Peirce on Abduction and Diagrams in Mathematical Reasoning

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Abstract

Questions regarding the nature and acquisition of mathematical knowledge are perhaps as old as mathematical thinking itself, while fundamental issues of mathematical ontology and epistemology have direct bearing on mathematical

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cognition. Several original contributions to logic and mathematics made by the American polymath, Charles Sanders Peirce, are of direct relevance to these fundamental issues. This chapter explores scientific reasoning as it relates to abduction, a name that Peirce coined for educated "guessing" of hypotheses, which he took to be "the first step of scientific reasoning" and the only creative one. Yet he also argued that all deductive reasoning is mathematical and that all mathematical reasoning is diagrammatic. Representation, especially in the form of a diagrammatic system of logic that Peirce developed, is explored here along with his logic of inquiry, most notably in terms of its manifestation as the logic of ingenuity. Originating in the field of engineering, here the diagram of a problem serves as a heuristic substitute for evaluating the actual situation, an approach that can be extended to other forms of practical reasoning such as ethical deliberation. This chapter also touches upon such diverse but related subjects as non-Euclidean geometry and nonclassical logic, with additional examples that help to elucidate cognitive elements of mathematical knowledge.

Keywords

Abduction · Diagram · Guess · Hypothesis · Ingenuity · Peirce · Reasoning

Introduction

Epistemological debates over the nature of mathematical knowledge and how it is acquired—if it is "acquired" rather than innate, at least on some level—have been waged ever since the first mathematicians began to think seriously about what it means to think mathematically. Is there some fundamental sense in which mathematical knowledge is innate, or is it something that must be taught and/or learned through experience and by example? The age-old question of whether mathematicians create or discover mathematical objects also relates to these fundamental issues of ontology and epistemology and ultimately to the essence of mathematical cognition with which this book is concerned. It turns out that the American polymath, Charles Sanders Peirce (1839–1914), founder of pragmatism and semiotics pioneer, had much to say of interest on these diverse yet related subjects.

The second of five children of Harvard mathematician Benjamin Peirce and Sarah Mills Peirce, Charles (Fig. 1) was considered a child prodigy and was home tutored by his father. He went on to receive his undergraduate degree from Harvard in 1859, whereupon he began to work on various projects with the US Coast and Geodetic Survey, where his father was also active and served as Superintendent from 1867 until 1874. In 1861 Charles began graduate studies at Harvard's Lawrence Scientific School, from which he received a B.Sc. degree in chemistry, the first awarded *summa* cum *laude*, in 1863.

At the Coast Survey, Peirce devoted himself primarily to geodesy and gravimetrics using pendulums, some of which he designed himself, to carry out accurate measurements of the Earth's gravitational field and in turn to provide an accurate

Fig. 1 Charles S. Peirce

description of its topology and overall shape. The accuracy of the pendulum measurements depended upon precise measurements of time. Meanwhile, Peirce was also doing additional work at the Harvard College Observatory, which resulted in his only book published during his lifetime, *Photometric Researches,* in 1878.

Peirce's Tiffany Watch

The following year, Peirce was aboard an overnight coastal steamer en route to New York when an incident occurred that he would later recount to his friend, the Harvard philosopher and psychologist William James. The story Peirce told James was a curious incident involving a stolen Tiffany watch and the remarkable means by which he was able to recover it. On his own account, he departed the boat in a rush early in the morning on June 21, 1879, leaving behind his coat and the watch, both of which were missing when he returned to retrieve them. The watch was no ordinary watch; it was a gold Tiffany chronometer that was on loan to Peirce from the Coast Survey to ensure the accuracy of his gravimetric studies. Peirce insisted that the captain of the ship line up all hands on deck, whom he proceeded to interrogate one by one. As he tells the story:

> I went from one end of the row to the other, and talked a little to each one, in as *dégagé* a manner as I could, about whatever he could talk about with interest, but would least expect me to bring forward, hoping that I might seem such a fool that I should be able to detect some symptom of his being the thief. When I had gone through the row, I turned and walked from them, though not away, and said to myself, "Not the least scintilla of light have I got to go upon." But thereupon my other self (for our own communings are always in dialogues,) said

Fig. 2 Pinkerton's postcard

PAWNBROKERS!

Please Stop If Offered, or Notify if Received.

Plain Gold Hunting Case Lever Watch, No. 04555, Charles Frodsham, maker. Stolen from State Room of Fall River Steamboat "Bristol," Saturday, June 21st, 1870.

$150. Will be paid for its recovery.

Send information to

ALLAN PINKERTON

June 23, 1870 06 Exchange Plaza, New York.

> to me, "But you simply must put your finger on the man. No matter if you have no reason, you must say whom you will think to be the thief." I made a little loop in my walk, which had not taken a minute, and as I turned toward them, all shadow of doubt had vanished. (Peirce 1929: 271)

Peirce offered his suspect $50, but to no avail. He then took a taxi straight to Pinkerton's, the detective agency, where he described the suspect and predicted that the thief would try to pawn the watch. Peirce then asked Pinkerton's to follow the man when he left the ship and, as soon as he had the pawn ticket, have him arrested, and reclaim the watch. When this plan failed, Pinkerton's sent a postcard (Fig. 2) to pawnbrokers in New York, Philadelphia, and Boston, offering $150 for return of the stolen watch. Within 24 hours the watch was located, and the pawnbroker who had received the watch described the man Peirce had previously identified "so graphically that no doubt was possible that it had been 'my man,'" as he later recalled (Peirce 1929: 275). For a more detailed account of this and other related events that also transpired at the time, in Peirce's own version of the story, see (Dauben 1995: 146–149).

When Peirce described these events many years later to James, he cited it as a perfect example of what he had come to regard as the essence of *abductive* reasoning, which he took to be an inclination to entertain hypotheses. This all served to explain, he believed, why it is that people so often guess right.

Abductive Reasoning

Peirce regarded what he called abduction as "the first step of scientific reasoning" (Peirce 1901: CP 7.218). By this he meant the way in which we formulate hypotheses, abduction being "all the operations by which theories and conceptions are engendered" (Peirce 1903: CP 5.590). Peirce explains it in somewhat greater detail as "the process of forming explanatory hypotheses. It is the only logical operation which introduces any new idea" (Peirce 1903: CP 5.172).

As Arthur Burks pointed out in an early article about Peirce and abduction still worth reading today (Burks 1946), Peirce initially spoke in terms of "hypothesis" but later adopted the word "abduction" as a type of reasoning distinct from either induction or deduction, citing Aristotle's discussion of it in the *Prior Analytics* once a single word is changed to correct what he maintained to be a corruption of the original Greek text (Peirce 1901: CP 7.250–251; Peirce 1903: CP 5.144). Sometimes, Burks points out, Peirce also referred to this type of reasoning as "retroduction" or "presumption" (Burks 1946: 271), and he seems to have ultimately preferred "retroduction" for the narrower notion of "reasoning from consequent to antecedent" (Peirce 1908: CP 6.469).

By whatever name, abduction is *only* the first step in scientific reasoning, because while it generates hypotheses, they do not lead to certainty. That can only be achieved, Peirce believed, through deductive reasoning, which deals with idealized objects. Moreover, the only way to determine the extent to which an abductive hypothesis is correct, or at least more in conformity with the facts of nature than any identified alternatives, is by making predictions (deduction) and then testing them by observation and experiment (induction). Peirce further explained these three kinds of elementary reasoning:

> The first, which I call *abduction* . . . consists in examining a mass of facts and in allowing these facts to suggest a theory. In this way we gain new ideas; but there is no force in the reasoning. . . . The second kind of reasoning is *deduction*, or necessary reasoning. It is applicable only to an ideal state of things, or to a state of things in so far as it may conform to an ideal. It merely gives a new aspect to the premises. . . . The third way of reasoning is *induction*, or experimental research. Its procedure is this. Abduction having suggested a theory, we employ *de*duction to deduce from that ideal theory a promiscuous variety of consequences to the effect that if we perform certain acts, we shall find ourselves confronted with certain experiences. We then proceed to try these experiments, and if the predictions of the theory are verified, we have a proportionate confidence that the experiments that remain to be tried will confirm the theory. I say that these three are the only elementary modes of reasoning there are. (Peirce c. 1905: CP 8.209)

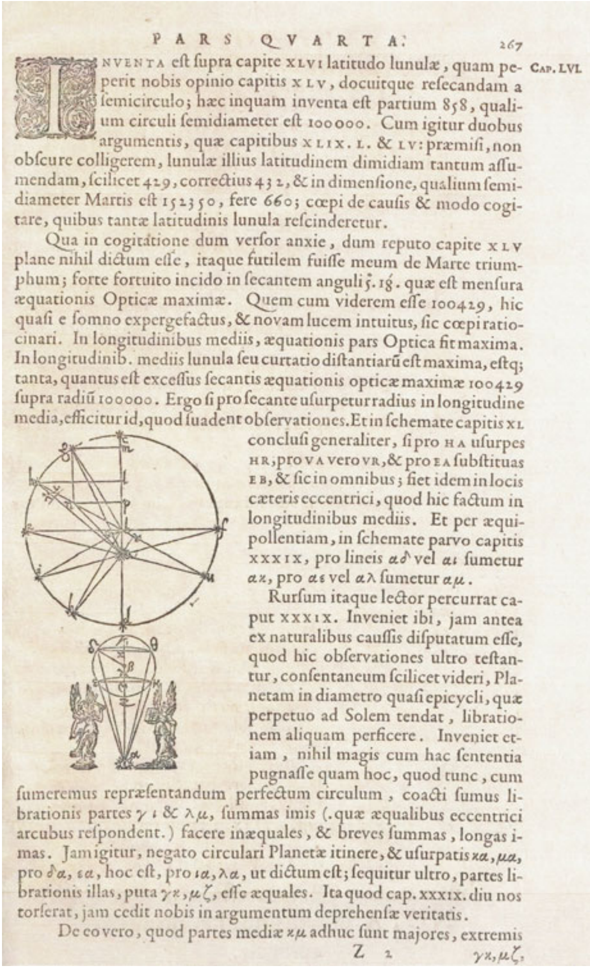
One of Peirce's favorite historical examples of abduction was the discovery by Johannes Kepler (1571–1630) that the orbit of Mars is an ellipse. In *The Watershed* (1960), Arthur Koestler elaborated on a treatment of Kepler that he had already given in *The Sleepwalkers* (1959), wherein he advanced a theory that likened creative discoverers to sleepwalkers, suggesting that even a great genius is never fully aware of what is actually involved in any truly creative act. In the case of Kepler, this was an easy conclusion to assert, given his own famous statement about his discovery of the elliptical orbit of Mars. Kepler gives a lengthy account of his step-by-step, painful progress toward unlocking the secret in his *Astronomia Nova* (1609), Chapter LVI, where he reveals his frustration in dealing with the planet and his eventual triumph:

