Envisioning Transformations—The Practice of Topology

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1 Introduction

Recently, philosophy of mathematics has broadened the scope of its inquiry, by paying closer attention to the actual work of mathematicians belonging to particular scientific communities. Accordingly, it is common today to refer to a shift in the philosophy of mathematics towards 'the philosophy of mathematical practice' (Mancosu 2008). In this perspective, mathematics is not seen as a timeless science dealing with immutable truths, but as a human enterprise embedded in history. In line with this practice-based approach, mathematics, as well as any other human enterprise, is not immutable but subject to change: refutations are part of it as well as proofs.[1] This point of view on mathematics has developed out of a dissatisfaction with the approaches typical of the philosophy of mathematics of the 20th century, mainly focusing on formal arguments and logical issues and whose principal aim was to provide mathematics with solid foundations. According to this view, philosophy of mathematics should not account for the production of mathematical knowledge but its sole concern is its 'final' justification. It is a logic-based

[1]Lakatos (1976) was one of the first to allow for the simultaneous presence of these opposite elements in mathematics into the philosophical discussion.

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B. Larvor (ed.), *Mathematical Cultures*, Trends in the History of Science,
DOI 10.1007/978-3-319-28582-5_3

philosophy of mathematics, devoted to the analysis of rigorous argumentation and to the definition of appropriate axiomatic systems.

To illustrate the limits of such an approach, Corfield has introduced the term "foundational filter" to describe how the exclusive focus on foundational issues has obscured many interesting features of the practice of mathematics: "But it is an unhappy idea. Not only does the foundational filter fail to detect the pulse of contemporary mathematics, it also screens off the past to us as not-yet-achieved" (Corfield 2003 p. 8). In Corfield's view, behind all forms of neo-logicism there is the (unhappy) assumption that such a filter must be applied if the aim is to see mathematics through the lens of philosophy. As a consequence, philosophy of mathematics has missed what should be one of its main targets, that is, contemporary mathematics as well as the history of mathematics. The task of the philosopher today should thus be precisely to dismantle the foundational filter. Once this is done, then mathematics appears as a complex object of research. Moreover, the way is paved for scholars who want to consider not only Western mathematics but also other forms of mathematics and mathematical cultures.[2]

In 2005 and 2008, Mancosu edited two volumes collecting essays that would challenge the logic-based approach to mathematics (Mancosu 2008; Mancosu et al. 2005). In his Introduction to the 2008 volume, he pointed out how philosophy of mathematics calls for a renovation: Lakatos was maybe the first scholar to react to the idea that philosophy of mathematics had to be considered as a foundation for mathematics, and he was followed by other "maverick" philosophers, such as Kitcher with his naturalism (Kitcher 1984).[3] According to Mancosu, the logic-based and the practice-based philosophies of mathematics are not opposed but complementary, the second leading to new analyses of the practice of mathematics through case studies.

However, as Larvor has pointed out in a recent article, the philosophy of mathematical practice

> remains somewhat under-theorized. Answers to the questions 'What is the philosophy of mathematical practice?' and 'How does one do it?' do not usually go far beyond the aspiration to study 'actual' mathematical activity and some now familiar complaints about other, better-established approaches to the philosophy of mathematics that employ formal models of mathematics and mathematical argument. […] It is not yet clear how the historical, sociological and psychological studies presented at conferences on the philosophy of mathematical practice can generate a significant challenge to the approaches that assume that formal logic can provide a philosophically adequate model of mathematical proof. (Larvor 2012, p. 2)

Larvor identifies here two conceptual gaps concerning the philosophy of mathematical practice. First, it is not clear what a philosophy of mathematical practice could be and how it is supposed to be pursued. If we assume that the object

[2]See for reference the works of Chemla (2005) or Høyrup (2005).

[3]The term "maverick" is taken from the Introduction of Aspray and Kitcher (1988).

of such an approach is the work of the mathematicians, what kind of object is it? Which activities, among the many mathematicians are involved in everyday should be considered as relevant? By which methods should they be studied? Second, it should be specified how historical, sociological and psychological investigations generate a significant challenge to the standard view that sees formal logic as an adequate model of mathematical proof.

We share Larvor's worries, and in this article we aim to give a tentative reply to both of them. Our first goal is to show how a new methodological approach to mathematics, based on the consideration of specific features of mathematical practice, will make interesting philosophical problems emerge from the consideration of the heterogeneous work of mathematicians. We will specify some possible objects of inquiry for a philosophy of mathematical practice. Considering these elements simultaneously would allow us to define the philosophy of mathematical practice as the analysis of the mathematicians' use of various available representations. This move would answer to the first conceptual gap that Larvor identifies. Moreover, we will also try to challenge the model of formal logic as adequate to account for proof in the particular case of topology and hint at a different one. We will claim that the practice of proving in topology is based on envisioning transformations on the appropriate representations of the objects of topology and manipulating them. In our view, this model for proof is different from the one proposed by the logic-based approach and far more consistent with the peculiarity of this field. As a case study, we will present the proof of Alexander's theorem, an important result in knot theory, which states that any link can be transformed in a braid.[4] The analysis of this case study will focus on the role played by the representations and on the cognitive work with and on them: justifications will be based essentially on visualizations and the control for rigor will be given by local criteria of validity established within the practice.

In Sect. 2, we will present the elements of the practice that in our view are the possible targets for the philosophy of mathematical practice. In Sect. 3, we will first introduce the main mathematical tools and then give a proof of Alexander's theorem. In Sect. 4, we will discuss the case study, and more generally the issue of what counts as a proof in topology. Finally, in Sect. 5, we will sum up our conclusion and hint at possible ways to further develop our research.

[4]Larvor mentions Alexander's theorem as an example of informal argument (Larvor 2012, p. 727), referring to Jones' presentation (Jones 1998, pp. 209–213). We will expand on that and present the case in detail relating it to our general framework. Elsewhere, we have defended an analogous approach to diagrammatic reasoning in mathematics by offering other case studies such as knot theory (De Toffoli and Giardino 2014) and low-dimensional topology (De Toffoli and Giardino 2015).

2 Defining the Target of the Philosophy of Mathematical Practice

In order to specify the object of inquiry of the philosophy of mathematical practice, we will identify the following three key possible targets. In our view, the philosophy of the mathematical practice should consider: (1) the collective dimension of the mathematical practice; (2) the influence on the mathematical practice of pre-existing cognitive capacities that get enhanced by expertise; and (3) the use of heterogeneous material representations. In the following, we will first discuss these three features separately, and then connect them to the notion of "permissible action", which has been proposed by Larvor (2012) as a new way of looking at inferences in the more general context of argumentation theory. This notion will be helpful to connect these three elements and better define the goal of our research.

