The Epistemological Subject(s) of Mathematics

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Abstract

Paying attention to the inner workings of mathematicians has led to a proliferation of new themes in the philosophy of mathematics. Several of these have to do with epistemology. Philosophers of mathematical practice, however, have not (yet) systematically engaged with general (analytic) epistemology. To be sure, there are some exceptions, but they are few and far between. In this chapter, I offer an explanation of why this might be the case and show how the situation could be remedied. I contend that it is only by conceiving the knowing subject(s) as embodied, fallible, and embedded in a specific context (along the lines of what has been done within social and feminist epistemology) that we can pursue an epistemology of mathematics sensitive to actual mathematical practice. I further suggest that this reconception of the knowing subject(s) does not force us to abandon the traditional framework of epistemology in which knowledge requires

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justified true belief. It does, however, lead to a fallible conception of mathematical justification that, among other things, makes Gettier cases possible. This shows that topics considered to be far removed from the interests of philosophers of mathematical practice might reveal to be relevant to them.

Keywords

Mathematical justification · Mathematical practice · Social epistemology · Feminist epistemology

1 Introduction

In a first analysis, it might seem that contemporary analytic epistemology[1] cannot be fruitfully applied to the philosophy of mathemtical practice. In this chapter, I suggest that this first impression is wrong-headed and that a systematic exchange between the two disciplines would be fruitful.[2]

Applying epistemology to mathematical practice might be thought as problematic for two reasons. First, the standard apparatus of general epistemology, in which (propositional) knowledge involves justification, truth, and belief, has been deemed to be overly narrow for capturing interesting features of mathematical practice.[3] Second, the application of epistemology to mathematics has been traditionally guided by the desire to eliminate the knowing subject as much as possible in order to get an objective, logical analysis of mathematical knowledge. Ideally, the resulting analysis would then be independent of actual mathematical practice.[4]

Although they might appear to be unrelated, these two reasons are connected. As I shall explain, it is only by reconceiving the knowing subject(s) of mathematics as embodied, fallible subjects embedded in specific mathematical contexts that the traditional analysis of knowledge (appropriately developed along the lines traced by social and feminist epistemologists) can be fruitfully applied to mathematical practice.

Let me start with some platitudes about general epistemology to bring out why its application to mathematical practice might seem uncalled for. The almost exclusive focus of epistemologists is directed toward *propositional attitudes,* that is, attitudes like *believing* and *knowing* that are related to propositions. Beliefs attach to propositions,

[1] Throughout the chapter, I will use *epistemology* to mean *analytic epistemology*. This clarification is needed in a context like the one of the present volume, which assembles heterogenous contributions belonging to different traditions. As I use the term, analytic epistemology is the theory of knowledge (and justification) in the analytic philosophical tradition of the Anglo-American world – a very different topic compared, for example, with *épistémologie* in the French tradition, which has to do with the critical and historical study of science and scientific knowledge.

[2] To be sure, there are significant exchanges already, albeit not systematic ones – I will turn to some of those in the conclusion. See, for example, Azzouni (2006), Easwaran (2015), and Tanswell and Kidd (2020).

[3] Lakatos (1976), Kitcher (1984), and Ferreirós (2016).

[4] See, for example, the discussion in Kitcher (1992).

and propositional knowledge is *knowledge that* such-and-such is the case, where this such-and-such is a proposition. Typical questions are: When is a belief that p (where p is a proposition) justified? When does it constitute knowledge? This seemingly innocuous starting point already clashes with a widespread commitment of philosophers interested in scientific and mathematical practice. This commitment is voiced explicitly by Reviel Netz in the context of his analysis of ancient Greek geometry:

> [W]hat unites a scientific community need not be a set of beliefs. Shared beliefs are much less common than shared practices. This will tend to be the case in general, because shared beliefs require shared practices, but not vice versa. [...] Whatever is an object of belief, whatever is verbalisable, will become visible to the practitioners. What you believe, you will sooner or later discuss; and what you discuss, especially in a cultural setting similar to the Greek, you will sooner or later debate. But the real undebated, and in a sense undebatable, aspect of any scientific enterprise is its non-verbal practices. (Netz 1999, 2)

