Formalism, History & Reflections: Math & Logic (translation)

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Formalism, History & Reflections: Math & Logic

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Abstract: During the twentieth century, the development of formal science and math foundation was marked by methodological advances. New approaches partialy owe their appearance to the abandonment of previous paradigms. To identify how these transitions occured, in this essay we observe some aspects regarding the works of Gödel and Tarski, who under certain interpretations restructured the epistemological bases of math's formalism. It'll be argued that the theoretical variability produced in abstract sciences upon the weakening of previous viewpoints forced shifting some components of the, until then, hegemonic programs.

Introduction

Developments in the realms of formal science and math's foundation during the twentieth century were determined by important changes in methods. Many of such discoveries and new schools only happened due to the disposal of what we could call a closed coherentist paradigm that, at the beginning of the twentieth century, as put by Rosado (2010), was replaced by viewpoints anticipating the needs for hierarchical metalinguistic approaches (pp. 28-29) allowing axiomatizations incapable of proving their self-consistency. To identify how such transitions happened, we will recall certain aspects of the works of Gödel and Tarski who, under some interpretations, restructured math's formalist epistemic grounds enabling the development of broad, theoretical, universal and abstract algebraic group analysis.

It will be held that theoretical variabilities produced in abstract sciences, upon weakened nineteenth century's (henceforth hilbertian) models partially abandoned some components of previous study programs proposed by Hilbert, due to the infeasibility of critical issues. Following M^{c}Elroy (2005), Hilbert was deeply occupied seeking proof's validity, studying the very process of math's reasoning and proof structures (p. 136).

Such variability may be reviewed based on both positivism and historicism. A neo-positivist approach will allow recycling viewpoints further onto Popper's prisms as, as stated by Lakatos (1968), this view equates evidence for supporting a scientific model with the rationality of holding its surrounding beliefs (p. 358). Hence, we can reinterpret the development of math pivoting upon gödelian theorems, indexing it as a falsificationist science. On the other hand, we can approximate a historicist analysis revolving around kuhnian proposals suggesting changes actually represent paradigm shifts.

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Previous Predominating Ideals

Before detailing this shift in math, let's note that what we have here called a hilbertian outlook dominated logicism but, even before, several other glances did coexist. Among such schools, different foundations incorporated idiosyncratic thoughts describing specific contrasts. As pointed by Mitchel (1988), for each one, only a portion of the information was valued, explained and modeled, while the rest was taken as settings (p. 59).

In such pre-normal scientific times, disparities in logicism and math's foundations ranged from purely utilitarian applications, up to philosophy and metaphysics. To Megara, Greece, we may track some important dialectic and stoic formalist developments.

There, the essence of critical distinctions was already appearing. Upon what he called the question of universals, Plato believed in the existence of purely metaphysical objects that did not interact as matter, space or time. These embodied essences. He proposed to observe nature divided into two universes: one of sensible things and appearances; and one of intelligible ideas and perfection. With his cavern allegory, he gave us his glimpse: our sensible world is always shadowed and imperfect because we perceive only instances of what would eventually come to be named platonic universals: ideas eternal, underlying and

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unconditioned, that may be truly contained only in our minds. Aurobindo (2000) also recalls Heraclitus had proposed that everything was born according to their logos: intelligent original forces different of human reasoning (pp. 88-89). All this had already began separating discourses from formal articulation, lexicography, syntaxis, and fundamental content.

Zeno of Citium was one of the first ones to emphasize a need for logic groundings. Philo, the dialectician, advanced proposing that material implications could falsify only false inferences from true premises. Megara's school also tried a kind of reduction to absurdity falsifying opposing ideas, establishing eristic proof methods. In the dialogues of Euclides of Megara we can see how Plato's perspective made ideas independent of sense data, consecrating the importance of symbolic abstractions (Luria, 1980, p. 17) and rejecting hedonic knowledge in favor of justification and truth.

Aristotle also marked the development of logic, from Middle East, to West, over centuries. One of his most distinguished topics was the one referring to future

Focusing on history, Kuhn recommends describing these epochs as pre-normal-science for a given paradigm.

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contingencies associated with the difficulties of free choice, as long as we consider that propositions retrospectively maintain their truth value towards the past. Aristotle's solution to this problem takes today a surprisingly profound meaning by implying precisely that there exist extra-systemic truths and sentences that fail to acquire truth values if we pretend that falseness and truth are mutually exclusive. As a counterpart, it is said that, faced with the Aristotelian problem, Diodorus the dialectician, teacher of Zeno and Philo, came up with what was called the master argument. With it, he suggested that there is nothing that is really possible and yet isn't, or will eventually be true.

Aristotle was a central figure when establishing the importance of syllogisms upon deductive procedures and, at the same time, of the role of what he named epagôgê over abstraction processes: from what was being observed, towards basic principles of scientific knowledge. With such proposals, the birth of logic was quite philosophically arranged: associated to ontology, metaphysics, ethics, rhetoric and even theology.

India, China and Arab nations have inexhaustible developments of which still today there is not much information in spanish. Universal numerical notations are direct inheritance; addressing mathematical equations algebraically; solving nonlinear equations with binomial theorems and induction, even digits and the absence of quantities measured through the numeral zero. Similarly, the first use of negative numbers and decimals dates back to ancient China.

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In ancient Greece, math largely revolved around geometry. The outstanding ideas of Pythagoras, Hipparchus, Thales, Apollonius and Euclid of Alexandria marked such developments. The work of Diofantes is particularly noteworthy. In his arithmetic, he already had established some methods to solve what later became known as diophantine equations. Only after two thousand years it was outlined why a general solution to such problems isn't feasible. Eratosthenes of Cyrene studied how to find the smallest primes of a given quantity. Evert & Piaget (1966) relate how ancient greek math foundations even went through their own crisis due to the discovery of irrational proportions and paradoxes by Zeno of Elea (p. 96), pioneer outlining philosophical problems and metaphysics emerging when considering operations and infinite quantities upon geometric, physic and mathematic analytics.

Though Aristotle discussed the excluded middle and non-contradiction, for centuries logic wouldn't approach math nor formalism, instead, it inquired upon problems of diverse nature, like free will and discursive argument fallacies. This type of thought was specially exacerbated in Western Europe during middle ages, when monastic studies gained centrality, allowing some of the first logical arguments justifying the existence of God. And among those, the ones of Saint Anselm and those of Saint Augustine

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inaugurated a tradition that would later descend to Gödel including Descartes, Spinoza, Leibniz, Hume, Kant, Hegel and Frege.

Through the reemergence of renewed arts and humanism during the renaissance, a search for logic formalizations kept slightly displaced, this time by increasingly empiricist considerations. In addition to new methods and instruments, theoretical developments, from the proposals of Copernicus, up to Newton, Huygens, Boyle, Snel and Hooke, arranged theoretical fields interested, overall, in understanding kinematics, technology, construction, art and economy. The foundations of probabilities, calculation and statistics were being laid. Eventually, the works of Leibniz, Lagrange, Euler and Gauss would initiate an approach to algebraic abstraction. As stated by Seligman (1962) logical and epistemological considerations that allude to Kant's doubt regarding feasibility of metaphysics (p. 5), and his criticism over any knowledge sustained merely from aprioristic and rationalistic approaches, both prove the technical spirit of the scientific revolution.