2.1 The Collective Dimension of the Mathematical Practice

The first element is an apparently trivial one: the consideration of the practice of mathematics refers to a specific 'mathematical culture', which has a collective dimension.[5] To analyze the work of the mathematicians implies in particular to look at the representational practices they share and the criteria of validity they adopt. Surprisingly enough, contrary to what has occurred in the philosophy of natural sciences, not much has been done to understand the collaborative aspects of the mathematical enterprise. The romantic and popular image of the mathematician as genius solving problems and proving theorems in isolation from the rest of the world does not reflect the actual practices of mathematics.[6] Mathematicians do not generally work independently from each other, discovering theorems in the solitude of their room. Especially nowadays, most of them pursue their research in laboratories and are part of communities, and as a community they share a set of ideas and assumptions and aim at finding results for a common set of open problems.[7] As Kitcher has proposed, a mathematical practice is formed by a quintuple consisting of the following components: (i) a language; (ii) a set of accepted statements; (iii) a set of accepted methods of reasoning; (iv) a set of questions to find answers to; and (v) a set of meta-mathematical views (Kitcher 1984). Very recently, Ferreiros has proposed to go beyond Kitcher's rather abstract framework, and focus on the obvious fact that there is no practice without practitioners (Ferreiros 2015). In order to provide an appropriate analysis of what regulates a mathematical practice, we need to include the resources and the abilities of a single mathematician in her or his

[5]The cycle of conferences that brought to this collection of essays was precisely devoted to pinpoint such a notion.

[6]See (Lawrence this volume) in this volume for a description of such a stereotype.

[7]This communities do not have necessarily to share the same location: contemporary technology allows for communities to form even if the experts are geographically apart.

interactions with her or his peers being part of the community. As Ferreiros sums up, nothing is gained by trying to study epistemology without a community of *agents*. In a similar fashion, Thurston, a Field medalist and one of the most influential low-dimensional topologists of the 20th century and therefore a practitioner himself, explained how the language and the culture of mathematics is divided into sub-fields, and each of these sub-fields—each of these groups of mathematicians—has its own jargon, a particular collection of mathematical ideas, and consequently a particular set of problems that are considered as relevant and in need for a solution (Thurston 1994). In Thurston's view, mathematicians pertaining to the same community share a "mental model". Should this "mental model" be the object of inquiry for the philosophy of mathematical practice? We will come back to this issue in the following section.

2.2 Pre-existing Cognitive Capacities and Expertise

As Ferreiros suggests, the practice of mathematics cannot be considered without the practitioners, that is, the community of mathematicians. But then, what kind of cognitive agents are they? What cognitive processes characterize the practice of mathematics? The label 'cognitive' is used in the literature in very different contexts, often with different meanings. According to part of the literature in cognitive science, human cognition refers to a few number of separable 'core-systems' that exist in our brain and activate very spontaneously in the interaction with the world, across tasks, ages, species and human cultures.[8] One of these systems would be related to sets, and to the numerical relationships of ordering, addition and subtraction.[9] However, this view of cognition as core knowledge contrasts with another approach to cognition that aims at considering the extent to which history and culture have shaped and modulated these systems of interactions. We argue that it is only at this level that it is possible to appreciate how the different sciences have developed out of pre-existing cognitive capacities. In the same spirit of Giaquinto's work, we agree that the epistemology of mathematics has to be constrained by results of research in cognitive science and mathematics education: a practice-based philosophy of mathematics must have "interdisciplinary roots" (Giaquinto 2007, p. v). In our view, a cognitive account of complex human activities such as mathematics involving high-level reasoning as well as elaborated systems of

[8]Empirical studies would provide evidence for four of these 'core' systems and hint at a fifth one: these systems work to represent (i) inanimate objects and their mechanical interactions, (ii) agents and their goal-directed actions, (iii) sets and their numerical relationships of ordering, addition, and subtraction, (iv) places in the spatial layout and their geometrical relationships, and possibly (v) members of one's own social group in relation to members of other groups thus guiding social interactions (see (Kinzler and Spelke 2007) for reference).

[9]We align with the literature by using the term 'set', but we specify that it should be intended in an informal sense. In our opinion, 'collection' would be a more appropriate term, but cognitive scientists do not seem to differentiate between the two. We thank José Ferreiros for having pointed out this terminological problem to us.

representation, cannot neglect the role of training and expertise with the various systems of representations. If we consider the cognitive aspects in the practice, then we will focus on two elements: (i) the cognitive capacities of mathematicians that come before mathematical education and (ii) the "mental models"—to use Thurston's expression again—that mathematicians build up in their training, in their collective enterprise. The challenge is then to understand on the one hand how expertise is built out of these pre-existing cognitive abilities, and on the other hand whether these latter might still have an influence on the mathematical practice. Nonetheless, to talk about mental models might be misleading because it risks overshadowing the role of systems of representation. What Thurston means by using this expression is not that these models are 'mental' because they do not need any kind of externalizations. As he claims, mathematicians "use wide channels of communication that go far beyond formal mathematical language. They use gestures, they draw pictures and diagrams, they make sound effects and use body language." (Thurston 1994, p. 166). Only some of these externalizations are also material, and therefore easily shared, inspected and reproduced. A selection of these channels of communication become stable and get organized in systems of representation whose use is controlled by the practitioners. This is what characterizes another possible target of research for the philosophy of mathematical practice that we will discuss in the following section.

2.3 Representations in and Across the Mathematical Practice

Practitioners are cognitive agents. We defend here the idea of cognition as 'distributed': cognitive processes are to be understood in terms of the propagation and transformations of representations, and cognitive events are not necessarily encompassed by the skin or skull of an individual.[10] They may be distributed in at least three senses: (i) across the members of a social group; (ii) because the operation of the cognitive system involves the coordination between internal and external (material or environmental) structure; (iii) through time in such a way that the products of earlier events can transform the nature of later events. This brings us to another crucial target for the study of the practice of mathematics, that is, the introduction and development of systems of representation that are indispensable for the practice, such as, symbols, notations and diagrams. The reference to a specific system of representation might in fact have an influence on the development of specific mental models. Material representations are introduced in a specific practice and, once they enter into the set of the available tools, they in turn influence the practice itself: they originate from the mathematicians' mental models and at the same time play a role in shaping them. Representations are cognitive tools, whose functioning depends in part from pre-existing cognitive abilities and in part from specific training.

[10]See for reference (Hutchins 2001).

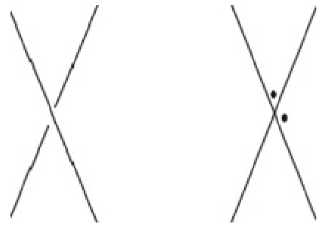

Fig. 1 Two examples of diagrams conventions displaying a crossing

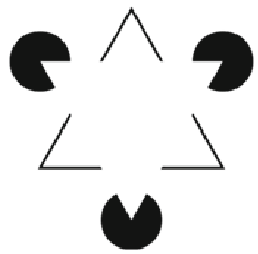

Fig. 2 Kanizsa's triangle

For example, consider two alternative notational conventions to indicate a crossing in a knot diagram, as in Fig. 1.[11] The two representations express the same information, but the first seems preferable, and in fact it is widely used while the second one is not. Why? Note that one of the segment in the first diagram is not continuous, since it breaks at the crossing and starts again after it. This break is very useful to suggest tridimensionality: the segment is *perceived* as continuous, going 'under' the other segment.[12] This interpretation is in fact consistent with the active grouping laws which have been investigated by *Gestaltpsychology* in visual perception. We have here a phenomenon that recalls the so-called Kanizsa's triangle (Kanizsa 1986), see Fig. 2.[13]

The notation thus exploits *good continuation*, one of the grouping laws belonging in Kanizsa's view to the *primary process* in visual perception, which is opposed to a more cognitive secondary process.[14] It is then thanks to grouping laws that, despite their 'skeletal nature', knot diagrams evoke the presence of the knot,

[11]The convention of indicating crossings by double points was used by early knot theorists, see for example (Alexander 1928).

[12]However, the second one has the advantage that when drawing a knot diagram, we can start with the associated planar graph and only later decide which strand goes under and which over.

