Erkenn (2014) 79:829–842
DOI 10.1007/s10670-013-9568-7

ORIGINAL ARTICLE

Forms and Roles of Diagrams in Knot Theory

Silvia De Toffoli · Valeria Giardino

Received: 20 November 2012 / Accepted: 19 October 2013 / Published online: 1 November 2013

Abstract The aim of this article is to explain why knot diagrams are an effective notation in topology. Their cognitive features and epistemic roles will be assessed. First, it will be argued that different interpretations of a figure give rise to different diagrams and as a consequence various levels of representation for knots will be identified. Second, it will be shown that knot diagrams are *dynamic* by pointing at the moves which are commonly applied to them. For this reason, experts must develop a specific form of enhanced *manipulative* imagination, in order to draw inferences from knot diagrams by performing *epistemic* actions. Moreover, it will be argued that knot diagrams not only can promote discovery, but also provide evidence. This case study is an experimentation ground to evaluate the role of space and action in making inferences by reasoning diagrammatically.

1 Introduction

In recent years, an interest has been growing among scholars towards the practice of mathematics, with the aim of explaining its internal mechanisms and methodologies. As Mancosu (2008, p. 2) summarizes, "the epistemology of mathematics needs to be extended well beyond its present confines to address epistemological issues having to do with fruitfulness, evidence, visualization, diagrammatic reasoning, understanding, explanation and other aspects of mathematical epistemology which are orthogonal to the problem of access to 'abstract objects'."

S. De Toffoli (✉)
Institut für Mathematik, Technische Universität Berlin, Straße des 17. Juni 136,
10623 Berlin, Germany
e-mail: toffoli@math.tu-berlin.de

V. Giardino
Institut Jean Nicod, CNRS-EHESS-ENS, Pavillon Jardin, Ecole Normale Supérieure,
29, rue d'Ulm, Paris 75005, France
e-mail: Valeria.Giardino@ens.fr

In this article, we aim at identifying and discussing the role of diagrammatic reasoning in one particular branch of mathematics: knot theory. As Mancosu continues, “Certain philosophical problems become salient only when the appropriate area of mathematics is taken into consideration.” Knot theory seems to us particularly well-suited to provide epistemological insights into one mathematical practice involving diagrams.

Knot theory is very rich in heterogeneous calculations and symbols, and for this reason it is an interesting case study to appreciate the productive role of notations. As Colyvan (2012, Ch. 8) has recently pointed out, developing an effective notation is crucial in the practice of mathematics: Notation has a role in advancing mathematics and contributes to mathematical understanding. Brown (1999, Ch. 6) claims that knot diagrams have computational power and therefore are a particularly good and effective notation. In accordance to this claim, we aim at unfolding the conditions for this computational power and more generally for knot diagrams’ effectiveness in prompting inference. First, we will draw attention to the procedures that knot diagrams make available. Muntersbjorn (2003, p. 167) has suggested that one of the principal heuristics governing the growth of mathematics is to make the implicit explicit, which most of the times means to make explicit “not premises, or propositions about mathematical objects, but rather procedures, or ways of engaging mathematical objects.” Following this line of thought, we will propose a framework in which a diagram is dynamic in the sense that it is related to procedures and possible moves. Second, we will argue that knot diagrams have at the same time diagrammatic and symbolic elements, and therefore the traditional dichotomy between visual and linguistic reasoning cannot capture them. This would help define a “more discriminating and more comprehensive” taxonomy for mathematical thinking, going beyond twofold divisions (Giaquinto 2007, p. 260).

In order to set up the framework we propose the following terminology. By *figure*, we mean a physical object, for instance drawn on a piece of paper or shown on a computer screen. A figure *per se* does not have a meaning; in order to become meaningful, it has to be considered inside a particular context of use, and therefore *interpreted* in such a context. It is only when the intention behind the figure is recognized that the figure is seen as a *representation* and as a consequence it becomes an illustration or a diagram. By *illustration*, we mean a *static* representation, which can be useful by conveying information in a single display, but where modifications are not well-defined. By *diagram*, we mean a *dynamic* representation, on which we can perform moves that can count as inferential procedures. Diagrams are dynamic inferential tools that are modified and reproduced by the experts for various epistemic purposes. They do not only represent strategies to solve problems but also give evidence for their solutions. We propose an *operational* account for knot diagrams, based on: (i) the *moves* allowed on them and (ii) the *space* they define.