Adding up, more or less metaphysical practical implications over physics and philosophy were common starting places upon the development of an interpretation that integrated logical formalism with math. During such transition period, it could be said that no point of view was

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dominant from any specific scientific lookout, yet a variety of positions coexisted, before finally delving onto logical foundations.

In this way, half-seventeenth-century's Leibniz established some basics for modern logic. Godwyn & Irvine, (2003) explain how, in search of terms under which it would be possible to produce verifications through syntactic reduction, and on to primitive tautologies, we can find upon his works some of the first formalizing concerns for math, added to his deep commitment with syllogisms that took it's own metaphysical, distinctive appeal (pp. 173-175).

Even without being the initiator of formal mathematical symbolist logic, in retrospective, a good part of his early developments influenced in one way or another, participated in, and inspired later developments towards that direction. Leibniz was a pioneer observing the needs of mathematical logics for using languages formally and unequivocally, considering any reasoning error could be captured in explicit operational mistakes. His logic started to

Couturat, early during the twentieth century, rediscovered much of Leibniz's logic. His revisions, and to those of Erdmann and Trendelenburg, began understanding what two hundred and fifty years before, Leibniz had been justifying while deeply interested in formalism. Based on the Aristotelian problem of future contingencies, he took on establishing free will's coupling to such contingencies, in a way that made every human decision look as if it had some explanation within laws of causality and God's will's framework.

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be formally arranged in what later was called statement logic, or propositional calculus: a way of formalizing operations that facilitates symbolic manipulations and has syntax. Sentences must be well formed to belong to such system, and from its truth values, truth values of other statements may also be deduced.

In the same way, at the beginning of the seventeenth century, George Peacock began to emphasize the great importance of approaching math algebraically, allowing with such entree deeper meanings. His gaze connected with a series of developments that, throughout the century, drove logical formalism onto consolidation, until it became the hegemonic paradigm regarding math's foundations. In such process, George Boole's round binary logic, and subsequent achievements established by Sanders-Pierce, Schröder and De Morgan played a central role in nomenclature, quantifying and theorems.

Just as statement logics operate in a general way upon their declarations, we call predicate calculus, or first-order logic, that which operates on propositions that are functional to subject's objects, and that can vary within the discursive domain. Logical statements can be built then by two separable-operable pieces, more or less independently. As put by Manchester (2009) it should be noted that western logic has patent propositional features

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considering valid arguments always keep direct relationship with premises leading to conclusions (p. 30).

This predicate calculus connects us with Sanders-Pierce's, De Morgan & Frege's developments, as they allow not having to isolate a specific subject, popularizing the use of logical quantifiers, allowing us to refer to the totality of elements as subjects of propositional functions. On to such improvements, growth diversity was stretched by Galois, Abel, & Cayley. And all this new ideas began forming what is now known as universal algebra, operating at an increasingly less numerical level, therefore generating more and more urgent pressures on to the requirements of formalizing theoretical modelling, following Peacock's observations, and gaining hegemonical establishment in an almost consensual manner from 1880s onwards. At that time, Frege, who had already outlined some first quantified logical calculation systems while establishing definite sums of specific inferential rules and a priori assumptions; attempted to axiomatically stone the roots of math and algebra during 1884, with Die Grundlagen der Arithmetik.

It was him who distinguished logical statements from propositional ones, since the former described mere facts that emerged from models (Tieszen, 2005, p. 238). Although unsuccessful in his efforts, the first attempt to conquer such expected logical settlement as such must be recognized. Yet, if we were to select a milestone that represented all these outlooks, arithmetic proposed by Peano really proves the impetus such ideals had had acquired. This is how, at the end of the nineteenth century, a closed, formalist, but finitary and coherentist view dominated the field of math and logic. Honoring the number of important results he discovered in physics, math, and geometry, let's associate Hilbert with this mathematical worldview. Hilbert established himself as one of its last, and most tenacious, representatives, especially during the year nineteen hundred. These were optimistic and ambitious times, Art Nouveau was emerging, first zeppelins flew and Tesla had a patent for a method pretending huge energy spread without cables.

Yet, foundational interpretations quickly entered into crisis. Many of math's developments had been able to elude a solid and formal basis, but this gradually became a challenge. As Capponi (1987) puts it, when the impulse and motivation are hampered by meaninglessness, man resorts to meditation and reflection, seeking logical and epistemological foundations (p. 7).

By the end of the nineteenth century, Cantor had developed a set theory that contained infinite elements. He established results regarding such sets' sizes, and those that formed by combining their elements. He also proved that there are discrete dimensions within infinite magnitudes associated to such sets. Controversial in diverse facets, his

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theory was beginning to generate foundational debates among finite-pro mathematicians who considered that objects were to be only constructible and finite.

Eventually, a group opposing to Cantor's ideas came to form the radical-reformist wing of constructivism in math: intuitionism. Brouwer initiated such overturn, founding math on the sole basis of intuitively constructible objects. And those perspectives actually led to different kinds of math since they weren't endowed with some of the crucial tautologies until then, like the law of double negation. This implied a schism between a minority that intended to rewrite and refund math upon an intuitionist system, and those that continued looking for a basis including developments like Cantor's.

Hilbert's program represents this last foundational dominant paradigm, around 1920s. His proposal combines three ways of understanding math, that we will try to describe in a partially independent manner for us to have clearer picture: formalism, finitarism and coherentism.

Formal Finitary Coherence

Formalism originally stood on the idea that math itself had no real meanings, but that it is a rather formal representation of what may be done. Though strongly

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related to logicism that places math upon formal logic's structures, it is presumable we would find deep differences between nowadays formalism, and what a nineteenth-centurish view would interpret as such, especially considering current perspectives wouldn't fit on to Hilbert's program. In any case, formalists views establish deductive methods we may operate almost typographically, taking into account that math doesn't make material attributions nor credits content. Such formalistic logic wasn't fully built until after tarskian proposals made the scope of logicist definability clear, clearly outlining how formal system's semantics may operate.

A somewhat more naïve, or more numerical formalism connects us with Hilbert's finitarian position. Finitarism seeks, to certain extent, enumerable math, but not only syntactically, it but also aspires to recursively

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There are two opposite ways conceiving linguistic semantics. On the one hand, semantic holism conceives meanings based on larger portions of that same language, maybe even all use cases that have ever been deployed. But another way of defining meaning is reductionism, considering that when a proposition establishes a reference, it does so in terms of different languages or object classes.

Both semantic positions have their projections in more or less formal languages. In logic, we consider that semantic notions at base impact the conceptions between propositional relational references, naming of objects, and essential meanings, both satisfaction and truth.

reduce any theorem with free variables, or, for example, operating on infinite cardinalities, up to primitive and concrete forms, where only finite quantities participate. As specified by Gödel (2006), Bernays argues that this view involves the possibility of speaking only of intuitive and pointable math objects, separating it from intuitionist proposals (p. 413) since it dismantles their most significant distinction, establishing that considerations about infinity in math are just generalizations of quantized finite procedures. If even the most abstract and conceptual math is a reversible induction of concrete operations on concrete figures, intuitionism cannot reject controversial deductions that come from it and, even more, it establishes a merely eccentric, superfluous distinction.