Being verbalizable, the objects of belief can be thought of as corresponding to sentences, which can be associated with propositions. But non-verbal practices, says Netz, must also be taken into account if we want to paint an accurate picture of scientific inquiries. It is safe to say that Netz's commitment can be associated with a widespread view among philosophers interested in mathematical practice, namely, that mathematical knowledge cannot be reduced to the knowledge of propositions alone.[5] Accordingly, all forms of knowledge should be considered, including *knowing how* to do something. The epistemologists' analysis is then seen as overly narrow and in need of broadening up beyond propositional attitudes.[6]

This point is well taken. Still, accepting that beliefs (and other mental states that take propositions as their content) are insufficient to accurately describe scientific or mathematical practices does not force us to marginalize them. We can still think of them as *central* to scientific practices. Indeed, this aligns with a practice-oriented tradition in the philosophy of science and mathematics – it will be enough to refer to Thomas Kuhn's (1962) description of scientific paradigms and Philip Kitcher's (1984) analysis of mathematical practices. More recently, José Ferreirós (2016) emphasized the importance of going beyond propositional attitudes to understand mathematical practices but admitted that "[m]athematics, as we understand it, is not just practical knowledge but has at its core theoretical knowledge" (Ferreirós 2016, 30). This theoretical knowledge, I submit, can be cashed out in propositional form. And in fact, according to Ferreirós, it is always possible, albeit in some cases just in retrospect, to associate a mathematical practice with a particular mathematical *theory* (*ibid.* 29).

The above considerations show that the fact that epistemology is almost exclusively focused on propositional attitudes does not thereby preclude its application to mathematical practice. What is more, recent efforts in epistemology are bringing into

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[5] This view is explicitly endorsed in Ferreirós (2016) and implicitly assumed in a plethora of works, such as the ones discussing (like Netz's) the roles of diagrams and notations in mathematics; see, for example (Chemla 2018; Manders 2008).

[6] Thanks to Gisele Secco whose criticism helped me realize the importance of this issue.

focus the analysis of *knowing how* and assessing the hypothesis that it is not always reducible to propositional knowledge, or *knowing that*.[7] These efforts could contribute to a better understanding of what unites scientific practices besides shared beliefs, to use Netz's expression in the quotation above. That is, philosophers of mathematical practice could take advantage of the tools developed to talk about knowledge how in general and apply them to their preferred domain.

The first obstruction to the application of epistemology to the study of mathematical practice is thus cleared. This much will suffice to convince the reader, I hope, that the traditional focus on propositional attitudes should not be a reason for philosophers of mathematical practice to brush aside epistemology. But this is by no means the sole obstruction.

Philosophers of mathematical practice might also be dissatisfied with how epistemology developed. Here is a cartoon version of the story. Going way back to Plato's *Theaetetus*, (propositional) knowledge has been thought of as something like justified true belief. And then, in the 1960s, Gettier (1963) came around and showed that justification, truth, and belief are not jointly sufficient for knowledge. This is because there are cases in which a subject is justified in believing a true proposition, but the proper connection between justification and truth is severed by some sort of epistemic luck that is incompatible with knowledge.

Suppose that I am justified in believing the proposition <Valeria is in Paris>. I am therefore justified in believing <Valeria is in Paris or p> for any proposition p since this is a simple logical consequence of <Valeria is in Paris>.[8] Therefore, even if I have no justification for believing that Valeria has a car, I am justified to believe proposition q = <Valeria is in Paris or she has a car>. As it turns out, unbeknown to me, Valeria just bought a new car and drove to Barcelona with it. In this case, q turns out to be true, but not for the reason I thought. I believe q, I am justified in believing in q, and q is true. But something has gone awry. I do not know q. Gettier cases like this one are cases in which a subject holds a justified true belief that does not constitute knowledge because some type of luck is at play.

For a while, after Gettier's publication in 1964, epistemologists set out to find an elusive fourth condition for knowledge. This new enterprise, however, did eventually stall. Many epistemologists decided to let go of knowledge to focus exclusively on justification. Others took knowledge as a primitive notion.[9]

The problem is that Gettier cases seem to be utterly irrelevant to mathematics. Even setting aside the vexed issue of how we should conceive the truth of mathematical propositions, Gettier cases seem to be impossible in mathematics because justification in mathematics has been traditionally associated with proofs, and these entail their conclusion, leaving no gap between justification and truth – at least if we

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[7] See, for example J.A. Carter and Pritchard (2015), Bengson and Moffett (2011), and Habgood-Coote (2019).