Fig. 3 Kepler, from (Figuier 1876, vol. 4: facing p. 49)

> While I was anxiously turning this thought over in my mind, reflecting that absolutely nothing was accomplished by chapter XLV, and consequently my triumph over Mars was futile, quite by chance I hit upon the secant of the angle 5°18′, which is the measure of the greatest optical equation. And when I saw that this was 100429, it was as if I was awakened from sleep to see a new light. (Kepler 1609: 267)

Kepler had tried everything, from circles moving on circles to ovals and ovoid egg shapes, hoping to find a path that would account for the positions of Mars that had been carefully observed by the Danish astronomer Tycho Brahe (1546–1601), who had recorded the positions of Mars over years of observations, resulting in the most accurate data then available. When Kepler finally realized the many mistakes he had made, seeing that mathematically the only possible shape the orbit of Mars could take—and likewise all of the other planets—must be an ellipse, it was, as he says, like having awoken from sleep and seeing the light, the answer clearly revealed (Figs. 3, 4 and 5).

But the secret was in the mathematics all along, not in the various given forms he had tried at first. The cognitive inspiration leapt out as soon as Kepler saw a connection only someone who had steeped himself in the numbers would have seen, and that "habit of thought," as Peirce might have said, in this case led Kepler not so much to guess right, but to work out the correct solution: the one that, in the end, having tried all other possibilities he could think of, led him to his final conclusion that the orbit Mars followed in its path around the sun must be an ellipse. As Peirce summarized:



PARS QVARTA. 267

INVENTA eſt ſupra capite XLVI latitudo lunulæ, quam pe- CAP. LVI.
perit nobis opinio capitis XLV, docuitque reſecandam a
ſemicirculo; hæc inquam inventa eſt partium 858, quali-
um circuli ſemidiameter eſt 100000. Cum igitur duobus
argumentis, quæ capitibus XLIX. L. & LV: præmiſi, non
obſcure colligerem, lunulæ illius latitudinem dimidiam tantum aſſu-
mendam, ſcilicet 429, correctius 432, & in dimenſione, qualium ſemi-
diameter Martis eſt 152350, fere 660; cœpi de cauſis & modo cogi-
tare, quibus tantæ latitudinis lunula reſcinderetur.

Qua in cogitatione dum verſor anxie, dum reputo capite XLV
plane nihil dictum eſſe, itaque futilem fuiſſe meum de Marte trium-
phum; forte fortuito incido in ſecantem anguli 5.18. quæ eſt menſura
æquationis Opticæ maximæ. Quem cum viderem eſſe 100429, hic
quaſi e ſomno expergefactus, & novam lucem intuitus, ſic cœpi ratio-
cinari. In longitudinibus mediis, æquationis pars Optica fit maxima.
In longitudinib. mediis lunula ſeu curtatio diſtantiarũ eſt maxima, eſtq;
tanta, quantus eſt exceſſus ſecantis æquationis opticæ maximæ 100429
ſupra radiũ 100000. Ergo ſi pro ſecante uſurpetur radius in longitudine
media, efficitur id, quod ſuadent obſervationes. Et in ſchemate capitis XL
concluſi generaliter, ſi pro HA uſurpes
HR, pro VA vero VR, & pro EA ſubſtituas
EB, & ſic in omnibus; fiet idem in locis
cæteris eccentrici, quod hic factum in
longitudinibus mediis. Et per æqui-
pollentiam, in ſchemate parvo capitis
XXXIX, pro lineis αδ vel αι ſumetur
ακ, pro αε vel αλ ſumetur αμ.

Rurſum itaque lector percurrat ca-
put XXXIX. Inveniet ibi, jam antea
ex naturalibus cauſſis diſputatum eſſe,
quod hic obſervationes ultro teſtan-
tur, conſentaneum ſcilicet videri, Pla-
netam in diametro quaſi epicycli, quæ
perpetuo ad Solem tendat, libratio-
nem aliquam perficere. Inveniet et-
iam, nihil magis cum hac ſententia
pugnaſſe quam hoc, quod tunc, cum
ſumeremus repræſentandum perfectum circulum, coacti ſumus li-
brationis partes γι & λμ, ſummas imis (.quæ æqualibus eccentrici
arcubus reſpondent.) facere inæquales, & breves ſummas, longas i-
mas. Jam igitur, negato circulari Planetæ itinere, & uſurpatis κα, μα,
pro δα, εα, hoc eſt, pro ια, λα, ut dictum eſt; ſequitur ultro, partes li-
brationis illas, puta γκ, μζ, eſſe æquales. Itaque quod cap. XXXIX. diu nos
torſerat, jam cedit nobis in argumentum deprehenſæ veritatis.

De eo vero, quod partes mediæ κμ adhuc ſunt majores, extremis
Z 2 γκ, μζ,

Fig. 4 *Astronomia Nova* (1609), Ch. 56: p. 267

Thus, never modifying his theory capriciously, but always with a sound and rational motive for just the modification he makes, it follows that when he finally reaches a modification—of most striking simplicity and rationality—which exactly satisfies the observations, it stands upon a totally different logical footing from what it would if it had been struck out at random, or the reader knows not how, and had been found to satisfy the observation. Kepler shows his keen logical sense in detailing the whole process by which he finally arrived at the true orbit. This is the greatest piece of Retroductive reasoning ever performed. (Peirce c. 1896: CP 1.74)

In fact, Kepler always regarded his discovery as a matter of divine providence for numerous reasons, including the fact that there was a serendipitous conjunction of three great astronomers in Prague in 1600: Kepler, Tycho, and Tycho's assistant, Longomontanus (1562–1647), who had just begun to focus his efforts on the orbit of

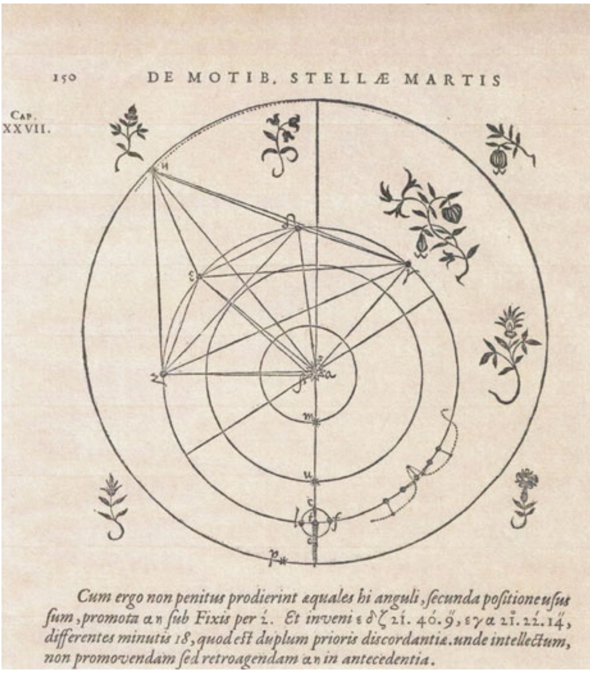


Fig. 5 Chapter 27, triangulation revealing the orbit of Mars (p. 150)

Mars as Kepler was about to do so as well. Kepler himself said of this wonderful coincidence that had it not been for Tycho and Longomontanus concentrating on the case of Mars when Kepler met them in Prague in 1600, he never would have made one of his most profound discoveries.

