus containing all the structural rules, and cocanonical rules for the logical expressions is called *cocanonical*. If it lacks the Cut rule but has the Identity axiom it's (`-C+A`) *semi-cocanonical*, if it's the other way around it's (`+C-A`) *semi-cocanonical*, and if it has neither it's (`-C-A`) *semi-cocanonical*.

Having presented our newly introduced target systems, we now move on to our main goal in this article: showing the connection of canonical and cocanonical systems with 3- or 4-valued non-deterministic matrices.

4 Non-deterministic semantics

In this section, we study the semantics of sequent calculi with canonical or cocanonical rules, borrowing and broadening the scope of some definitions found in reference texts from the literature on non-deterministic semantics, such as [4]. For our purposes below, we will use non-deterministic semantics to interpret logical connectives in structures known as Qmatrices, Pmatrices, and Bmatrices, that generalize regular logical matrices—as found, e.g. in [6].

The origin of these structures is found in the discussion of logical consequence as induced by logical matrices. Recall that, when studying regular matrices and their connection to logic, it is customary to distinguish a set of designated values, later used in defining the notion of a model of a formula, and the notion of a model of a sequent. Some years ago in works such as [16], [7], and [5] scholars defined, respectively, Qmatrices, Pmatrices, and Bmatrices by generalizing logical matrices in distinguish not one but two sets of values. Lots of interesting things can be said of them, and for details we defer the reader to the source material and, in particular, to the very elucidating and recent [5].

In what follows, we will try to exploit the best of two worlds, by using the logical power of Bmatrices, while also allowing connectives to have non-deterministic interpretations.

Definition 4.1. A *non-deterministic Bmatrix* (NBmatrix, for short) $\mathbf{M}$ for $\mathcal{L}$ consists of a nonempty set $\mathcal{V}_\mathbf{M}$ of truth-values, a pair of sets $\mathsf{Y}_\mathbf{M}, \mathsf{N}_\mathbf{M} \subseteq \mathcal{V}_\mathbf{M}$ of designated and antidesignated truth-values, respectively, and a function $\diamond_\mathbf{M}$ from $\mathcal{V}_\mathbf{M}$ to $\wp(\mathcal{V}_\mathbf{M}) \setminus \emptyset$, for every connective $\diamond$ of $\mathcal{L}$.

Definition 4.2. A *valuation* in a NBmatrix $\mathbf{M}$ for $\mathcal{L}$ ($\mathbf{M}$-valuation, for short) is a function v from $FOR(\mathcal{L})$ to $\mathcal{V}_\mathbf{M}$ such that, for every complex formula $\diamond(\varphi_1, ..., \varphi_n)$, we have that $v(\diamond(\varphi_1, ..., \varphi_n)) \in \diamond_\mathbf{M}(v(\varphi_1), ..., v(\varphi_n))$.

Definition 4.3. Let $\mathbf{M}$ be a NBmatrix. A $\mathbf{M}$-valuation v is a *model* of a sequent $\Gamma \Rightarrow \Delta$ if and only if whenever $v(\gamma) \in \mathcal{V}_\mathbf{M} \setminus \mathsf{N}_\mathbf{M}$ for all $\gamma \in \Gamma$, then $v(\delta) \in \mathsf{Y}_\mathbf{M}$ for some $\delta \in \Delta$. Similarly, a $\mathbf{M}$-valuation is a model of a set of sequents S if and only if it is a model of every $s \in S$. Finally, a sequent $\Gamma \Rightarrow \Delta$ is valid in $\mathbf{M}$ (symbolized as $\vDash_\mathbf{M} \Gamma \Rightarrow \Delta$) if and only if every $\mathbf{M}$-valuation is a model of it.

In the present article, we will work with NBmatrices defined over subsets of a four-valued set of truth values $\mathcal{V}_\mathbf{M} \subseteq \{t, \top, \bot, f\}$, with the further assumptions that $\mathsf{Y}_\mathbf{M} \subseteq \{t, \top\}$, and $\mathsf{N}_\mathbf{M} \subseteq \{\top, f\}$.[3] For the sake of clarity, we refer by the *classical satisfiability* of a sequent or set of sequents to the phenomenon of having a model, when formulas are only allowed to take values in $\{t, f\}$.

Let $\Gamma \Rightarrow \Delta$ consist only of propositional variables in $p_1, ..., p_n$, then a valuation v over this set can be portrayed as a tuple $\langle x_1, ..., x_n \rangle$ where $v(p_i) = x_i \in \{t, \top, \bot, f\}$. With this succinct notation in mind, we can talk about valuations and their relations with canonical and cocanonical rules, with the hope of extracting semantics for each n-ary connective $\diamond$ of $\mathcal{L}$ out of the corresponding rules and the values assigned, in each valuation, to their auxiliary or coauxiliary formulas.

Definition 4.4. A tuple $\langle x_1, ..., x_n \rangle$ *fulfills a canonical rule* r for an n-ary connective $\diamond$ of $\mathcal{L}$ if and only if it is a model of each of its premises. Analogously, it *fulfills a cocanonical rule* r for an n-ary connective $\diamond$ of $\mathcal{L}$ if and only if it isn't a model of its conclusion.

[3]For our present purposes, this can be solely taken as a mere stipulation. However, see [5] or [8] to expand on the dispensability of more values.