For deflationism, asserting a proposition is true does not add information to the proposition itself. In this way, we adopt a deflationary view when we make the assumption that truth references are just redundancies of language. Tarski developed a theory where truths are defined in metalanguages and where language itself never has enough expressive resources referring to essential definitions of metatheoretical truths. Thus, essential truth always demands an underlying explanation, something sometimes interpreted as deflationary reductionism. This viewpoint implies that reductionism, rather than atomizing, is ad infinitum. Each metatheory has its own meta-metatheory. Certain truths are then defined as first level non-referenceable. Saving a number of paradoxes, it implies meaning is never fully comprehensible, just as Platonic universals can never be fully perceived in the sensical world.

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Finitism established a foundationalist scope since, at most, it seeks math's totality profoundly proportionate to arithmetical amounts of hypothetically feasible measures. We may visualize finitism as a primitive, exacerbated formalism, where meaning is still attributed to math, and deductive methods not only have rigid syntactic mechanics, but rather operate with meaning, based arithmetics going back to the last theorem of concrete quantities. This view eventually came in contrast with a whole spectrum of interpretations and systems that have transfinite quantities or magnitude, including proofs of induction on infinite measurements.

Hilbert's third position is deeply understood and concealed in nineteenth-century's conceptions of math. We will call coherentism to the tacit claim aimed to prove math's consistency from where it is built, as if it was a closed circuit from which no extra-systemic allusions emerged. Closed coherentism also seems to be linked to an idiosyncratic tacit spirit: an ambitious, discipline-unifying vigor that was complemented by several decidability and completeness expectations implicit in Hilbert's program. Assuming axiomatic systems were capable of deciding truth or falsity for every enunciated proposition, the program tried to find a conclusive and justified framework combining assumptions allowing all math to be represented without formal paradoxes. Closed coherentism had shared all this positivist alienation, even supposing that such a categorical system would be capable of expressing its own consistency and proving it. From such viewpoints, after finding the axiomatization, math's development could fully direct towards expanding expressive power so that more and more propositions could be judged by a complete, unified system.

Hilbert vehemently supported hegemonic outlooks conjecturing completeness, and aspired unification. His points of view are treasured within his famous twenty-three problems proposal, extending through deep differences with Poincaré, Brouwer and Weyl, and can only be fully understood as a counterpart to the current intuitionist proposal of the time. Quine (2002) reminds us that despite all this, it's important to emphasize that formalism associated to Hilbert, though abandoning meanings for underlying notation, like intuitionism, seeked to move away from Platonic universals (pp. 53-55). It was Tarski who solidified

Referring to nomenclature, a descriptivist position associates nominal designations as objects that meet characteristics interlocutors attribute to names. Such was the position of Russell and Frege. Nominal descriptivism generated some controversy due to the laxitude of its designators, especially when resorting to qualified modes such as doxastic and truth deductions or refutations, using knowledge, possibilities, or the need for certain assertions.

by Arturo Espinosa
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a semantic base, banning theoretical access to specific, essential, meta-elements. He was born a year after the twenty-three problems were raised.

Tarski developed prolific works around logic foundations achieving more transparent distinctions between colloquial languages and formal ones: a pending issue until then. Seeking to eradicate antinomies, he delved onto adapting formal language's roots, confirming a framework that, when tested for self-consistency, made closed coherentism impossible if encompassing all math.

Regarding Hilbert's approach, we should recall that he was not the creator nor founder of the interpretation that we, here, term after him. Still, he is a great representative of the position we have therefore been calling hilbertian. Thus, a milestone in the history of math occurred on August 8, 1900, during the International Math Congress at Sorbonne University, when Hilbert, famously declaring that there is no math ignorabimus, presented a list of challenges to be addressed for future math; a month later he declared «wir müssen wissen, wir werden wissen» (we must know and we will know). Of these problems, the first required precision about sizes and kinds infinity may take, the second, a proof that arithmetical a priori assumptions never contradict themselves.

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Completing a Base for Math & Logic

At the beginning of the twentieth century, an outstanding publication of Russell & Whitehead notably captured and ordered axiom bases for what we have here endorsed as a nineteenth-centurish archetype. During almost a century, logic-math's efforts had developed around proposals hoping to solidify their foundations. In 1918, for example, Bernays, young Hilbert's collaborator, had successfully established how sentential logic could prove every truth expressible by itself. But, as Parsons (2007) puts it, only after concluding the incapability of a priori kantian empiricist logical dichotomies, and after

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Epistemic transitions might be better understood using Kuhn's ideas. He constructed comprehensive analytics regarding how we may interpret what he called scientific revolutions. And he observed science as an activity whose models adapt not only due to available information, but also based on a diversity of conjunctural matters, ultimately subjective, depending on simplicity, scope and context: the problems of scientific interests and hegemonic worldviews.

Kuhn suggested a progression going from pre-normal science on to normal science, crisis, revolutions and post-revolution periods. He also named paradigm shifts, the replacements of one model on to another, making a remarkable difference with Popper's ideas. For Kuhn, paradigms are usually not measurable: to develop objective counterpoints between scientific worldviews is hence, impractical. Actually, at the beginning of the twentieth century, the famous principles of Russell and Whitehead exhibited the first insights that Kuhn might had qualified as anomalous.

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understanding the main limitations of such aspirations, foundational problems moved him on to reformulating his own epistemic positions (pp. 136-137).

Avoiding paradoxes, Russell and Whitehead had added a debated axiom called reducibility. For the sake of consistency, this axiom – pardoning my redundancy – certainly limited the scope of their axiomatization limiting the depth of functional definitions explicitly to the predicate. Still, even this formalization was marked by the imprint of our hilbertian gaze: finitarian and enclosed in coherentism. Russell-Whitehead's proposal quickly became well known. Yet, with its limitations and what now seems a somewhat complex nomenclature, it did not become the frame of reference while instead, the axiomatic system known today as Zermelo-Fraenkel's set theory turned out to be an hegemonic reference, managing to prove arithmetical consistencies despite incompleteness and the inability to prove self-consistency.

Regarding Russell-Whitehead's proposals, it should be noted that Hilbert expressly stated his opposition for their axiom of reduction. Stating it had to be proven that it was consistent with the rest of the system, Hilbert made clear that it was necessary to look for an axiomatization that would eventually prove its own consistency and that would not arbitrarily and unnecessarily limit its own expressive scope while doing so. Such aspirations became unsustainable as we can prove formal systems that are sufficiently comprehensive not only have arbitrary limits: moreover, those verifying their own consistency are inconsistent.