Mathematics as the Epitome of Abduction

Michael Hoffmann, in considering various problems with Peirce's theory of abduction, clarifies what Peirce had in mind in regarding abductive reasoning as a form of guessing:

> [W]hile deduction is apodictic and truth preserving reasoning (cf. Peirce 1902c, CP 4.233), abduction only infers guesses from guesses. When we have found in theorematic reasoning a new perspective, or when we have "added" something else to our diagram and "the conclusion appears" (Peirce 1909, NEM 3: 870), this conclusion is as apodictically true as any corollarial deduction. In abduction, however, if we guess that a certain curve A might describe our measured data of marathon races, and if we infer from this guess that "there is reason to suspect that A is true," it is one thing to prove a theorem and another to formulate it, even if it might again be necessary for proving a theorem to formulate further theorems. Thus, it would make sense to describe the first task as theorematic deduction and the second task as abduction. With regard to abduction in mathematics, the reader is invited to try some experiments with an example developed by Otte (1998) .... If you click at the figures

> presented there, you can move certain points of geometrical diagrams. You will find yourself confronted with surprising mathematical "facts" that will give you an idea of abduction in mathematics. (Hoffmann 1999: 293–294)

This raises immediately for anyone interested in mathematical cognition the difference between formulating a theorem in mathematics and proving it. Formulation of a theorem involves an abductive process of perhaps guessing that a particular situation plausibly holds. The deductive part of the process then produces a proof that the hypothesis is correct, establishing it as a matter of fact, as a theorem. This is what Peirce had in mind by "theorematic reasoning," in contrast to "corollarial deduction":

> Deductions are of two kinds, which I call *corollarial* and *theorematic*. The corollarial are those reasonings by which all corollaries and the majority of what are called theorems are deduced; the theorematic are those by which the major theorems are deduced. If you take the thesis of a corollary,—i.e. the proposition to be proved, and carefully analyze its meaning, by substituting for each term its definition, you will find that its truth follows, in a straightforward manner, from previous propositions similarly analyzed. But when it comes to proving a major theorem, you will very often find you have need of a *lemma*, which is a demonstrable proposition about something outside the subject of inquiry; and even if a lemma does not have to be demonstrated, it is necessary to introduce the definition of something which the thesis of the theorem does not contemplate. (Peirce 1901: CP 7.204)

The key difference in mathematical reasoning is the introduction of a lemma from "outside the subject of inquiry" in accordance with the "form of inference" for abduction, as partially quoted above by Hoffmann: "The surprising fact, C, is observed. But if A were true, C would be a matter of course. Hence, there is reason to suspect that A is true" (Peirce 1903: CP 5.189). Peirce goes on to acknowledge that this requires recognition, in the wake of the surprising observation, that it would follow necessarily if a certain hypothetical premiss were true. This is the abductive step, the leap or conjecture by which "the well-prepared mind [like Kepler's] has wonderfully soon guessed each secret of nature" (Peirce 1908: CP 6.476).

The history of mathematics has numerous examples of such hypotheses conjectured or believed to be true but lacking a cogent proof. One of the most renowned of these is the famous "last theorem" of Pierre de Fermat (1607–1665), which was finally proved by the British mathematician Andrew Wiles and published in 1995. For more about Wiles and Fermat, see (Aczel 1997). As he was reading a Latin translation of the *Arithmetica* by the Greek mathematician Diophantus, Fermat recorded comments in the margins in his copy of the book. At one point, when he had reached a particular problem in the *Arithmetica*, II.8, he was inspired to consider something even more general. Problem II.8 asks, given a square, that it be divided into the sum of two smaller squares. This reduces arithmetically to the problem, given a number c^2, to find numbers a and b such that $c^2 = a^2 + b^2$. For powers greater than 2, Fermat conjectured there were no possible solutions. Unfortunately, his proof was too long, he said, to fit into the space available in the margin of his copy of Diophantus, as he duly noted. Figure 6 is the title page and page 61 with Fermat's annotation noting his famous conjecture, from the 1670 edition of the translation from the Greek into Latin by Claude-Gaspard Bachet. The 1670 edition was published by Fermat's son, Samuel Fermat, with his father's marginal annotations as they appeared in his copy, an edition of 1637, of the Bachet translation first published in 1621.

Another example of a "guess" that demonstrates the powerful intuition of a mathematician is the equally famous continuum hypothesis, first advanced by the German mathematician Georg Cantor (1845–1918). This concerns a matter in which Peirce himself was particularly interested and on which he held views quite inconsistent with those of Cantor. Basically, the continuum hypothesis states that if the cardinal number of the set of all natural numbers N is $\aleph_0$ (aleph-null), then the cardinality of the set of all real numbers is of the next higher cardinality, $2^{\aleph_0} = \aleph_1$, such that there are no sets of numbers of cardinality between them. This was something that Cantor first formulated in the 1890s and spent the rest of his life trying to prove. It was a guess that he was certain for multiple reasons must be true, but one that he could never establish (Dauben 1979). It remains unproven to this day.

Diagrams and Abductive Reasoning

Diagrams have also been essential to the advance of mathematics throughout its history. In 1703, when Oxford University Press published a dual-language edition of all then-known works of the Greek mathematician Euclid, it issued a magnificent quarto volume, *Euclidis quae supersunt omnia* (Euclid's Collected Works; lit. Everything that Survives of Euclid), by the Savilian Professor of Astronomy, David Gregory (1659–1708). The university engraver, Michael Burghers, was called upon to design an appropriate frontispiece. In doing so, Burghers drew on a famous story recounted by Vitruvius (*De architectura*, Book VI) about the Socratic philosopher Aristippus, shipwrecked off the island of Rhodes. Encountering geometric diagrams drawn in the sand, he exclaims: "*hominum enim vestigia video*" (I see a vestige of man). In Burghers' frontispiece (Fig. 7), with mathematics representing the epitome of human reason, there are three geometric diagrams, and the one to which the foremost figure points with his foot is the diagram for a particular theorem in Euclid's *Elements*, Proposition I.32.

It may be worth noting that Oxford actually used this same frontispiece on two subsequent occasions: first, less than a decade later when it published a Latin translation of the *Conics* of Apollonius (*Apollonii Pergaei conicorum*) by Edmund Halley, Savilian Professor of Geometry, in 1710, and again in 1792 when it published a dual-language edition in Greek and Latin, this time the collected works of Archimedes (*Archimedis quae supersunt omnia*) in an edition by Giuseppe Torelli. However, the diagrams suitable for Gregory's Euclid would not do for volumes on conic sections or the works of Archimedes, so appropriate diagrams were substituted (Figs. 8, 9 and 10).

For Peirce, each diagram would have represented a different sort of reasoning, a means of exploring certain hypotheses about parallel lines, conic sections, or the properties of spirals. In the case of the diagram for Euclid's Proposition I.32


DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX,
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.
CVM COMMENTARIIS C. G. BACHETI V. C.
& obseruationibus D. P. de FERMAT Senatoris Tolosani.
Accessit Doctrinæ Analyticæ inuentum nouum, collectum ex varijs eiusdem D. de FERMAT Epistolis.
OBLOQVITVR NVMERIS SEPTEM DISCRIMINA VOCVM
TOLOSÆ,
Excudebat BERNARDVS BOSC, è Regione Collegij Societatis Iesu.
M. DC. LXX.

Arithmeticorum Liber II. 61
QVÆSTIO VIII.
OBSERVATIO DOMINI PETRI DE FERMAT.
QVÆSTIO IX.

OBSERVATIO DOMINI PETRI DE FERMAT.
Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos & generaliter nullam in infinitum vltra quadratum potestatem in duos eiusdem nominis fas est diuidere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.

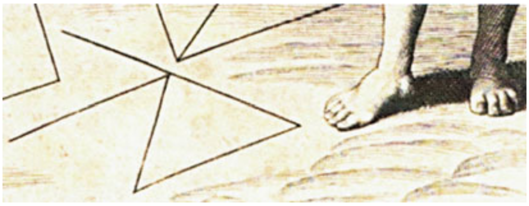
Fig. 6 (Top left) Title page of the edition of Claude-Gaspard Bachet's Latin translation of Diophantus' *Arithmetica*, in the edition of 1670 incorporating Fermat's marginal note on page 61 (top right), with the "Observatio" enlarged (below), showing the famous marginal note of Fermat about his "proof" of what came to be known as "Fermat's last theorem"

(Fig. 11), its accompanying text reads as follows: "In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles" (Heath 1908: 316).

It is not the proposition or its proof that matters here, but the postulate on which its proof depends, namely, the famous fifth postulate of Euclid's *Elements*, which asserts: "That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles" (Heath 1908: 202). For further discussions of the problematic nature of Euclid's fifth postulate, see (Heath 1908: 202–220; Trudeau 2001: 118–153; Gray 2007: 79–88).

Fig. 7 David Gregory: *Euclidis quae supersunt omnia* (1703): frontispiece

Fig. 8 Euclid (1703)



On the simpler equivalent of Postulate 5, the “parallel postulate” first advanced by Proclus in his commentary on Euclid and later by Playfair—that given a line and a point not on the given line, only one line can be drawn through the given point that is parallel to the given line—see (Grabiner 2009: 4). Note that in Euclid’s version of the postulate, there is no reference to parallel lines, which may be his reason for not wishing to assume a simpler version of this hypothesis like those advanced by Proclus and Playfair.

There are many things that are odd about this appearing as a postulate in Euclid’s geometry. All the other postulates are immediately self-evident, e.g., that a straight line may be drawn between any two points or that all right angles are equal. But the fifth is *not* self-evident, and for more than two millennia, mathematicians tried to prove that the parallel postulate was in fact derivable from the other definitions and axioms of Euclidean geometry. Euclid himself may have tried to prove it and, failing

Fig. 9 Apollonius (1710)

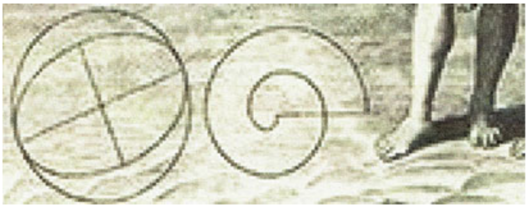


Fig. 10 Archimedes (1792)

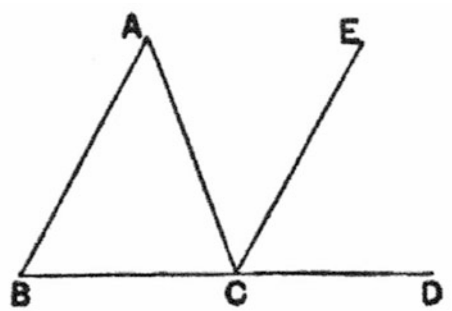


Fig. 11 Thomas Heath's diagram accompanying his translation of Euclid's Proposition I.32 (Heath 1908: 317)

to do so, included it instead as a postulate because it was necessary to prove important basic theorems in his geometry, especially those dealing with parallel lines in Book I, like Propositions 1.31 and 1.32, and later theorems that in turn use either of these propositions.

