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Coordination and Harmony in Bilateral Logic

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Abstract

Ian Rumfitt (2000) developed a *bilateralist* account of logic in which the meaning of the connectives is given by conditions on asserted and rejected sentences. An additional set of inference rules, the *coordination principles*, determines the interaction of assertion and rejection. Fernando Ferreira (2008) found this account defective, as Rumfitt must state the coordination principles for arbitrary complex sentences. Rumfitt (2008) has a reply, but we argue that the problem runs deeper than he acknowledged and is in fact related to the challenge of establishing proof-theoretic harmony. We motivate a distinctively *bilateral* criterion for harmony and show how the bilateralist can meet it. This also resolves Ferreira's complaint.

1. Bilateralism

Logical inferentialists claim that the meaning of the logical constants is given by their use in inferences. To know, for example, the meaning of 'and' is to know what to infer from sentences containing 'and' and to know how to infer such sentences. The meaning of 'and' is thus given by the standard natural deduction rules governing '$\wedge$', allowing one to infer p and q from $p \wedge q$ and *vice versa*. When contrasted with truth-conditional semantics—according to which the meaning of 'and' is given by the Tarskian truth function for '$\wedge$'—the view is also known as *proof-theoretic semantics*.

Inferentialism has become associated with logical revisionism, as it seems to rule out classical logic. As Arthur Prior (1960) notes, not just *any* rules confer a coherent meaning onto a connective. His example is the connective *tonk* whose (putative) meaning is given by the following rules.

$$\text{(tonk I.)}\ \frac{A}{A\ \text{tonk}\ B} \qquad \text{(tonk E.)}\ \frac{A\ \text{tonk}\ B}{B}$$

Clearly, a language containing *tonk* is trivial. Assuming that logical consequence is transitive, any sentence can be derived from each other sentence by applying *tonk*-Introduction followed by *tonk*-Elimination. Thus, the above rules do not confer a coherent meaning. With *tonk*'s shadow looming, inferentialists cannot be sure that whatever inference rules they lay down succeed at conferring meaning. Prior claims to have sunk inferentialism, but inferentialists have responded by motivating criteria for the acceptability of sets of inference rules that rule out the problematic cases. Roughly, rules for introducing and eliminating a (putative) connective ought to be *balanced* so one cannot get out more from the elimination of a connective than is required for its introduction. Prior's *tonk* clearly fails this test. The project of explicating this idea has come to be known as finding a criterion for *proof-theoretic harmony* (Dummett, 1991).

Many non-equivalent criteria for harmony have been proposed (see Steinberger, 2013). All standard ones appear to entail that the classical rules for negation are disharmonious and hence must be rejected. Thus, it seems, the project of proof-theoretic semantics is incompatible with classical logic.

Rumfitt (2000) suggests that *bilateralism* can reconcile inferentialism with classical logic. Bilateralists claim that the received, *unilateral* view on logic errs by defining the meaning of the connectives solely in terms of assertion. Bilateralists, by contrast, define meaning in terms of both assertion and rejection. For negation specifically, bilateralists endorse the following *operational rules* (so called as they relate to an operator, negation). As is customary, + is a sign for assertion and − a sign for rejection.

$$(+\neg\text{I.})\ \frac{-A}{+\neg A} \qquad (+\neg\text{E.})\ \frac{+\neg A}{-A} \qquad (-\neg\text{I.})\ \frac{+A}{-\neg A} \qquad (-\neg\text{E.})\ \frac{-\neg A}{+A}$$

That is, one can move between assertion and rejection by introducing or eliminating a negation. Note that the rule for asserted elimination is the inverse of the rule for asserted introduction and the rule for rejected elimination is the inverse of the rule for rejected introduction. Thus, by all known standards of harmony, these rules are harmonious—trivially, what one gets from an elimination is what is required for an introduction.

