

## Technical Notes

We here state some basic results used in the main body of “Abstraction and Grounding”, forthcoming in *Philosophy and Public Affairs*. We will assume that each of the pluralities we discuss is indexed to an ordinal. For the purposes of constructing explanatory arguments, we will also assume that we have a first-order language  $\mathcal{L}$  with identity, names for every element of each of the pluralities of individuals under discussion, names for every natural number, and a relational predicate symbol for each of the relations-in-extension among the pluralities that we will discuss. We write  $aa$  and  $bb$  for non-empty pluralities, and  $\emptyset\emptyset$  for the empty plurality (if there is one). Let

$$T^+(aa) = \{\ulcorner c = c' \urcorner \mid c = c' \wedge (c, c' \in aa)\} \quad T^-(aa) = \{\ulcorner c \neq c' \urcorner \mid c \neq c' \wedge (c, c' \in aa)\}$$

and  $T(aa) = T^+(aa) \cup T^-(aa)$ .

Intuitively,  $T(aa)$  is the set of *truths* concerning identities and distinctnesses among  $aa$ .

In what follows, we will refer to an indexed collection using standard notation, writing  $(x_i)_{i < \alpha}$  for  $\{x_i \mid i < \alpha\}$ . To avoid clutter, we will write  $(x_i)$ , omitting the subscripted restriction ‘ $i < \alpha$ ’ entirely. We indicate co-indexed sets by using the same subscripts. Where there are two subscripts, the first subscript may sometimes depend on the second subscript, and these abbreviations may be embedded. Some examples:

**Abbreviation    Expansion**

$(x_i)$	$x_0, x_1, \dots$
$(\Delta_i \Rightarrow \phi_i)$	$\Delta_0 \Rightarrow \phi_0; \Delta_1 \Rightarrow \phi_1, \dots$
$(x_{ij})$	$x_{00}, x_{10}, \dots \quad x_{01}, x_{11}, \dots, \quad x_{0j}, x_{1j}, \dots, x_{ij}, \dots, \quad , \dots$

The notions of a *relevant derivation* of the formula  $\phi$  from the set of formulas  $\Delta$  and of  $\Rightarrow$  are defined as in Appendix A. We will be sloppy about use-mention distinctions when more care will not improve clarity.

Where  $f \in aa \otimes bb$ , let the *domain* of  $f$  be the plurality  $\mathcal{D}(f)$ , such that  $a \in \mathcal{D}(f)$  iff  $f(a, b)$ , for some  $b$ ; and let the *range* of  $f$  be the plurality  $\mathcal{R}(f)$  such that  $b \in \mathcal{R}(f)$  iff  $f(a, b)$ , for some  $a$ .

**Proposition 1** Let  $f \in aa \otimes bb$ ,  $f \neq \emptyset$ , and  $\neg f(a, b)$ . Let  $\mathcal{D}(f) \setminus \{a\} = (a_i)$  and  $\mathcal{R}(f) \setminus b = (b_j)$ . Then  $(a \neq a_i), (b \neq b_j) \Rightarrow \neg f(a, b)$ .

*Proof* We may suppose (wlog) that  $a \in \mathcal{D}(f)$ ,  $b \in \mathcal{R}(f)$ ,  $f(a, b_1)$ , and  $f(a_1, b)$ , so that  $\neg f(x, y)$  if grounded in the same way as  $\neg((x = a \wedge y = b_1) \vee (x = a_1 \wedge y = b) \vee (x = a_i \wedge y = b_i))$ , for  $i \geq 2$ . Then, since  $\neg f(a, b)$ ,

$$\neg(a = a \wedge b = b_1), \neg(a = a_1 \wedge b = b), (\neg(a = a_i \wedge b = b_i)) \Rightarrow \neg f(a, b).$$

The result follows by CUT, since

$$b \neq b_1 \Rightarrow \neg(a = a \wedge b = b_1) \quad a \neq a_1 \Rightarrow \neg(a = a_1 \wedge b = b) \quad (a \neq a_i, b \neq b_1 \Rightarrow \neg(a = a_i \wedge b = b_i)).$$

□

**Proposition 2** Let  $f \in aa \otimes bb$ , and  $f(a, b)$ . Then  $a = a, b = b \Rightarrow f(a, b)$ .

**Proposition 3** Suppose  $f \in aa \otimes bb$ , and  $f : aa \xrightarrow[\text{onto}]{1-1} bb$ . Then,

1. For some  $S \subseteq T(aa)$ ,  $S, T(bb) \Rightarrow f$  is 1-1; and
2. For some  $S \subseteq T(bb)$ ,  $S, T(aa) \Rightarrow f$  is functional.