[8] Note that this "or" is inclusive. It is the standard disjunctive connective of classical logic: "A or B" is true unless both A and B are false.

[9] This lead to *knowledge first* approaches in epistemology; see Williamson (2000).

take mathematical propositions to be conditional on their premises (i.e., the axioms). Moreover, mathematical propositions are necessary, and thus it is unclear how epistemic luck could enter the picture.

Furthermore, several of the analyses of knowledge developed by epistemologists after Gettier did explicitly exclude mathematical knowledge. For one, Alvin Goldman's causal theory of knowledge is designed to apply to empirical beliefs exclusively.[10] And this is not considered to be a problem at all:

> My concern will be with knowledge of empirical propositions only, since I think that the traditional analysis is adequate for knowledge of nonempirical truths. (Goldman 1967, 357)

The causal theory was then replaced by *reliabilism* (Goldman 1979). Very roughly, this view says that a belief is justified if and only if it is produced by a reliable belief-producing process. Reliabilism applies to mathematics, but it is not immediately clear how it can help us with the specificities of *mathematical* justification, especially if this is connected with the notion of proof. This is because it delivers an overly idealized account of mathematical beliefs based on proofs. Since the premises of genuine proofs entail their conclusions, it is plausible to think that beliefs justified by genuine proofs will be produced by an ideally reliable process. But this is certainly not what happens in practice. A broadly reliabilist framework could, however, be used to paint a faithful picture of mathematical practice. However, to do that, we would have to move away from an infallible conception of mathematical justification and begin to explain how our belief-forming processes at play in mathematical practice are in fact reliable, even if not ideally reliable – for an effort in that direction, see Avigad (2021).

Other attempts to characterize knowledge invoke modal notions (e.g., sensitivity and safety) to get at some sort of robustness that is intuitively peculiar to knowledge. Modal notions do not, however, apply to necessary truths in non-trivial ways.[11]

This cartoon story makes it clear that the concerns of general epistemologists do not meet the ones of philosophers of mathematics, let alone the ones of philosophers of mathematical practice. What is more, in the background of many epistemological discussions about knowledge lurked the problem of skepticism. How can we revindicate the fact that we know most things we think we know? Like that we have hands, that water is H_2O, or that Trenton is the capital of New Jersey. And here, too, the focus is on ordinary beliefs rather than mathematical beliefs.[12]

One strategy to show that epistemology still has something to offer to philosophers of mathematical practice is to observe that there are other topics besides the mainstream ones. I mentioned already recent efforts toward the analysis of

[10] It is therefore somewhat bizarre that the famous *problem of access to abstract objects* originally formulated in Benacerraf (1973) involved Goldman's causal theory of knowledge (more on this later).

[11] There are ways to apply them to mathematics, however. See, for example, Clarke-Doane (2020).

[12] This has not always been the case. For instance, Descartes's most radical form of skepticism involved mathematical beliefs, but these do not feature in most contemporary discussions.

knowledge how. I will later indicate a few other such themes. My main goal, however, is to suggest something stronger. Namely, that even these traditional topics that seem so far from the concerns of a practice-oriented philosophy of mathematics could be applied to it. Among other things, I will draw on previous work (De Toffoli 2021a, 2022) to show that Gettier cases are indeed possible in mathematics, and that thinking about them can be instructive to philosophers of mathematical practice. Gettier cases, I will indicate, arise when we substitute the traditional infallible notion of mathematical justification associated with proofs with a fallible counterpart. This substitution is natural when we put at the center of our epistemological inquiry fallible human subjects rather than highly idealized ones.

Summing up, it is not hard to explain why it seems that mathematics is not relevant to general epistemology. Moreover, when epistemology has been applied to mathematics, it is not the mathematics that philosophers of mathematical practice seek to investigate – but rather a mathematics conceived as a realm of necessary truths, the knowledge of which can be gained a priori. To my mind, this gives a plausible (if only partial) answer to why the philosophy of mathematical practice and epistemology have been overlooking each other, with just a few (but notable) exceptions. I want to suggest that it is time for a change. My goal here is to prepare the terrain for further work at the intersection of the philosophy of mathematical practice and epistemology by showing that we can fruitfully think about mathematical practice by adopting long-established epistemological frameworks.