Diagramming the Parallel Postulate

As it turns out, the parallel postulate is *not* provable from the other axioms of Euclid's geometry and thus on Peircean terms may be taken as a hypothesis to be investigated, the examination of which should depend upon diagrams like those drawn by Burghers in his frontispiece for Wallace's edition of Euclid's geometry. In fact, to enable exactly this sort of investigation, Peirce even provided a diagram for the parallel postulate (Fig. 12) as part of his development of existential graphs in a long manuscript called "Logical Tracts No. 2" (Peirce 1903: CP 4.471, Fig. 120).

Perhaps in all of pure mathematics, there is no more famous hypothesis than this one, whose history spans from the Greeks to the present. As Judith Grabiner explains:

> The historical focus on the fifth postulate came because it felt more like the kind of thing that gets proved. It is not self-evident, it requires a diagram even to explain, so it might have seemed more as though it should be a theorem. In any case, there is a tradition of attempted proofs throughout the Greek and then Islamic and then eighteenth-century mathematical worlds. (Grabiner 2009: 4)

Kant's views on geometry and space may have influenced Peirce's thinking, as well:

> Kant argued that we need the intuition of space to prove theorems in geometry. This is because it is in space that we make the constructions necessary to prove theorems. And what theorem did Kant use as an example? The sum of the angles of a triangle is equal to two right angles, a result whose proof requires the truth of the parallel postulate. (Kant, "Of space," cited in (Grabiner 2009: 12))

In 1806, Joseph-Louis Lagrange was among the last of the great mathematicians to try and prove the parallel postulate. Although never published, a paper he wrote

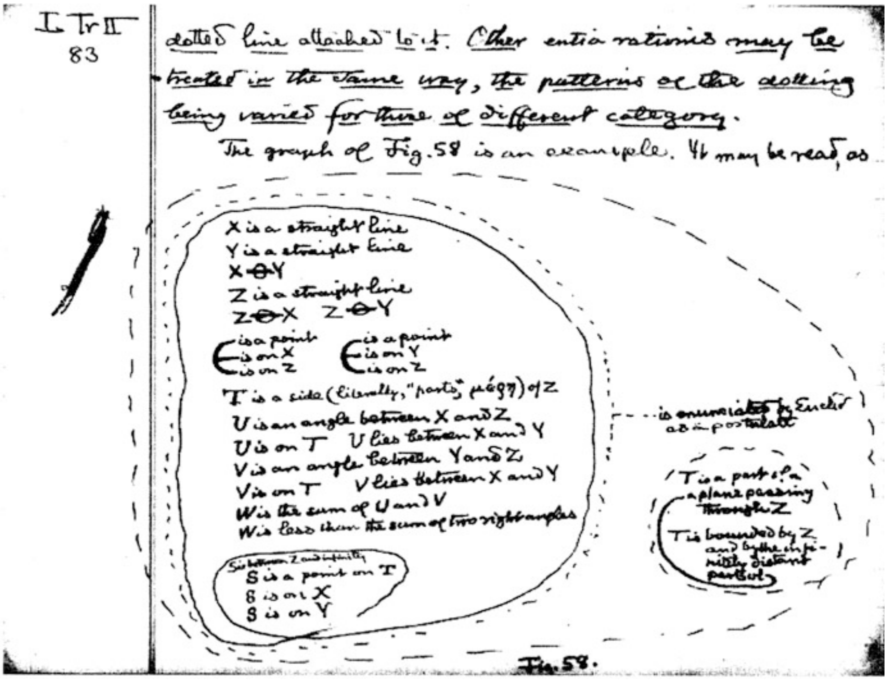


Fig. 12 Peirce's holograph diagram; Peirce papers, Houghton Library, Harvard University

on this subject was presented to the *Institute de France*, one in which he argued that the postulate should not in fact be assumed but ought to be provable from the definitions, axioms, and postulates of Euclidean geometry. Why was Lagrange so interested in proving the parallel postulate rather than simply accepting it as a necessary assumption? Again, Judith Grabiner explains:

> Because there was so much at stake. Because space, for Newtonian physics, has to be uniform, infinite, and Euclidean, and because metaphysical principles like that of sufficient reason and optimality were seen both as Euclidean and as essential to eighteenth-century thought. How could all of this rest on a mere assumption? So, many eighteenth-century thinkers believed that it was crucial to shore up the foundations of Euclid's geometry, and we can place Lagrange's manuscript in the historical context of the many attempts in the eighteenth century to cure this "blemish" in Euclid by proving Postulate 5. (Grabiner 2009: 14)

Early in the nineteenth century, several mathematicians suspected that consistent geometries could be devised without the parallel postulate, among them the German Carl Friedrich Gauss, but he was so concerned that no one would take such ideas seriously that he withheld them from publication. Instead, the two earliest published pioneers of non-Euclidean geometry were well out of the mainstream of European mathematics, namely, the Russian Nikolai Ivanovich Lobachevsky (1792–1856) and the Hungarian János Bolyai (1802–1860). By midcentury, other mathematicians like Bernhard Riemann (1826–1866) and Eugenio Beltrami (1835–1900) had presented non-Euclidean geometries in ways that many mathematicians could follow and begin to appreciate. Among them was Charles Sanders Peirce.

Peirce and Non-Euclidean Geometry

In a letter to William James in 1909, Peirce urged him to learn about non-Euclidean geometry (Eisele 1975: 151). Why was this a matter of such importance to Peirce? In part because, on a cognitive level, he was convinced that it was relevant to what we know about physical space and even the very laws of nature. Peirce had come to appreciate non-Euclidean geometry, perhaps first through the work of Lobachevsky, which was translated into English by the American mathematician George Bruce Halsted in 1891 as *Geometrical Researches on the Theory of Parallels*. Peirce was asked to review this for *The Nation* in 1892. In doing so, Peirce wrote that Lobachevsky's "overthrow of the axioms of geometry . . . must lead to a new conception of nature, less mechanical than that which has guided the steps of science since Newton's discovery" (Peirce 1892: CP 8.91).

Peirce predicted that the revolution in mathematics caused by the discovery and acceptance of non-Euclidean geometry would lead to a parallel revolution in metaphysics, namely, the end of a belief in mechanistic determinism. From the axiomatization of Euclidean geometry, including Postulate 5, it is possible to prove by deductive means alone that, for example, the sum of the angles of any triangle equals two right angles. This is the purport of Euclid's Proposition I:32. But as Shannon Dea puts it:

> Lobachevsky's and Riemann's combined discoveries showed that the sum of a triangle's angles does not necessarily equal the sum of two right angles. In doing so, they revealed the question of the actual sum of a triangle's angles to be an empirical question subject to measurement. (Dea 2008: 615)

As Dea continues, this permitted Peirce "to carve out a space for chance and vagueness in the physical universe," one that was more in keeping with the idea of a world in which the laws of nature were not fixed but might have changed and might still be changing over time, one that "more easily conformed to an evolving universe characterized by stochastic causation than the static universe of mechanistic determinism." By Dea's count, in various writings between 1891 and 1893, Peirce repeated his contention that the discovery of non-Euclidean geometry "spells the end of mechanistic determinism" (Dea 2008: 615).

How does all this relate to Peirce and his interest in diagrammatic reasoning? Peirce agreed with Kant in believing that mathematics consisted of "the study of schematic shapes in the form of diagrams, where observation and testing bring out the new relations between the parts" (Eisele 1975: 151). In a short piece written in the spring of 1890, "The Non–Euclidean Geometry Made Easy," Peirce included a diagram (Fig. 13) to illustrate Lobachevsky's "imaginary geometry," although for Peirce the image was a diagram in space.

He then went on to draw several important conclusions:

> We have an *a priori* or natural idea of space, which by some kind of evolution has come to be very closely in accord with observations. But we find in regard to our natural ideas, in general, that while they do accord in some measure with fact, they by no means do so to such a point that we can dispense with correcting them by comparison with observations. (Peirce 1890: W 8: 25)

Peirce's many years of gravitational research "swinging pendulums" for the US Coast and Geodetic Survey placed him in an excellent position to appreciate the curved geometry of the Earth's surface. That being the case, he surely would have welcomed the experiment that Einstein had suggested to demonstrate the correctness of another triumph of abduction, his general theory of relativity, confirming his prediction of the gravitational curvature of space itself as verified by two expeditions to observe a solar eclipse in 1919 (Landau 2019; Gilmore and Tausch-Pebody 2020). Peirce likewise advocated further empirical study to ascertain the real shape of the universe:

> Thus, the postulates of geometry must go into the number of things approximately true. It may be thousands of years before men find out whether the sum of the three angles of a triangle is greater or less than 180 degrees; but the presumption is, it is one or the other. (Peirce c. 1893: CP 1.130)

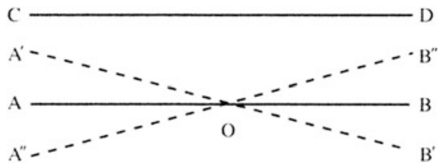


Fig. 13 Peirce's diagram illustrating Lobachevski's non-Euclidean geometry. (Peirce 1890: W 8: 25)

For Peirce, the cognitive elements in all this are related to diagrammatic reasoning and the role that diagrams play in the study of space, as well as the possibility that rejecting the famous Postulate 5 of Euclid's *Elements* does not lead to inconsistent geometries. For a recent, detailed study of Euclid examined diagrammatically, including analysis of the syntax and semantics of diagrams, see (Miller 2007). Moreover, the discovery of non-Euclidean geometries linked mathematics to Peirce's rejection of mechanistic determinism. Just as for Lagrange proving the parallel postulate was essential to affirming the order of the Newtonian view of absolute space and time and a wholly deterministic clockwork universe, Peirce saw in the triumph of non-Euclidean geometry a non-Newtonian world in which even the most fundamental "laws" of nature were subject to evolution, to change, a view that he called *tychism*.