Definition 4.5. Let **G** be a (semi-)canonical sequent calculus for $\mathcal{L}$. The NBmatrix $\mathbf{M_G}$ associated to it is defined such that $\mathcal{V}_\mathbf{M}$ should be identified with $\{t, \top, \bot, f\}$ if **G** is (`-C-A`) semi-canonical, with $\{t, \top, f\}$ if **G** is (`-C+A`) semi-canonical, with $\{t, \bot, f\}$ if **G** is (`+C-A`) semi-canonical, and with $\{t, f\}$ if **G** is canonical. Furthermore, in all cases $\mathsf{Y}_\mathbf{M} = \{t, \top\} \cap \mathcal{V}_\mathbf{M}$, and $\mathsf{N}_\mathbf{M} = \{\top, f\} \cap \mathcal{V}_\mathbf{M}$. Finally, for every n-ary connective $\diamond$ of $\mathcal{L}$:

(i) $t \in \diamond_{\mathbf{M_G}}(x_1, ..., x_n)$ if and only if $\langle x_1, ..., x_n \rangle$ doesn't fulfill any left rule of **G** for $\diamond$;

(ii) $f \in \diamond_{\mathbf{M_G}}(x_1, ..., x_n)$ if and only if $\langle x_1, ..., x_n \rangle$ doesn't fulfill any right rule of **G** for $\diamond$;

(iii) $\bot \in \diamond_{\mathbf{M_G}}(x_1, ..., x_n)$ if and only if $\langle x_1, ..., x_n \rangle$ doesn't fulfill any rule of **G** for $\diamond$;

(iv) $\top \in \diamond_{\mathbf{M_G}}(x_1, ..., x_n)$, for every $x_1, ..., x_n$.

Naturally, if $\bot \notin \mathcal{V}_\mathbf{M}$, then disregard clause (iii), and similarly if $\top \notin \mathcal{V}_\mathbf{M}$, disregard clause (iv).

The following definition is the crux of the much desired well-behaved character of left and right introduction rules for connectives, in the context of sequent calculi. For a thorough discussion thereof, see, e.g. [3] and the articles referred below.

Definition 4.6. A (semi-)canonical system **G** is called *coherent* if $S_1 \cup S_2$ is a set of sequents that is classically unsatisfiable whenever **G** includes two canonical rules of the form $S_1/ \Rightarrow \diamond(p_1, ..., p_n)$ and $S_2/\diamond(p_1, ..., p_n) \Rightarrow$, for all n-ary connectives $\diamond$ in $\mathcal{L}$.

In a nutshell, in the presence of Identity and Cut, coherence prevents us from triviality. Roughly speaking, if the system is not coherent, assuming Identity for all the auxiliary formulas in the premises of both rules, we can start a derivation with the instances needed to apply the two canonical rules and, after applying both, we can use Cut and trivialize the system.

For further illustration, let's observe a typical example of a pair of canonical rules that are not coherent, those for the infamous *tonk* from [18]:

$$\frac{\psi, \Gamma \Rightarrow \Delta}{\Gamma, \varphi \odot \psi \Rightarrow \Delta}\,\mathrm{L}\odot \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \odot \psi}\,\mathrm{R}\odot$$

It's easy to check that the premises of both rules are not unsatisfiable. Let's see what happens with the non-deterministic semantics in calculi containing only these two rules, together with Cut, Identity, or both.

In the first case, we have a canonical sequent calculus. The tables obtained from these rules are the following:

$\odot$	t	f
t	$\{t\}$	$\{\}$
f	$\{f, t\}$	$\{f\}$

So, in a canonical system, we cannot obtain well-defined semantics since there is an empty output for one operation. Similarly, if we add Cut we still get some empty outputs. However, if we add Identity but not Cut, we have a semantics for the connective $\odot$:

$\odot$	t	$\top$	f
t	$\{t, \top\}$	$\{\top\}$	$\{\top\}$
$\top$	$\{t, \top\}$	$\{\top\}$	$\{\top\}$
f	$\{t, \top, f\}$	$\{\top, f\}$	$\{\top, f\}$

The fact that the absence of Cut allows for the presence of incoherent pairs of rules is represented in the formulation of Theorems 5.4 and 5.5.

Proposition 4.7. *For every coherent canonical or (*`+C-A`*) semi-canonical sequent calculus* $\mathbf{G}$*,* $\mathbf{M_G}$ *is well-defined.*

Proof. See Proposition 4 in [3] and Theorem 5 in [12]. ☐

It's easily seen that, regardless of coherence, if $\mathbf{G}$ is a (`-C+A`) semi-canonical or a (`-C-A`) semi-canonical sequent calculus, then $\mathbf{M_G}$ is well-defined.

	$\neg$
t	$\{\top, f\}$
$\top$	$\{\top\}$
$\bot$	$\{t, \top, \bot, f\}$
f	$\{t, \top\}$

$\wedge$	t	$\top$	$\bot$	f
t	$\{t, \top\}$	$\{\top\}$	$\{t, \top, \bot, f\}$	$\{\top, f\}$
$\top$	$\{\top\}$	$\{\top\}$	$\{\top, f\}$	$\{\top, f\}$
$\bot$	$\{t, \top, \bot, f\}$	$\{\top, f\}$	$\{t, \top, \bot, f\}$	$\{\top, f\}$
f	$\{\top, f\}$	$\{\top, f\}$	$\{\top, f\}$	$\{\top, f\}$

$\vee$	t	$\top$	$\bot$	f
t	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top\}$
$\top$	$\{t, \top\}$	$\{\top\}$	$\{t, \top\}$	$\{\top\}$
$\bot$	$\{t, \top\}$	$\{t, \top\}$	$\{t, \top, \bot, f\}$	$\{t, \top, \bot, f\}$
f	$\{t, \top\}$	$\{\top\}$	$\{\top, f\}$	$\{\top, f\}$

Figure 4: Four-valued truth tables emerging from the semi-canonical calculus

To exemplify the effect of Definitions 4.5 and 4.6, observe in Figure 4 the four-valued truth tables. Canonical calculi counting with the logical rules from Figure 2, together with both Identity and Cut, will render their restriction to the rows and columns labeled by t and f. Thus, the (`-C-A`) semi-canonical calculus will render the full four-valued truth tables from the aforementioned Figure. In turn, the (`+C-A`) semi-canonical case will render the $\{t, \bot, f\}$-reduct and the (`-C+A`) semi-canonical one will render the $\{t, \top, f\}$-reduct thereof, which are isomorphic to the so-called three-valued Schütte and strong Schütte truth tables inspired in [19], and taken from [10] and [11]—respectively.