Here is the plan for the chapter. Section 2, which is the bulk of the chapter, is devoted to exploring how to characterize the knowing subject(s) if we want to pursue a successful epistemology of mathematical practice. I will indicate some points of contact with the characterization of the subjects by social and feminist epistemologists. In Sect. 3, I turn to the traditional analysis of knowledge and discuss how it can be applied to mathematics. I build on previous work (De Toffoli 2021a) to sketch a fallibilist account of mathematical justification that makes it plausible to think that Gettier cases could arise in this context. I then sum up the discussion and point to further epistemological themes relevant to mathematical practice.

2 With or Without Subjects?

I start my analysis of the knowing subject(s) by discussing how the epistemology of mathematics (and of science more generally) has been conceived by Karl Popper and others as an enterprise aimed at the study of "knowledge in the objective sense" in which subjects do not figure at all. Afterward, I consider how highly idealized subjects entered the picture. I then sketch a different type of subject: a fallible subject whose rationality is bounded. It is only with this type of subject in mind that we can pursue a *human* epistemology of mathematics. But in order for this epistemology to be tethered to actual practice, subjects should not be considered in isolation but connected with other subjects and embedded in a specific mathematical context. I then explore the possibility of indexing knowledge to groups rather than to individuals – as has been done by some social epistemologists. I also consider how this reconception of the knowing subject(s) could bring philosophers of mathematical practice close to feminist epistemologists. I wind up the section by canvassing the importance of conceiving epistemic subjects as not entirely passive, but as active members of specific mathematical practices.

2.1 The Epistemology Without a Subject

Descartes, Locke, and Hume are illustrious among the so-called *belief-philosophers*. This is a label introduced by Popper to describe those philosophers that are "interested in our subjective beliefs, and their basis or origin" (1968, 334). According to Popper, epistemology is a theory of scientific knowledge in which the knowing subject plays no role:

> Knowledge in the objective sense is knowledge without a knower: it is knowledge without a knowing subject. (Popper 1968, 335)

In Popper's view, there are three worlds. The world of material objects, the world of subjective ideas, and a third world, which is the world of *objective contents of thought*. Driven by a similar motivation of separating subjective ideas from objective non-material items, Frege had already introduced a *third realm*.[13]

General epistemology – to be clear, the one pursued by belief-philosophers such as Descartes and later by analytic epistemologists – focuses on the second world and, according to Popper, should be reclassified as psychology or sociology, being utterly irrelevant to his conception of epistemology, which refers exclusively to the third world. According to him, epistemology relates to items such as scientific problems, conjectures, and theories. These items are created by human animals but, most importantly, are *autonomous*. Crucially, Popper's objective knowledge is not constrained by the cognitive abilities of any subject. Indeed, it even includes pieces of knowledge that no human subject will ever acquire.

Why was Popper so adamant in rejecting the knowing subject? It is because he wanted, like Frege before him, to shun *subjectivism* at all costs.[14] In Frege's case, the threat was *psychologism*,[15] and what had to be saved was mathematics. Frege assumed that *ideas* (i.e., things belonging to the second realm) would inevitably vary between subjects. If mathematical entities were ideas, then we would be forced

[13] Frege's third realm, however, differently from Popper's third world, only accommodates necessary immutable truths, leaving out the laws of science.

[14] As Susan Haack (1979) explains, it is not trivial to pin down the exact locus of the negative feature of the type of subjectivism Popper had in mind. First, the objective/subjective dichotomy can be conceived in several different ways. Second, the notion of *intersubjectivity* can be placed in between, thus leaving the dichotomy to give space to something more nuanced.

[15] For an analysis of the reasons (some of which are institutional in nature) why psychologism started to be conceived as a major sin in the German philosophy departments of the beginning of the twentieth century, see Kusch (1995).

to distinguish, for example, between *my* Pythagorean theorem and *your* Pythagorean theorem. But that is absurd.[16] Popper's worries have a similar flavor.