Peirce and Nonclassical Logic

A similar situation exists in the field of logic, where at least since Plato and Aristotle two primary "laws of thought" have been recognized: noncontradiction as "not both A and not-A" and excluded middle as "either A or not-A," where "A" is any proposition whatsoever and "not-A" is its negation. Systems of logic are considered to be "classical" if they conform to these two "laws," which taken together constitute the principle of bivalence: No proposition is both true and false, and every proposition is either true or false. However, "nonclassical" systems of logic have emerged in recent decades that dispense with one or the other. For example, "paraconsistent" or "dialethic" logics permit some contradictions as a way of accommodating certain paradoxes, while "fuzzy" logics facilitate approximate reasoning by assigning a range of values rather than only two.

Throughout his life, Peirce considered himself to be first and foremost a logician, and he is widely credited with several significant advances in the field including the independent invention of quantification, the first use of the truth-table method to define two- and three-valued operators, and the introduction of a particular formulation of excluded middle that is now called "Peirce's law." This name for it is ironic, because although Peirce steadfastly maintained the inadmissibility of contradictions, he recognized that excluded middle is not without exceptions:

> Logic requires us, with reference to each question we have in hand, to hope some definite answer to it may be true. That *hope* with reference to *each case* as it comes up is, by a *saltus*

> [leap], stated by logicians as a *law* concerning *all cases*, namely, the law of excluded middle. (Peirce n.d.: NEM 4: xiii)

In fact, his landmark paper, “On the Algebra of Logic: A Contribution to the Philosophy of Notation” (Peirce 1885: CP 3.359–403), “anticipated the development of mathematical logic by about 40 years” by identifying five “icons” that effectively serve as a set of axioms for classical logic, the last of which is the “law” that now bears his name. Moreover, “his axiomatization of 1885, omitting Peirce’s Law, which he included as a last resort to prove the completeness of Classical Propositional Logic, hides the nucleus of an axiomatization of Intuitionistic Propositional Logic” (Oostra 2013: 20–22; own translation). This is another nonclassical system, which is analogous to non-Euclidean geometry in that its only difference from classical logic is the omission of a single “postulate,” namely, excluded middle and its corollaries such as Peirce’s “law” and double negation elimination. L. E. J. Brouwer initially outlined intuitionism in 1907, while his student Arend Heyting first formalized its logic in 1930, which is also commonly referred to as constructive logic because of its original philosophical motivation:

> [A] characteristic feature of intuitionism is the requirement that the notion of truth of a proposition should be explained in terms of the notion of proof, or verification, rather than as correspondence with some sort of mind-independent realm of mathematical objects; from this one concludes that not every sentence is either true or false. (Raatikainen 2004: 131)

In other words, constructivism in general and intuitionism in particular seek to account for *subjective* or *epistemological* indeterminacy: There are some propositions that cannot currently be evaluated as either true or false because *knowledge* is indeterminate, since there is not (yet) a proof one way or the other. This contrasts with Peirce’s primary reason for being skeptical of excluded middle, which he stated in two slightly different ways:

> To speak of *the* actual state of things implies a great assumption, namely that there is a perfectly definite body of propositions which, if we could only find them out, are the truth, and that everything is really either true or in positive conflict with the truth. This assumption, called the principle of excluded middle, I consider utterly unwarranted, and do not believe it. (Peirce 1893: NEM 3:758)
>
> No doubt there is an assumption involved in speaking of *the* actual state of things . . . namely, the assumption that reality is so determinate as to verify or falsify every possible proposition. This is called the *principle of excluded middle*. . . . I do not believe it is strictly true. (Peirce 1893: NEM 3:759–760)

In other words, Peirce sought to account for *objective* or *ontological* indeterminacy: There are some propositions that are neither true nor false because *reality* is indeterminate. As he wrote years later in his Logic Notebook, “every proposition, S is P, is either true, or false, or else S has a lower mode of being such that it can neither be determinately P, nor determinately not-P, but is at the limit between P and not P” (Peirce 1909: MS 339). This is consistent not only with his tychism but also with his overarching *synechism*: “that tendency of philosophical thought which insists upon the idea of continuity: as of prime importance in philosophy and, in particular, upon the necessity of hypotheses involving true continuity" (Peirce 1902: CP 6.169).

For Peirce, "true continuity" contrasts with "a *pseudo-continuum* as that which modern writers on the theory of functions call a *continuum*" (Peirce 1908: CP 6.176). He thus rejected the growing consensus among the mathematicians of his time, led by Richard Dedekind and Georg Cantor, that all the real numbers together constitute a continuum. Their conception was grounded in set theory, such that a continuum could be built up from a sufficiently large transfinite multitude of distinct parts, while Peirce viewed the whole as more fundamental, having parts that are indefinite unless and until they are deliberately marked off for some purpose (Schmidt 2020). The upshot is that he steadfastly maintained the reality of infinitesimals, a stance that has arguably been vindicated by the subsequent development of synthetic differential geometry and smooth infinitesimal analysis, the logic of which—i.e., the logic of true continuity—is intuitionistic (Bell 2006: 294–297). Consequently, had Peirce followed through on his remarkably prescient insights and fully developed such a system, it might instead be known today as *synechistic* logic.

Moving Pictures of Thought

In any case, Peirce himself ultimately considered his most important contribution to logic to be the development of a "diagrammatic syntax" for propositions and a set of transformation rules for carrying out deductive inferences from them, a system that he dubbed "existential graphs." He had three objectives in mind for it: "to afford a method (1) as *simple* as possible (that is to say, with as small a number of arbitrary conventions as possible), for representing propositions (2) as *iconically*, or diagrammatically and (3) as *analytically* as possible" (Peirce 1908: CP 4.561n).

A blank sheet stands for the continuum of all true propositions, any of which may be explicitly "scribed" on it as a graph-instance consisting of a single letter in the "Alpha" version for propositional logic or of names denoting abstract general concepts and heavy lines denoting concrete indefinite individuals ("something") in the "Beta" version for first-order predicate logic. Each name has one, two, or three "pegs" where a heavy line may be attached, signifying the attribution of the concept to that individual. The number of pegs associated with the name corresponds to the "valency" of the concept as monadic ("redness"), dyadic ("killing"), or triadic ("giving"); see Fig. 14.

There is no limitation on the number of graph-instances that may be scribed on the sheet, which signifies the primitive relation of *coexistence* such that juxtaposing multiple graph-instances expresses the conjunction of the propositions that they represent. There is also no limitation on the number of branches that may be added to a heavy line, which corresponds to the primitive relation of *identity* such that each branch attached to a name attributes another concept to the same individual. Coexistence and identity are thus *continuous* relations, and they are also symmetrical: "A and B" is logically equivalent to "B and A," while "some S is P" is logically equivalent to "some P is S."

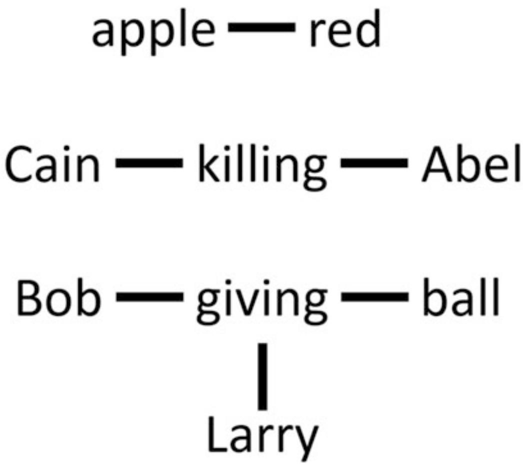


Fig. 14 Existential graphs for “some apple is red,” “Cain killed Abel,” and “Bob gives a ball to Larry”

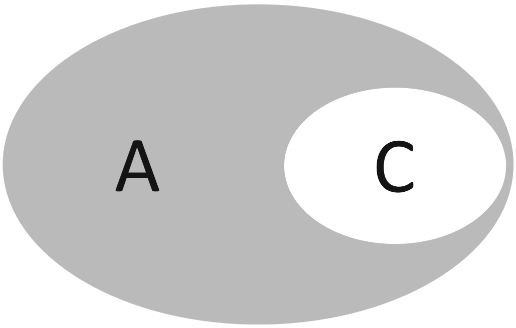


Fig. 15 Existential graph for “if A then C”

A third primitive relation is required, and for Peirce it is the most fundamental of all: “The first relation of logic, that of antecedent and consequent, is unsymmetrical. Now an unsymmetrical relation cannot result from any combination of symmetrical relations alone” (Peirce 1905: NEM 3:821). This is what he usually called “consequence,” now typically referred to as “implication.” It is represented in existential graphs by a “scroll,” which is “a curved line without contrary flexure and returning into itself after once crossing itself, and thus forming an outer and an inner ‘close.’ . . . In the outer I scribed the Antecedent [A], in the inner the Consequent [C]” (Peirce 1906: CP 4.564); see Fig. 15.

The continuity of the scroll itself thus reflects the continuity of the inference from the antecedent to the consequent, and disjunction is derived from it by simply adding more loops with inner closes. Any graph-instance, including the blank, is always interchangeable with a scroll that has an empty outer close and that same graph-instance in its inner close.