Moving forward to our contributions, we now establish a similar phenomenon as the one portrayed above, but regarding cocanonical systems.

Definition 4.8. Let $\mathbf{G}$ be a (semi-)cocanonical sequent calculus for $\mathcal{L}$. The NBmatrix $\mathbf{M_G}$ associated to it is defined such that $\mathcal{V}_\mathbf{M}$ should be identified with $\{t, \top, \bot, f\}$ if $\mathbf{G}$ is (`-C-A`) semi-cocanonical, with $\{t, \top, f\}$ if $\mathbf{G}$ is (`+C-A`) semi-cocanonical, with $\{t, \bot, f\}$ if $\mathbf{G}$ is (`-C+A`) semi-cocanonical, and with $\{t, f\}$ if $\mathbf{G}$ is cocanonical. Furthermore, in all cases $\mathsf{Y}_\mathbf{M} = \{t, \top\} \cap \mathcal{V}_\mathbf{M}$, and $\mathsf{N}_\mathbf{M} = \{\top, f\} \cap \mathcal{V}_\mathbf{M}$. Finally, for every n-ary connective $\diamond$ of $\mathcal{L}$:

(i) $t \notin \diamond_{\mathbf{M_G}}(x_1, ..., x_n)$ if and only if $\langle x_1, ..., x_n \rangle$ fulfills a right rule of $\mathbf{G}$ for $\diamond$;

(ii) $f \notin \diamond_{\mathbf{M_G}}(x_1, ..., x_n)$ if and only if $\langle x_1, ..., x_n \rangle$ fulfills a left rule of $\mathbf{G}$ for $\diamond$;

(iii) $\top \notin \diamond_{\mathbf{M_G}}(x_1, ..., x_n)$ if and only if $\langle x_1, ..., x_n \rangle$ fulfills any rule of $\mathbf{G}$ for $\diamond$;

(iv) $\bot \in \diamond_{\mathbf{M_G}}(x_1, ..., x_n)$, for every $x_1, ..., x_n$.

Naturally, if $\top \notin \mathcal{V}_{\mathbf{M}}$, then disregard clause (iii), and similarly if $\bot \notin \mathcal{V}_{\mathbf{M}}$, disregard clause (iv).

As we did with canonical rules, we provide a technical approach to the well-behaved nature of left and right elimination rules for connectives in the context of sequent calculi.

Definition 4.9. A (semi-)cocanonical system $\mathbf{G}$ is called *cocoherent* if always either s_1 or s_2 is classically satisfied whenever $\mathbf{G}$ includes two cocanonical rules of the form $\Rightarrow \diamond(p_1, ..., p_n)/s_1$ and $\diamond(p_1, ..., p_n) \Rightarrow /s_2$, for all n-ary connectives $\diamond$ in $\mathcal{L}$.[4]

To understand the effect of this well-behavedness criterion for generalized elimination rules, notice that in the presence of Identity and Cut, cocoherence prevents us from triviality. Roughly speaking, if the system is not cocoherent, assuming Identity for all the coauxiliary formulas in the premises of both rules, we can start a derivation with the instances needed to apply the two cocanonical rules and, after applying both, we can use Cut and trivialize the system.

For an additional illustration of this criterion, let's observe an example in a pair of non-cocoherent cocanonical rules—which, incidentally, are the inverses of the rules for *tonk* discussed above:

$$\frac{\varphi \odot \psi, \Gamma \Rightarrow \Delta}{\psi, \Gamma \Rightarrow \Delta}\text{EL}\odot \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \odot \psi}{\Gamma \Rightarrow \Delta, \varphi}\text{ER}\odot$$

It can be easily checked that it's possible for the conclusions of both rules to be jointly not classically satisfied. Let's see what happens with the non-deterministic semantics in a calculus containing only these two rules, and Cut or Identity or both.

In the first case, assuming we have Identity and Cut, we have a cocanonical sequent calculus. The tables obtained from these rules are the following:

$\odot$	t	f
t	$\{t\}$	$\{t, f\}$
f	$\{\}$	$\{f\}$

So, in cocanonical systems, we cannot obtain well-defined semantics in the presence of non-cocoherent pairs of rules, since there is at least an empty output for one operation. Similarly, if we add only Identity we still get some empty outputs. However, if we add Cut but not Identity, we have a semantics for the connective $\odot$:

[4]Please, notice the scope of the quantification in the previous definition. There, the "always" means that for each valuation, it classically satisfies either one or the other of the sequents in question, which is different to saying that one of the sequents is classically satisfied in all valuations, or the other sequent is classically satisfied in all valuations. In other words, no tuple $\langle x_1, .., x_n \rangle$ is a classical counterexample to both s_1 and s_2, as referred to in the definition.