[13]The figure is taken from Wikimedia Commons, the free media repository.

[14]Other grouping laws belonging to the primary process are the following: *vicinity, same attribute* (like color, shape, size or orientation), *alignment, symmetry, parallelism, convexity, closure, constant width, amodal completion, T-junctions, X-junctions, Y-junctions*. See for reference (Kanizsa 1986).

and furthermore trigger our imagination in finding ways of modifying it.[15] This example shows how we can have different presentations of the same mathematical content, and that certain of them trigger visual capacities which are available even before mathematical training. Our view is that much more philosophical work needs to be done from this perspective on the role of alternative representations and notations, a topic that has been neglected by the logic-based approaches. We will return to this issue in the discussion of our case study.

2.4 One Useful Strategy: Tracking Permissible Actions

To sum up, the objects of inquiry we propose for the philosophy of mathematical practice are the following: (1) the collective dimension of mathematical practice; (2) the cognitive capacities of the practitioners deriving from pre-existing abilities but nurtured by expertise; (3) the use of material representations.

Consider now what Jones, a Field medalist, claims about mathematicians' confidence in their results, despite the well known foundational problems: "I remember being worried by Russell's paradox as a youngster, and am still worried by it, but I hope to demonstrate […] that it is not at all difficult to live with that worry while having complete confidence in one's mathematics (Jones 1998, p. 203). Therefore, the question for the philosophy of mathematical practice is the following: if it is true that from the point of view of the practitioners the confidence on one's mathematics is not based on 'logic' or foundations, what grounds does one have for it? How can this confidence be based on the 'practice'? Our suggestion is that the collective dimension of mathematical practice plays a crucial role in controlling the *permissible actions* in a particular domain. As we will see, this brings to the definition of new (local) criteria of validity, which calls for a reformulation of our inherited notion of mathematical rigor. In Larvor's view, it is possible to interpret inferences as *actions*. If this is the case, then we do not have to consider abstract categories, the form and the content of an argument, but a list of many and various concrete objects of inferential actions: diagrams, models, expressions in special notations, and so on (Larvor 2012, p. 723). As he explains, "we can say something in the direction of explaining how informal arguments work as arguments: they are rigorous if they conform to the controls on *permissible actions* in that domain" (Larvor 2012, p. 724, emphasis added). We will adopt the notion of permissible action to define how inferential and epistemic actions in topology are controlled by the practice. Permissible actions help in defining what counts as mathematical practice, because: (i) they are accepted in a collective dimension; (ii) they rely on the cognitive abilities of the practitioners and finally (iii) they refer to the use of stable systems of representations. This notion seems thus to encompass

[15]Choosing among different possible notations is a very deep and complex matter in the practice of mathematics. In knot theory, many different notations are needed and there are no 'more natural' ones. See for reference (Brown 1999) as a starting point and our previous study on knot diagrams (De Toffoli and Giardino 2014).

the three elements of the mathematical practice that we have defined above and that are in our view of philosophical interest. To become a practitioner means to learn to operate correctly on the representations, that is, to perform the appropriate actions. In previous works we focused on the use of diagrams and pictures in particular mathematical domains—knot theory and low-dimensional topology—by analyzing their forms and epistemic roles (De Toffoli and Giardino 2014, 2015). In the present article, we introduce mathematical braids and present the proof of Alexander's theorem, a deep result connecting braids to knots. Braids have been very important for the study knots but are also theoretically interesting in themselves, since their investigation encompasses geometry, topology and abstract algebra.

3 Case Study

In order to present Alexander's theorem we first introduce some mathematical preliminary concepts. The aim is to convey the mathematical results without entering in too technical details. Then, we will present the proof, which connects two mathematical domains: knot theory and braid theory.[16] In order to take full advantage of the case study it would be useful to keep in mind the points we identified above as characterizing the philosophically relevant aspects of the practice of mathematics.

3.1 Braids and Braid Groups

Since we will be interested in connecting braids to knots, let us first briefly introduce knots.[17]

Definition 1 A *knot* is a closed simple curve in space. A *link* is a collection of knots.

In Fig. 3 you can see an example of a knot and of a link. Note that knots are a particular kind of links, i.e., links with just one component. Aligning to the typical jargon of knot theorists, we will from now on talk generally about knots to refer to both knots and links, unless the difference between knots and links is at issue.

Knots are considered up to ambient isotopies: we are not interested in the particular geometric form of a knot but on how it is knotted. An important result is that every knot (and every link) has a diagram, a two dimensional projection of it

[16] A good reference for the study of mathematical knots is (Adams 1994) and one for the study of braids is (Murasugi and Kurpita 1999).

[17] See (De Toffoli and Giardino 2014) for a philosophical discussion on knot theory and knot diagrams.

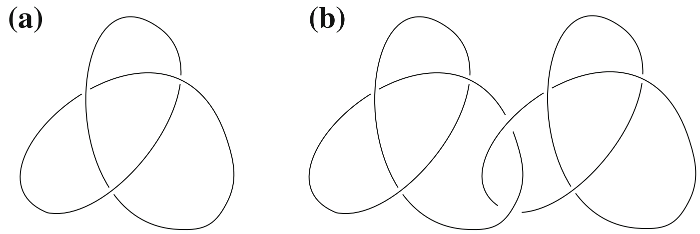

Fig. 3 Knots and links

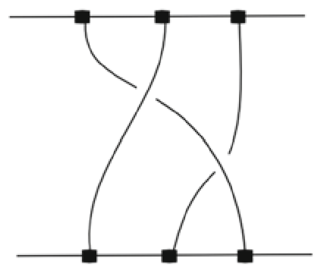

Fig. 4 A braid diagram

with some regularity constraints (examples of diagrams are in Fig. 3).[18] As mentioned before, specific conventions are used for knot diagrams, in particular broken lines suggest crossings (see Fig. 1) so as to indicate which strand goes under and which goes over. These conventions are important because they allow us to efficiently manipulate knot diagrams.[19]

Alexander's theorem is a famous result that connects mathematical knots to braids. Braids were introduced by Artin around 1930 and have been studied in relation to knots. Around 1984 Jones discovered by using braids the now famous *Jones polynomial*, a knot invariant.[20] Jones arrived at defining his polynomial for knots in a purely algebraic fashion, by studying specific algebraic structures form a statistical mechanical point of view. It is only through the presentation of braid groups (as we will define them) that he later realized the possibility of applying his results to braids and then to knots (Jones 1985).

As knots are abstractions of physical knots, braids are abstractions of physical braids made with hair or strings. We can imagine a braid as formed by n strings starting at a horizontal line and going down, maybe tangling, until they reach another horizontal line. For example, in Fig. 4 a braid with 3 strings is represented. The representation is actually a *braid diagram*, i.e., a projection of a braid in a plane with certain clarity restrictions. A braid (or knot) diagram is straightforwardly interpreted as representing a three-dimensional set of curves. In fact, as seen in the case of knots, the convention at crossings makes it intuitive which strand goes under and which over.

[18]See (Cromwell 2004, p. 52).

[19]See (De Toffoli and Giardino 2014).

[20]For this result, Jones was awarded the Field medal in 1990.