We argue that these definitions are not stipulative, but in line with the jargon typical of this specific practice. Experts have acquired a form of imagination that prompt them to re-draw diagrams and calculate with them, performing *epistemic* actions (Kirsh and Maglio 1994).[1] This imagination derives from our interaction

[1] Kirsh and Maglio distinguish between *pragmatic* actions, i.e.“actions performed to bring one physically closer to a goal”, and *epistemic* actions, i.e. “actions performed to uncover information that is hidden or hard to compute mentally”, by examining their role in *Tetris*, a real-time, interactive video game.

with concrete objects and our familiarity with manipulating them. In topology, which is informally referred to as 'rubber-band geometry', a practitioner develops the ability to imagine continuous deformations. Manipulations of topological objects are guided by the consideration of concrete manipulations that would be performed on rubber or other deformable material.[2]

In Sect. 2, we will introduce knot theory and identify different representational 'levels' for knots, with various forms and epistemic roles. In Sect. 3 we will consider knot diagrams 'in action', by presenting examples of diagrammatically defined invariants. In Sect. 4 we will discuss our case study in the light of our considerations about diagrammatic reasoning and notations. In Sect. 5 we will sum up possible conclusions and hint at future directions of research.

2 Knot Theory and Its Representations

2.1 Mathematical Knots in a Nutshell

We introduce mathematical knots preserving a rigorous presentation without entering into details not essential to our analysis.[3]

Definition 1 A *knot* is a smooth closed simple curve in the Euclidean 3-dimensional space.[4]

We can think of mathematical knots as abstractions of physical knots: They have no thickness, since a curve has just one dimension, and they are closed—the ends of the curve are glued together. A knot is not only a curve, but a curve *in space*, i.e. an *embedding* of a circle in $\mathbb{R}^3$. From an *intrinsic* point of view, any knot is topologically equivalent to a circle, however from an *extrinsic* point of view knots may be different from each other in the sense that it may be impossible to unravel a knot in $\mathbb{R}^3$ without cutting and gluing so as to form a circle or a different knot. Definition 1 is not enough to determine which knots are *equivalent*.

Definition 2 Two knots are *equivalent* if there is an ambient isotopy[5] transforming one into the other. A *knot type* is a class of equivalent knots.

[2] We are developing an account of the peculiarities of the practice of low-dimensional topology with particular focus on proving, using different kind of diagrams and visual material in general (De Toffoli and Giardino, forthcoming).

[3] See Adams (1994) or Lickorish (1997) for introductory manuals.

[4] The vast majority of knot theory only deals with *tame* knots, i.e. knots that admit a diagram with only a finite number of intersection points. This restriction is meant to ban so-called *wild* knots, which are "monsters" in Lakatos' terminology. Every tame knot is equivalent to a smooth knot, that is why it is common to consider only smooth knots or other equivalent categories. A simple curve is a curve without self-intersections.

[5] Two knots K_1 and K_2 are *ambient isotopic* if there exists a continuous map: $h : \mathbb{R}^3 \times [0, 1] \to \mathbb{R}^3$, with $h_t(x) := h(x, t)$, such that (i) $h_t : \mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism for all t, (ii) $h_0 = \text{id}$ and (iii) $h_1(K_1) = K_2$. Ambient isotopies model the deformations on knots that we can perform without cutting and then pasting the two cut ends.

This equivalence relation is very intuitive: It captures the intention of describing the ‘knottedness of knots’ and not their particular geometric shape, by focusing on topological properties that remain unchanged under deformations like stretching them or moving them around. In knot theory the term ‘knot’ is often interchanged with ‘knot type’. Even if this causes no confusion to the working mathematician (Lickorish 1997, p. 2), our assumption is that it is philosophically relevant to unveil conceptual differences by disambiguating terms as much as possible.

2.2 From Illustrations to Diagrams

The main problem in knot theory is to recognize whether two knots are equivalent. This can be a challenge even for the simplest knot type, called the *unknot*—that is, as the name suggests, a not knotted knot type.

Figure 1 shows three two-dimensional figures which are easily interpreted as three-dimensional objects thanks to the fact that the curves are represented as having thickness and being sensitive to light. In the proposed framework, they are *illustrations* of knots. We can think of illustrations as *pictures* of knots, taken from different points of view, some of which give rise to chaotic representations. For this reason, despite following some general standards—such as to suggest three-dimensionality—and being a first informal access to knots, illustrations are not sufficiently constrained and are in some cases mathematically useless. With the exception of Fig. 1a that clearly represents the unknot, it is difficult to recognize whether this is also the case for the other two. Consider Fig. 1b. One way to determine whether the underlying knot is knotted or not would be to take a rope and give it the form depicted; then, after gluing its ends, check if it remains knotted by moving it around. Nonetheless, if we consider Fig. 1c, even this naive strategy cannot be undertaken. In fact, this illustration hides the relevant information.