$\odot$	t	$\bot$	f
t	$\{t, \bot\}$	$\{t, \bot\}$	$\{t, \bot, f\}$
$\bot$	$\{\bot\}$	$\{\bot\}$	$\{\bot, f\}$
f	$\{\bot\}$	$\{\bot\}$	$\{\bot, f\}$

The fact that the absence of Identity allows for the presence of non-cocoherent pairs of rules is represented in the formulation of Theorems 5.9 and 5.10.

Proposition 4.10. *For every cocoherent cocanonical or (*`-C+A`*) semi-cocanonical sequent calculus* $\mathbf{G}$*,* $\mathbf{M_G}$ *is well-defined.*

Proof. We only need to guarantee that the non-deterministic truth function for every n-ary connective $\diamond$ of $\mathcal{L}$ in $\mathbf{M_G}$ is non-empty, for every tuple $\langle x_1, ..., x_n \rangle$. Assume, then, that some is indeed empty. This implies that there's a n-tuple $\langle x_1, ..., x_n \rangle$ such that $\langle x_1, ..., x_n \rangle$ fulfills both rules of $\mathbf{G}$ for $\diamond$, which entails that $\mathbf{G}$ is not cocoherent—thereby concluding the proof. $\square$

Similarly, regardless of cocoherence, if $\mathbf{G}$ is a (`+C-A`) semi-canonical or a (`-C-A`) semi-canonical sequent calculus, then $\mathbf{M_G}$ is well-defined.

	$\neg$
t	$\{\bot, f\}$
$\top$	$\{t, \top, \bot, f\}$
$\bot$	$\{\bot\}$
f	$\{t, \bot\}$

$\wedge$	t	$\top$	$\bot$	f
t	$\{t, \bot\}$	$\{t, \top, \bot, f\}$	$\{t, \bot\}$	$\{\bot, f\}$
$\top$	$\{t, \top, \bot, f\}$	$\{t, \top, \bot, f\}$	$\{\bot, f\}$	$\{\bot, f\}$
$\bot$	$\{\bot\}$	$\{\bot, f\}$	$\{\bot\}$	$\{\bot, f\}$
f	$\{\bot, f\}$	$\{\bot, f\}$	$\{\bot, f\}$	$\{\bot, f\}$

$\vee$	t	$\top$	$\bot$	f
t	$\{t, \bot\}$	$\{t, \bot\}$	$\{t, \bot\}$	$\{t, \bot\}$
$\top$	$\{t, \bot\}$	$\{t, \top, \bot, f\}$	$\{t, \bot\}$	$\{t, \top, \bot, f\}$
$\bot$	$\{t, \bot\}$	$\{t, \top\}$	$\{\bot\}$	$\{\bot\}$
f	$\{t, \bot\}$	$\{t, \top, \bot, f\}$	$\{\bot\}$	$\{\bot, f\}$

Figure 5: Four-valued truth tables emerging from the semi-cocanonical calculus

Once again, to illustrate the effect of Definitions 4.8 and 4.9, observe the four-valued truth tables in Figure 5. Cocanonical calculi counting with the logical rules from Figure 3, together with both Identity and Cut, will render their restriction to the rows and columns labeled by t and f. Thus, the (`-C-A`) semi-cocanonical calculus will render the full four-valued truth tables from the figure mentioned above. In turn, the (`+C-A`) semi-cocanonical case will render the $\{t, \bot, f\}$-reduct and the (`-C+A`) semi-cocanonical one will render the $\{t, \top, f\}$-reduct thereof, which are isomorphic to the so-called three-valued strong Schütte and Schütte truth tables from [10] and [11]—notice the duality with the case of the canonical calculi discussed a few paragraphs before.

At last, with all these tools at our disposal, we can proceed to the main contribution of this article, that is, the proof that these 3- and 4-valued non-deterministic semantics can provide sound and complete representations of the consequence relations of canonical and cocanonical systems alike.

5 Characterization results

Definition 5.1. Let $\mathbf{G}$ be a sequent calculus. A sequent $\Gamma \Rightarrow \Delta$ is *derivable* in it, (symbolized as $\vdash_{\mathbf{G}} \Gamma \Rightarrow \Delta$) if and only if there is a finite rooted tree whose topmost leaves are instances of the axioms of $\mathbf{G}$, where each node is the result of an application of the rules of $\mathbf{G}$ to the immediately previous nodes, and the root is s.

Definition 5.2. Let $\mathbf{G}$ be a sequent calculus, S a set of sequents, and s a sequent. The rule S/s is *derivable* in it (symbolized as $S \vdash_{\mathbf{G}}^{seq} s$) if and only if s is derivable in the sequent calculus resulting from adding all $s' \in S$ as axioms to $\mathbf{G}$.

Definition 5.3. Let $\mathbf{M}$ be a NBmatrix, S a set of sequents, and s a sequent. The rule S/s is valid in it (symbolized as $S \vDash_{\mathbf{M}}^{seq} s$) if and only if every $\mathbf{M}$-valuation that is a model of S, also is a model of s.

Notice that all the relations above are substitution-invariant Tarskian consequence relations.

Theorem 5.4. *For every coherent canonical or (*`+C-A`*) semi-canonical sequent calculus* $\mathbf{G}$*,* $\vdash_{\mathbf{G}} = \vDash_{\mathbf{M_G}}$ *and* $\vdash_{\mathbf{G}}^{seq} = \vDash_{\mathbf{M_G}}^{seq}$*.*

Proof. See Theorem 4.7 in [3] and Theorem 6 in [13]. □

Theorem 5.5. *For every (*`-C+A`*) semi-canonical or (*`-C-A`*) semi-canonical sequent calculus* $\mathbf{G}$*,* $\vdash_{\mathbf{G}} = \vDash_{\mathbf{M_G}}$ *and* $\vdash_{\mathbf{G}}^{seq} = \vDash_{\mathbf{M_G}}^{seq}$*.*

Proof. See Theorems 4 and 7 in [13]. □

These are already well-known established results about canonical Gentzen-style sequent calculi, but in what follows we generalize them for cocanonical deduction systems of various sorts. We do this by generalizing the reduction trees technique, as discussed e.g. in [10] and many other places.