Moreover, by applying $(+\neg\text{E.})$ and $(-\neg\text{E.})$ in succession one can immediately eliminate double negations, vindicating a hallmark of classical logic. To obtain the classical meaning for negation, the bilateralist must furthermore derive classical *reductio*. For this, she needs to say something more about how assertion and rejection interact. She does so by laying down the following *coordination principles* (so called as they do not define an operator, but coordinate the fundamental speech acts).

$$(\text{Rejection})\ \frac{+A \qquad -A}{\bot} \qquad (\text{SR}_1)\ \frac{\begin{matrix}[+A]\\ \vdots\\ \bot\end{matrix}}{-A} \qquad (\text{SR}_2)\ \frac{\begin{matrix}[-A]\\ \vdots\\ \bot\end{matrix}}{+A}$$

The (Rejection) principle states that asserting and rejecting the same sentence is absurd. The *Smileian reductio* principles, named after Timothy Smiley (1996), state that when asserting a sentence is absurd, one may infer its rejection, and when its rejection is absurd, one may infer its assertion.*

Using the coordination principles, Rumfitt (2000) demonstrates that given any two sentences A and B, A entails B classically if and only if $+A$ entails $+B$ in his bilateral logic, thereby vindicating classical logic as the logic of assertion. However, Ferreira (2008) identifies a problem with the coordination principles. He demonstrates that Smileian *reductio* must be stated for *all* sentences A, including complex sentences containing logical connectives. Having Smileian

*In bilateral logic, it is customary to treat $\bot$ as a punctuation mark indicating that a logical dead end has been reached (see Tennant, 1999; Rumfitt, 2000). It is hence not a sentence and not signed with + or −.

reductio for only atomic sentences A will not result in the classical meaning of negation. But, Ferreira continues, stipulating the coordination principles for complex sentences is unprincipled. The bilateralist claims to have given the four rules above as the meaning-conferring rules for negation. If she now insists to also have the coordination principles for complex sentences having negation as their main operator, the meaning of negation *also* depends on the coordination principles.

Bilateralism has recently garnered increased attention (e.g. Incurvati and Schlöder, 2019; Kürbis, 2019), but the foundational issue raised by Ferreira has not been settled. In this paper, we argue that having coordination principles for only atomic sentences is indeed the way to go for bilateralism. Not only is Ferreira's observation astute (and a response by Rumfitt, 2008, is unsuccessful), having the coordination principles for arbitrarily complex sentences allows for *tonk*-like connectives (Section 2). Despite appearances, this does not sink bilateralism. We suggest a revision of Rumfitt's system that vindicates the bilateral defence of classical logic (Section 3) and conclude by discussing where this leaves the dialectic (Section 4).

2. Coordination and Harmony

Ferreira shows that Rumfitt needs to stipulate the Smileian *reductio* principle for all sentences, not just atomic ones. But this is a problem, as 'the sense of a molecular [complex] sentence must be fully determined by the introduction and elimination rules of its principal connective (given the conditions for asserting and denying the ingredient sentences)' and thus 'the co-ordination principles for arbitrary sentences must not be postulated. They should rather follow from the rules and the co-ordination at the atomic level.' (Ferreira, 2008, p. 1057).

Rumfitt (2008) responds that Ferreira has overlooked an asymmetry between (Rejection) and Smileian *reductio.* While the former must be stated for atoms only, there is nothing wrong with having Smileian *reductio* for complex sentences. Rumfitt says that a coordination principle is *preserved* if: when its atomic instances are stipulated, its complex instances are derivable. He agrees that (Rejection) must be preserved in order for bilateralist logic to be coherent.

> I took the preservation of [(Rejection)] to be a precondition for the connectives to possess coherent bilateral senses. ... If a complete sentence is to have a determinate bilateral sense, then asserting it and denying it must be contradictory speech acts. Accordingly, we should require ... that asserting and denying any given complex sentence are contradictory acts, given that it is contradictory at once to assert and deny any given atomic sentence. (Rumfitt 2008, p. 1060)

However, Rumfitt continues, one can also formalise *intuitionistic* logic bilaterally (see Kürbis, 2016), which requires dropping Smileian *reductio* (among other modifications). Both the intuitionistic and the classical meanings of the connectives are coherent, so unlike (Rejection), Smileian *reductio* is not a necessary condition for a coherent bilateral specification of sense. Thus, Smileian *reductio* is merely

> the hallmark of classicism, not of bilateralism. ... Since [Smileian *reductio*] is not required for the determinacy of bilateral sense, I see

> no reason why someone who wishes to formalize classical logic in a bilateral style should not lay down [Smileian *reductio*] as a substantive logical law that applies to all sentences. (Rumfitt 2008, p. 1061)

But then, so much worse for the classical logician! If she needs to 'lay down' Smileian *reductio*, she has not overcome the problem that motivated her move to bilateralism. Her problem was that the unilateral rules for classical negation do not appear to be harmonious. Her response was to state *bilateral*, harmonious rules for negation. But if in addition to these rules, she needs *further* principles governing the meaning of sentences containing negation, the problem of harmony re-rears its head. She needs to demonstrate that her rules for negation *and* her coordination principles for sentences having negation as their main operator are *together* harmonious, as they all contribute to the meaning of negation. Rumfitt has not done so.