Popper's attitude toward epistemology is not unique to him. It is endorsed by other players in the twentieth-century philosophical landscape. The "epistemology without a subject," as Ferreirós (2016, 11) labels it, found fertile ground in the realm of mathematics, which, after all, was where Frege's inquiries took place. In France, Jean Cavaillès (1937, 1938) developed a philosophy of mathematics focused on concepts, rather than on subjects.[17]

Focusing on mathematics is congenial to an epistemology without a subject because mathematics is prima facie a realm of necessary truths that admit indefeasible justification.[18] After all, it is reasonable to think that we could not be wrong that $1 + 1 = 2$. This infallibility indicates that there are no relevant individual differences in mathematics. The main assumption of philosophers of mathematics was that mathematical justification (and knowledge) is underwritten by proofs, which were conceived as deductions from axioms. The focus thus was to justify mathematical theories rather than mathematical beliefs. The epistemology without a subject ignored individual beliefs and the inner working of mathematics and focused on very general issues.[19]

If the application of epistemology to mathematics led to an *epistemology without a subject*, it is not surprising that philosophers of mathematical practice felt the need to look elsewhere for philosophical inspiration. But this is not the end of the story. Besides efforts toward the justification of mathematical theories, in the second half of the twentieth century, philosophers of mathematics started tackling different epistemological issues. And this time, subjects were needed. The problem, however, is that these subjects, as we are about to see, were too idealized to be taken as the subjects of inquiry for a philosophy of mathematical practice.

2.2 Ideal Rationality

In 1973, Benacerraf published what would become one of the most influential papers in the philosophy of mathematics of the twentieth century, "Mathematical Truth" (1973). The paper poses a dilemma. If mathematics is, as it seems to be, about mathematical objects, and if mathematical objects are abstract, then how is knowledge of such objects possible at all? A quandary arises from combining a view of mathematical objects as not spatiotemporally located (and thus causally inert) with

[16] To be sure, this is a simplification. Frege critique of psychologism is sophisticated and developed across multiple works.

[17] See Sinaceur (2019).

[18] And in fact, the most virulent form of psychologism, against which Frege's criticism were levelled, arises from the application of psychology to logic and mathematics.

[19] Think, for example, about conventionalism and Quine's holistic empiricism. For an elaboration on these ideas, see De Toffoli (2021a).

the causal theory of knowledge. *Benacerraf's dilemma*, as it is now commonly referred to, has generated a massive amount of scholarship. The causal theory of knowledge is surpassed by subtler alternatives, but Benacerraf's dilemma has proven to be robust with respect to different epistemological theories of knowledge.

Benacerraf's dilemma has to do with individual knowledge, and thus it introduces individual subjects into the epistemology of mathematics. But the problem it raises is very general and has to do with metaphysical considerations that practicing mathematicians are generally oblivious to. From such a general vantage point, the knowledge that 2 is a prime number poses no more and no fewer problems than the knowledge of Fermat's Last Theorem. This is because Benacerraf's dilemma focuses exclusively on our knowledge of mathematical objects *in general*. In other words, the type of discussion of mathematical knowledge it generates does not require considering any sophisticated mathematics at all.

What is more, mathematical reasoning is not relevant at all to Benacerraf's type of questions. The only propositions that matter for such an epistemological inquiry are the starting points. Given the axioms, the rest follows unproblematically by means of deductions, or so the story goes. That is, given the axioms, we can infallibly know all the rest of mathematics by proofs. If there is a fallibilist component at all in our justification of mathematical facts, it is relegated to our beliefs about the axioms. Any other type of fallibility would have to do with human (in)competence. It should be borne in mind that nobody denies that we are fallible and at times make mistakes – what is rejected is that such fallibility is of any interest in the epistemology of mathematics.

These considerations do not square with a philosophy of mathematics interested in mathematical practice. Mathematicians seldom concern themselves with the axioms and are generally indifferent to the choice of a particular formal system.[20] More crucially still, the subjects for which only the starting points matter are logical omniscient subjects.

Note that although it is ill-advised to study mathematical practice by conceiving subjects as logically omniscient, this is not the case for other domains. In the context of formal epistemology, it is perfectly suitable:

> [I]n many standard applications of the Bayesian machinery, the assumption [of logical omniscence] is natural. Suppose, for instance, we're trying to use it to decide between two hypotheses about the chance with which a coin will land heads when it is tossed. [. . .] In this case, logical omniscience tells us that we must be certain that, if the coin didn't land heads on the first toss, it landed tails; it tells us to be at least as confident that it landed heads on the first toss as we are that it landed heads on the first two tosses; and so on. In these cases, logical omniscience seems a reasonable requirement. And it was such cases where Bayesianism first found application. (Pettigrew 2021, 9992)

In cases such as the one described in the quote above, it is plausible to think that we are forced to assign the same probability to all logically equivalent propositions, on pain of irrationality. And then, Richard Pettigrew explains, it is natural to extend

[20] See Burgess (2015, 145) illuminating discussion of different types of *indifference* in mathematical practice.

further such a requirement. Logical omniscience is a requirement that arises from the Dutch Book Arguments, which govern the norms that regulate rational credences and were developed by Frank P. Ramsey and Bruno de Finetti.