*Proof* For each  $a_i, a_j \in aa$ ,  $b_k \in bb$ , let  $\phi_{ijk} = (f(a_i, b_k) \wedge f(a_j, b_k) \rightarrow a_i = a_j)$ . If  $a_i \neq a_j$ , then, since  $f$  is 1-1, either  $f(a_i) \neq b_k$  or  $f(a_j) \neq b_k$ . Suppose (wlog)  $f(a_i) \neq b_k$ . By P1, for some  $S \subseteq T(aa)$ ,  $S, (b_k \neq b'_m) \Rightarrow f(a_i \neq b_k) \Rightarrow \phi_{ijk}$ , for  $bb \setminus \{b\} = b'_1, b'_2, \dots$ . If  $a_i = a_j$ , then, by P2,  $a_i = a_i, b_k = b_k \Rightarrow \phi_{ijk}$ . Now, for each  $b_k \in bb$ , there are  $a_i, a_j \in aa$  such that  $a_i = a_j$  and  $f(a_i, b)$ , and so  $a_i = a_i, b_k = b_k \Rightarrow \phi_{ijk}$ . And, for each  $b_k, b'_k \in bb$  such that  $b_k \neq b'_k$ , there are  $a_i, a_j \in aa$  such that  $a_i \neq a_j$ , and thus  $S, (b_k \neq b'_k) \Rightarrow \phi_{ijk}$ . So,  $S, T(bb) \Rightarrow (\forall a_i, a_j \in aa)(\forall b_k \in bb)\phi_{ijk} = f$  is 1-1, for some  $T \subseteq T(aa)$ . This proves (1). An exactly similar argument yields (2).

□

**Proposition 4** Suppose  $f : aa \xrightarrow[onto]{1-1} bb$ , and let  $aa = (a_i)$  and  $bb = (b_i)$ . Then

1.  $(a_i = a_i)(b_j = b_j) \Rightarrow f$  is onto; and
2.  $(a_i = a_i)(b_j = b_j) \Rightarrow f$  is total.

*Proof* Let  $b_j \in bb$ . Then, for some  $a_{k_j} \in aa$ ,  $a_{k_j} = a_{k_j}, b_j = b_j \Rightarrow (\exists a \in aa)f(a) = b$ . So,  $(a_{k_j} = a_{k_j}), (b_j = b_j) \Rightarrow (\forall b \in bb)(\exists a \in aa)f(a) = b$ . Since  $f$  is total,  $(a_{k_j} = a_{k_j}) = (a_i = a_i)$ . This proves (1). A similar argument proves (2).

□

**Proposition 5** Suppose  $f(aa \xrightarrow[onto]{1-1} bb)$ . Then:

1.  $T(aa), T(bb) \Rightarrow f : aa \xrightarrow[onto]{1-1} bb$ ;
2.  $T(aa) \Rightarrow f : aa \xrightarrow[onto]{1-1} aa$ ; and
3.  $T(aa) \Rightarrow aa \approx aa$ .

*Proof* (1) follows by P3 and P4. (2) follows immediately from (1), and (3) from (2).

□

**Proposition 6**

1. Suppose  $f \in aa \otimes bb$ ,  $a_i, a_j \in aa$ ,  $b_k \in bb$ ,  $f(a_i, b_k), f(a_j, b_k)$ , and  $a_i \neq a_j$ . Then
  - (a)  $a_i = a_i, a_j = a_j, b_k = b_k, a_i \neq a_j \Rightarrow \neg(f(a_i, b_k) \wedge f(a_j, b_k) \rightarrow a_i = a_j)$ ; and
  - (b)  $a_i = a_i, a_j = a_j, b_k = b_k, a_i \neq a_j \Rightarrow \neg(f \text{ is 1-1})$ .
2. Suppose  $f \in aa \otimes bb$ ,  $b_i, b_j \in bb$ ,  $a_k \in aa$ ,  $f(a_k, b_i), f(a_k, b_j)$ , and  $b_i \neq b_j$ . Then
  - (a)  $b_i = b_i, b_j = b_j, a_k = a_k, b_i \neq b_j \Rightarrow \neg(f(a_k, b_i) \wedge f(a_k, b_j) \rightarrow b_i = b_j)$ ; and
  - (b)  $b_i = b_i, b_j = b_j, a_k = a_k, b_i \neq b_j \Rightarrow \neg(f \text{ is functional})$ .