The interest of our contribution here lies in the fact that reduction trees are not usually implemented for calculi with elimination or, in our terms, cocanonical rules. One of the reasons for this is that the case of cocanonical rules presents a much more complex situation compared with the case where we only deal with introductions. Before moving on, we should mention that many of the techniques employed here are inspired by ideas appearing in [11], whereas others are innovations of our particular proof.

For starters, we let $\mathbf{G}$ be a (semi-)cocanonical system, and we assume we have cocanonical right rules for the connectives in $C^r = \{\diamond_{r_1}, ..., \diamond_{r_n}\}$ and cocanonical left rules for the connectives $C^l = \{\diamond_{l_1}, ..., \diamond_{l_m}\}$, In addition, we assume the availability of an enumeration of all the formulas of the language $\mathcal{L}$, that we refer to as $F_0, F_1, F_2, ...$

The purpose of the construction below is to check whether a certain sequent $\Gamma \Rightarrow \Delta$ follows from a finite set of sequents S. For this purpose, we write down our target sequent $\Gamma \Rightarrow \Delta$ as the root of the tree—denoting it, for future reference, by $\Gamma_0 \Rightarrow \Delta_0$. The technique now consists of building one tree in steps in such a way that at each step n we transform the topmost sequents $\Gamma_n \Rightarrow \Delta_n$ of each branch of the tree.[5] The tree so built will be called a *reduction tree*.

As a matter of clarification, we will say later that a sequent $\Gamma \Rightarrow \Delta$ is in the Weakening-closure (W-closure, for short) of a set of sequents S if there exists a sequent $\Sigma \Rightarrow \Pi$ in S such that $\Sigma \subset \Gamma$ and $\Pi \subset \Delta$. Now, if in the reduction

[5] To simplify the notation, by $\Gamma_n \Rightarrow \Delta_n$ we refer to possibly different sequents, i.e. to the different topmost sequents belonging to each branch of the tree at the step n.

tree mentioned above, the topmost sequent of a branch of the tree belongs to the W-closure of S (or the W-closure of some instance of Id, if Id is in **G**), then we say that the branch is closed—otherwise, it's open. If all of the branches of a tree are closed we say the tree is closed, otherwise it's open. The tree is built according to the following instructions:[6]

Step $\mathbf{n = 3m}$ If **G** contains Cut, extend the tree with two branches, one of them having $F_m, \Gamma_n \Rightarrow \Delta_n$ and the other $\Gamma_n \Rightarrow \Delta_n, F_m$. If not, erase this instruction from the construction.

Step $\mathbf{n = 3m + 1}$ Now we provide instructions corresponding to all the right cocanonical rules, in order. We start with the cocanonical rule for the k-ary connective $\diamond_{r_1}$. Recall that cocanonical rules can have coauxiliary formulas, so if some such formulas occur on the left or the right of the conclusion then we will refer to them as $\psi_{l_1}, ..., \psi_{l_s}$ and $\psi_{r_1}, ..., \psi_{r_t}$ respectively. Thus, we have four possible situations, depending on whether this rule:

Case (I): Has coauxiliary formulas only on the right

Case (II): Has coauxiliary formulas only on the left

Case (III): Has no coauxiliary formulas

Case (IV): Has coauxiliary formulas on both sides

For the first case, assume the application of the rule has the following form:

$$\frac{\Gamma \Rightarrow \Delta, \diamond_{r_1}(\psi_1, ..., \psi_k)}{\Gamma \Rightarrow \Delta, \psi_{r_1}, ...\psi_{r_t}}$$

We should consider some subcases: (i) $\{\psi_{r_1}, ...\psi_{r_t}) = \{\psi_1, ..., \psi_k\}$, (ii) $\{\psi_{r_1}, ..., \psi_{r_t}\} \subset \{\psi_1, ..., \psi_k\}$. For each, we need to consider particular reduction instructions. Assume $\Gamma_n = \{\gamma_1, ..., \gamma_i\}$, $\Delta_n = \{\delta_1, ..., \delta_j\}$, so $j = |\Delta_n|$ and $i = |\Gamma_n|$ (with $j = 0$ or $i = 0$, respectively, when they are empty).

Case (I.i): If $j = 0$, go to the next connective in C^r (if there are no more connectives to consider in C^r go to the next step). If $j > 0$, let's define $\Delta_{\diamond_{r_1}} = \{\diamond_{r_1}(\psi_1, ..., \psi_k) : \psi_1, ..., \psi_k \in \Delta_n\}$. Finally, extend the tree by adding $\Gamma_n \Rightarrow \Delta_{\diamond_{r_1}}, \Delta_n$. Basically, what this instruction demands is to construct all the possible k-ary sequences of formulas that can be obtained from the formulas appearing on the right of the sequent we are reducing (Δ_n), and extending the tree by adding one copy $\diamond_{r_1}(\psi_1, ...\psi_k)$ on the right for each of them.