Indeed, when we stipulate Smileian *reductio* for all complex sentences, we find *tonk*-like operators. The (putative) meaning of the unary connective *bink* (bilateral tonk) is given by the following inference rules.

$$(+\text{bink I.})\ \frac{+A \qquad -A}{+\text{bink}\,A} \qquad (+\text{bink E.}_1)\ \frac{+\text{bink}\,A}{+A} \qquad (+\text{bink E.}_2)\ \frac{+\text{bink}\,A}{-A}$$

$$(-\text{bink I.})\ \frac{-A}{-\text{bink}\,A} \qquad (-\text{bink E.})\ \frac{-\text{bink}\,A}{-A}$$

Clearly, the rules for asserted elimination are the inverse of the rules for asserted introduction and the rules for rejected elimination are the inverse of the rules for rejected introduction. Thus, according to the received standards of harmony (by which Rumfitt judges his rules for negation to be acceptable), the rules for *bink* are harmonious. However, given Rumfitt's coordination principles, having *bink* means that all sentences are rejected.

$$\dfrac{\dfrac{\dfrac{[+\text{bink}\,A]^1}{+A}\,(+\text{bink E.}_1) \qquad \dfrac{[+\text{bink}\,A]^1}{-A}\,(+\text{bink E.}_2)}{-\text{bink}\,A}\,(\text{SR}_1)^1}{-A}\,(-\text{bink E.})$$

It follows from this derivation that there is a proof of $-A$ for any sentence A, trivialising the consequence relation. So *bink* is incoherent like Prior's *tonk*.

Besides harmony, Rumfitt provided us with a new criterion for the admissibility of inference rules: that they preserve (Rejection). That is, to show that *bink* is admissible by Rumfitt's lights, we also need to show that if we have (Rejection) for atomic sentences only, we can still show that any application of (Rejection) to a sentence with *bink* as its main operator can be rewritten to an application of (Rejection) to the bare sentence. This is simple, as whenever we have $+\text{bink}\,A$ and $-\text{bink}\,A$, the above rules immediately allow us to infer $+A$ and $-A$. Thus, *bink* meets Rumfitt's standards, but is incoherent.

We can consider another connective *blink* that demonstrates the importance of preserving (Rejection).

$$(+\text{blink I.})\ \frac{+A}{+A\ \text{ blink }\ B} \qquad (+\text{blink E.})\ \frac{+A\ \text{ blink }\ B}{+A}$$

$$(-\text{blink I.})\ \frac{-B}{-A \text{ blink } B} \qquad (-\text{blink E.})\ \frac{-A \text{ blink } B}{-B}$$

Again, the eliminations are the inverses of the introductions, so these rules meet the received criteria of harmony. But, again, trivialisation ensues.

$$\dfrac{\dfrac{\dfrac{+A}{+A \text{ blink } B}\,(+\text{blink I.}) \qquad \dfrac{[-B]^1}{-A \text{ blink } B}\,(+\text{blink I.})}{\bot}\,(\text{Rejection})}{+B}\,(\text{SR}_2)^1$$

Thus, for any two sentences A and B, one can infer $+B$ from $+A$, rendering *blink* incoherent. However, *blink* does not preserve (Rejection). That is, if one only has (Rejection) for atomic sentences, one cannot show that $+p \text{ blink } q$ and $-p \text{ blink } q$ entail $\bot$. This is because from the former one can only get to $+p$ and from the latter only to $-q$.

What do we learn from *bink* and *blink*? The received standards of harmony, by which Rumfitt judges his operational rules, were all developed for the standard *unilateral* systems of deductions. In these systems, what one can do with a connective is determined by operational rules. In a bilateral system, however, coordination principles permit further inferences. By simply 'laying down' Smileian *reductio* (or another coordination principle) 'as a substantive logical law that applies to all sentences', as Rumfitt suggests, we can endow apparently harmonious connectives like *bink* (or *blink*) with *tonk*ish powers.