To provide more controlled representations, *knot diagrams* are introduced. In order to draw a diagram for a knot, we project it on a plane, by keeping track, for each crossing in the projection, of which arc goes over and which under. Not all projection directions are allowed, since the projection must be *regular*, i.e. (i) the intersection points are in finite number, (ii) they are transversal and (iii) no more than two arcs meet at a time. These requirements prevent the emergence of messy representations. For example, Fig. 1c cannot correspond to a diagram—even if we

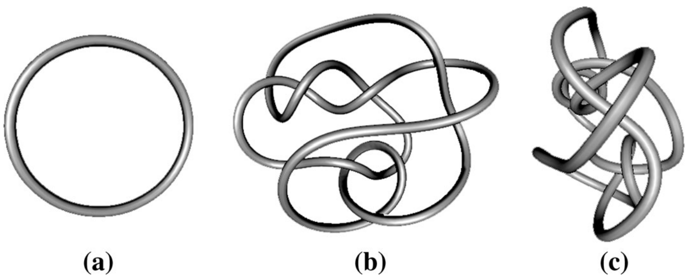


Fig. 1 Illustrations of the unknot

remove thickness and add the appropriate interruptions at crossings—as it violates the conditions that intersections are transversal and only two arcs meets at a given intersection point.

Definition 3 A *knot diagram* is a smooth regular projection of a knot onto a plane with relative height information at the intersection points.

Definition 4 A knot diagram D *represents a knot* K if and only if D is a regular projection of K. A knot diagram D *represents a knot type* $\mathcal{K}$ if and only if there exists a knot K of type $\mathcal{K}$ such that D represents K.

For every knot, and therefore for every knot type, there exists a diagram representing it (Cromwell 2004, Theorem 3.3.2). However, the same knot type admits different diagrams: This is due to the choice of (i) one particular representative knot from the equivalence class and (ii) a direction of projection. Moreover, because of (ii), the same knot admits different diagrams. Conversely, two different knots of the same knot type may be represented by the same diagram due to the fact that by projecting we lose information.

Knot diagrams are in a sense *privileged* points of view on knots and knot types: They display only a certain number of properties by selecting the relevant ones. If illustrations are analogous to pictures, diagrams are like *maps*. In order to draw a map, it is required to define conventions that would make it legible. Nevertheless, these conventions are not completely arbitrary: In the case of knot diagrams they are intended to suggest three-dimensionality. We interpret knot diagrams as 'almost' entirely on a plane: Despite their being two-dimensional, the interruptions of the segments, as for example in the diagrams of Fig. 2, are seen as occlusions. This is extremely useful because it allows us to treat these diagrams as quasi-concrete objects: We imagine possible transformations on them that would leave the corresponding knot type unchanged. It is this interpretation that guides us among the diagrams representing equivalent knots.

Unlike illustrations, knot diagrams have the twofold role of representing *and* being mathematical objects in themselves: They are not only representations of knot and knot types, but also images of functions $p(K)$ with relative height information at the intersection points, where p is a regular projection onto a plane and K is a knot. Because they are representations, we can modify them, leaving the knot type unchanged; because they are mathematically defined objects, we can operate mathematically and calculate with them and draw conclusions in a reliable way.

Theoretically, a single diagram is sufficient to infer all the properties of a knot type, since it determines it univocally. Notwithstanding, in the practice it is not possible to

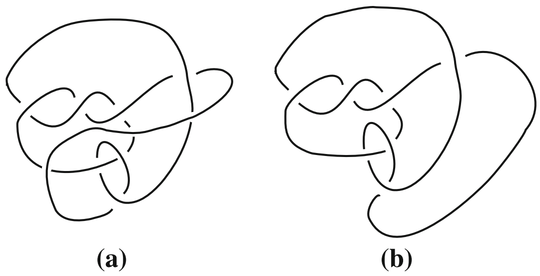


Fig. 2 Two diagrams representing the unknot

extract all the properties of a knot type from one diagram: Each diagram is a point of view on a knot type and therefore displays only certain properties of it. For example, both diagrams in Fig. 2 hide the property of being unknotted of the corresponding knot type. We can transform the first into the second by 'pulling' the middle arc down. This move alone only allows us to conclude that both diagrams represent equivalent knots; to see that they actually represent the unknot we would have to apply further similar moves. In the following, we present a formalization for these moves.