Case (I.ii): If $j = 0$, go to the next connective in C^r (if there are no more connectives to consider in C^r go to the next step). If $j > 0$, the difference with the previous case is that since the coauxiliary formulas are

[6]Each step n in the construction depends on m, which is the subscript of the enumeration of formulas. Notice that m is a variable and we generate all the steps by considering in order each m (starting from 0). For each m, we have 3 different steps.

not all the formulas in $\psi_1, ...\psi_k$, we need to consider not only all the k-ary sequences that can be obtained from the formulas in Δ_n but also the sequences containing arbitrary formulas in the positions of $\diamond_{r_1}(\psi_1, ...\psi_k)$ that do not correspond to the positions in which any coauxiliary formula appears. So, the idea is to build all such sequences by using the formulas in $\{F_0, ..., F_m\}$. In this way, we use the formulas in the general enumeration to employ them at each step as a set of arbitrary formulas. After building all these sequences, we extend the tree by adding $\diamond_{r_1}(\psi_1, ..., \psi_k)$ on the right.[7] Since Case (II) is similar to this one, we move on to the next case.

Case (III): Since there are no coauxiliary formulas, let's define $\Delta_{\diamond_{r_1}}$ as above and extend the tree with $\Gamma_n \Rightarrow \Delta_{\diamond_{r_1}}, \Delta_n$. Here we are in the limit case of the procedure we have described for Case (I.ii). Since all the formulas disappear once we apply the rule, we need to consider all the k-ary sequences containing arbitrary formulas. As in the aforementioned case, we employ the enumeration of formulas up to m, $\{F_0, ..., F_m\}$ to consider in order all of the possible k-ary combinations of formulas.

Case (IV): Since this case can be adapted from Case (I) and (II), we leave it to the reader as an exercise.

Once we finish the corresponding Case for $\diamond_{r_1}$, we repeat in order this procedure for each connective in C^r. When there are no more connectives to consider in C^r we move on to the next step.

Step $\mathbf{n = 3m + 2}$ In this step, we need to consider all the left cocanonical rules in order. Since this step is completely symmetrical to the right rules, we leave it to the reader as an exercise.

Stop condition If all the topmost sequents $\Gamma_n \Rightarrow \Delta_n$ of the tree are in the W-closure of S (or in the W-closure of some instance of Id, if Id is in $\mathbf{G}$), stop the process.

This procedure serves as a schema for any (semi-)cocanonical sequent calculus $\mathbf{G}$. However, given a particular calculus, perhaps there can be more efficient instructions. We leave these peculiarities aside to keep the construction as general as possible.

Lemma 5.6. *For every (semi-)cocanonical sequent calculus* $\mathbf{G}$*, set of sequents* S*, and sequent* $\Gamma \Rightarrow \Delta$*, if the corresponding reduction tree is closed, then* $S \vdash^{seq}_{\mathbf{G}} \Gamma \Rightarrow \Delta$.

Proof. Assume we start with a sequent $\Gamma \Rightarrow \Delta$, we apply the construction described, and its reduction tree is closed. This means that all of the sequents in the tips of the tree are in the W-closure of S (or of Identity, if $\mathbf{G}$ has it), and therefore can be obtained from S by successive applications of Weakening. Thus, we need to prove that there is a proof of $\Gamma \Rightarrow \Delta$ from the tips of the tree. Firstly notice that the stop conditions make all the closed branches finite. Thus, by König lemma, every closed tree is finite. Secondly, notice that the instructions given to build the tree are exactly the rules of the calculus read from top to bottom, plus some structural rules. So it's straightforward to transform any

[7] We omit here the formal way of defining these sequences since it would require too much notation for describing something that is otherwise so easy to understand.

closed tree into a derivation in **G**, just by applying the rules corresponding to each instruction. □

Lemma 5.7. *For every cocoherent cocanonical or (*`-C+A`*) semi-cocanonical sequent calculus* **G***, and every set of sequents S and sequent $\Gamma \Rightarrow \Delta$, if its reduction tree is open, then there is a valuation v in* $\mathbf{M_G}$ *which is a model of all the sequents in S but is not a model of $\Gamma \Rightarrow \Delta$, i.e. $S \nvDash^{seq}_{\mathbf{M_G}} \Gamma \Rightarrow \Delta$.*

Proof. Take any open branch of an open tree, and consider the result of collecting in Γ_ω all the formulas appearing on the left of any sequent of this branch and in Δ_ω all the formulas appearing on the right of any sequent of this branch. Recall, that no sequent appearing on any node of this branch is in the W-closure of the theory S or in the W-closure of Identity. Now, let v be a valuation such that:

$$v(\varphi) = \begin{cases} t & \text{if } \varphi \in \Gamma_\omega \text{ and } \varphi \notin \Delta_\omega \\ f & \text{if } \varphi \notin \Gamma_\omega \text{ and } \varphi \in \Delta_\omega \\ \top & \text{if } \varphi \notin \Gamma_\omega \text{ and } \varphi \notin \Delta_\omega \end{cases}$$

Notice that the third clause is only relevant if **G** lacks the general form of Cut, and cannot happen otherwise. Now, we show by induction on the complexity of the φ that the valuation v we just designed is one of the valuations respecting $\mathbf{M_G}$.

The base case corresponds to φ being a propositional variable and is verified by definition. For the inductive step, let us consider a formula $\varphi = \diamond(\psi_1, ..., \psi_n)$. There are three jointly exhaustive and mutually exclusive possibilities. Again, notice that the third case is only relevant if **G** lacks the general form of Cut, and cannot happen otherwise.