But even in the context of Bayesian epistemology, logical omniscience can become a *problem* since it clashes with our ordinary judgments of rationality. And in fact, there are several attempts to *solve* the problem of logical omniscience and to apply formal epistemology to logical learning and other cases of bounded rationality.[21] But these efforts have not considered in detail sophisticated mathematics, which arguably is the main target of study for philosophers of mathematical practice.

2.3 Non-Ideal Rationality

If logical omniscience is a requirement of rationality, it is not a very useful one to evaluate the epistemic achievements and falls of *human* subjects. Nobody would be able to pass the bar. As David Christensen (2004) explains:

> When we call someone "irrational," we are saying that he is deficient *relative to a contextually appropriate standard*, which need not be—and typically is not—the standard of absolute rational perfection. (152, *emphasis added*)

Developing *contextually appropriate standards* allows us to distinguish between different types of failures. One thing is a careful mathematician who makes a subtle mistake. An entirely different case is the one of an absent-minded mathematician who comes up with a bogus proof for some result or another. As Daniel Greco and Brian Hedden (2016) explain:

> [I]t is important to distinguish being irrational in the sense of falling short of the rational ideal from being irrational in the sense of being less rational than (most of) the rest of us. (3677)

The type of rationality we are concerned with when analyzing mathematical practice is *non-ideal,* and it is going to be indexed to a particular mathematical context. The challenge is then to focus on specific mathematical practices and spell out the context-dependent norms that regulate such practices. And this is perfectly in line with a philosophy of mathematics interested in mathematical practice.

As a matter of fact, several works produced by philosophers of mathematical practice can be understood as a reply to such an epistemological challenge. As a way of example, consider the multiple studies on diagrams in mathematics. Albeit focusing on different practices and endorsing different methodologies, most of these studies aim at unveiling the (often implicit) norms governing the use of diagrams in different contexts. For example, there is ample literature discussing the epistemic role of diagrams in Euclidean geometry and the way in which their use was regulated (Netz 1999; Manders 2008; Macbeth 2010; Panza 2012). Diagrams from other historical practices have been considered as well – in this regard, the work

[21] For a recent endeavor in that direction, see Pettigrew (2021).

of Karine Chemla (2018) on diagrams in ancient Chinese mathematics is a prime example. Examples of studies considering diagrams in contemporary mathematics are Carter (2019), De Toffoli and Giardino (2014), and De Toffoli (2023).

These studies put at the center of their inquiry fallible, limited subjects with a specific cognitive makeup and inhabiting particular contexts. It is precisely because of their cognitive shortcomings that representational choices (such as the ones to use diagrams) are relevant in practice.

2.4 Social Epistemology

Putting at the center of epistemological inquiries non-ideal subjects embedded in a specific mathematical practice obliges us to look at the social dimension of mathematics. In point of fact, contrary to the romantic image of the mathematician working in isolation, most members of a mathematical practice interact systematically with each other and with the broader mathematical community. It is for this reason that the epistemic norms at play in mathematics present a social dimension.

For instance, what counts as a proof is not fixed once and for all: what is acceptable in one context might not be acceptable in another. This does not mean that we must renounce to a context-independent notion of logical validity, but that arguments put forward as proofs cannot be evaluated independently from the audience to which they are addressed. By way of example, in topology but not in other branches of mathematics, arguments involving sequences of visualizations might be acceptable as proofs.[22]

Moreover, it has been pointed out that some of the norms governing justification in mathematics guarantee the possibility of the mathematical community to perform a self-monitoring activity on the results it produces, thus aiding single mathematicians to overcome their individual limitations.[23] These considerations point to the fact that to paint a faithful picture of mathematical practice, the social dimension of mathematics should be considered. To do so, it is possible to import into the philosophy of mathematics some of the tools that have been developed by social epistemologists.