2.3 Possible Moves

Many different geometric shapes are possible for knot diagrams. Nevertheless, in general knot diagrams, like knots, are considered up to topological equivalence.

Definition 5 Two knot diagrams are *equivalent* if there is an ambient isotopy of the plane transforming one into the other. A *topological knot diagram* is a class of equivalent knot diagrams.

Definition 6 A topological knot diagram $\mathcal{D}$ *represents a knot* K if and only if there exists a diagram D in the class $\mathcal{D}$ such that D represents K. A topological knot diagram $\mathcal{D}$ *represents a knot type* $\mathcal{K}$ if and only if there exists K of type $\mathcal{K}$ and D in the class of $\mathcal{D}$ such that D represents K.

By these definitions, the diagrams in Fig. 3 not only represent the same knot type but are actually equivalent diagrams, that is the same topological diagram.

Despite the fact that knot theorists commonly deal with topological knot diagrams, they also need to discriminate among equivalent diagrams through their different geometric properties. To work with topological diagrams, experts have to choose one geometric representative and this choice is not arbitrary. For example, some are more convenient than others: All three diagrams in Fig. 3 are equivalent—and thus the same if considered topologically—but the diagram in Fig. 3c is confusing because it presents useless geometric properties—that is why a diagram such as this is never used in the practice. Moreover, there are particular geometric properties, such as the presence of symmetries, which can be used to infer properties of the represented knot type. In the same example, the diagrams in Fig. 3a and in Fig. 3b, which are the same diagram also if considered geometrically, display a clear symmetry of order four, while the diagram in Fig. 3c does not. It is the context of use that determines whether a diagram is considered as embedded in a geometrical or a topological space.

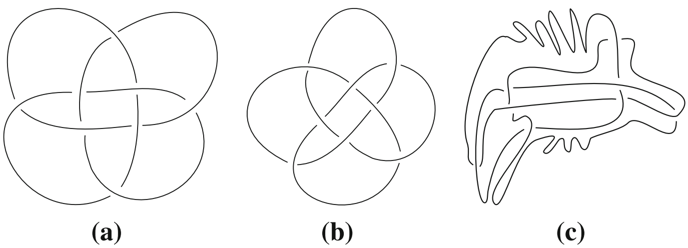


Fig. 3 Three equivalent diagrams of the knot type 8_{18}

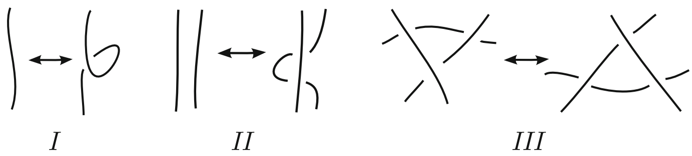


Fig. 4 The three Reidemeister moves

We define now a series of moves on topological knot diagrams that connect different topological diagrams representing the same knot type: the three *Reidemeister moves*. These moves are *local modifications*, as depicted in Fig. 4.

Move *I* consists in inserting or deleting a buttonhole, Move *II* in sliding an arc under another and Move *III* in sliding an arc under a crossing. If one interprets the elements of the figures correctly, as overlapping threads constituting a knot diagram, it is easy to see that these moves do not alter the underlying knot type. Reidemeister's Theorem (Cromwell 2004, Theorem 3.8.1) shows that through a finite sequence of these three moves we can connect any two diagrams representing the same knot type. For example, the move used to go from the diagram in Fig. 2a to the one in Fig. 2b, can be decomposed in three Moves *III*. Of course, since these moves are defined on topological knot diagrams, a representative of a class can be modified also via ambient isotopies, but these are part of the definition of the space in which the diagram is embedded. Reidemeister's Theorem allows us to define an equivalence relation $\sim_R$ between knot diagrams considered topologically—and thus between classes of geometric diagrams.

Definition 7 Two topological knot diagrams are *Reidemeister-equivalent* if there is a finite sequence of Reidermeister moves transforming one into the other.

This dynamics defined on diagrams is what actually connects them to knot types.[6] From the Reidemeister's Theorem we obtain the following proposition, which enables us to identify a knot type with a class of topological knot diagrams.