Case (I): $\diamond(\psi_1, ..., \psi_n) \in \Gamma_\omega$ and $\diamond(\psi_1, ..., \psi_n) \notin \Delta_\omega$

Case (II): $\diamond(\psi_1, ..., \psi_n) \notin \Gamma_\omega$ and $\diamond(\psi_1, ..., \psi_n) \in \Delta_\omega$

Case (III): $\diamond(\psi_1, ..., \psi_n) \notin \Gamma_\omega$ and $\diamond(\psi_1, ..., \psi_n) \notin \Delta_\omega$

For the first case, assume that $\diamond(\psi_1, ..., \psi_n) \in \Gamma_\omega$ with the intention of proving that v respects $\mathbf{M_G}$, we consider some subcases: (I.i) **G** counts with a left cocanonical rule, (I.ii) **G** counts with a right cocanonical rule,

Case (I.i): If **G** doesn't count with a right cocanonical rule for $\diamond$, then by Definition 4.8, we know that a v such that $v(\diamond(\psi_1, ..., \psi_n)) = t$ is a valuation respecting $\mathbf{M_G}$. If **G** counts with a right cocanonical rule for $\diamond$, then it should have couaxiliary formulas. Otherwise, we would have that $\diamond(\psi_1, ..., \psi_n) \in \Delta_\omega$, contradicting our assumption. Furthermore, for any such rule let's refer to its coauxiliary formulas as $\psi_{l_1}, ..., \psi_{l_n}$ and $\psi_{r_1}, ..., \psi_{r_m}$. We know that $\psi_{l_1}, ..., \psi_{l_n}$ and $\psi_{r_1}, ..., \psi_{r_m}$ cannot be found, respectively, in Γ_ω and Δ_ω. Otherwise, we would have that $\diamond(\psi_1, ..., \psi_n) \in \Delta_\omega$, contradicting our assumption. Thus, wherever $\psi_1, ..., \psi_n$ are found in Γ_ω and Δ_ω if at all, their values according to v don't constitute a tuple that appropriately fulfills any right cocanonical rule for $\diamond$. Thus, by Definition 4.8, we know that a v such that $v(\diamond(\psi_1, ..., \psi_n)) = t$ is a valuation respecting $\mathbf{M_G}$. Case (I.ii) is analogous to this one, and so is left to the reader as an exercise.

Since Case II of the inductive step is analogous to Case I, to conclude we prove the third case. For that, we assume that $\diamond(\psi_1, ..., \psi_n) \notin \Gamma_\omega \cup \Delta_\omega$ with the intention of proving that if $v(\diamond(\psi_1, ..., \psi_n)) = \top$, then v is a valuation respecting $\mathbf{M_G}$. Once again, we should consider some cases: (III.i) $\mathbf{G}$ counts with a left cocanonical rule and (III.ii) $\mathbf{G}$ counts with a right cocanonical rule.

Case (III.i): It can only be the case that there are some coauxiliary formulas in the left cocanonical rules for $\diamond$, to which we refer as $\psi_{l_1}, ..., \psi_{l_n}$ and $\psi_{r_1}, ..., \psi_{r_m}$, depending on whether they appear on the left or right of the conclusion of the left cocanonical rule for $\diamond$. By the reduction instruction for the left cocanonical rules for $\diamond$, we know that $\psi_{l_1}, ..., \psi_{l_n} \notin \Gamma_\omega$ and $\psi_{r_1}, ..., \psi_{r_m} \notin \Delta_\omega$—otherwise, $\diamond(\psi_1, ..., \psi_n)$ would be in Γ_ω, contradicting our assumptions. By the inductive hypothesis, and Definition 4.8, we know that the valuation distribution for $\psi_{l_1}, ..., \psi_{l_n}$ and $\psi_{r_1}, ..., \psi_{r_m}$ is not a constituent of any tuple that appropriately fulfills any left cocanonical rule for $\diamond$. If $\mathbf{G}$ doesn't count with a right cocanonical rule for $\diamond$, then a v such that $v(\diamond(\psi_1, ..., \psi_n)) = \top$ is a valuation respecting $\mathbf{M_G}$. Otherwise, we are in Case (III.ii) which is completely analogous to this one. Note, in particular, that if the same $\psi_{l_1}, ..., \psi_{l_n}$ and $\psi_{r_1}, ..., \psi_{r_m}$ mentioned before are coauxliary formulas for any right cocanonical rule for $\diamond$, then by similar reasoning as before we know that $\psi_{l_1}, ..., \psi_{l_n} \notin \Gamma_\omega$ and $\psi_{r_1}, ..., \psi_{r_m} \notin \Delta_\omega$—otherwise, $\diamond(\psi_1, ..., \psi_n)$ would be in Δ_ω, contradicting our assumptions. By the inductive hypothesis, and Definition 4.8, we know that the valuation distribution for $\psi_{l_1}, ..., \psi_{l_n}$ and $\psi_{r_1}, ..., \psi_{r_m}$ is not a constituent of any tuple that appropriately fulfills any right cocanonical rule for $\diamond$. Thus, a v such that $v(\diamond(\psi_1, ..., \psi_n)) = \top$ is a valuation respecting $\mathbf{M_G}$.