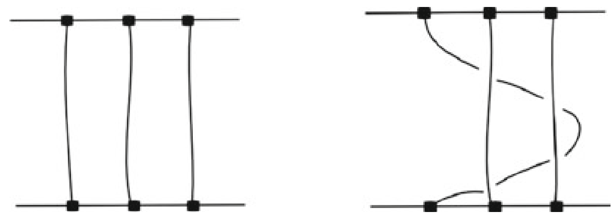

Fig. 5 Equivalent braids

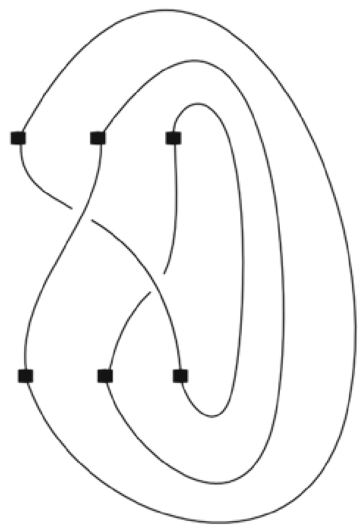

Fig. 6 The closure of a braid

Note that the strings have to go down monotonically: they never have a tangent vector that is parallel to the horizontal lines.

As for knots, braids are considered up to ambient isotopies. We are not doing metric geometry but topology: quantitative considerations are not relevant here. In order to identify a braid we just need to know how its strings are tangled together. So, for example the braids in Fig. 5 are equivalent, we say that they are the same braid.

From a braid we can consider its *closure* which will be a link: we just connect the points in the upper horizontal line with the ones in the lower horizontal line, as in Fig. 6.[21] Closing a braid as above we obtain a link with a certain number of components; only in specific cases will we obtain a knot (only if after the closure, all the strands are connected).

The main difference between knots and braids is that these latter allow for a straightforward algebraic interpretation. First of all, we can distinguish braids according to their number of strings (and thus of starting and ending points): a *n-braid* is a braid formed by *n* strings. Braids with a certain number of strings can be

[21]It is possible to close the braids in other ways so that we can obtain different knots, but this is not relevant here.

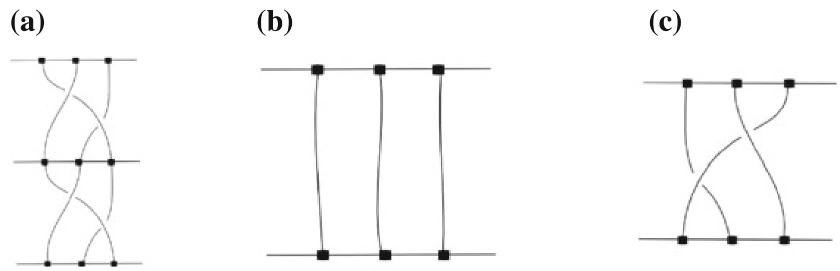

Fig. 7 The braid group

composed, i.e., we can consider to join two braids, just by identifying the end points of the first with the starting points of the second, in order to form a new one. With this *operation*, the *n*-braids form a group.

First of all, let us recall what an abstract group is.

Definition 2 A *group* $(G, \cdot)$ is a set G with an operation $\cdot$ that sends two elements $a,\ b \in G$ to their composition $a \cdot b$. The following axioms must be satisfied:

1. *Closure* $\forall\ a,\ b \in G,\ a \cdot b \in G.$
2. *Associativity* $\forall\ a,\ b,\ c \in G,\ (a \cdot b) \cdot c = a \cdot (b \cdot c).$
3. *Identity Element* $\exists\ id \in G$ such that $\forall\ a \in G,\ a \cdot id = id \cdot a = a.$
4. *Inverse Element* $\forall\ a \in G,\ \exists\ a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = id.$

So, a crucial aspect of braids, as mentioned, is that they form an algebraic structure:

Theorem 1 *The n-braids form a group, for all n. This is* B_n, *the braid group of order n.*

Intuitively, it is very easy to see that all the braids with a fixed number of strings form a group. In fact, we can take as set all the *n*-braids and as the operation the one consisting in attaching a braid on the bottom of another, as in Fig. 6. In truth, we have to slightly modify the operation defined above. In fact, we defined the composition on diagrams and not on braids, which are equivalent classes of diagrams. But it is easy to extend the operation to braids as well.[22]

In order to check that B_n is actually a group we need to check that all the group axioms are satisfied. This is readily done:

1. *Closure* The composition of two braids with *n* strings is certainly another braid with *n* strings.
2. *Associativity* From the definition of our braid operation, its associativity follows.

[22]In order to extend the operation to braids we would need to verify that by composing different diagrams of the same braid, we obtain the same braid (which is a straightforward result, which is omitted here).

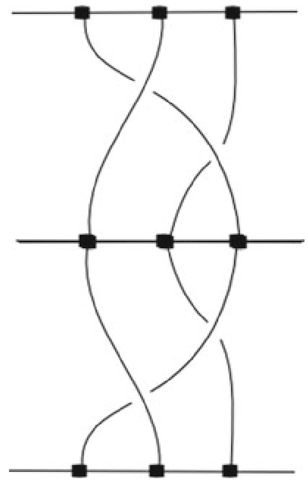

Fig. 8 The composition of an element and its inverse

3. *Identity Element* The identity, as in Fig. 7b, is just the trivial braid where all the strands are straight and untangled.
4. *Inverse Element* The inverse of a given braid is its mirror image. In Fig. 7c the inverse of the braid in Fig. 4 is represented. It is clear that combining the two we obtain the trivial braid, see Fig. 8.

The fact that B_n is a group is a deep result, in particular because it implies that each braid is an element of this group. We can present B_n with a set of generators and relations. Then, it will be possible to decompose any n-braid as a composition of the generators and their inverses. We can identify $n - 1$ generating braids: $\sigma_1, \ldots \sigma_{n-1}$, where σ_i is the braid with only a simple twist of the ith strand on the $(i + 1)$th strand. In Fig. 9 are represented σ_1 and σ_2 as generators of B_3. It is straightforward that these braids actually generate all the braid group. In fact, all braids can be decomposed into single twists. Therefore, by composing the σ_is and their inverse we can create any braid. The generators and their inverses are the atomic building blocks with which we can build any braid.

Figure 10 represents the braid in Fig. 4 as the composition of the generator σ_1 and the inverse of the generator σ_2.

So, B_n, the braid group in n strands, is generated by n-1 simple braids. Nevertheless, B_n is not a free group: some relations have to be satisfied. These are of two kinds:

Fig. 9 The two generators of B_3

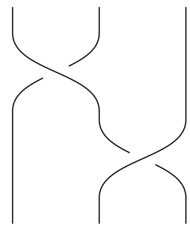

Fig. 10 The braid $\sigma_1\sigma_2^{-1}$

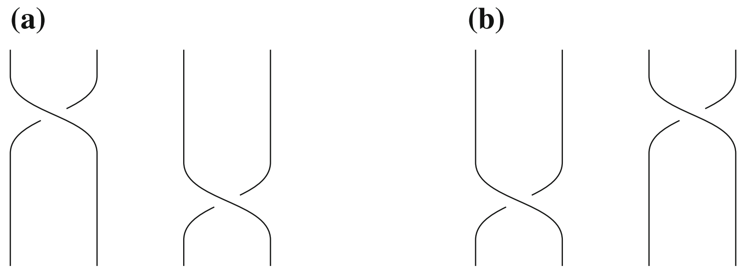

Fig. 11 Equivalent braids

1. $\sigma_i\sigma_k = \sigma_k\sigma_i$ if $|i-k| \geq 2$.
2. $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ if $1 \leq i \leq n-2$.