*Proof* By P2, we have  $a_i = a_i, b_k = b_k \Rightarrow f(a_i, b_k)$  and

$$a_j = a_j, b_k = b_k \Rightarrow f(a_j, b_k).$$

(1a) follows by an application of CUT. (1b) follows immediately from (1a). The proof of (2) is similar.

□

### Proposition 7

1. Suppose  $f \in aa \otimes bb$ ,  $f$  is nonempty,  $b \in bb$ , and, for all  $a \in aa$ ,  $\neg f(a, b)$ . Then, letting  $bb \setminus \{b\} = (b'_j)$ ,  $T^-(aa), (b \neq b'_j) \Rightarrow \neg(f \text{ is onto } bb)$ .
2. Suppose  $f \in aa \otimes bb$ ,  $f$  is nonempty,  $a \in aa$ , and, for all  $b \in bb$ ,  $\neg f(a, b)$ . Then, letting  $aa \setminus \{a\} = (a'_i)$ ,  $T^-(bb), (a \neq a'_i) \Rightarrow \neg(f \text{ is total on } aa)$ .

*Proof* For each  $a_i \in aa$ , P1 implies  $(a_i \neq a'_i), (b \neq b'_j) \Rightarrow \neg f(a_i, b)$ , where  $aa \setminus \{a_i\} = (a'_i)$ . So,

$$T^-(aa), (b \neq b'_j) \Rightarrow \neg(\exists a \in aa)f(a, b) \Rightarrow \neg(\forall b \in bb)(\exists a \in aa)f(a, b) \Rightarrow \neg(f \text{ is onto } bb).$$

This proves (1). The proof of (2) is similar.

□

Let  $B$  be any individual not in  $\mathbb{N}^+$ . Inductively define  $aa_n$  for  $n \in \mathbb{N}^+$  so that  $aa_1 = B, B$  and  $aa_{n+1} = aa_n \cup n, n$ .

### Proposition 8

1. Suppose  $m, n \in \mathbb{N}$ ,  $m > n$ , and  $aa_n \not\approx aa_m$ . Then  $T(aa_m) \Rightarrow aa_n \not\approx aa_m$ .
2. Suppose  $m, n \in \mathbb{N}$ ,  $m > n$ , and  $aa_m \not\approx aa_n$ . Then  $T(aa_m) \Rightarrow aa_m \not\approx aa_n$ .

*Proof* To prove (1), note that, since  $\neg f : aa_n \xrightarrow[onto]{1-1} aa_m$  for every non-empty  $f \in aa_n \otimes aa_m$ , P6 and P7 imply that  $S \Rightarrow \neg f : aa_n \xrightarrow[onto]{1-1} aa_m$ , for some  $S \subseteq T(aa_m)$ . For the empty function  $\emptyset$ ,  $B = B \Rightarrow \neg \emptyset(B, B) \Rightarrow \neg(\emptyset : aa_n \xrightarrow[onto]{1-1} aa_m)$ . So, we have  $S \Rightarrow aa_n \not\approx aa_m$ , for some  $S \subseteq T(aa_m)$ . Now, there is a  $g \in aa_n \otimes aa_m$  such that  $g(B, b)$  for each  $b \in aa_m$ . Moreover,  $g$  is not functional, since  $m > n$ . So, by P6, for each  $a_i, a_j \in aa_m$ , where  $a_i \neq a_j$ ,  $a_i = a_i, a_j = a_j, B = B, a_i \neq a_j \Rightarrow \neg(g \text{ is functional})$ . By AMALGAMATION,  $T(aa_m) \Rightarrow aa_n \not\approx aa_m$ . (2) is proved similarly, using a  $g \in aa_m \otimes aa_n$  that is a constant, non-injective function.