All these contexts refer us to the moment in which creative processes trigger investigations and subsequent justification search. Culturally, the need to justify theories, just like language and most human processes, emerges and can be understood through the devices of variation and evolutionary selection (Neuman, 2003, pp. 33). Facing math foundation's difficulties according to hilbertian viewpoints

The works of Frege, Russell and early Wittgenstein had a significant impact on neopositivism, an image reading processes under the development of science, while conceiving its emerging philosophy, during the first half of the twentieth century, and distinguishing between contexts of justification and discoveries.

These neopositivist ways of conceiving sciences facilitated linguistic scoping around statements, proposing scientific demarcations: in order to have meaning and be scientific, statements must be verifiable (or falsifiable). Hence, for example, we find a distinction between abstract universals referring to rather general natural laws, versus observations referring to empirical sensical records. On their part, theoretical statements are the base of deductive theories, associated with justification contexts. Some of the most prominent participants of neopositivism were Carnap, Hempel & Schlick. Notable criticisms have been raised by Kuhn, Popper, Lakatos, Quine, Feyerabend & Putnam.

and pretensions, after incompleteness discoveries, several kinds of analytics emerged. And such variability produced proliferous divergent developments, subsequently proving applicability and suitability over years. Best adapted proposals started to be selected by those who study and made use of math, consolidating and bringing more collaborations and developments back in to theoretical grounds.

As for the paradigm shift that began with Gödel's developments, the context of the discovery may be associated with arguments justifying math's problems, seeking to induce and show completeness for formal systems based on their axiomatization. In such line of research, Hilbert's proposals, together with Ackermann's, led Gödel to select his doctoral thesis.

Unlike (elementary) geometry, from arithmetics, it's possible to prove the existence of truths that don't have theoretical verifications, and it isn't possible to develop arithmetic systems confirming them (Tarski, 1994, pp. 127-128). However, in 1929, Gödel developed the completeness theorem, crystallizing quantified logic in first order as complete and finitary: the totality of its deductions can eventually be obtained computationally, through rules of syntax.

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Complete quantified logic seemed an important approach towards a more general proof for completeness because there are no (tauto)logical truths (with first-order quantifiers) that aren't logically deducible. Later, a qualitative leap in math found an unexpected angle upon Hilbert's second problem's interpretations.

Seeking an understanding of such development's implications, it seems convenient to take a look at logic's expressive capacities and dimensions. Tarski observed that when the definition of some meanings is representable in that same theory, contradictions emerge. We can partly deduce that this moved him on to establishing a theory where semantic definitions were established upon other theories with increasing essential expressiveness. This requirement transcends even up to unrestricted logic because it doesn't pretend math's realism, differentiating itself from intuitionism of constructivist aspirations. As long as unambiguous conventional formalisms remain, Tarski's semantics become infinite regressions over truth and meaning. Following Lakatos (1968), this does not imply we should take a cynical nor skeptical position towards math: we can honestly defend its furtheron defeatable knowledge based on our subjective, intuitive psychology (pp. 22-23).

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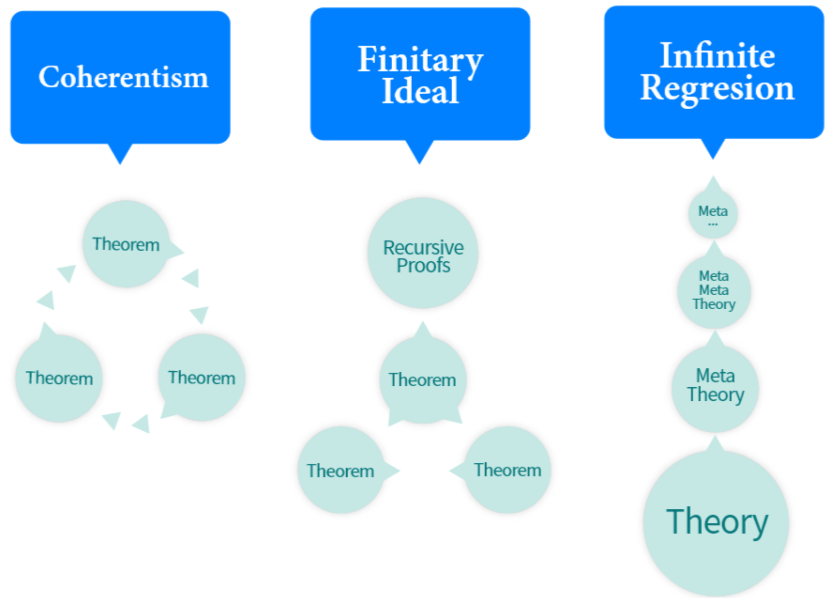

Figure 1. Epistemic foundations for formal math axiomatizations.

Besides meaning definitions, first-order logic has syntactic restrictions, especially in relation to predicates and quantifying variables. We can think of infinite logics depending on the complexity of their propositions. Sentential logic doesn't quantify variables (it is zero ordered), first order logic may quantify objects from the predicate up to one degree, second order logic can quantify over clauses or portions upon the predicate, which would be a second degree, etc. The limit to which this progression tends is a higher-ordered logic in which there are no syntactical restrictions on the predicates, nor on their nested quantifiers.

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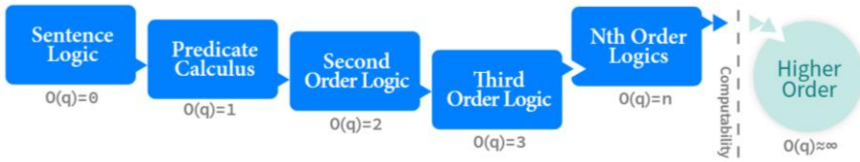

Figure 2. Order of formal logic.

Still, first-order logic's limits haven't limited its usage due to its finitude and results such as the Gödel's completeness. It is a somewhat docile and grounded language, at least more than higher ordered logic. Furthermore, allowing infinite representable axioms, it's expressive power may be enlarged.

Collapsing Aide & Theoretical Progress

Let's look at Gödel's second incompleteness

We may relate to Gödel observing how Hilbert's claims led him on to phenomena incompatible with hegemonic paradigms. Such was the context of discovery, independent of scientific theoretical methods, incorporating historical and conjunctural assumptions: leading to new models. Hereafter, within contextual justification, theoretical exploration gathered objective evidence. From problem emergence in previous models, on to acceptance, assimilation and building mathematical knowledge (Kuhn, 1998, p. 31): driving context-justifying intents for nineteenth-century's aspirations: pushing new discoveries and emerging understandings. Now endowed with confirmational reductionism commensurable against preceding paradigms, emerging viewpoints' insight surpassed adequacy levels beyond descriptivist redefinitions, therefore, enforcing to admit self-limiting epistemic doubt.

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theorem as a paradigmatic shift for math, inaugurating new analytics, overcoming previous formalization research while seeking a unique, complete mathemathical foundation. Such paradigm shift became strengthened by Tarski's semantic foundations, confirming that some of the problems proposed by Hilbert in 1900, such as the tenth and first, simply couldn't be analyzed seeking for a strict and straightforward answer.

Propositional treatment's meaning and philosophies began to clash into distinct routes, allowing multidirectional depths. In order to attribute propositional meaning, an informal, plural and intersubjective realism was developed, admitting enactive imprecisions whereas subtracting itself from deciding upon the totality of assertions; meanwhile, a more formal deflationism of extra-syntactic value also arose (Putnam, 2000, pp. 134-136).