These relations are easily interpreted geometrically. In Fig. 11 we see that if the strands do not tangle each other, it is the same if one generating twist comes before or after another. In particular, we observe that in B_4 the following relation holds: $\sigma_1\sigma_3 = \sigma_3\sigma_1$. It is intuitive to see that the transformation that connects the diagram in Fig. 11a to the one in Fig. 11b does not alter the type of the braid, i.e., how it is tangled.

In Fig. 12 we see that $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$: in the diagram of Fig. 12a, imagine moving down the first strand and up the second in order to transform it into the diagram in Fig. 12b. Note that this kind of transformations are easily captured in a

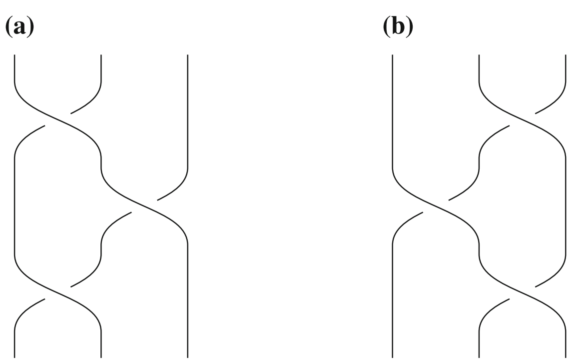

Fig. 12 Equivalent braids

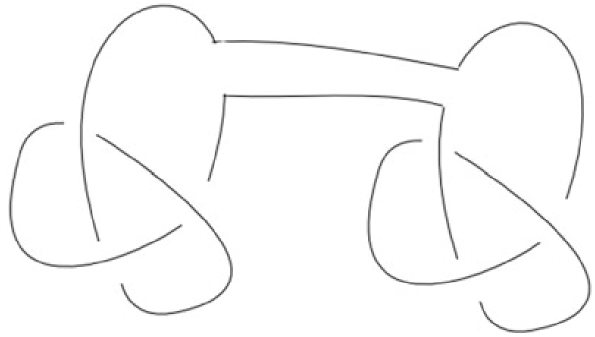

Fig. 13 Connected sum of two trefoil knots

video. In fact, in order to see that two braids are equivalent we have to imagine a continuous deformation taking one into the other.[23]

In conclusion, we have:

$$B_n = \langle \sigma_1, \ldots \sigma_{n-1} | \ \sigma_i\sigma_k = \sigma_k\sigma_i \text{ if } |i-k| \geq 2, \ \sigma_i\sigma_{i+1}\sigma_i \\ = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ if } 1 \leq i \leq n-2 \rangle.$$

It is clear that this algebraic treatment of braids opens the door to a series of results. We can do algebra and get results on braids. For example, the question whether two braids are equivalent is translated into the algebraic language as whether two *words*, i.e., two formal expressions on the generators and their inverses, represent the same element in a group. This does not mean that the latter question is easier, but we are offered another possible way to look for an answer. Note that for knots, nothing of this sort is given. Intuitively, a knot diagram is more chaotic than a braid diagram. In fact, it is hard to give a syntactic description of knots and this will be unavoidably dependent on arbitrary decisions (for example, if we want to *decode* a knot diagram we have to choose a starting point). That is one of the reasons for the importance of the theorem that we will present in the following section.

Moreover, as we have seen, braids form a group with the composition of braids as operation. For knots, this is not the case. We can still define an operation on knots: *connected sum*. This operation allows us to join two knots together as in Fig. 13, but it does not have an inverse.[24] With the connected sum, knots form a *monoid*, which is a 'poorer' structure compared to the one formed by a group.

3.2 Alexander's Theorem

We introduce now Alexander's theorem and give a proof that follows the original one, which can be found in (Alexander 1923). In order to make the proof more

[23] A video would be very effective to show this isotopy. In the discussion, we will assess the informative value of videos for mathematics and for topology.

[24] See (Lickorish 1997, Chap. 2) for reference.

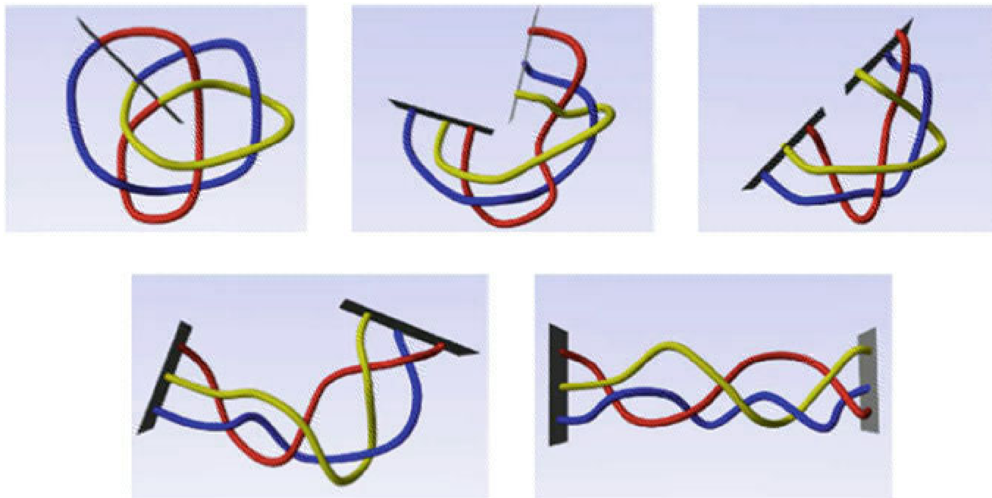

Fig. 14 Opening a knot

accessible we have introduced some illustrations. Moreover, we complete Alexander's proof with the visual strategy developed by Dalvit (2011). Some images are still frames taken from her video about braids (Dalvit 2013). Note that the original proof contains no pictures. We will come back to this issue in the discussion.

Theorem 2 (Alexander) *Every knot can be represented (not uniquely) as a closed braid.*

Proof First we start with a knot K. The original proof deals more generally with links, but the gist of it remains if we consider only links of one component, i.e., knots. Then we assume that K is well-behaved, it has a finite diagram, that is, we exclude pathological cases of *wild* links, i.e., links whose diagrams would have an infinite number of crossings (this is a standard procedure.)

We want to prove that K is ambient isotopic, i.e., equivalent, to a closed braid B. Remember that in this context ambient isotopic knots are considered equal.

A crucial passage in the proof consists in noticing that B can be described as a knot such that there exists an axis around which the knot always goes in the same direction (clockwise or anticlockwise) (see for example Fig. 15a). More generally, if such an axis exists for an arbitrary knot, then we can consider a half plane with the axis as boundary and intersecting the knot in just n points. Afterwards, cutting along the plane we form a braid, as in the sequence of figures displayed in Fig. 14.[25] This shows that given a knot, if we find such an axis, then it is possible to transform it to a braid without changing its *knot type*, i.e., how it is knotted.

[25]We thank Ester Dalvit for having given us permission to reproduce the images in Figs. 14 and 17 from (Dalvit 2013).