□

Let  $S_1 = \{\ulcorner B = B \urcorner\}$ , and, for  $n \in \mathbb{N}^+$ , let

$$S_{n+1} = \{\ulcorner B = B \urcorner, \ulcorner B \neq 1 \urcorner, \ulcorner 1 \neq B \urcorner, \dots, \ulcorner B \neq n \urcorner, \ulcorner n \neq B \urcorner\}.$$

**Proposition 9** For all  $m, n \in \mathbb{N}^+$ ,  $m < n$ :

1.  $S_m \Rightarrow m = m$ ;

2.  $S_n \Rightarrow n \neq m$ ; and
3.  $S_n \Rightarrow m \neq n$ .

*Proof* We prove the result by induction. The basis case for (1) follows immediately from P5, since  $T(aa_1) = \ulcorner B = B \urcorner$ , and so  $B = B \Rightarrow aa_1 \approx aa_1 \Rightarrow 1 = 1$ . The result in the basis cases for (2) and (3) follows from P8 and the basis case of (1). For the induction step, assume that each of (1)-(3) are true for each  $k < m, j < n$ . To see that (1) is true for  $n$ , notice that P5 implies  $T(aa_n) \Rightarrow n = n$ , and every member  $\phi$  of  $T(aa_n) \setminus S_n$  has one of the forms  $\ulcorner j = j \urcorner$ ,  $\ulcorner j \neq j' \urcorner$ , or  $\ulcorner j' \neq j \urcorner$  for some  $j, j' < n, j > j'$ . By IH,  $S_j \Rightarrow k = k, S_j \Rightarrow j \neq j'$ , and  $S_j \Rightarrow j' \neq j$ . Also,  $S_j \subseteq S_n$ . So, CUT yields (1). The arguments for (2) and (3) are similar, using P8 in place of P5.

□

Since the specification of explanatory inferences and the grounding principles in (deRosset and Linnebo, ming, §5) are exactly parallel, and strict ground, like  $\Rightarrow$  is closed under CUT, Proposition 1 in (deRosset and Linnebo, ming, §5) can be proved by substituting ' $<$ ' for ' $\Rightarrow$ ' in the proof of P9.

**Proposition 10** Suppose that  $\emptyset\emptyset$  is an empty plurality, i.e.,  $(\forall x)x \notin \emptyset\emptyset$ . For all  $n \in \mathbb{N}^+$ :

1.  $\emptyset \Rightarrow 0 = 0$ ;
2.  $\emptyset \Rightarrow 0 \neq n$ ; and
3.  $\emptyset \Rightarrow n \neq 0$ .

*Proof*  $\emptyset\emptyset \otimes \emptyset\emptyset$  has exactly one member, the empty function  $f$ , and we have  $\emptyset \Rightarrow (\forall x \in \emptyset\emptyset)\phi$ , for any  $\phi$ . So,  $\emptyset \Rightarrow f : \emptyset\emptyset \xrightarrow[\text{onto}]{1-1} \emptyset\emptyset \Rightarrow 0 = 0$ , yielding (1). Let  $bb_{n+1} = 0, 1, \dots, n$ , for  $n \in \mathbb{N}$ . To show (2), note that  $\emptyset\emptyset \otimes bb_n$  has exactly one member, the empty relation  $f$ . Also, we have  $\emptyset \Rightarrow \neg(\exists a \in \emptyset\emptyset)\phi$ , for all  $\phi$ . So,

$$\begin{aligned} \emptyset &\Rightarrow \neg(\exists a \in \emptyset\emptyset)f(a, 0) \Rightarrow \neg(\forall b \in bb_n)(\exists a \in \emptyset\emptyset)f(a, b) \Rightarrow \neg(f \text{ is onto } bb_n) \\ &\Rightarrow \neg(f : \emptyset\emptyset \xrightarrow[\text{onto}]{1-1} bb_n) \Rightarrow \neg(\exists g \in \emptyset\emptyset \otimes bb_n)(g : \emptyset\emptyset \xrightarrow[\text{onto}]{1-1} bb_n) \Rightarrow \emptyset\emptyset \not\approx bb_n \Rightarrow 0 \neq n. \end{aligned}$$

The proof of (3) is similar, using the failure of the empty relation in  $bb_n \otimes \emptyset\emptyset$  to be total on  $bb_n$  in place of the failure of the empty relation in  $\emptyset\emptyset \otimes bb_n$  to be onto  $bb_n$ .