After Gödel's second theorem, sufficiently complex models' deductive depths were to be studied knowing that they'd never ensure an axiomatic base. The shadow of a doubt hanging over all the theoretical construct, and establishing that at any moment, an axiomatic collapse due to some distant unadverted contradiction could actually happen became a permanent certainty. Apparently, it was only after this new backgrounding the range of math-logic theories expanded upon new dimensions. There was no more interest upon deducing and consolidating one unique formal system, while it began to make sense to wonder what types of deductions can be allowed with different models and assumptions. Why and how can a higher order theory prove the consistency of a lower one? To what extent, and under what requirements and assumptions can the consistency of systems greater and lesser than first-order arithmetics be verified? How and what types of axiomatic theories verify their own consistency? All these questions were absent in the previous hilbertian approach.

Gödel observed that within any axiomatization allowing Peano arithmetics, self-consistency proofs triggered paradoxical contradictions. Hence, he concluded that consistency proofs were incompatible with such axiomatizations. Gödel neither proved nor disproved the original problem posed by Hilbert: it is possible to prove arithmetic consistency standing on broader frameworks. Yet, his work unraveled deep flaws at math's heart and logics founding it.

Between 1923 and 1931 many things happened: with faith in solvability, Hilbert & Ackermann proposed their entscheidungsproblem, stopping problem; Banach & Tarski published paradoxical consequences for spheres' topologies based on the axiom of choice; Enigma was built; Flemming accidentally rediscovered penicillin; Wittgenstein insisted for his last time on the need for clear, appropriate, precise and unambiguous logicophilosophicus symbolisms excluding pseudo-propositions; Bohr's instrumentalism prevailed over Einstein's realism at Solvay; Russell & Whitehead published a second edition of Principia Mathemathica, modifying reducibility with a proposal as much, or even more limiting and questioned; Heissemberg introduced uncertainty principles; Reichebach founded Berlin Circle, and Thomas Alva Edison died.

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Gödel's incompleteness theorems managed to interpret math's coherentist formalism anomalies. In particular, they explained why until 1920s, despite persistent attempts, it had not been possible to establish a complete and contradictionless axiomatization that went beyond Peano's arithmetic and included vector spaces, sets, and free group theories. At that time, even some corners of science had begun colliding with formalist antinomies.

Bertrand Russell already in 1903 had sent a letter to Frege where he explained how his, about to publish, logicist foundations led to some formal contradictions. Plus, we must recall all the problems that arose when allowing intratheoretical meaning access.

With such developments, it seemed that seeking unified axiomatizations became less and less feasible. An important result had been studied by Skolem, establishing that enumerability for some of the characteristics respecting describable objects depended on the axiomatization upon which such enumerations and descriptions were being built. We can connect this accumulation of problems related to the context of discovery with what Kuhn established as a paradigm crisis. This hardened delving into what Hilbert had proposed. Such anomalies started to force math into irregular explanations avoiding certain fields of analysis. Hence, the ad hoc axiom of Principia Mathemathica was widely criticized, especially by Wittgenstein & Hilbert, since

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it resulted in an operation of reducible functions prohibited from redounding within their own argument.

During 1931, drawing on revolutionary arithmetizations for numerical methods, Gödel formalized a self-referential paradox through which he constructed a sentence which's insusceptibility to be proven, and which's verification from the system itself, respectively implied its veracity (extrasystem) and falsehood. This settled that formal systems may reach complexity enough to express true but undecidable propositions that become impossible to prove in within. This is the first incompleteness theorem.

Five years later, during 1936, Alfred Tarski defined some limitations for numerical methods of the kind Gödel used in his incompleteness developments. In particular, the indefinability theorem limits formalist adequacies establishing that any arithmetization cannot successfully express its own underlying semantic concepts. Thus, phrases such as Gödel's are constrained within the upper class of the system

Recognizing the possibility of being talking about different elements, Frege and Russell proposed lax designations (Kripke, 2005, pp. 11-17), knowing they were to lose meaning if there were actually no objects satisfying the nominal references. Due to such difficulties, coherentist finitarism was becoming unfeasible.

based on a picture
mathshistory.st-andrews.ac.uk/Biographies/Frege

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Hence, they may refer to systemic evidence (provability) but not directly to meaning's falsity or veracity. These last ones belong to sublevels of inaccessible metatheories.

Though discovering the opposite, we may understand Gödel's first theorem as originally trying to prove system's completeness from axioms. Yet instead, Gödel stated a second theorem transforming sufficiently complex formal deductive systems' interpretations.

Gödel's results have an undoubtedly philosophical scope, but the projection of each result onto a variety of problems not studied by him must be analytically scrutinized with special care; for example, the second theorem implies that convention's consistency are strong assumptions (Wang, 1995, p.209-212), arithmetically undecidable. Specifically, if an axiomatization allows inferring the conclusion established by the first incompleteness theorem, it will not be able to deduce or axiomatize its own

Adding explanations against findings may inductively support disciplinary theoretical models. Then, observations are theoretically explained. Formalist axiomatic models like Hilbert's, made hypothetically mechanizable imitations of several cognitive processes, theorizing in testing-frameworks, deductive figures emerging from rules and presuppositions (Magnani, 2009, p. 379- 380).

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consistency. Otherwise, in virtue of such consistency, it would be able to also falsify, through that same first theorem, the sentence to which it is refering, simultaneously proving it. This contradiction that denies any proof of consistency for certain systems is Gödel's second theorem, implying an epistemological topologic change for math until then conceived.

Falsifiable Empiric Math

Our mind is able to conceive inconceivable concepts. Therefore, arithmetic or math principles might actually not be consistent. However, theoretical inductions of consistency and completeness had developed since that time, based on what had been observed. Models' capacities and successful predictions had increased seeming limitless and thus, it had been taken for granted that an axiomatization with such features could be built without contradictions. Presumed proofs of consistency and completeness just pended to be found, leaving behind contradictory axiomatizations.

Hilbert's search leaned towards an axiomatic scheme rather than a finite number of axioms. This may be interpreted as a general rule defining infinite axioms and can be thought as an enumeration whose infinite limit is precisely the whole scheme. Therefore, implicitly, Hilbert

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also aspired to prove that operations and theorems, at infinity's limit, derived from finite math-structures.

Figure 3. Axiomatic schemes orders.

Moving on, it is convenient to have an interpretation of math and logic suitable for neopositivistic outlooks. On the one hand, there is controversy about whether math is a science; whether its object of study is something real. But giving an answer to such questions seems out of the reach of our scope. Yet, we may make transitory assumptions considering we could produce empirical interpretations of math and logic as formal sciences, expressly taking in consideration the paradigmatic shift we have been reviewing. All this because, since neopositivism, an important natural resistance for taking math as science denotes it seems to have no distinctions and/or materializations between protocolary statements and what constitutes proof for theoretical ones. Referred

Comprehension is a classic form of axiomatic schematization. It defines an infinite variety of sets based on logically definable properties, avoiding paradox, while demanding element belongingness to known sets. Schematics may be of first-order, but still, higher-order logic (with quantifiers that have been nested not only upon objects, but upon functions within the predicate) can, in a single axiom, express sentences that would require infinitely many first-order statements.