There are already some efforts in this direction. For example, Kenny Easwaran (2009) has argued that proofs should be *transferable* – that is, roughly, that they should convince the relevant experts of their correctness without any appeal to testimony.[24] He then explains:

> Philosophers of mathematics have long focused on individual epistemic norms for mathematics, and deductive logic has been very useful in helping to understand these norms.

[22] See De Toffoli (2021b).

[23] See Easwaran (2009) and De Toffoli (2021a).

[24] In my view, transferability can also be understood in terms of the a priori – given a minimal, fallible conception of a priori justification. In this volume, Danielle Macbeth (2021, 13) also connects transferability to the a priori: "The proof is, as it is sometimes put, "transferable" (Easwaran 2009). Indeed, it is in just this sense that mathematical knowledge is a priori: It is a priori not because it is infallible but because and insofar as it does not rely on empirical evidence, whether that of one's own senses or that given on the testimony of another."

> However, transferability is a social norm—it can help the community develop a better grasp on the knowledge of its members, even though it may not have any advantages for the individual considered in isolation. (Easwaran 2009, 356–357)

Although general epistemology has traditionally focused on individual subjects in isolation, social epistemology is growing very rapidly. This is a branch of epistemology that puts particular emphasis on the fact that the subjects under inquiry are embedded in specific social contexts.

Sanford Goldberg (2018) ventured so far as to trace the very source of epistemic normativity to our social nature. In his view, interacting with others has many benefits, but it also puts us in a vulnerable position. That is why we are entitled to require specific standards of the doxastic lives of others – and these entitlements give rise to the normativity of epistemological notions such as justification and knowledge. His proposal applies to beliefs in general, but it would be worth considering how it could be modulated in specific for the mathematical case.

One respect in which social epistemology could be applied to the study of mathematical practice is the following. Social epistemologists are not only interested in how social norms and social contexts do influence epistemic notions such as individual justification and knowledge, but also in how such notions *themselves* can be indexed to a plurality of subjects. This suggestion could be evaluated and tested by considering specific cases from the practice of mathematics. For example, several proposals have been put forward on how to characterize group beliefs (Gilbert 2004; Lackey 2016, 2020). The literature can be broadly divided into two camps: the one formed by those who endorse an *inflationary* approach and the one formed by those defending a *deflationary* approach. Briefly, according to the latter but not to the former, group beliefs are reduced to the beliefs of the members of the group.

Looking at the social structure of mathematics could offer important data in support of one or the other position. More specifically, contemporary mathematics offers us examples in which it is helpful to think of groups of practitioners themselves, rather than individual mathematicians, as epistemic subjects. *Big proofs*, that is, proofs that are not within reach of a single mathematician, are an ever more common phenomenon in contemporary mathematics – besides computer-assisted proofs, prime examples of big proofs are proofs that involve large collaborations. A famous example of the latter is the proof (or the multiple proofs) of the theorem of the classification of finite simple groups.[25] Focusing on such a case or similar ones could, for instance, be used to evaluate and support the inflationary approach to justified group belief.

2.5 Feminist Epistemology

I started this section by surveying the *epistemology without a subject* and, passing through an epistemology considering logical omniscient subjects, I ended with the consideration of fallible, limited subjects inhabiting specific contexts. I then suggested that subjects are better conceived as entertaining systematic relations with each other

[25] See Habgood-Coote and Tanswell (2021).

rather than as operating in isolation. Finally, I indicated that, in some cases, it might even be useful to think of multiple subjects rather than individuals as the relevant subject to which to index belief, justification, and knowledge.

Moving from a very abstract and disembodied way of thinking about the epistemology of mathematics to a more concrete and embedded way is in line with the methodological guidelines of feminist epistemology. And, in fact, I want to suggest that the philosophy of mathematical practice and the epistemology approaches that are most suitable for it share some joint tenets with feminist epistemology.