Indeed, *bink* does *not* preserve Smileian *reductio* (we provide a proof in the Appendix). And the derivation of triviality requires that Smileian *reductio* be applied to a complex sentence involving *bink*, which cannot be reduced to applying Smileian *reductio* to a simpler sentence. With Smileian *reductio* only applicable to atomic sentences, *bink* is harmless. Thus, whether a connective is *tonk*-like in a bilateral system is not only a matter of harmony between its operational rules, but *also* determined by the relation between its operational rules and the coordination principles.

Thus, what is needed is a specifically bilateral criterion of harmony, one that takes into consideration *both* the relation between introduction and elimination rules of the same sign, on the one hand, and the relation between operational rules and coordination principles, on the other.[†] The bilateralist need look no further than preserving her coordination principles. If the operational rules defining a connective preserve all coordination principles, we can be sure that the coordination principles do *not* contribute to the meaning of the connective (this much is uncontroversial between Rumfitt and Ferreira; we concur). If additionally the connective's operational rules are harmonious according to the received, unilateral standards, then they confer a coherent meaning. Thus, given some notion of unilateral harmony, we may define *bilateral harmony* as follows.

> **Bilateral harmony:** A connective **c** is bilaterally harmonious iff$_{\text{Def}}$ (i) (+**c**I.) and (+**c**E.) are unilaterally harmonious; (ii) (−**c**I.) and (−**c**E.) are unilaterally harmonious; (iii) all coordination principles

†Others (e.g. Francez, 2014; Kürbis, 2021) have found bilateral *tonk*s and suggested adjustments to the unilateral harmony criteria to rule them out. These criteria are not suited to address Ferreira's challenge and we are the first to motivate a bilateral harmony criterion based on the observation that non-atomic coordination principles are problematic.

are preserved by the rules for **c** (i.e. when all coordination principles are restricted to atomic sentences, all their instances for sentences containing **c** as their main operator are derivable).

We are not committing to any particular notion of unilateral harmony here. This is an advantage. The debate on unilateral harmony is not settled and the received criteria are up for revision. Our definition allows the bilateral logician to follow any revision.

However, it would seem that this suggestion only means more trouble for the bilateralist logician. For it follows from Ferreira's (2008) result that Rumfitt's system for classical bilateral logic is inharmonious, as Smileian *reductio* is not preserved. The bilateralist can overcome this. In the following section, we show how to obtain a harmonious system that is equivalent to Rumfitt's. *A fortiori*, this system is immune to Ferreira's objection.

3. Harmony restored

Ferreira demonstrates that in Rumfitt's system with Smileian *reductio* restricted to atomic sentences, one cannot derive that $-(A \wedge \neg A)$; the same argument shows that there is no derivation of $+(A \vee \neg A)$. Thus the laws of non-contradiction and excluded middle fail and, with them, the bilateral defence of classical logic. Or so it seems, as this conclusion must be qualified. One can derive other versions of the law of excluded middle in Rumfitt's system with Smileian *reductio* restricted to atoms. By a straightforward derivation, for instance, one obtains a negative-implicative version: $+(A \to B) \to ((\neg A \to B) \to B)$. We present the formal proof, as well as the proofs of all other formal results in this section, in the Appendix.

Moreover, Rumfitt's operational rules for negation and the conditional are bilaterally harmonious for his coordination principles. The following rewriting scheme shows that any application of (SR_1) to a sentence having negation as its main operator can be re-written to an application of (SR_2) to the same sentence without negation.

$$
\dfrac{\begin{array}{c}[+\neg A]^1 \\ \mathcal{D} \\ \bot\end{array}}{-\neg A}\,(\mathrm{SR}_1)^1
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\begin{array}{c}\dfrac{[-A]^1}{+\neg A}\,(+\neg\mathrm{I.}) \\ \mathcal{D} \\ \bot\end{array}}{+A}\,(\mathrm{SR}_2)^1}{-\neg A}\,(-\neg\mathrm{I.})
$$

The case where (SR_2) is applied to a sentence having negation as its main operator is analogous. A similar rewriting scheme shows that Rumfitt's operational rules for the conditional preserve Smileian *reductio*. Combined with Rumfitt's demonstration that his rules are unilaterally harmonious and preserve (Rejection), it follows that his operational rules for negation and the conditional are bilaterally harmonious. As all of Frege's axioms for the propositional calculus can be derived from them, the bilateralist can have a harmonious system for classical logic on the signature $\{\neg, \to\}$.