Finally, notice that this valuation is a model of the set of sequents S and also is a counterexample of the root sequent $\Gamma \Rightarrow \Delta$. For the latter, given $\Gamma \subseteq \Gamma_\omega$ and $\Delta \subseteq \Delta_\omega$, we know that $v(\gamma) = t$ and $v(\delta) = f$, for all $\gamma \in \Gamma$, and $\delta \in \Delta$. For the former, observe that for each $\Sigma_i \Rightarrow \Pi_i \in S$ there must be a $\sigma_j \in \Sigma_i$ such that $\sigma_j \notin \Gamma_\omega$ or a $\pi_k \in \Pi_i$ such that $\pi_k \notin \Delta_\omega$. Otherwise, the branch would be closed, given that there would be a node in the W-closure of some sequent in S. Thus, it must be the case that for each $\Sigma_i \Rightarrow \Pi_i \in S$ there must be a $\sigma_j \in \Sigma_i$ or a $\pi_k \in \Pi_i$ such that $v(\sigma_j) \in \{\top, f\}$ or $v(\pi_k) \in \{t, \top\}$. Thus, v is guaranteed to be a model of all the sequents in S. □

Lemma 5.8. *For every (*`+C-A`*) semi-cocanonical or (*`-C-A`*) semi-cocanonical sequent calculus* $\mathbf{G}$*, and every set of sequents* S *and sequent* $\Gamma \Rightarrow \Delta$*, if its reduction tree is open, then there is a valuation* v *in* $\mathbf{M_G}$ *which is a model of all the sequents in* S *but is not a model of* $\Gamma \Rightarrow \Delta$*, i.e.* $S \not\models^{seq}_{\mathbf{M_G}} \Gamma \Rightarrow \Delta$*.*

Proof. Similar to the previous lemma, thus left to the reader as an exercise. □

Theorem 5.9. *For every cocoherent cocanonical or (*`-C+A`*) semi-cocanonical sequent calculus* $\mathbf{G}$*,* $\vdash_{\mathbf{G}} = \models_{\mathbf{M_G}}$ *and* $\vdash^{seq}_{\mathbf{G}} = \models^{seq}_{\mathbf{M_G}}$*.*

Proof. From Lemmas 5.6 and 5.7. □

Theorem 5.10. *For every (*`+C-A`*) semi-cocanonical or (*`-C-A`*) semi-cocanonical sequent calculus* $\mathbf{G}$*,* $\vdash_{\mathbf{G}} = \models_{\mathbf{M_G}}$ *and* $\vdash^{seq}_{\mathbf{G}} = \models^{seq}_{\mathbf{M_G}}$*.*

Proof. From Lemmas 5.6 and 5.8. □

6 Final remarks

To conclude, we provide some reflections on possible extensions of semi-canonical and semi-cocanonical systems, obtained by adding restricted forms of the Identity axiom and the Cut rule. Thus, consider the following definitions.

Definition 6.1. Let $\diamond$ be any n-ary connective. An application of the restricted form of the Identity axiom for $\diamond$ that we call the axiom *Identity for* $\diamond$ has the following form:

$$\frac{}{\Gamma, \sigma(\diamond(p_1, ..., p_n)) \Rightarrow \Delta, \sigma(\diamond(p_1, ..., p_n))} \text{ Id for } \diamond$$

Definition 6.2. Let $\diamond$ be any n-ary connective. An application of the restricted form of the Cut rule for $\diamond$ that we call the rule *Cut for* $\diamond$ has the following form:

$$\frac{\Gamma \Rightarrow \sigma(\diamond(p_1, ..., p_n)), \Delta \qquad \Pi, \sigma(\diamond(p_1, ..., p_n)) \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow, \Delta, \Sigma} \text{ Cut for } \diamond$$

It is worth mentioning that since fully canonical or cocanonical systems contain unrestricted forms of Identity and Cut, it follows that the restricted forms are derivable. Thus, it's only the corresponding Identity-free or Cut-free semi-canonical or semi-cocanonical calculi that it is interesting to extend with these restricted forms. In this vein, we provide some pointers to understand how the corresponding non-deterministic semantics would look like, in each case.

For semi-canonical calculi, Definition 4.5 would need a little tweak. If we assume $\mathbf{G}$ is a semi-canonical sequent calculus for $\mathcal{L}$ extended by some Id for $\diamond$ or Cut for $\diamond$ rules, then $\mathbf{M_G}$ is defined is just like above, plus an additional caveat. This addendum would say something like: if $\mathbf{G}$ counts with a restricted form of the the Identity axiom or the Cut rule for $\diamond$ then, *only* for $\diamond$ formulas, disregard clauses (iii) or (iv)—respectively. Analogously, for semi-cocanonical calculi, Definition 4.8 would require a slight modification. In such a case, the addendum would say: if $\mathbf{G}$ counts with a restricted form of the Cut rule or the Identity axiom for $\diamond$ then, *only* for $\diamond$ formulas, disregard clauses (iii) or (iv)—respectively.

With these modifications, characterization results along the lines of Theorems 5.4, 5.5, 5.9, and 5.10 can be proved, establishing that for each corresponding extended system $\mathbf{G}$, it's the case that $\vdash_{\mathbf{G}} = \vDash_{\mathbf{M_G}}$ and $\vdash^{seq}_{\mathbf{G}} = \vDash^{seq}_{\mathbf{M_G}}$. Interestingly, the proofs of such extended theorems are similar to and, in fact, nothing more than modifications of the proofs of the previously mentioned ones.

It should be mentioned, that the only interesting detail of these adapted proofs corresponds to the case of extensions of (`-C+A`) semi-cocanonical or (`-C-A`) semi-cocanonical system, with potentially various restricted forms of the Cut rule. Namely, to cover the case where we have a sequent calculus $\mathbf{G}$ with Cut for $\diamond$ rules, with $\diamond \in \{\diamond_1, ..., \diamond_n\}$, instructions to build the corresponding reduction tree would need to be also modified. In particular, we would need to extend the tree with n steps to account for each of these potential applications of restricted Cut rules. However, given this would constitute nothing but a mere generalization of the procedure for the general and unrestricted form of Cut that we mention in our proof above, we leave a full description of these details to the interested reader as an exercise.

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