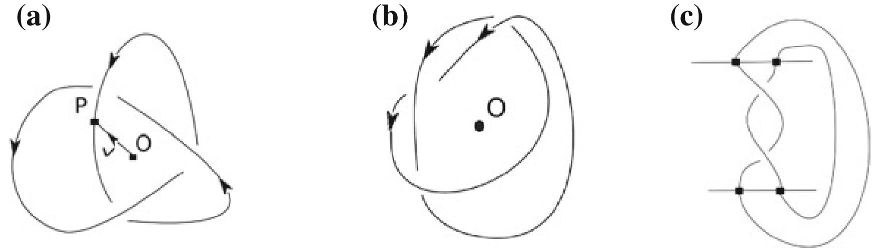

Fig. 15 The trefoil knot as a 2-braid

Therefore, we need to prove that such an axis exists for any knot. In fact, given such an axis, we can always put the knot in the form of a closed braid, as in Fig. 15.

Now, consider the problem at a diagrammatic level. Let $\mathscr{D}_K$ be a diagram for K and $\mathscr{D}_B$ be a diagram for B. We want to show that they are equivalent, i.e., there exists a series of diagrammatic transformations that do not alter the underlying knot type and that convert one into the other—in Alexander's words, these are "*legitimate operations*" (Alexander 1923, p. 94, emphasis added).

Choose a point O in the plane of $\mathscr{D}_K$, so that O is not collinear with any segment of $\mathscr{D}_K$ (we can consider the diagram $\mathscr{D}_K$ to be formed by little straight segments, i.e., piece-wise linear). This point is the projection of the axis that we defined above. Consider now another point P moving on $\mathscr{D}_K$ and the vector $v = \overrightarrow{OP}$ (see Fig. 14). When P goes through it, v will turn sometimes in one direction around O and sometimes around the other direction. If we transform $\mathscr{D}_K$ so that P will turn only around one direction, then we are done.

So, let us consider a portion s of $\mathscr{D}_K$, not containing more than a crossing, that goes in the opposite direction. Let us call A and B the endpoint of s, then we can choose a point C such that the point O lays inside the triangle ABC. Now replace AB with the two segments AC and CB (of course keeping the crossings information). Using this move, we transform all the portions of the knot going in the wrong direction by "throwing them over one's shoulder" (Jones 1998, p. 211). We can imagine a similar move on smooth curves, and not straight segments.[26] Basically, we have to identify a portion of the knot that is turning in the wrong way and throw it to the other side of the axis. For example, in Fig. 16 is depicted a diagram that has just one piece going in the wrong direction.[27] After this move, all the portions of the diagram in Fig. 16 go in the same direction around the point O.

Similar moves are better visualized through a video. In fact, we can isolate the portions of the knot that turn in the wrong direction and modify it continuously so that they will turn in the right direction. In Fig. 17 you can see some still shots from the video *Braids. A movie.* by Dalvit (2013): some portions of the knot are turning in the wrong direction. Intuitively, the move consists in replacing a portion of the

[26]It is a deep result that for knot theory working on the category of smooth curves is equivalent to working in the *PL* category of piece-wise linear segments.

[27]This example is taken from (Jones 1998, p. 211) with some modifications.

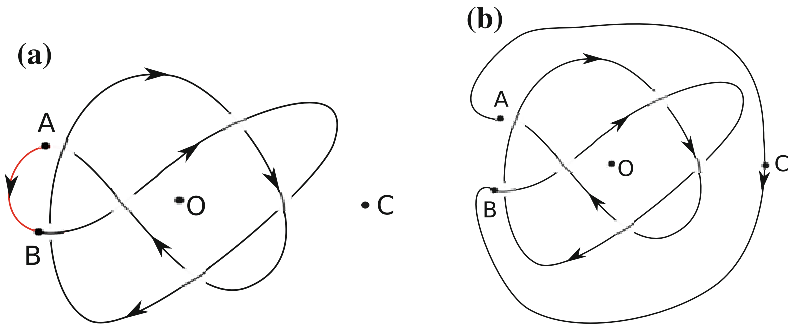

Fig. 16 Alexander's move

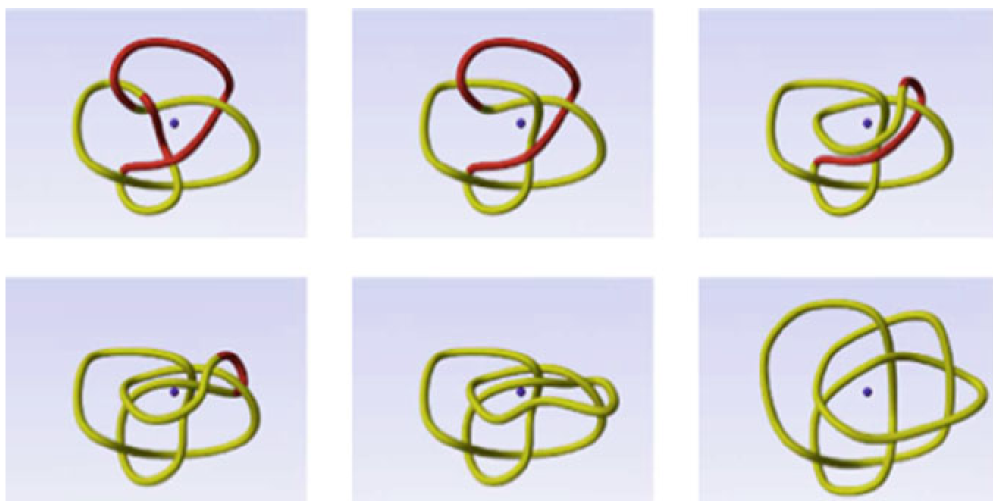

Fig. 17 Transforming the knot

knot that goes in the opposite direction by throwing it in the other side of the point O so that it goes in the right direction. This has to be done carefully, without introducing new entanglements.

In Alexander's words: "the transformation of $\mathscr{D}_K$ *obviously* corresponds to an isotopic transformation of the space curve L" (Alexander 1923, p. 94, emphasis added and notation changed). Repeating the process, we can eliminate each segment that was going in the wrong direction. At one point we reach a projection with the desired property. Therefore the diagram of our knot K is transformed in a diagram of a braid B. Note that it is quite straightforward to extend this result to a link L. In fact, we do the same procedure for each of the components of L, and of course we make them all turn around the axis in the same direction. QED.

It is easy to check that starting with two different knots, we obtain different braids. In fact, the main point in Alexander's proof is to apply transformations that do not alter the knot type. Nevertheless, different braids can give the same knot.

Another important theorem due to Markov[28] defines moves on braids which tell exactly which braids give the same knot. Alexander's result becomes even stronger in the light of Markov's theorem. In fact, joining these two theorems not only do we know how to transform any knot in a braid but also how to 'move' exactly from knots to braids and back.

4 Discussion of the Case Study

Following Alexander's proof, many issues arise. We will apply the methodological guidelines we outlined in Sect. 2 to address the following questions in relation to our case study. What is the role of the community of practitioners in defining the criteria of validity for the proof? How is rigor assured? We might also ask what cognitive abilities contribute to the efficacy of the informal setting of the proof, in particular in relation to the specific role assumed by visualization. As mentioned before, the original proof published by Alexander does not present any picture but at the same time it is considered a 'visual' proof. Why is it so? Moreover, the material pictures play a crucial 'role, but one objection could be that experts are able to exploit their visualization capacities without drawing them. What is then the role of material specific representations? How much do topologists rely on them to reason about the topological objects? In the following sections, we will discuss these issues.