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to such viewpoints, it becomes of interest to construct some model allowing math to be interpreted in a way in which theories are being verified through observations.

Empiric sciences are firstly referred to empirical objects. For math, a conflict immediately appears. To overcome such difficulties, though logic and math are overstood as itinerating within abstract and theoretical spheres, we should assume this echoes an epistemic and structural component inherent to formalized representable systems, given some setting (with shareable conventions). That is to say, under our interpretation, the real or empirical object to which math refers will be a rigid, abstract, universal, formal definition, coming from assumptions with rules of inference (axioms) and always leading to certain inevitable, repeatable consequences, as characteristics. The syntactic formality of such definition is important, becoming necessary to root ourselves on settlements and adequate criteria such as those purposed by Tarski. His infinite regression offer may be interpreted as deflationary, allowing math's logic to be read through empirical interpretations, if we allow ourselves to interpret truths respecting formal theories.

If closed coherentism searched for consistency, it was precisely because it was based on clearly reviewable and strict formal systems. When Russell and Whitehead proposed math's principles, they established a closed

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convention upon which formal deductions followed unambiguous methodological rigorizations. Axiomatic contradictions would have been taken as design problems, furthering research upon an appropriate axiomatization. Standing on such premises, upon seemingly limitless math theorems' capacities, Hilbert established his famous second problem, inducing a theoretical conjecture awaiting for justification.

It is generally considered that differences between math and other sciences are given by the fact that mathematical refutations do not need empirical corroborations and the usage of one axiomatic system over another one is decided, in principle, by intuitive considerations, entirely arbitrary (Agassi, 2014, p. 78), the so-called psychologism. Yet, for us to elaborate our neopositivist vision, we need to admit possible validities for any feasible axiomatization. Abstracting meaning, we could specify incipient terminologies talking of formal bodies emerging, or not, from upon choosing specific a priori

As put by Feyerabend (1986), the definition of criteria for keeping theories has nothing to say about ‹rationality›. He gives Lakatos' example, suggesting new measurements while reminding that accepting any apparently irrational developments might turn science irrational (pp. 187-188). Recalling Kuhn, we keep in mind that theoretical progressions, if rational successions, involve circumstantial accommodations and breaks, social and even political-economic commitments, that open way for later stages.

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axioms (as long as we have rules to list and/or index them). Such apriorisms produce theorems as a consequence. Functionally, emerging systems then have different way to approach reality necessarily depending on modeled problems' natures. The latter is a technical or applied consideration.

If the formal hierarchies collapse (due to inconsistency) when incorporating a specific combination of apriorisms upon the base, we may interpret we are facing empirical protocolary observations: such apriorisms are incompatible with each other and don't allow emerging formal bodies (no system is generated). Between consistency and completeness, this didn't seem conceivable at the end of nineteenth century. Still, in the opposing scenario, we shall not yet establish irrevocable statements since Gödel's second theorem prevents us from confirming consistency.

For math's foundations, the need to rearrange appeared due to an increase of frustrated ambitions and a growing pressure from physics, incipient cybernetics and computer sciences. Such process was facing empirical protocolary observations and failed theoretical construct emergences, such as Frege's. Just as each theorem or consequence implies its negation may be established as an a priori basis together with the rest involved in deriving it in a theory that forcefully collapses; also, the lack of

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paradigmatic shifts would have been incompatible with several conjunctions, historical processes and socio-cultural qualifications. Consequently, we can say that this came to happen when theoretical decontextualizations anomalously piled up, or however we refer to such accumulating discrepancies of consecutive interpretative, impractical, unproductive and unpredictive deficits of hegemonic models, upon emerging series of organizable empirical phenomena.

From Gödel onwards, math became uncertain and, though developing over formal bodies, for the sake of incompleteness, it doesn't irrevocably establish any statement beyond presumed consistency. With such ideas he was able to admit systems not only incomplete, but of undecidable consistency. When formal bodies lead to consequences, incorporating their negation a priori produced hierarchical collapsation. Hence, apriorisms constructed in these systems became inconsistent, reducible to absurd: data empirically protocolizable for our current math interpretation. Formalizing such ideas goes beyond our scope, but at this point it seems very clear. Any consequence of an emerging formal body can be translated (axiomatizing its negation) into an empiric protocolary observation: a system whose hierarchy we know for sure that necessarily collapses.

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For such cases, given the principle of explosion, a consistent system cannot be achieved even if we were to incorporate more a priori assumptions trying to repair the issue. Moreover, this specific construct for formal sciences isn't subject to Gödel's doubt: our empirical formalization of collapsed theorems is true even if it was discovered that basic assumptions would have collapsed by themselves, without the need to add the negation of our selected proposition. In a literal sense, the proposal is falsationist, an interpretation for math and logic not available before the gödelian paradigm shift.

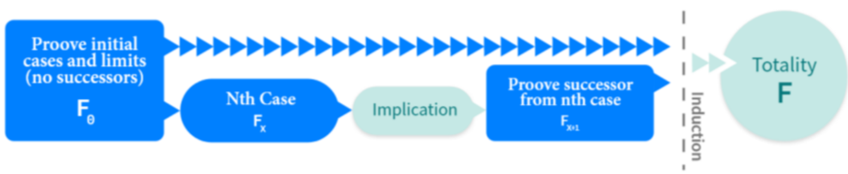

Figure 4. Schematization inductive order.

The previous approach throws us directly into Popper's understanding of science. He added alternative viewpoints for science, seemingly shedding light onto how math proceeds deductively and

We may add some final shades to falsifiability. Without refuting his perspective, interpreting Wittgenstein just towards tautological certainty and not into describing the state of things, Anscombe (1965) suggests detaching logic from definitions referring to whether things are or not. His viewpoints also contrast divergent criterions when evaluating propositional meanings: positivist verifications against adequate referential sufficiency (pp. 150-155). Hence, falsifiable proposals imply two directly associated distinctions.

why Hilbert's induction of consistency failed to successfully adapt to empirical realities of axiomatizations, requiring then a kuhnian paradigm shift. Following Popper, scientific work should approach problems looking for instances where theories don't hold, in our example, when hierarchic systems collapse. If we proceed in such manner, Popper ensures that discarding statements becomes a deductive task, tollendo tollens.

Consequent Axiom Selection

Let's take as an example the axiom of choice (allowing to select between two indistinct variables without having

Falsificationism admits that meta-analytical guidelines, such as Gödel's, describe some level's state of affairs: epistemic things that interest logic, for example. Our here-built position visualizes tautologies and hierarchical collapses as descriptive facts regarding the state of things, implying that consequences deduced from any set of basic apriorisms is equivalent to empiric protocolary observations that, endowed with appropriate corrections, inherit adequacy meanings from their semantic references. Still, we cannot forget the possibility of formal bodies expressing meaningful but undecidable sentences.