The problematic uses of Smileian *reductio* are to assume $[+A \wedge B]$ to conclude $-(A \wedge B)$ and to assume $[-A \vee B]$ to conclude $+A \vee B$. Ferreira's result shows that, in Rumfitt's system, such applications of (SR_1) and (SR_2) cannot be rewritten so that Smileian *reductio* is applied to simpler sentences. However, this is not a problem with Smileian *reductio*, but with Rumfitt's operational rules for conjunction. Consider the following alternative rules governing rejections of conjunctions.

$$(-\wedge\mathrm{I.})^i \; \dfrac{\begin{matrix}[+A]^i\\ \vdots \\ -B\end{matrix}}{-A\wedge B} \qquad\qquad (-\wedge\mathrm{E.})\; \dfrac{-A\wedge B \qquad +A}{-B}$$

These are easily seen to be unilaterally harmonious and to preserve (Rejection). Combined with the standard rules for assertions of conjunctions (i.e. inferring $+A\wedge B$ from $+A$ and $+B$, and *vice versa*), we can show by the following rewriting scheme that Smileian *reductio* is preserved. (The rewriting scheme for (SR_2) is analogous.)

$$\dfrac{\begin{matrix}[+(A\wedge B)]^1\\ \mathcal{D}\\ \bot\end{matrix}}{-(A\wedge B)}\,(\mathrm{SR}_1)^1 \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{matrix}\dfrac{[+A]^1 \qquad [+B]^2}{+(A\wedge B)}\,(+\wedge\ \mathrm{I.})\\ \mathcal{D}\\ \bot\end{matrix}}{-B}\,(\mathrm{SR}_1)^2}{-(A\wedge B)}\,(-\wedge\ \mathrm{I.})^1$$

These rules for rejections of conjunctions have independently been discussed by Nils Kürbis (2016) in the context of bilateral intuitionistic logic. They may appear to be incomplete, as classical conjunction is symmetric. So one also requires the version of $(-\wedge\mathrm{I.})$ where one derives $-A$ from $[+B]$ to conclude $-A\wedge B$, and the version of $(-\wedge\mathrm{E.})$ where from $-A\wedge B$ and $+B$ one concludes $-A$ (see Kürbis, 2019, ch. 6). But these rules are derivable from just the two above.[‡]

The analogous move works for disjunction. That is, we obtain a bilaterally harmonious set of rules for disjunction by replacing Rumfitt's rules governing assertions of disjunctions by the following.

$$(+\vee\mathrm{I.})^i \; \dfrac{\begin{matrix}[-A]^i\\ \vdots \\ +B\end{matrix}}{+A\vee B} \qquad\qquad (+\vee\mathrm{E.})\; \dfrac{+A\vee B \qquad -A}{+B}$$

Thus, the bilateralist can provide a bilaterally harmonious, classical calculus for Rumfitt's full language if she revises his rules for rejections of conjunctions and

[‡] The proofs go as follows:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}[+B]^1\\ \vdots\\ -A\end{matrix} \qquad [+A]^2}{\bot}\,(\mathrm{Rejection})}{-B}\,(\mathrm{SR}_1)^1}{-(A\wedge B)}\,(-\wedge\mathrm{I.})^2 \qquad\qquad \dfrac{\dfrac{\dfrac{-A\wedge B \qquad [+A]^1}{-B}\,(-\wedge\mathrm{E.}) \qquad +B}{\bot}\,(\mathrm{Rejection})}{-A}\,(\mathrm{SR}_1)^1$$

Although these use Smileian *reductio*, this is not a problem as it is not applied to a sentence whose main operator is conjunction.

assertions of disjunctions. How this squares with Ferreira's result is subtle. His argument rests on a sound model theory for Rumfitt's calculus without any version of Smileian *reductio*. Our revised rules for conjunction and disjunction rules are unsound for this model theory and hence immune to Ferreira's argument. Accordingly, deriving Rumfitt's rules from ours requires Smileian *reductio* for atoms, and deriving ours from his requires it for complex sentences.