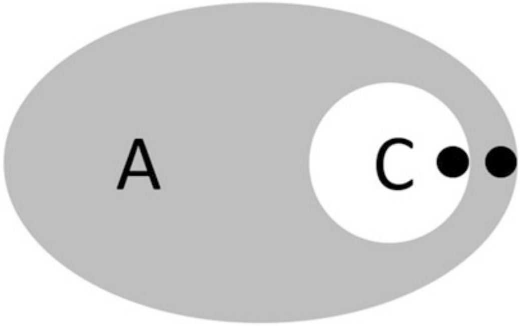


Fig. 16 Existential graph for "not both A and not-C"

On the other hand, the scroll also serves as a discontinuity or topical singularity that interrupts the sheet. With that in mind, Peirce eventually realized that *shading* oddly enclosed areas is the best way to distinguish them from unenclosed and evenly enclosed areas, which are unshaded: simpler than counting lines, more iconic for conveying that they are different surfaces, and more analytical because shaded areas correspond to a universe of possibility rather than actuality (Peirce 1906: CP 4.576–581). The permissible transformations of graph-instances, which correspond to rules of inference, are then summarized as follows:

1. Erasure—In an unshaded area, any graph-instance or portion of a line may be deleted.
2. Insertion—In a shaded area, any graph-instance may be added and any lines may be joined.
3. Iteration—Any graph-instance already scribed may be reproduced identically in the same area or in a more enclosed area, and any unattached end of a line may be extended into a more enclosed area.
4. Deiteration—Any graph-instance that could have resulted from iteration may be deleted.

For classical logic, a continuous scroll is logically equivalent to nested "cuts," which are simple oval lines or shaded/unshaded areas that represent negation. Peirce derived this from a scroll whose consequent is "a proposition implying that every proposition is true," resulting in "a black spot entirely filling the close in which it is," which "may be drawn invisibly small" (Peirce 1906: CP 4.454–456). However, at one point he retracted this last provision and instead advocated retaining a small blackened inner close attached to the cut as a reminder of its theoretical basis (CP 4.564n, c. 1906). Moreover, he called it an "error" and an "inaccuracy" to analyze "if A then C" as no different from "not both A and not-C"; see Fig. 16:

> For in reasoning, at least, when we first affirm, or affirmatively judge, the conjugate of premisses, the judgment of the conclusion has not yet been performed. There then follows a real movement of thought in the mind, in which that judgment of the conclusion comes to

> pass. Now surely, speaking of the same A and B as above, it were absurd to say that a real change of A into a sequent B consists in a state of things that should consist in there not being an A without a B. For in such a state of things there would be no change at all. (Peirce 1908: MS 300)

It is interesting to note that in intuitionistic logic, negation is likewise defined as the implication of falsity, and although “if A then C” implies “not both A and not-C,” the inference in the other direction is invalid. In fact, simply by explicitly distinguishing a scroll for implication from nested cuts for double negation, existential graphs can be employed in accordance with intuitionistic logic rather than classical logic (Oostra 2010, 2011). Either way, implementing a series of diagrammatic transformations to draw a deductive conclusion from a set of premisses serves as “a moving-picture of Thought” (Peirce 1906: CP 4.11). For a more detailed exposition of existential graphs, see (Roberts 1992).

Representation and Semeiotic

The parallel postulate and the “law” of excluded middle are presuppositions that, contrary to conventional ways of thinking as shaped by the Euclidean geometry and classical logic of experience, may nevertheless give way to useful alternatives when subjected to careful “experimentation.” Such activities might be in physical terms as was the case with the eclipse observations confirming the curvature of space, or the kinds of imaginative exercises that Peirce considered in geometry of the Lobachevskian sort, or reasoning in accordance with classical vs. intuitionistic logic by employing existential graphs with different rules.

These specific examples relate to the problem of representation more generally. Representation is intrinsic to human life and essential for human communication. Words, gestures, symbols, signs, and diagrams, among other means of conveying information, involve mental, informational, and computational models, the intent of which is to capture the relevant characteristics of the reality that such constructs are intended to explore or express.

In ordinary day-to-day activities, it is a matter of determining informally what characters and characteristics are to be conveyed. However, it is not only in mathematics, logic, and the sciences generally, whether pure or applied like engineering, but in virtually any intellectual discipline, including philosophy, that it is necessary to incorporate appropriate assumptions and simplifications. There are two common strategies for doing this, namely, abstraction and idealization. Abstraction involves bracketing out and ignoring certain aspects of reality in order to gain a better understanding of the other aspects, while idealization involves replacing a complex aspect of reality with a necessarily simplified version. Thus, such strategies entail that no representation can be completely accurate, and this raises the question of how they relate to reality. For Peirce, this is the domain of semeiotic—his preferred spelling for the science that studies signs and signification (Peirce 1903: CP

1.191). As he expressed it: “A sign is something which stands to somebody for something in some respect or capacity” (Peirce 1897: CP 2.228).

Most modern semiotic theories, such as Saussure’s, have been dyadic, emphasizing only two components, namely, the signifier and that which is signified, the relationship holding between these being essentially arbitrary. Their limitations became evident with the emergence of deconstruction and other aspects of postmodernism that effectively preclude objective meaning. Peirce developed an alternative, founding what eventually came to be known as pragmatism, involving a theory of signs that is *triadic*, emphasizing the irreducible relation of three correlates: the sign itself, that which it signifies, and the effect that it produces, the last brought about by a “habit of interpretation.” He aligned these elements—sign, object, and interpretant—with three fundamental categories that he identified, initially through exploring whatever is or could be present to the mind (phenomenology) and then by reasoning about the underlying nature of reality (scientific metaphysics). Peirce named these categories Firstness, Secondness, and Thirdnesss (Richmond 2005).

Firstness is quality, feeling, possibility, spontaneity, and vagueness; Secondness is reaction, difference, actuality, persistence, and particularity; Thirdnesss is mediation, generality, purpose, regularity, and order. For example, in Peircean semeiotic an icon relates to its object through some kind of resemblance (Firstness), an index due to a physical or other direct connection (Secondness), and a symbol by means of conventions or rules, the habits of interpreters that ensure their being so understood (Thirdnesss) (Peirce 1911: EP 2:461). Hence a statue is an icon, a weathervane is an index, and a word or a sentence is a symbol.

It should never be forgotten that for Peirce, representation—semiosis—is closely connected with reality, since he held that “all this universe is perfused with signs, if it is not composed exclusively of signs” (Peirce 1906: CP 5.448n). Because thought itself consists of signs, inquiry is seen by Peirce as a deliberate, collaborative endeavor to process such thoughts in a way that results in genuine knowledge. Scientific inquiry involves all three basic modes of inference: abduction as the formulation of an explanatory hypothesis, deduction as the explication of what else would be the case if that hypothesis is correct, and induction as the examination of whether those consequences ever fail to materialize in an appropriate experiment, which would falsify the hypothesis.

Diagrammatic Logic

Peirce further held that all deductive reasoning is mathematical and that all mathematical reasoning is diagrammatic, i.e., it proceeds by creating, manipulating, and observing an icon meant to reflect the form of the significant relations among the parts of the object of interest. Mathematicians need to discern, as best they can, which relations are indeed significant in order to devise suitable icons of their form. While the word “diagram” may typically be associated with some sort of picture, Peirce considered a diagram more broadly as a mental image, possibly changing or developing, of such a thing as it represents. Although a physical sketch or model may be employed as an aid, the essential thing to be performed is the act of imagining (Peirce 1906: NEM 4:219n1). While diagrams embody formal relations, it is not essential that they do so visually. Thus, while a geometric figure is surely a kind of diagram, so too are algebraic expressions.

What makes diagrammatic reasoning powerful is that, although it essentially involves deductive inference—for there is nothing in the conclusion that was not already present in some way in the premisses—it can, nonetheless, reveal something that was not initially evident, or at least not clearly so. Yet a diagram must necessarily be provisional, as it always includes abstractions and idealizations chosen by whomever is employing it. In that choosing, it is active, that is, it involves creativity: "It is necessary that something should be DONE. In geometry, subsidiary lines are drawn. In algebra permissible transformations are made" (Peirce 1902: CP 4.233). Such modifications or transformations are, of course, not completely arbitrary, since they must conform to the precepts of the representational system involved, which then will also dictate their outcomes.

In this regard, Peirce wrote that "all reasonings turn upon the idea that if one exerts certain kinds of volition, one will undergo in return certain compulsory perceptions" (Peirce 1905: CP 5.9). Still, an important question remains: While nature corroborates or falsifies a theory through encounters in the real world, what is the normative aspect of a hypothetical one? Peirce suggests an answer:

> [C]ertain modes of transformation of Diagrams . . . have become recognized as permissible. Very likely the recognition descends from some former Induction, remarkably strong owing to the cheapness of mere mental experimentation. Some circumstance connected with the purpose which first prompted the construction of the diagram contributes to the determination of the permissible transformation that actually gets performed. (Peirce 1906: NEM 4: 318)

In other words, which transformative moves are legitimate becomes apparent mainly through the persistent activity of the intellect, which is far less costly or time-consuming than a genuinely inductive investigation because it does not deal with a course of experience, but rather with whether or not a certain state of things can be imagined (Peirce 1902: CP 2.778). How one proceeds in an individual case of diagramming is subject to constraints, depends on one's intentions, and indeed involves the entire train of thought, which from the semiotic standpoint is a continuous stream of signs. To be effective, such a diagram must incorporate the features that are relevant to achieving the end being pursued, while most other considerations need to be bracketed and ignored.