4.1 Revising the Criteria of Validity

In the first part of the article, we addressed the first of Larvor's worries and proposed a methodological framework with three possible targets for the philosophy of mathematical practice. At this point, we have to address the second of his worries and assess how the study of a specific mathematical practice can provide new insights in the practice of proving that would go against the standard notion of proof as inherited from the logic-based approach to mathematics. It is easy to see that the practice of topology presents many examples of proofs that cannot easily be reconciled with the notion of proof as a syntactic object constituted by a sequence of sentences. For example, consider Alexander's proof and in particular the way one transforms a link into a braid. This transformation requires envisioning modifications on the representations used (or imagined). In Alexander's words, these transformations "obviously" correspond to isotopic transformations: as part of the proving process, instead of taking into account a sequence of sentences, one has to envisage a sequence of continuous transformations. Therefore, the reasoning involved in this specific proof cannot be identified exclusively with propositional reasoning, and even less with formal reasoning.

[28]See (Murasugi and Kurpita 1999, Chap. 9).

It could be suggested that if this proof is not based on formal reasoning, then it is based on visualization. In fact, the proof is recognized as 'visual': the topologist needs to 'see' the transformations in order to understand it. We partly agree with this claim, but we find it potentially misleading. In our view, the appeal to vision is not enough to characterize Alexander's proof in relation to standard proofs in the logic-based approach. In fact, the relevant reasoning is rather based on some form of imagination that exploits pre-existing cognitive capacities related to the manipulation of concrete spatio-temporal objects and is enhanced by mathematical expertise. In cases such as this, topologists are required to imagine a series of possible transformations on the relevant representations. The rules for such manipulations are given by the appropriate interpretation. For example, in the case of Alexander's proof, a correct interpretation of knot diagrams will allow practitioners to manipulate these diagrams in order to find an axis around which the knot goes in the same direction, that is, to perform the appropriate continuous transformations. In particular, it will be possible to operate the "throw over the shoulder" trick, as described above. The practitioners share this form of reasoning and teach it to students. Of course, it is epistemologically relevant to envision transformations on the representations, since these transformations can count as reasons for reaching new valid conclusions. The "mental model" Thurston refers to should be understood in such a context, as familiarity with transforming the material representations and at the same time as control on the mathematical meaning and the inferential weight of each of these "legitimate operations", to use Alexander's term.

In our reconstruction, Alexander's proof is characterized by the the following features. First, it counts as *justification*. Of course, as in any other proof, not all passages have to be justified. As mentioned in the previous sections, the community to which the proof is addressed shares some background knowledge concerning the use of the available systems of representation. Moreover, standards of justifications are assumed as well: the community defines the 'permissible actions' on the representations. We will go back to this issue in the next section. Second, in order to follow the proof, mathematicians envisage transformations of and on the diagrams. Their interaction with the representations is essential: the figures are not static, they have to be used dynamically so as to trigger a form of imagination that allows mathematicians to draw inferences. Elsewhere, we have defined this cognitive capacity *manipulative imagination* (De Toffoli and Giardino 2014). This form of imagination is based on a widening of our spatial perception together with our physical intuitions of space, but needs to be trained in the specific collective practice. For Alexander's theorem, the crux of the proof consists in identifying the right transformations that allow us to find an axis around which the knot turns in just one direction. It is only through visualization that we know that this transformation gives us an isotopic knot, and it is left to our intuition to prove that this transformation is always possible and that it is not an infinite process. Alexander does not really gives us any other justification: this reasoning plays an epistemic role. His proof is not an isolated case in topology, so we conjecture that other proofs in the same sub-field or neighboring ones can be characterized by similar features.

4.2 Operating (Legitimately) on the Notation

Let us now focus on what we mean here by claiming that the key to the proof is to envisage transformations on some material (or mental, if the topologist is trained enough) representations. Jones (1998) compares Alexander's theorem to a very technical and abstract result in algebra, von Neumann's density theorem. Alexander's theorem (Theorem *A*) is accessible to a non-technical audience; on the contrary, von Neumann's theorem (Theorem *vN*) requires a substantive technical background even only to understand its claim. According to Jones, this contrast is due to the inevitably different role of formalism in each of the two theorems. In his description, a careful analysis of these two proofs reveals that the proof of Theorem *A*, if properly formalized, would be much longer than that of Theorem *vN*. This is because one would have to be precise about the kinds of continuous deformations allowed, and to construct the functions required for the transformation described by Alexander and exemplified in Fig. 16 would be a hard task. Nonetheless, Theorem *A* is easier because it concerns a very concrete situation, and we can rely on our full intuition about three-dimensional space. This claim is crucial and in line with our discussion. The method of proving consisting in manipulating representations in order to learn something new about the topological objects clashes with the standard notion of mathematical argument: we are not used to think about mathematics in this way. Nonetheless, the proof is accepted as valid by the community of practitioners. The practice of proving that is common in this branch of mathematics is in fact quite distant from almost that of every other area because it largely relies on 'seeing' topological objects, which amounts to envisaging transformations on the representations available for them. In this framework, to convey such visualizations counts already as justification.

But if this is true, then what kind of informal arguments count as reasons and what kind of proofs are accepted? As suggested in the previous section, and by adopting the methodological framework presented above, we will focus on what Alexander defines as "legitimate operations" (Alexander 1923, p. 94). This notion is close to that of Larvor's "permissible actions". The main point is that it is necessary to identify for each practice the inferential steps in the relevant arguments. In the case of the practice of proving in which Alexander's proof is embedded, the inferential steps are made by manipulating the representations. More generally, they are epistemic actions performed or imagined on the available representations. Of course, this is done under the constant control of the practitioners. The legitimate operations are parts of their mental model, and can be considered as reliable to gain new knowledge about the object of research. Moreover, this leads us to consider the representations used as a very peculiar sort of notation, which allows performing permissible actions.

To illustrate this point, let us consider again the video *Braids* (Dalvit 2013), which has been recently produced to allow understanding Alexander's theorem and, more generally, some basic notions about braids. We have used some still images from it in presenting Alexander's proof. Videos for other kinds of mathematical practices would risk obscuring the relevant passages in the proof. Think for example of a proof in Euclidean geometry: one should understand how to construct a figure from the previous one, and a sequences of figures would probably serve this purpose better.[29] In contrast, in the case of Alexander's theorem and more generally of low-dimensional topology, videos can be very informative and effective, precisely because they easily convey continuous transformations.[30]

Nonetheless, by claiming that it is crucial to envisage transformations on the available material representations, we do not have to mistaken the material figures for the imagination process. Actual pictures trigger our imagination and help us see modifications on them, but for the people who are already acquainted with a practice on pictures of a certain type (e.g., links or braids) it is perhaps not necessary any more to actually draw all the pictures. As previously mentioned, the original proof by Alexander did not contain any single figure (Alexander 1923). For the experts, what matters is the spatial configurations that are displayed by the figures and not their appearances. This is not in contrast with our interpretation of the proving process in Alexander's proof. Any trained topologist reading it would find no difficulties in imagining the appropriate representations and envisaging the required transformations on them. When a mental model is stable, there is no need to draw explicitly all the figures that are part of it—as well as there is no need to make all the background (propositional) knowledge explicit.

4.3 Moving from One Representation to Another

Another important aspect of Alexander's theorem is that it allows interpreting the same mathematical information in different contexts. In most cases, the 'translation' from one mathematical representation to another coming from a different field enhances our knowledge of the mathematical subject. Mathematicians indeed move between various systems of representation and various notations, that is, between different 'mathematical languages'. As we know, Alexander's theorem shows that every knot can be put in braid form. This is a strong result because, as we have seen, braids allow for a straightforward algebraic treatment. For example, as we have already mentioned, the introduction of Jones polynomials has developed out of this 'translation' of knots into braids.