4. Conclusion

Ferreira (2008) was right to demand of the bilateral logician that *all* her coordination principles be preserved by her operational rules. When this is not so, we must take the coordination principles to contribute to the meaning of the connective whose operational rules do not preserve them. This calls into question whether the connective is characterised harmoniously and so whether its putative meaning is coherent. We have backed up this line of reasoning by laying out two *tonk*-like connectives: one not preserving Rumfitt's coordination principle (Rejection) and another not preserving Smileian *reductio*.

To make good on any claims of harmony, the bilateralist logician should apply a distinctively *bilateral* criterion of harmony according to which operational rules must be unilaterally harmonious *and* preserve all coordination principles. This criterion is well motivated: if a coordination principle is not preserved, it contributes to the meaning of an operator in a way that is not controlled by unilateral standards of harmony, which may render the meaning incoherent. We traced Ferreira's problem to some of Rumfitt's particular rules for disjunction and conjunction being disharmonious and demonstrated that there are properly harmonious alternatives. Thus, the bilateralist logician can indeed have classical negation while meeting even stringent standards of proof-theoretic harmony.

Rumfitt, for his part, was right to call Smileian *reductio* a hallmark of classicism. Recent work has seen bilateral versions of intuitionism (Kürbis, 2016) and an extension of the bilateral idea to *multilateral* logics (Incurvati and Schlöder, 2019) that each come with their own 'hallmark' coordination principles. But being a hallmark of some logic or another does not make for an exemption from harmony. Our standard of harmony can be applied to these variants and generalisations and it remains to be seen whether they can meet it.*

Appendix

In this Appendix, we provide formal proofs for the technical results claimed above.

Bink does not preserve Smilean reductio*:*

The proof uses the Kripke models introduced in (Ferreira 2008).

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Definition. An **F-Model** is a tuple $M = (W, R, v_+, v_-)$ where (W, R) is a non-empty poset and v_+, v_- are valuation functions from the set of propositional atoms to $\mathcal{P}(W)$ such that:

i) $\forall w, w' \in W$, wRw' and $w \in v_+(p) \Longrightarrow w' \in v_+(p)$
ii) $\forall w, w' \in W$, wRw' and $w \in v_-(p) \Longrightarrow w' \in v_-(p)$
iii) $v_-(p) \cap v_+(p) = \emptyset$ for all atoms p.

Satisfaction in an F-model is defined as follows: (See Ferreira 2008, p. 1055 for the satisfaction conditions of the Boolean connectives).

$M, w \models_+ p$ iff $w \in v_+(p)$
$M, w \models_- p$ iff $w \in v_-(p)$
$M, w \models_+ \mathit{bink}A$ iff $M, w \models_+ A$ and $M, w \models_- A$
$M, w \models_- \mathit{bink}A$ iff $M, w \models_- A$

Write $M, w \models +A$ as a shorthand for $M, w \models_+ A$ and $M, w \models -A$ as a shorthand for $M, w \models_- A$.

Proof that bink does not preserve Smileian reductio. As Ferreira (2008, p. 1055f) demonstrated, Rumfitt's calculus without coordination principles is sound for the class of F-models. Moreover, we can show that the rules for *bink* are sound for the class of all F-models. That is, if some node w of some F-model $\mathcal{V}$ satisfies the premises of a *bink* rule, then it satisfies its conclusion. The method is uniform, so we only present the argument for (+*bink* I.). Suppose that some node w of some F-model $\mathcal{V}$ is such that $\mathcal{V}, w \models +A$ and $\mathcal{V}, w \models -A$. Then $\mathcal{V}, w \models +\mathit{bink}\ A$ by the third satisfaction clause. The soundness of the other *bink* rules is shown analogously.

Now, consider the F-Model $\mathcal{M}$ with $W = \{w_0, w_1, w_2\}$, $R = \{(w_0, w_0), (w_0, w_1), (w_1, w_1), (w_0, w_2), (w_2, w_2)\}$ and where $v_+(p) = \{w_1\}$ for all atoms p, $v_-(q) = \{w_2\}$ for all atoms q. Assume towards a contradiction that the rules for *bink* preserve Smileian *reductio*. As shown by Ferreira (2008, p. 1056), this model satisfies Smileian *reductio* for atomic sentences. Thus, by the assumption and the soundness of *bink*, it follows that this model satisfies Smileian *reductio* for all sentences. The derivation in §2 that applies Smileian *reductio* to $+\mathit{bink}\,p$ shows that $\vdash -p$. By Soundness and because $\mathcal{M}$ satisfies Smileian *reductio*, it follows that $\mathcal{M}, w \models -p$ for all w. But by definition, $\mathcal{M}, w_1 \not\models -p$. Contradiction. By *reductio*, the rules for *bink* do not preserve Smileian *reductio*.