Proposition 1 *Let K and K′ be two knots. Then, $K \sim K'$ if and only if there exists D representing K and D′ representing K′ such that D is in the class $\mathcal{D}$, D′ is in the class $\mathcal{D}'$ and $\mathcal{D} \sim_R \mathcal{D}'$.*

2.4 Different Spaces

We propose now a classification for knot diagrams according to the different moves that are allowed on them, so as to make explicit the various possible interpretations that are common in the practice.

Let $\mathcal{D}_0$ be the kind of knot diagram interpreted in a geometric space, where lengths and angles count, as in Definition 3. Modifications leaving Euclidean properties invariant are still allowed, i.e. rigid motion. For example, Fig. 3a and in Fig. 3b represent the

6 It is also possible to partially translate diagrams and moves on them into codes (Adams 1994, Ch. 2). Many of such codes (like the *Dowker Code*) have been developed to the aim of using computers in order to classify knots. However, the possibility of translating a knot type or diagram into a code, that is their potential inter-translatability, does not tell us anything about the way in which diagrams are interpreted and effectively used in the practice of knot theory.

same diagram of type $\mathcal{D}_0$. Let $\mathcal{D}_1$ be the standard kind of knot diagram considered in a topological space, where more modifications, i.e. ambient isotopies of the plane, can be performed on a diagram without transforming it into a distinct one. A $\mathcal{D}_1$ kind of diagram is a class of diagrams of $\mathcal{D}_0$ kind by the equivalence relation $\sim$ introduced in Definition 5. Let $\mathcal{D}_2$ be the kind of knot diagram on which also the Reidemeister moves are allowed: A $\mathcal{D}_2$ kind of diagram is a diagram up to a further extended set of modifications. More precisely, it is a class of diagrams of $\mathcal{D}_1$ kind by the equivalence $\sim_R$ introduced in Definition 7. Proposition 1 tells us that a knot type can be identified with a $\mathcal{D}_2$ kind of diagram. In these different definitions of knot diagrams, more and more objects are grouped together in the same class and treated as equal. A diagram is always generic until a certain degree, identified by its belonging to one of the previously defined classes. More in general, based on various sets of moves, which can also be different from the ones mentioned here[7], we can create hierarchies of diagrams.

In turn, these diagrams form different spaces. We define the space $\mathcal{S}_0$ as the one formed by all diagrams of $\mathcal{D}_0$ kind and similarly the spaces $\mathcal{S}_1$ and $\mathcal{S}_2$ as formed by all diagrams of $\mathcal{D}_1$ and $\mathcal{D}_2$ kind respectively.

Then the following holds: $\mathcal{S}_1 \simeq \mathcal{S}_0/\{Ambient\,isotopies\} = \mathcal{S}_0/\sim$. The space $\mathcal{S}_1$ can be seen as a quotient[8] of $\mathcal{S}_0$: All elements which can be connected by ambient isotopies are identified. Furthermore we have: $\mathcal{S}_2 \simeq \mathcal{S}_1/\{Reidemeister\,moves\} = \mathcal{S}_1/\sim_R$. The space $\mathcal{S}_2$ can also be obtained from $\mathcal{S}_0$:$\mathcal{S}_2 \simeq (\mathcal{S}_0/\{Ambient\,isotopies\})/\{Reidemeister\,moves\} = (\mathcal{S}_0/\sim)/\sim_R$. There is a bijection between the space $\mathcal{S}_2$ and the space of knot types.

If diagrams emerge from a figure only once their possible moves, and consequently their space, are fixed, then their use is connected to their dynamics. Experts perform actions on diagrams by re-drawing them in appropriate ways, according to the way they interpret them. For this reason, novices need to train their imagination in order to recognize the various possible moves on diagrams, and then be able to effectively use them. Moreover, these manipulations are similar to the manipulations we can perform on concrete objects, but instead of having a pragmatic aim, they have an *epistemic* one (Kirsh and Maglio 1994). The use of diagrams triggers a form of *manipulative* imagination that gets *enhanced* by the practice. Thanks to this imagination, knot diagrams become an effective notation to make operations and calculations: According to specific aims, we can form sequences of diagrams connected by specific moves.

3 Diagrams in Action: Knot Invariants

On the basis of the above classification, it is possible to study knot types from diagrams. One way is to define *invariants* via diagrams. An invariant is a mathematical element associated to knots that depends only on the knot type, e.g. a number or an algebraic structure. We will present examples of diagrammatically defined invariants, in order to show that (i) knot diagrams can have an evidential role and (ii) it is

[7] The moves presented here are basic in knot theory. However, other moves can be defined for more specific aims. See for example the Kirby calculus for surgery equivalences (Kirby 1978).