Operations upon diagrams, whether external or imaginary, take the place of the experiments upon actual things that are performed in scientific research (Peirce 1905: CP 4.530). Moreover, diagrams and representational systems are artifacts that people design. As Michael Hoffmann puts it: ". . . *seeing a solution* presupposes *seeing a problem* . . . The central idea of this kind of reasoning is that we *see* problems when we try to represent what we know about something . . . We have to *represent* what we know – or think to know – in order to *see*, first, its limitations and, second, new possibilities" (Hoffmann 2006: 3–4). This hints at very broad applications that Peirce begins to take up in the normative sciences: theoretical esthetics, theoretical ethics, and logic broadly conceived as semeiotic.

Inquiry and Ingenuity

For Peirce, normative science is "the science of the laws of conformity of things to ends," such that "esthetics considers those things whose ends are to embody qualities of feeling, ethics those things whose ends lie in action, and logic those things whose end is to represent something . . . That is right action which is in conformity to ends which we are prepared deliberately to adopt" (Peirce 1903: CP 5.129–130). He eventually came to view logic as a form of ethics, because thought is a form of conduct, and self-control is essential to both thinking well and acting well. This is important because while neither the past nor the immediate present is affected by the will, the future is susceptible to being influenced to some extent through deliberate thought and action.

Accordingly, although the logic of inquiry applies most directly to science, it also manifests as a logic of ingenuity. Schmidt (2016) coined this term in a series of magazine articles written for an audience of practicing structural engineers, observing that the English words "ingenuity" and "engineer" have the same etymological root. While Peirce largely concentrated on the logic of inquiry, he did some work as a consulting engineer in the 1890s and wrote about the logic of ingenuity as described here (without calling it that) during the same time frame (e.g., Peirce 1898: CP 3.559). In modern engineering practice, the logic of ingenuity is the process of (abductively) creating a diagrammatic representation of a problem and its proposed solution and then (deductively) working out the necessary consequences, such that this serves as an adequate substitute for (inductively) evaluating the actual situation. Here abduction constitutes the creative process that leads to the selection of one preliminary solution from multiple candidates. Deduction corresponds to the analyses that indicate the expected behavior of that design, given certain idealized presuppositions. Induction operates over time as an engineer learns from experience to develop competence, proficiency, and eventually expertise, which manifests as the ability to make better abductions.

Hence both inquiry and ingenuity employ signs in the interest of making that which is indeterminate more determinate. However, neither can make the indeterminate *fully* determinate, because according to Peirce the only *complete* sign is the entirety of reality itself, consisting of continuous and complex systems of relations. It follows that the fallibility of any current understanding must be acknowledged, since uncertainty in representation is constrained by reality and can never be completely eliminated.

Determination occurs primarily as *discovery* in science, conforming representation to reality, but as *decision* in engineering, conforming reality to representation. Moreover, while Peirce insisted on grounding science firmly in fact and reason, he was equally adamant that practical matters should be governed primarily by instinct and sentiment. When it comes to “topics of vital importance”—an expression William James once used in a letter to Peirce intended to encourage him to avoid his usual penchant for logical and metaphysical analyses in a series of upcoming lectures, which James thought impenetrable to most audiences—people must rely on their existing beliefs, which turn out to be none other than their established habits of feeling, action, and thought.

Moreover, while science is often regarded as a collaborative endeavor by a large community of investigators working together over an extended period of time to discover general laws and the like, for a single individual under ordinary conditions, “the sole object of inquiry is the settlement of opinion. We may fancy that this is not enough for us, and that we seek, not merely an opinion, but a true opinion. But put this fancy to the test, and it proves groundless; for as soon as a firm belief is reached we are entirely satisfied” (Peirce 1877: CP 5.375). To phrase this somewhat differently, it could be said on the one hand that truth is the goal of inquiry only in the long run, since ongoing interaction with nature prevents people from ever being permanently satisfied with their beliefs as they repeatedly confront evidence that some of them are false. On the other hand, the goal of ingenuity is something that we may hope to achieve in the short term: solving a problem despite incomplete knowledge. In both cases, the upshot of most thinking is, as in engineering, an exercise of volition (Peirce 1878: CP 5.397).

Logic is often most closely identified with strictly deductive reasoning and ingenuity with mere cleverness, but it was Peirce’s tendency to think of logic in the broader sense as the norms of thought in general, and one can think of ingenuity in the narrower sense as the distinctive essence of the practice of engineering. While they are structurally analogous, scientific and engineering reasoning are widely understood as pursuing very different ends. Rather than the discovery of a universal theory with general application, an engineer typically works toward the design of a particular artifact for a specific purpose. Much as science is viewed as an especially systematic way of *knowing*, engineering may thus be viewed as a particularly systematic way of *willing* (Schmidt 2013).

Modeling and Analysis

Especially in large, complex projects as opposed to mass-produced products, the abductive aspect of engineering involves developing an idealized model of the artifact and its immediate environment. The deductive aspect is processing this model in accordance with idealized assumptions, often facilitated by a computer. Finally, the inductive aspect is interpreting the results by comparing them with idealized rules, codes, and various industry standards. As previously remarked, Peirce argued that the logic of inquiry in science is ordinarily self-correcting in the long run, such that the existent world will confront a persistent investigator with unpleasant surprises if a hypothesis is inconsistent with how it really operates. When this happens in engineering, there tends to be a high cost in dollars or even lives, so that the logic of ingenuity quasi-necessarily involves the assessment of the model rather than of the artifact itself. If the conclusion of such an analysis is not

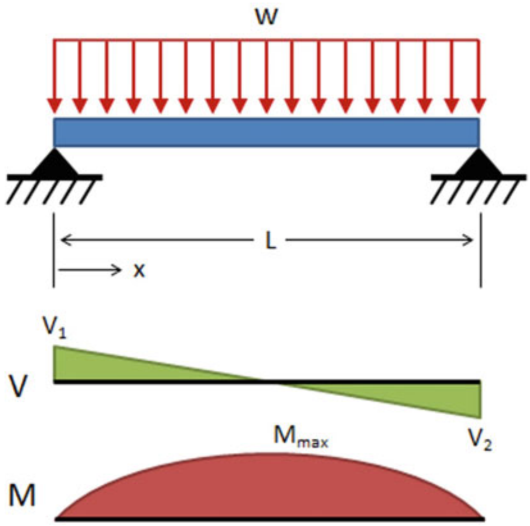


Fig. 17 Top: Diagram of a simply supported beam with uniformly distributed loading. Middle: Diagram representing the shear along the length of the beam. Bottom: Diagram representing the moment along the length of the beam

acceptable, then it is necessary to revise the model as well as the corresponding design and then carry out another analysis.

In short, the engineer must deem everything to be satisfactory before moving on to drafting instructions for constructing the artifact. Therefore, in formulating a model, it is crucial that the engineer discerns which relations are truly significant and only then devises a suitable icon of their form accordingly. This requires good judgment grounded in considerable experience because it is rarely feasible to incorporate *all* aspects of the situation into an engineering model. In fact, for a complex system, it is not even feasible to incorporate all *relevant* aspects. Despite being the creator of the model and presumably familiar with it in all its details and in most cases having an approximate idea of what to expect, the engineer will not necessarily be able to anticipate all its results and their consequences in advance, and revisions may well be needed.

This can be true even for the simplest design scenarios. The problem of determining the static forces associated with a simply supported beam under a uniform loading so that it can be sized appropriately is one of the first that students confront in any structural engineering curriculum. It is relatively easy to describe and captures the central aspects of the logic of ingenuity. Three kinds of diagrams are commonly employed (Fig. 17).

The first diagram represents the beam itself, how it is attached at its two ends, and the loading that is applied to it. The triangle at each end is an icon of a *pin* that restrains horizontal and vertical movement but not rotation. The connected series of downward-pointing arrows symbolizes a *distributed* loading, and their constant length indicates its constant magnitude (w), which results in the *reactions* that the beam transfers to whatever is supporting it ($V_1 = V_2$). The second diagram represents the *shear* (V) along the length of the beam (L), which is the vertical force at each location (x) whose equation is the integral of the distributed loading (w). The third diagram represents the *moment* (M) along the length of the beam (L), which is the bending force at each location (x) whose equation is the integral of the shear (V). These relations reflect how the diagrammatic nature of mathematics is not limited to geometry, but also encompasses algebra.

A real structure almost never contains even one beam that conforms to the assumptions underlying this analysis. There is no such thing as a frictionless pin or roller. Floors and roofs typically bear the weights of discrete furniture, equipment, people, etc. Nevertheless, by calibrating the code provisions that establish uniform loadings for different kinds of occupancies and incorporate factors of safety into the calculated member capacities, effective and economical structures are routinely built in accordance with such a simplified procedure.

Practical and Ethical Reasoning

To sum up, since diagrams and representational systems are artifacts that humans design, it should not be surprising that engineers routinely employ them. But it should be added that the logic of ingenuity may apply to the consideration of *any* potential activity that could be undertaken voluntarily. This logic thus extends beyond engineering into the much broader domain of ethics, a subject hardly touched upon by Peirce until relatively late in his career. In a lecture titled "What Makes a Reasoning Sound?" (Peirce 1903: EP 2:242–257), he answered his own question by drawing a parallel with what makes an action morally right. Regarding this, Campos (2015) summarizes six stages identified by Peirce as directly relating logic to ethics:

1. Affirming *ideals* that together constitute a worldview and shape one's character.
2. Establishing an *intention* to behave in accordance with those ideals.
3. Formulating *rules of conduct*, "practical maxims for what ought to be done in circumstances that fall under a more or less vague description".
4. Making a *resolution* for how to act if and when a specific occasion arises that is foreseen through the use of "semiotic imagination—the ability to create and transform signs—guided by practical knowledge of what paths events may follow".
5. Converting this resolution into a *determination*, an abiding disposition that is "capable of effectively guiding conduct".
6. Engaging in critical *review* of one's actions in relation to all the above, which produces approval or disapproval of the former and sometimes revision of the latter.