Derivation of the negative-implicative version of LEM

Rumfitt's rules for $\rightarrow$ are the following.

$$(+\rightarrow\text{I.})^i\ \frac{\begin{array}{c}[+A]^i\\ \mathcal{D}\\ +B\end{array}}{+A \rightarrow B} \qquad (+\rightarrow\text{E.})\ \frac{+A \rightarrow B \qquad +A}{+B}$$

$$(-\rightarrow\text{I.})\ \frac{+A \qquad -B}{-A \rightarrow B} \qquad (-\rightarrow\text{E.}_1)\ \frac{-A \rightarrow B}{+A} \qquad (-\rightarrow\text{E.}_2)\ \frac{-A \rightarrow B}{-B}$$

The following derivation shows that $+(A \rightarrow B) \rightarrow ((\neg A \rightarrow B) \rightarrow B)$ is a theorem in the negative-implicative fragment of Rumfitt's system.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[+\neg A \to B]^3 \quad \dfrac{\dfrac{\dfrac{\dfrac{[+A \to B]^4 \quad [+A]^1}{+B}(+\to\text{E.}) \quad [-B]^2}{\bot}(\text{Rejection})}{-A}(\text{SR}_1)^1}{+\neg A}(+\neg\text{I.})}{+B}(+\to\text{E.}) \quad [-B]^2}{\bot}(\text{Rejection})}{+B}(\text{SR}_2)^2}{+(\neg A \to B) \to B}(+\to\text{I.})^3}{+(A \to B) \to ((\neg A \to B) \to B)}(+\to\text{I.})^4$$

Rumfitt's rules for → preserve Smilean reductio

The following rewrtiting schemes show how to reduce the complexity of applications of Smilean *reductio* to sentences with → as main operator. We use the discharge index 0 to label empty discharges (as, e.g., in the label $(\text{SR}_2)^0$ where $+B$ is concluded from a contradiction reachable without using the assumption $-B$).

$$\dfrac{\begin{matrix}[+A \to B]^1\\ \mathcal{D}\\ \bot\end{matrix}}{-A \to B}(\text{SR}_1)^1 \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{matrix}\dfrac{\dfrac{\dfrac{[+A]^1 \quad [-A]^2}{\bot}(\text{Rejection})}{+B}(\text{SR}_2)^0}{+A \to B}(+\to\text{I.})^1\\ \mathcal{D}\\ \bot\end{matrix}}{+A}(\text{SR}_2)^2 \quad \dfrac{\begin{matrix}\dfrac{[+B]^3}{+A \to B}(+\to\text{I.})^0\\ \mathcal{D}\\ \bot\end{matrix}}{-B}(\text{SR}_1)^3}{-A \to B}(-\to\text{I.})$$

$$\dfrac{\begin{matrix}[-A \to B]^1\\ \mathcal{D}\\ \bot\end{matrix}}{+A \to B}(\text{SR}_2)^1 \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\dfrac{[+A]^1 \quad [-B]^2}{-A \to B}(-\to\text{I.})\\ \mathcal{D}\\ \bot\end{matrix}}{-A}(\text{SR}_1)^1 \quad [+A]^3}{\bot}(\text{Rejection})}{+B}(\text{SR}_2)^2}{+A \to B}(+\to\text{I.})^3$$

Our new rules for conjunction preserve (Rejection)

We keep with Rumfitt's rules for the introduction and elimination of ∧ under +, which are as follows.

$$(+\wedge\text{I.})\ \dfrac{+A \qquad +B}{+A \wedge B} \qquad (+\wedge\text{E.})_1\ \dfrac{+A \wedge B}{+A} \qquad (+\wedge\text{E.})_2\ \dfrac{+A \wedge B}{+B}$$

Our new rules govern the introduction and elimination of ∧ under −. Using them, the following rewriting scheme shows how to reduce the complexity of applications of (Rejection) to sentences with ∧ as main operator.