[8] Let X be a topological space and $\sim$ an equivalence relation on it. The quotient space $Y = X/\sim$ is defined to be the set of equivalence classes of elements of X.

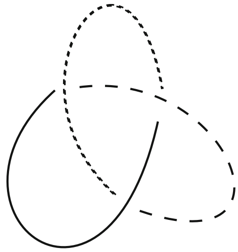


Fig. 5 A coloring of a minimal diagram of the trefoil knot

necessary to use more than one diagram representing the same knot type in order to appreciate its different properties. To define an invariant, we can start by defining a property on a $\mathcal{D}_1$ kind of knot diagram and then make sure that it is preserved under Reidemeister Moves; this is a crucial consequence of Reidemeister's Theorem.

3.1 3-Colorability

Consider a $\mathcal{D}_1$ kind of knot diagram. We fix three colors and color each of its arcs. The diagram is *3-colorable* if it admits a coloring such that: (i) each arc is colored by one color, (ii) the three arcs that meet at each crossing are all of the same color or all of different ones, (iii) at least two different colors are used. Undoubtedly, the minimal diagram[9] of the unknot, which has no crossings, does not admit such a coloring. On the contrary, a minimal diagram of the trefoil knot is 3-colorable, as depicted in Fig. 5—where the colors are represented with various dotted lines.

To prove that 3-colorability is a property of the knot type and not of one particular diagram, we have to check that it is preserved under Reidemeister moves. Consider inserting locally a buttonhole, that is performing a Move *I* in one direction. If the diagram is 3-colorable it remains so. We just keep the coloring: The new crossing will satisfy condition (ii)—the three arcs will have the same color. If the diagram is not 3-colorable it cannot become so. Two arcs of the new crossing are actually the same and therefore have the same color: This implies that also the third arc must be colored in the same way. The unfilled condition remains unfilled after such a move, since it is a local modification and cannot influence the coloring of the diagram near other crossings. Similarly, the other Reidemeister moves preserve the 3-colorability (Adams 1994, pp. 24–25).

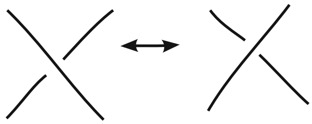
The trefoil knot is 3-colorable, while the unknot is not, and thus they cannot be equivalent. Therefore, this invariant allows us to prove the existence of non-trivial knots, i.e. knots which are not equivalent to the unknot. Despite being intuitively clear, the existence of such knots needs to be proven.

3.2 Unknotting Number

Consider another invariant: the unknotting number of a knot type, i.e. the minimal number of crossing changes in a diagram representing that knot type in order to

[9] The crossing number of a knot diagram is the number of its crossings. Let $\mathcal{K}$ be a knot type. The crossing number $C(\mathcal{K})$ of $\mathcal{K}$ is the minimum over the crossing numbers of all the diagrams representing it. A minimal diagram is a diagram presenting $C(\mathcal{K})$ crossings.

Fig. 6 Switching a crossing



transform it into a diagram representing the unknot. By crossing changes we mean a local 'switch' of a crossing, like the one in Fig. 6.

For example, the unknotting number of the trefoil knot is one. If we switch a random crossing from the diagram in Fig. 5, we obtain a diagram of the unknot, and thus the trefoil knot has either 1 or 0 as unknotting number. Nevertheless, since the trefoil is a non-trivial knot, its unknotting number cannot be zero. In order to calculate the unknotting number, experts have to perform a specific move different from the ones defined above.

In Fig. 7 are two non-equivalent diagrams of the same knot type: Fig. 7a is a minimal diagram (with 10 crossings) with unknotting number three and Fig. 7b is a non-minimal diagram (with 14 crossings) of the same knot type with unknotting number two. In order to appreciate the crossing number of the corresponding knot type, we need to look at a minimal diagram like the one in Fig. 7a. Nevertheless, such a diagram cannot give us information about the unknotting number, which can be proved to be less than or equal to two, by experimenting with the diagram in Fig. 7b. If we switch the marked crossings we obtain a non-trivial diagram of the unknot.[10]

Non-equivalent diagrams representing the same knot type unveil different properties of it. Not only do we need to choose a good geometric representative of a diagram of $\mathcal{D}_1$ kind, and therefore be able to 'move' between equivalent $\sim$ diagrams, but we also have to consider non-equivalent diagrams representing the same knot type, and therefore be able to 'move' between equivalent $\sim_R$ diagrams.