Other similar cases can be given. Carter has recently shown the interest of working on a kind of 'semiotics' of mathematics, by relying on the work of Peirce (Carter 2010). She takes into account the role of diagrams in the practice of proving in free probability theory. In her case study, she proposes to consider diagrams as "iconic", because they display properties that can be used to formulate algebraic analogues. Thanks to the diagrams, a practitioner is allowed to go from them to an

[29]There are actually exceptions, but we have no time to discuss them here.

[30]Another example is Sullivan, Francis, and Levy's video *The Optiverse* (Sullivan et al. 1998). Through the video, one can concretely see a sphere eversion that is geometrically optimal in the sense that it minimizes the elastic bending energy (see for reference Sullivan (1999)).

algebraic description and back. This move allows the practitioners to make experiments on the available diagrams and then calculate algebraically their results. It is interesting to notice that in Carter's case as well the diagrams do not appear in the 'official' published version of the article, despite their crucial contribution. Nevertheless, in Carter's case, the diagrams are not part of the 'mental model' of a mathematician working in free probability theory. In particular, they are visual tools which are contributing in suggesting definitions and proof strategies and they function as "frameworks" in parts of the proofs (Carter 2010). On the contrary, the diagrams in Alexander's proof are part and parcel of the reasoning and they are indispensable for understanding.

Starikova has analyzed another case study from contemporary mathematics from a relatively recent mathematical subject: geometric group theory. She discusses how the representation of groups by using Cayley graphs made it possible to discover new geometric properties of groups (Starikova 2010, 2012). In her case study, groups are represented as graphs. Thanks to the consideration of the graphs as metric spaces, many geometric properties of groups are revealed. As a result, it is shown that many combinatorial problems can be solved through the application of geometry and topology to the graphs and by their means to groups.

It is helpful here to refer to an unpublished paper of Manders that is also behind Starikova's analysis (Manders et al. 1999). In this article, Manders takes into account the contribution of Descartes' *Géométrie* compared to Euclid's plane geometry. He gives particular stress to the introduction of the algebraic notation. More generally, in his view, the practitioners often produce and respond to artifacts, which can be of different sorts: natural language expressions, Euclidean diagrams, and algebraic or logical formulas. Mathematical practice can thus be defined as the control of the "selective responses" to given information, where response is meant to be "emphasizing" some properties of an object while "neglecting" others. According to Manders, artifacts help implementing and controlling these selective responses, and therefore their analysis is crucial if the target is the practice of mathematics in question. Moreover, selective responses are often applied from other domains. Think of the introduction of algebraic notation to apply algebraic methods to geometry. In Descartes' geometry, geometric problems are solved through solving algebraic equations, which represent the geometric curves. Also in this case, the idea is that by using different representations of the same content, new properties might be appreciated.

The potential advantage of moving from one representation to another clarifies the importance of notation, which is a crucial feature characterizing the mathematical practice and deserves philosophical attention. In his recent introduction to philosophy of mathematics, Colyvan devotes one whole chapter to notation, and explains that we could think of mathematics as a language (Colyvan 2012); if we do that, then we easily realize that "good notation is far from trivial" (Colyvan 2012, p. 156). Colyvan criticizes the standard approaches by claiming that one cannot dismiss the idea that notation can help to reveal unknown mathematical facts. In his view, discovery can be notation-driven. We are inclined to agree with Colyvan and, as mentioned in the previous section, to consider notation in a very broad sense. In our view, we can think of the diagrams in free probability theory as well as of Caley graphs as particular notations, once the legitimate operations that can be applied on them are taken into account. These diagrams are of course not simply useful sketches, but the very elements of a system of representations in which manipulations rules are more or less explicitly defined. Moving from one notation to another for the same mathematical content is indeed a good strategy to discover new relations. The effectiveness of applying such a strategy reveals the richness of mathematics as “the theory of formal patterns”, as Thurston among others has proposed to define it (Thurston 1994, p. 162). Mathematicians are in a constant negotiation between the introduction of a specific notation and the definition of their abstract objects, some properties of which do not seem to emerge before a good notation for them is introduced.

5 Conclusions

In this article, we have proposed to consider the philosophy of mathematical practice as an inquiry concerning the community of mathematicians as cognitive agents who share specific systems of representations on which operations (permissible actions) are performed. In our case study, we showed how the actual practice of proving in braid theory can involve a form of reasoning that cannot be reduced to formal statements without completely altering the proof. Reasoning in this field is based on pre-existing cognitive capacities—mathematicians imagine a series of possible manipulations on the representations they use—and is modulated by expertise. This form of reasoning is shared by the experts: it is the kind of reasoning that one has to master to become a practitioner. Moreover, the actions allowed on the representations—what Alexander calls “legitimate operations”—as well as the representations themselves, are epistemologically relevant, since they are integral parts both of the reasoning and of the justification provided. This is in line with the idea that cognition is distributed among the practitioners of a sub-field and that there is a constant feedback between their mental models and the representations they use. Moreover, this example shows how the interplay between different disciplines—knot theory, braid theory and algebra—through the consideration of the relations between alternative systems of representation enhances understanding and help to clarify the mathematical meaning. We are moving in a framework consistent with Kitcher’s naturalism, which considers also the role of the human cognitive agents and the artifacts they produce, as recommended by Ferreiros.

Our suggestion is that such a framework, based on the consideration of the permissible actions, can be applied to other areas of mathematics and other practices of proving as well. We hypothesize that the ability to envision transformations on the representations can be recognized as a characteristic feature of other and even more algebraic areas of mathematics. Consider for example dealing with an algebraic equation. It would certainly be possible to envisage some permissible actions on it as well, for example by taking a part of the right-hand side of the equation to the left-hand side, thus applying appropriate and legitimate operations. Our proposal for further research is to explore other proving practices in such a dynamic framework for mathematical inference and proof, with the aim of identifying analogies and differences. Of course, this would bring us far from the logic-based approach to mathematics, which is not concerned with mathematics as it is actually done by experts, but with possible axiomatizations or rational reconstructions for it.

Logic is not the unique core of mathematics and other systems of representations —such as the ones based on figures—are not only heuristically relevant but can have an epistemic role. Therefore, they deserve philosophical attention. One consequence is that mathematical rigor will be achieved via different criteria of validity and not through an universal logic-based criterion. To be able to appreciate these different local criteria of validity it is then necessary to consider many different practices, each having its own dynamics, with the risk of ending up with an explosion of studies. As Larvor suggests for informal proofs, it is possible that the list of objects of inferential actions is very long and very varied. However, this inversion of route should not be perceived as a methodological limit: on the contrary, the bright side is that the philosophy of mathematical practice would aim at adhering as much as possible to the real observable practices and would work towards the appreciation of the astonishing richness of forms that the mathematical practice can assume. The philosophy of mathematical practice can thus lead to new and mostly unexplored territories of research.

Acknowledgments We want to thank in particular Brendan Larvor for the organization of the cycle of conferences on Mathematical Cultures. A previous version of this paper was presented at the first meeting in 2010. We thank the participants for useful feedback. Thanks to John M. Sullivan for his comments. We also thank Ester Dalvit for having given us permission to reproduce part of her work and for her useful feedback.

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