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ +A\wedge B\end{matrix}\qquad \begin{matrix}\mathcal{D}_2\\ -A\wedge B\end{matrix}}{\bot}\,\text{(Rejection)} \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_2\\ -A\wedge B\end{matrix}\qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ +A\wedge B\end{matrix}}{+A}\,(+\wedge\text{E.})_1}{-B}\,(-\wedge\text{E.}) \qquad \dfrac{\begin{matrix}\mathcal{D}_2\\ +A\wedge B\end{matrix}}{+B}\,(+\wedge\text{E.})_2}{\bot}\,\text{(Rejection)}$$

Our new rules for disjunction preserve (Rejection)

We keep with Rumfitt's rules for the introduction and elimination of $\vee$ under $-$, which are as follows.

$$(-\vee\text{I.})\ \frac{-A \qquad -B}{-A\vee B} \qquad (-\wedge\text{E.})_1\ \frac{-A\vee B}{-A} \qquad (-\vee\text{E.})_2\ \frac{-A\vee B}{-B}$$

Our new rules govern the introduction and elimination of $\vee$ under $+$. Using them, the following rewriting scheme shows how to reduce the complexity of applications of (Rejection) to sentences with $\vee$ as main operator.

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ +A\vee B\end{matrix}\qquad \begin{matrix}\mathcal{D}_2\\ -A\vee B\end{matrix}}{\bot}\,\text{(Rejection)} \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ +A\vee B\end{matrix}\qquad \dfrac{\begin{matrix}\mathcal{D}_2\\ -A\vee B\end{matrix}}{-A}\,(-\wedge\text{E.})_1}{+B}\,(+\vee\text{E.}) \qquad \dfrac{\begin{matrix}\mathcal{D}_2\\ -A\vee B\end{matrix}}{-B}\,(-\wedge\text{E.})_2}{\bot}\,\text{(Rejection)}$$

Our new rules for disjunction preserve Smilean reductio

The following rewriting schemes show how to reduce the complexity of applications of Smilean *reductio* to sentences with $\vee$ as main operator.

$$\dfrac{\begin{matrix}[+A\vee B]^1\\ \mathcal{D}\\ \bot\end{matrix}}{-A\vee B}\,(\text{SR}_1)^1 \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{matrix}\dfrac{\dfrac{\dfrac{[-A]^1 \qquad [+A]^2}{\bot}\,\text{(Rejection)}}{+B}\,(\text{SR}_2)^0}{+A\vee B}\,(+\vee\text{I.})^1\\ \mathcal{D}\\ \bot\end{matrix}}{-A}\,(\text{SR}_2)^2 \qquad \dfrac{\begin{matrix}\dfrac{[+B]^3}{+A\vee B}\,(+\vee\text{I.})^0\\ \mathcal{D}\\ \bot\end{matrix}}{-B}\,(\text{SR}_1)^3}{-A\vee B}\,(-\vee\text{I.})$$

$$\dfrac{\begin{matrix}[-A\vee B]^1\\ \mathcal{D}\\ \bot\end{matrix}}{+A\vee B}\,(\text{SR}_2)^1 \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{matrix}\dfrac{[-A]^2 \qquad [-B]^1}{-A\vee B}\,(-\vee\text{I.})\\ \mathcal{D}\\ \bot\end{matrix}}{+B}\,(\text{SR}_1)^1}{+A\vee B}\,(+\vee\ \text{I.})^2$$

Our new rules are unsound for Ferreira's model theory

Ferreira (2008) shows that Rumfitt's rules for negation are sound for the class of F-models defined above. Assume towards a contradiction that our conjunction and disjunction rules are sound for them as well. Consider the following derivations.

$$\dfrac{\dfrac{[-A]^1}{+\neg A}\ (+\neg\text{I.})}{+A \vee \neg A}\ (+\vee\text{I.})^1 \qquad\qquad \dfrac{\dfrac{[+\neg A]^1}{-A}\ (+\neg\text{E.})}{-\neg A \wedge A}\ (-\wedge\text{I.})^1$$

By assumption, it follows that for any F-model $\mathcal{U}$ and any node w in it, $\mathcal{U}, w \models +A \vee \neg A$ and $\mathcal{U}, w \models -\neg A \wedge A$. But this is clearly false, since the model $\mathcal{M}$ we used above to show that *bink* does not preserve Smileian *reductio* is such that $\mathcal{M}, w_0 \not\models +A \vee \neg A$ and $\mathcal{M}, w_0 \not\models +A \vee \neg A$ (see Ferreira 2008, p. 1056).

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