4 Discussion of the Case Study

4.1 Knot Diagrams Require Interpretation

The meaning of a knot diagram is fixed by its context of use: Diagrams are the results of the interpretation of a figure, depending on the moves that are allowed on them and at the same time on the space in which they are embedded. Once we establish the appropriate moves, we fix the ambient space, thus determining the different equivalence relations. The context of use does not have to be pre-defined. This is not a "damaging ambiguity"[11]; on the contrary, it expresses the richness of

[10] Bleiler [1984] proved that the unknotting number of a knot type is not necessarily appreciable from a minimal diagram of it. In this specific case, he proved that the unknotting number of the knot type 10_8 is exactly two.

[11] Shin and Lemon use this term to refer to Euler's belief that the same kind of visual containment relation among areas used in the case of two universal statements can be used as well in the case of two existential statements; this is not correct and the employed representation raises a "damaging ambiguity" (Shin and Lemon 2008).

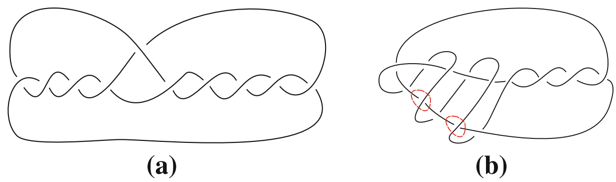


Fig. 7 Two non-equivalent diagrams of the knot type 10_8

this notation and explains why it is effective in promoting inference. The indetermination of meaning makes different interpretations available, and therefore allows attending to various properties and moves. In order to 'see' a diagram in a figure we have to recognize the relevant information and be aware of the possible modifications. For this reason, the effectiveness of a diagram increases with expertise: Only experts are able to fully exploit the richness of the different possible meanings that a figure can acquire.

Giaquinto (2007, p. 261) identifies *aspect-shifting* as a crucial operation of spatial thinking in mathematics. By shifting aspects, the same figure is arranged differently depending on alternative interpretations of its elements. For example, in Plato's *Meno*, in order to find a square with double area of a given one, we build another square having the diagonal of the original one as a side. To prove that the new square meets the requirement, we draw its two diagonals and then we have to recognize that the triangles in which it is now divided are (i) a quarter of it and (ii) a half of the original square. In our examples, we also appreciate aspect-shifting, but it comes in a different form. While in Plato's case we recognize a local rearrangement of parts, in knot diagrams we observe a global re-interpretation of the space in which the diagram is embedded. This interpretation precedes actual transformations or moves and is the condition for them.

4.2 Knot Diagrams are Dynamic

A knot diagram prompts possible manipulations. The rules of motions for the space defined by knot diagrams are given in the interpretation step: Our interaction with the diagrams is in this sense pivotal. This case is a new example of what Giaquinto (2007, p. 263) defines as *visualizing motion*. In his view, "cognitively speaking" this falls under the heading of image transformations. These operations have epistemic value because they promote discovery by bringing our attention to new information. To visualize motion means to be able to understand what will be the outcome of a certain global or local move. On the one hand, to recognize the equivalence of the diagrams in Fig. 3a, c we need to visualize a continuous global motion transforming the latter into the first. On the other hand, to interpret correctly the Reidemeister moves as not altering the represented knot type, we need to visualize local motions.

Therefore, the dynamic nature of knot diagrams involves a form of manipulative imagination that gets enhanced through training by transposing our manipulative capacities from concrete objects to this notation. As Giaquinto (2007, p. 264) suggests, "even symbol movements can have a haptic feel, something perhaps reflected in the metaphor of symbol manipulation". Knot diagrams are a notation where it is possible to identify this haptic feel. In this way, diagrams become tools for and objects of 'experiment', on which experts perform epistemic actions, which correspond to inferential steps in an argument.

4.3 Knot Diagrams are a Good Notation

In the following, we specify the reasons why knot diagrams can be considered as a good notation. First, they count as a notation because they are a system of symbols representing mathematical concepts. Second, they are a good notation because they have inferential and computational power: Drawing sequences of diagrams allows experts to effectively make inferences and calculations. Their use implies not only aspect-shifting but also visualizing motion which lead to draw new diagrams connected to a given one. Knot diagrams are "trans-configurational" as defined by Macbeth (2012), because drawing inferences through them requires re-drawing.