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explicit algorithms at hand). If we add the axiom of choice to some system, even if we don't have an explicit incorporative construction, whenever we take a set of sets, even if it is an infinity of them, we now take for granted that it is possible to select and refer to one element of each of them (Monk, 1969, pp. 116-117). Though today not so debated, its nature is closely linked to Hilbert's first problem and its incorporation in the first axiomatizations was heavily discussed: it was moving away from constructivist, finitarist and intuitionist considerations.

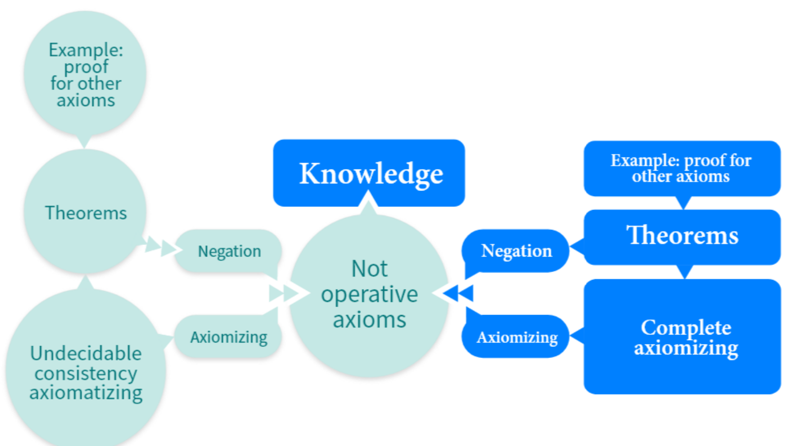

Figure 5. Some instances for math's falsification.

Now, taking or not taking this assumption as true has consequences in the arising system. In practical terms, two different mathematics are allowed. The axiom of choice is an a priori assumption in certain axiomatizations in which it is not subject to proof and, may or may not be assumed to be true. Although the flexibility it gives to mathematical constructions seems reasonable, it allows us to deduce

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particular and somewhat counterintuitive theorems. A specific case is the Banach-Tarski paradox. The axiom of choice allows us to devise a method to disassemble the volume of a sphere into infinite groups of points, groups that in turn may be reassembled into two spheres that are identical to the original.

As we have been reviewing, since positivism, real propositions are those that are subject to verification or falsification (Giaquinto, 1983, p. 126). Our falsifying proposal implies that if we axiomize the negation of Banach-Tarski's paradox, together with the axiom of choice (and another quota of specific axioms), we obtain an empirical datum: it will not be possible to establish an operable formal system since the inclusion of both is reduced to something absurd. It is this type of observations that constitute protocolary, observational statements in our empiricist theory for axiomatizations. Now, the procedure is falsificationist, on the basis that not all systems can prove their consistency, and actually, they can't even prove all true sentences they handle. This requires that observation statements are taken as a reference for real or empirical phenomena, even at metaphysical or epistemic levels, and that in our empiric interpretation for math, we can associate with collapsing models.

Parallel to observational statements, neopositivistic views also point out that, given these kind of result, we may begin to abduct theoretical statements that determine the best hypothetical explanation for such phenomena. Going back to our previous example, it becomes possible to better outline some features for the axiom of choice, deepening into its consequences, even if considering some weaker and stronger forms of choice. When incorporated, both theoretical statements and choice allow establishing and indicating concepts that are not directly defined, but whose existence could be inferred if we needed to. For the axiom of choice, the existence of these concepts can be proven within the system (like infinitely decomposing a sphere's volume); for protocolary statements, induced into an underlying theoretical model (like math's consistency intents).

In 1979, Snapper established how, since they are incorporated by virtue of their content and not their form, focusing our attention on axiomatic formalities, tautologizing Platonic logic-realism foundations could make us interpret critically both axioms of choice and induction (pp. 207-209). We can maintain presumed axiomatic consistency incorporating such apriorisms if we syntactically delimit the capacity of arithmetizations within Tarski's indefinability theorem's limits. Any

To explain theoretical and protocolary statements, we take the axiom of choice, but the observation of inconsistency allows us to induce theoretical content, illuminating an epistemic field similar to Plato's universals (logos), a space materialized from analytical interactions.

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direct reference to the veracity of sentences must be expressed in metalanguages that allow greater expressivity, preventing axiomatic schemes from directly pointing semantic designations.

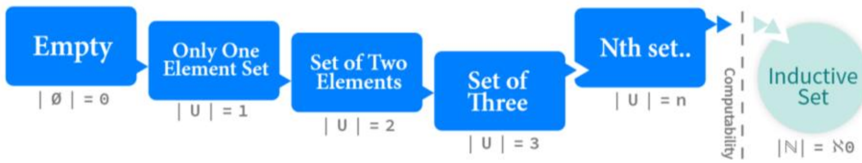

Figure 6. Inductive axiom order.

It is worth considering that there are various formalizations relating the axiom of choice (though weaker), preventing the negation of Gödel's 1929 completeness. However, denying incompleteness theorems isn't consistent with elementary axiomatizations such as Skolem's finitary arithmetic foundations. In other words, this doesn't depend on the axiom of choice. If we believe there is a consistent math overcoming, for example, elementary function arithmetics and using iterative reduction, then we must recognize that given enough systemic

As Hilbert's gaze didn't find a complete and consistent system; the paradigm shift inaugurated new perspectives. This changed mathematical methodologies, from restrictive, purely syntactic proposals, as proof theories seeking foundations; on to semantic, unrestricted and group outlooks marked by Tarski's developments around logic and truth theory (Burdman & Feferman, 2005, p. 123). With the latter, it became feasible to construct systems operating over different axiomatic assemblies.

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complexity, it will become impossible to verify the consistency of such same system. From mathematical falsificationism, we observe that an inoperable system emerges when axiomatizing the negation of incompleteness together with the rest of the axioms that derive it (without necessarily including the axiom of choice among them). After all, as we have discussed, it was for this reason that Hilbert's initial program lost hope since, as Von Neumann (1995) puts it, if math systems aren't contradictory, they cannot be defined with procedures upon those systems themselves (p. 623).

Falsifiable Empiric Math

From an historical viewpoint, we can't conclude that sheer delving into math inevitably led to incompleteness theorems. It all seems to connect with a multitude of conjunctural factors, for example, incidental perspectives emerging in logic, philosophy and natural sciences such as physics. Besides, intuitionist controversies and Hilbert's hegemonization played critical roles. Problems like those raised by Hilbert and Ackermann together with more radical positions as Brouwer's were vital influences for Gödel's works (Dawson, 1997, p. 37).

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We should recognize that searching to solidify math's foundations perhaps seemed to end up deteriorating them, mathematically proving ignorabimus. But closed hilbertian formalism had been weakening through recursive failures of successive foundationalist attempts to specify whole maths without contradictions, using coherentist, closed and finitary logicist platforms. These and other anomalies were becoming increasingly difficult to assimilate into math, and only up to certain extent. In parallel, partly due to the proliferation of different formal systems that sought to respond to Hilbert's challenge: on one hand, different math perspectives began to take shape; on the other, Gödel's work pushed on to rethinking Hilbert's program.