□

**Proposition 11** Suppose that  $\emptyset\emptyset$  is an empty plurality, i.e.,  $(\forall x)x \notin \emptyset\emptyset$ . For all  $n, m \in \mathbb{N}$ , where  $n \neq m$ ,  $\emptyset \Rightarrow n = n$  and  $\emptyset \Rightarrow n \neq m$ .

Let  $bb_{n+1}$  be defined as in the proof of Prop 10, and recall that  $\text{PREC}(xx, yy)$  abbreviates

$$(\exists yy' \subseteq yy)(\exists y \in yy)(\forall z \in yy)(z \in yy' \leftrightarrow z \neq y) \wedge xx \approx yy).$$

Given an empty plurality  $\emptyset\emptyset$  and the explanatory inferences for quantifications restricted to a plurality, it is easy to see that there will be explanatory arguments witnessing  $\Delta \Rightarrow$

$\text{PREC}(bb_k, bb_{k+1}) \Rightarrow P(k, k+1)$  for all  $k \in \mathbb{N}^+$ , where all members of  $\Delta$  have one of the forms:  $n = n$ ,  $n \neq m$ , or  $bb_{n+1} \approx bb_{n+1}$ , for some  $n, m \in \mathbb{N}$ . Similarly, it is easy to see that there will be explanatory arguments witnessing  $\Delta \Rightarrow \text{PREC}(\emptyset\emptyset, bb_1) \Rightarrow P(0, 1)$ , where  $\Delta$ 's members are all either  $0 = 0$  or  $\emptyset\emptyset \approx \emptyset\emptyset$ . So, the application of Prop 11 yields:

**Proposition 12** Suppose that  $\emptyset\emptyset$  is an empty plurality, i.e.,  $(\forall x)x \notin \emptyset\emptyset$ . For all  $n, m \in \mathbb{N}$ , where  $n + 1 \neq m$ ,  $\emptyset \Rightarrow P(n, n + 1)$  and  $\emptyset \Rightarrow \neg P(n, m)$ .

Now we can sketch how all of the facts expressible in second-order Peano arithmetic are grounded, assuming the existence of an empty plurality  $\emptyset\emptyset$ . Second-order Peano arithmetic can be formulated in a language,  $\mathcal{L}_{\text{PA}2}$ , whose only primitive predicates are ‘=’ and ‘P’ (Boolos, 1995). We wish to proceed to show, by induction on syntactic complexity, that, so long as there is an empty plurality  $\emptyset\emptyset$ , for every formula  $\varphi$  of  $\mathcal{L}_{\text{PA}2}$  (relative to a variable assignment), either  $\varphi$  or  $\neg\varphi$  is derivable in an explanatory argument from the empty set of premises and so zero-grounded. (To simplify the exposition, we elide the variable assignments and talk directly about natural numbers and relations-in-extension based on these.) Propositions 11 and 12 ensure that the claim holds for atomic formulas involving ‘=’ and ‘P’. The same goes for atomic formulas involving plural membership or predication of a relation-in-extension (cf. Appendix A). The induction step for disjunction, conjunction, and negation is straightforward. As for the quantifiers, the key is first to define the plurality  $nn$  of all natural numbers as the least plurality containing 0 and closed under the successor relation. (This plurality exists according to our Critical Plural Logic by its axiom of Infinity; see Appendix B.) We can now use plurality-restricted quantifiers of the form ‘ $(\exists x \in nn)$ ’ and ‘ $(\forall x \in nn)$ ’ to interpret the first-order quantifiers of  $\mathcal{L}_{\text{PA}2}$ , and analogously for quantification over pluralities and relations-in-extension. The induction step for true existential generalizations and negated universal generalizations involving these quantifiers is straightforward, while that of true universal generalizations (or negated existential generalizations) requires that true *plurality-restricted* generalizations of these forms can be derived in explanatory arguments from the collection of their instances (or negated instances).

## References

- Boolos, G. (1995). Frege’s theorem and the Peano postulates. *Bulletin of Symbolic Logic*, 1(3):317–326. Reprinted in Boolos (1998).
- Boolos, G. (1998). *Logic, Logic, and Logic*. Harvard University Press, Cambridge, MA.
- deRosset, L. and Linnebo, O. (forthcoming). Abstraction and Grounding. *Philosophy and Phenomenological Research*.