For example, to prove that the diagram in Fig. 2a represents the unknot, we have to draw other diagrams by applying moves on it until we get to a diagram that we recognize as representing the unknot. Of course, to go from the diagram in Fig. 2a to the one in Fig. 2b we do not need to write intermediate steps because, by using manipulative imagination, we recognize that the represented knot type is the same. This is because the requested modification, even if composed by three Reidemeister moves, can be visualized mentally as a single modification.

In the actual practice, all knot diagrams are incomplete, even if mathematically one knot diagram identifies one knot type: Each diagram shows only certain properties of the knot type it represents. In fact, as for calculating the unknotting number, in the practice we may have to use non-equivalent diagrams of the same knot type in order to study its different properties. As Brown (1999, p. 96) sums up, "the moral to be drawn from knot theory is that knots (and other mathematical entities) are like this: they, too, have indefinitely many different kinds of attributes, and sometimes we only uncover them as we find new ways of representing them."

This case study supports an approach to mathematics according to which mathematical symbols are intimately linked to the concepts they represent. De Cruz and De Smedt (2013, p. 4) claim that "symbols are not merely used to express mathematical concepts" but are "constitutive of the concepts themselves. Mathematical symbols enable us to perform mathematical operations that we would not be able to do in the mind alone, they are epistemic actions." In our case study, we showed that knot diagrams both represent objects and allow for procedures.

4.4 Knot Diagrams can Provide Evidence

We have seen that invariants may be diagrammatically defined, and thus a knot diagram can provide justification for a conclusion about a knot type. The kind of evidence produced is accepted in the practice, and even if it might not be a necessary justification, it is a sufficient one. By 3-colorability, it is possible to prove the existence of non-trivial knots. This is done through knot diagrams by adding structure to them.

We mention here another example of an invariant where knot diagrams provide evidence and in particular they allow for a connection between diagrammatic and algebraic reasoning: the *knot group*.[12] For every knot, we can define a group. The *Wirtinger presentation* for it can be written directly from a diagram by interpreting each arc of the diagram as a generator and each crossing as a relation. Different Wirtinger presentations will therefore correspond to non-equivalent diagrams, but two diagrams of the same $\mathcal{D}_2$ kind, and thus representing the same knot type, will give rise to two Wirtinger presentations of isomorphic groups—because this group is an invariant of the knot type. The knot group can be used to distinguish knots and in particular to give an alternative proof of the existence of non-trivial knots. Moreover, it shows that diagrammatic and algebraic reasoning can be related since knot diagrams can be interpreted algebraically. This reveals that these diagrams can be used as syntactic devices.

5 Conclusions

To summarize, we have analyzed a diagrammatic practice and we have argued that it is fruitful for conjecturing and discovering and moreover that it encompasses proving. Knot diagrams require different interpretations and are an effective notation on which epistemic actions can be performed.

We envisage two directions for future research. First, we plan to apply the proposed framework and terminology to other diagrammatic practices of knot theory, for example the use of knot diagrams as a notation in polynomial calculations. Furthermore, we will evaluate how to modulate the framework in order to encompass other visual practices of topology and of other fields of mathematics. Second, our long-term objective is to assess whether an operational framework along these lines could be applied to other forms of diagrammatic reasoning outside mathematics (Giardino 2013). This would show that diagrams in general are not only visual prompts but have dynamic features that involve aspect shifting and visualizing motion. Therefore, their use would presuppose a complex synthesis of many different cognitive capacities, from unlearned ones to others requiring expertise.

Acknowledgments We wish to thank Francesco Berto and Achille Varzi for having made our collaboration possible. We presented our work on knot diagrams in various occasions and received helpful feedback. In particular, we are thankful to Anouk Barberousse, Roberto Casati, José Ferreiros, Marcus Giaquinto, Hannes Leitgeb, Øystein Linnebo, John Mumma, Marco Panza, and John Sullivan. We thank three anonymous referees for their thoughtful comments and suggestions that helped us improve a first version of the article. Silvia De Toffoli acknowledges support from the *Berlin Mathematical School* and from the DFG Collaborative Research Center *SFB/TR 109 "Discretization in Geometry and Dynamics"*. Valeria Giardino is grateful to the *Spanish Ministry of Education* and to the *Exzellenzcluster 264 - TOPOI* for supporting her research.

[12] See Lickorish (1997, Ch. 11) for a detailed explanation.

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