Peirce wrote of the fourth step, "This resolution is of the nature of a plan, or, as one might almost say, a diagram" (Peirce 1903: EP 2:246). Prompted by this hint, Campos suggests that each of the others is likewise analogous to an aspect of diagrammatic reasoning:

1. Ideals correspond to a set of "framing hypotheses" that comprise a representational system.
2. An intention corresponds to the purpose of the exercise.
3. Rules of conduct correspond to "heuristic[s] . . . that direct an inquirer to employ a certain method of solution depending on the general type of problem under investigation."
4. A resolution corresponds to "a mathematical model that may be formulated and investigated abstractly but is intended to apply to a concrete state of things."
5. A determination corresponds to the intellectual virtue of judgment that eventually emerges from "mathematical experience."
6. Review corresponds to observation of the results of diagram manipulation.

Such an analysis is consistent with the claim that engineering is an especially systematic way of willing, because if this is so, then the former's distinctive reasoning process should be paradigmatic for the latter. However, engineering mostly deals with material phenomena, which Peirce conceptualized as "inveterate habits becoming physical laws" (Peirce 1891: CP 6.25), while the behavior of people is always subject to change because their habits are far more malleable, with the result that people are far less predictable. Consequently, rather than the *quantitative* models that are routinely employed in engineering, practical reasoning and ethical deliberation involve formulating and evaluating more *qualitative* representations such as narratives.

Nevertheless, the key to success in these various domains is the same: having the ability to discern the significant aspects of reality and consistently capture them before definitively selecting a way forward from among multiple viable options. Indeed, the logic of ingenuity—whether in engineering, in science more generally, or in any other endeavor whatsoever—is itself a carefully cultivated habit that facilitates imagining possibilities, assessing alternatives, and selecting one of them to actualize. From this standpoint, there is a sense in which all reasoning is an implementation of diagrammatic reasoning, demonstrating the relevance of mathematical cognition far beyond the boundaries of mathematics itself.

Guessing Right

Returning now to the example which introduced this reflection on the place of diagrammatic reasoning in mathematics and which has focused especially on how abduction plays a crucial role there as in all science and, ultimately, in all thought, consider again Peirce's "guess" as to who had stolen his Tiffany watch. In the following passage, Peirce outlines what he considers to be the principles guiding anyone who arrives at a promising hypothesis:

> Underlying all such principles there is a fundamental and primary abduction, a hypothesis which we must embrace at the outset, however destitute of evidentiary support it may be. That hypothesis is that the facts in hand admit of rationalization, and of rationalization by us. . . . [N]o new truth can come from induction or from deduction . . .. It can only come from abduction; and abduction is, after all, nothing but guessing. We are therefore bound to hope that, although the possible explanations of our facts may be strictly innumerable, yet our mind will be able . . . to guess the sole true explanation of them. *That* we are bound to assume, independently of any evidence that it is true. Animated by that hope, we are to proceed to the construction of a hypothesis. (Peirce 1901: CP 7.219)

It would seem, then, that the fundamental principle of hypothesis formation is a bare "hope" that one will "guess" the very hypothesis which will explain the facts and so lead to the solution of the problem at hand. While it has perhaps taken modern science some time to catch up with this notion that "guessing" is indeed an essential element in our thought and subsequent behavior, one finds contemporary thinkers acknowledging the necessity and power of this natural first step in all inquiry. For example, Usama Fayyad, a noted American data scientist, has remarked, "Sometimes guessing is the best you can do. In the real word, we guess all the time and its serves us well." In fact, guessing would seem to precede even human thinking, for as Fayyad goes on to explain: "The brain has a good error rate. But, the point is, you can function with that error rate. Animals do a lot guesswork" (Bearman 2003). Of course, Peirce's unique contribution is to recognize that this function is a third form of inference as important as deduction and induction.

It appears that this natural tendency to guess right—at least some of the time—would *a fortiori* be the case for even quotidian thought and action. As Peirce notes in the quotation just above, "the facts at hand admit of rationalization, and of rationalization by us" such that "our mind will be able . . . to guess the sole true explanation of them." This seems as applicable to anyone in science as it was to Peirce's recovery of his expensive Tiffany watch, the valuable chronometer that he was determined to find, however unlikely the means. The guess that led to its recovery, like abductive hypotheses in mathematics and science, was not "lucky" but "educated." As discussed above, the researcher in any given discipline will tend to be well prepared by experience and training to make fruitful guesses, abductions which can be structured as hypotheses to be tested. Through deliberate training, people can become more and more "attuned to the truth of things" (Peirce 1908: CP 6.476), especially when concentrating on a particular field of inquiry.

Near the conclusion of his 1903 Harvard Lectures, Peirce introduced what he called three "cotary propositions," notions meant to "sharpen" the meaning of pragmatism as he had outlined it. Especially in consideration of abduction, the last of these is the most telling:

> The third cotary proposition is that abductive inference shades into perceptual judgment without any sharp line of demarcation between them; or, in other words, our first premises, the perceptual judgments, are to be regarded as an extreme case of abductive inferences, from which they differ in being absolutely beyond criticism. The abductive suggestion comes to us like a flash. It is an act of insight, although of extremely fallible insight. It is true that the different elements of the hypothesis were in our minds before; but it is the idea

> of putting together what we had never before dreamed of putting together which flashes the new suggestion before our contemplation. (Peirce 1903: CP 5.181)

This passage seems especially apposite for the question at hand. For if the order of "abductive inference" and "perceptual judgments" is reversed in the text above, it might be said of the Tiffany watch example that Peirce's perceptual judgments shaded into an abductive inference, an "abductive suggestion," the correct guess about the watch thief coming to him in a "flash." Moreover, "Not the smallest advance can be made in knowledge beyond the stage of vacant staring, without making an abduction at every step" (Peirce 1901: LOS 900).

The final question that must be answered is this: If making educated guesses is crucial to the advance of all knowledge, including mathematics, why is it that we so often guess "right"? For Peirce, "It is a primary hypothesis underlying all abduction that the human mind is akin to the truth in the sense that in a finite number of guesses it will light upon the correct hypothesis" (Peirce 1901: CP 7.220). He believed that this is what the history of science had demonstrated time and again. This in turn depended upon what he termed an "instinct for truth":

> [T]he history of science proves that when the phenomena were properly analyzed, upon fundamental points, at least, it has seldom been necessary to try more than two or three hypotheses made by clear genius before the right one was found. . . . For the existence of a natural instinct for truth is, after all, the sheet-anchor of science. From the instinctive, we pass to reasoned, marks of truth in the hypothesis. (ibid)

There was a strong evolutionary component to this way of conceiving the world and the capacity of the human mind to fathom its deepest workings:

> This feature of the laws of nature is evidence that whatever power it be that is behind them is behind the constitution of human reason, which has such a surprising facility in finding them out. Nature is conformed to general formulae, which really determine how future events shall turn, and these formulae are of such a character that human reason is closely allied to them. Add to this that Nature was not made a long time ago but is even now in the process of being brought about, and is every day growing more wonderfully admirable for human reason. (Peirce 1901: LOS 888–89)

Human cognition is naturally attuned, Peirce believed, to the material world that science and mathematics are able to reveal and model so successfully because they are intimately connected with each other. This is linked to the human mind's "inward power of knowing," and in a revealing discussion of the arguments Hume had advanced against miracles and the working of natural laws in the physical world, Peirce took this a step farther:

> The mind of man has been formed under the action of the laws of nature, and therefore it is not so very surprising to find that its constitution is such that, when we can get rid of caprices, idiosyncrasies, and other perturbations, its thoughts naturally show a tendency to agree with the laws of nature. (Peirce 1901: LOS 901)

Peirce then went on to assert that the human mind exhibits a “magnetic turning toward the truth.” As the case of Kepler so often reminded him, its compass is forged in the experience of each lifetime, especially the experience of a scientist interacting with the world that in turn shaped the mind that was following a cognitive instinct. It was not that just anyone’s guess would do, but that as Louis Pasteur had said and Peirce himself quoted, “chance favors only the prepared mind.”

Conclusion

This is exactly the point that Peirce believed his recovery of the Tiffany watch demonstrated so dramatically and which he took to be typical of scientific discovery generally: the mind’s intuitive compass always pointing toward the truth. Human cognitive faculties have been attuned by nature to seek out the truths of the physical world, even if they are often hidden in obscurity. Kepler was a paradigmatic example for Peirce in that, even after having rejected the correct solution of the planetary orbits as ellipses, he eventually came back to accept the fact that in the end, the mathematics allowed only this one solution; and however much he might have resisted it, resistance eventually yielded to acceptance. It was the principle of Sherlock Holmes in action that when all else has been eliminated, then whatever remains, however improbable, must be the truth. The ellipses, he knew, would require a new physics of the heavens, as Kepler said, and a new mathematics as well; but in pointing the way, he was among the great leaders of modern science.

This, it might be said, was the ultimate cosmic principle for Peirce. In the course of all his writing about abductive reasoning, the mind’s ability to guess right was the crux of his belief in the advance of science and the inevitable increase in positive knowledge with the passage of time, no matter how much it might be detoured along the way. The mathematical cognition of the human mind includes an intuitive gift for finding the correct solutions to increasingly complex problems. This is akin to a practiced engineer’s ability to anticipate in advance the general outcome of a given model prior to going to the expense and trouble of constructing what has been designed. It is the element in every creative act that brings the researcher, whether in pure mathematics or in applied areas of science, to the ever-closer approximation of a true understanding of the workings of the world.

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