Currently, everybody tends to exceed their generosity visualizing logic and math as irrefutable fields (Quine, 1998,

Returning to Kuhn's historicism, in relation to what we mentioned, paradigmatic models tend to be incommensurable and it's difficult to identify a point of rational analysis capable of fully relating them. However, the presumed expression of a model where an axiomatization (endowed with arithmetics) prooving itself complete and capable of answering any expressible question became shadowed upon the emerging proposal that infinitely many axiomatizations allow different conclusions based on variability assumptions even capable of proving other systems' axiomatizations. But even so, both conceptions correspond to different human outlooks when studying phenomena. Indeed, there are tools upon math's nature that still may be treated, conceived and handled from a hilbertian point of view without even considering an analysis of the quality of the deductive model from which it is being resolved.

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p. 169). As counterpoint, here we have outlined that there are epistemic devices to extract knowledge from math in an empirical manner, even interpreting math itself as an empiricizable refutation process. But even so, we cannot ignore the fact that math is fated to uncertainty and epistemic doubt, safeguarding consistency. This is a doubt about which, unlike natural sciences, there is no doubt.

Consequently, of our three hilbertian positions outlined at the beggining, we can identify connected but differentiated destinies. Closed coherentism was somehow dismantled, leaving hope for unified epistemic determinism as a mere anecdotical history of math, relegating its position to a primitive number of limited axiomatizations related to intuitionist constructionism and/or finitarist foundationalism. Semantics from then on, seems to always refer to underlying theories and systems endowed with

Due to math's nature, it isn't entirely clear whether Kuhn would have called this a revolution, but the new paradigm clearly broke the previous one, proposing to study systems separately, resigning to incompleteness and not ensuring consistency for huge portions of them. Therefore Hilbert's program lagged behind, at least for systems complex enough to build arithmetic.

Historically, theoretically transforming math's interpretations completed. Variability that began with installing a new model, ended with its replacement by another one. In this case, a multisystemic formalism is maybe less ambitious, but studies with more precision while explaining anomalies that could never have been understood previously.

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enough expressive richness. Peano arithmetic cannot prove itself consistent since consistency tests always belong to a broader system, at least with widely specific assumptions like transfinite induction.

Something finitarist today reappears, for example, in computer enumeration, lexicographic computation and programmable arithmetic, bringing us closer to a formalism of finitist recursions. In this direction, transformed and strengthened, if beyond arithmetic, formalism had to abandon all hope of intercepting something real, remaining locked in a symbolic, platonic, insurmountable rationalism that lends itself only to those who venture speculating on absolutely dubious analog interpretations.

After all, during 1930s, a crisis producing interpretive paradigm shifts where math couldn't pretend full self-consistency proof, eventually allowed math's disciplines turning into multisystemic studies. That is, going deeper into model theory, proof theory, reverse math, type theory, ordinal analysis, universal algebra and representation theories. Plus, searching for comprehensive axiomatizations ceased to make much sense when multiple parallel axiom systems consolidated, from quantifier-free arithmetics, on to Quine-Morse system. Though our intuition insists complex math up to date is correct, or rather, insists that portions of reality can be analogically modeled by any of such math axiomatizations without contradictions, from

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then on, it's clear that if it's about models allowing arithmetic, it's always an act of faith.

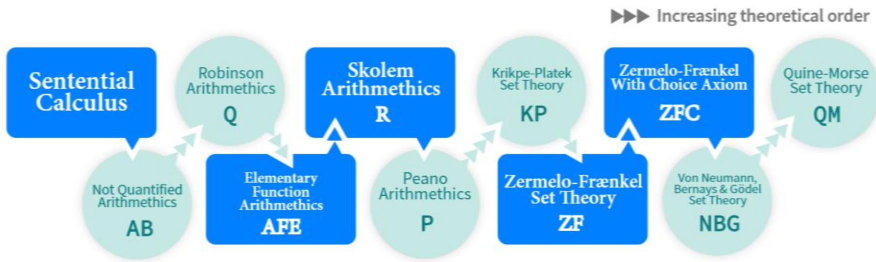

Figure 6. Some ordered axiomatizations.

This new view of formal systems was solidly installed upon Gödel-Tarski's discoveries and advancements, opening perspectives on to axiomatic diversity, understanding the unsustainability of some interests until then, and giving foundations an escape door.

After Gödel ran into the most unsolvable point of Hilbert's proposal, there have been several instances upon which theories appearing after such paradigm shift have given us better understandings of axioms. Thus, the consistency of Zermelo-Frænkel's set theory, which, though not verifiable from within itself, nor below, and even less towards finitary constructions, can still be confirmed from superior frameworks (but with its own assumptions) such as the system of Quine & Morse. It has also been possible to prove the consistency of Peano axioms from quite humble systems. However, according to Monk (1976), these proofs cannot be internalized in Peano's own arithmetic, requiring a transfinite level of induction, over-natural (p. 299).

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The acceptance of such maneuvers deserves a separate note since, returning us to the dispute over Cantor, stretching the metaphor, it seems to cross the limits of recursive primitive arithmetics presuming that Plato's universals can be grasped in the palm of our hands to make them talk. It is to be considered, however, that this is an achieved treat when stretching systems such as Zermelo-Frænkel's into deducing transfinite inductions while avoiding choice.

In this way, although proofs could be understood literally as solutions to Hilbert's second problem, they are equally impeded by Gödel's theorem, or at least require a priori axiomatizations of transcendent inductions on to recursive accesses. So far, we cannot ensure that reasoning based on assumptions from which they are built is

When the new ideas established themselves, they tended to spread quickly. Evolutionarily, they were put on test under different circumstances for which they exhibited all their capacity, expanding. Neopositivism calls this the context of justification, referring in some way to how formal considerations are articulated, within a logical or scientific framework, thus justifying in this case, the adoption of a new formalism. In the case of math's falsification, there is certain overlapping with the context of discovery upon Gödel's works. However, the model ends up justifying itself with Tarski's semantic theory.

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consistent. That is why this is an act of faith, one referring both to the type of math we are willing to admit, and on to the combination of assumptions in which hope will be placed, assuming them to be consistent and valid a priori.

Main Ideas: Summary

☑ *The time-path through which math has been clarifying how its foundations may be understood has been very long. This process goes through a turning point early in 1900s.*

☑ *Math that builds arithmetic makes its deductions knowing that the fundamental premises' consistency (that of axioms) cannot be proven nor axiomatized.*

☑ *First-order logic forms a complete, closed loop.*

☑ *Transfinite induction generalizes over infinite quantities in the same way that finite induction generalizes over numbers.*

☑ *Formal systems' semantics always provide unattainable meta-spaces from theory, where truth and other concepts such as property satisfaction are defined.*

☑ *The formal and/or abstract sciences are deductive and seem to have different nature compared to empirical sciences.*

☑ *We can bring into falsifying scientific terms typographical systems with strict rules, considering as empirical observations any axiomatic combinations that collapse due to inconsistency.*

☑ *We can build a variety of formal systems that may open from different assumptions, and have different expressivities.*

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