To be sure, feminist epistemology is a cluster of heterogenous views rather than a unified position.[26] Notwithstanding the great variety among the particular theories, they all share some core commitments. As Helen Longino (1998) has emphasized, one of such core commitments is to put at the center of investigation non-ideal subjects situated in a specific context:

> The shift from a transcendent and disembodied subject to empirical, embodied, and differentiated subjects is often represented as a loss, encouraging a representation of the aspects of situatedness as interfering with knowledge or cognitive access. [. . .] however, we might better think of them as focusing devices, or cognitive resources, directing our attention to features of that which we seek to know that we would otherwise overlook. (335)

Indeed, as I have suggested, without this de-idealization of the knowing subject(s), it would be impossible to appreciate the epistemic norms governing actual mathematical practice.

Longino also individuates another aspect that unites feminist approaches to epistemology: the consideration of social factors. Subjects are often considered to be interdependent. In her words,

> One further consequence of acknowledging the embodied character of knowers that feminists have explored is the dependence or interdependence of knowers. (*ibid.,* 335)

The methodology of feminist epistemologists is thus seen to converge with the one of social epistemologists.

It is therefore clear that an epistemology suitable to the study of mathematical practice conceives of the knowing subject(s) in a way that is in line with how these subject(s) are conceived within feminist epistemology. This is not surprising. After all, some of the feminist approaches, like Longino's own approach,[27] emerge precisely from what can be called the *practice turn in the philosophy of science*[28] – a trend that started well before[29] the *practice turn in the philosophy of mathematics*.[30]

[26] See Garavaso (2018).

[27] See Longino (2002).

[28] See, for example, Soler et al. (2014).

[29] Its beginning could be pinned to Kuhn's (1962) influential work on scientific revolutions; later on, emphasis on practice was prominent among historians and sociologists of science.

[30] Although there are important precursor, like (Lakatos 1976), the philosophy of mathematical practice took off much later. See, for example, Mancosu (2008).

2.6 Subjects or Agents?

Before moving on to the analysis of knowledge, I want to address a last issue on the nature of epistemic subjects. Ferreirós favors talking of *agents* rather than *subjects*. Although I share some of Ferreirós's motivations, I will take sides against his proposal.

Ferreirós strives to emphasize the active nature of mathematicians producing and sharing knowledge in a specific mathematical practice. The problem is that etymologically, "subject" has a passive connotation ("sub-jacere": to throw under). In his words:

> I dare say that the *oblivion of action* was the worst defect of traditional epistemology, its main shortcoming that, in my view, has made impossible a well-grounded account of human knowledge. Even perception, which modern philosophers from Descartes and Locke onward understood as a passive reception of impressions, is in the light of neurobiology and cognitive science a complex system that results from the interplay of sensory input and motor output (not anything like a primitive faculty of the mind, as philosophers tended to think). Input and output, sensation and action–without this feedback, one cannot even make sense of perception, let alone the further complexities of knowledge production. (11)

I agree with Ferreirós that traditionally epistemic subjects have been conceived of as entirely passive – and that this is ill-suited to study mathematical practice. However, naturalized epistemologists do nowadays consider the lessons of cognitive science and are aware of the important feedback between sensory input and motor action. Moreover, epistemologists that are sympathetic to the embodied cognition movements and its cognates are certainly happy to consider the subjects of epistemology as embodied. As we saw, this is also one of the main tenets of feminist epistemology:

> One of the consistent themes running through the feminist rethinking of the subject of knowledge is the insistence on its embodiment. (Longino 1998)

Still, there is an important sense in which the *agents of ethics* and the *subjects of epistemology* are better kept separate. What we believe is not *up to us* in the same way as what we do. This is clear for perception: most of my perceptual beliefs are formed without conscious deliberations. Trivially, I might influence what my perceptual beliefs are by controlling what sensory stimuli I expose myself to. For instance, if I turn my head right and keep my eyes open, which I can control, I will inevitably see out of the window and form the belief that the sun is shining. There might also be other forms of indirect control. However, even for reflective beliefs such as mathematical beliefs, it is plausible to think that they often result automatically from exposure to the relevant evidence.[31] There are various positions in the doxastic voluntarism debate that indicate several ways in which we might be

[31] For instance, this would be in line with Kornblith (2012).

able to have some kind of control over our beliefs,[32] but they generally take for granted the difference between belief and action.

When I go through a proof, and I understand it, or when I read a compelling argument, I cannot help but form a belief.[33] There is no direct choice there. But action is different. If I see a child drowning in a pond, I can choose between saving him and ruining my beautiful Italian boots or watching him drowning and preserving the boots.[34]

Crucially, saying that belief and action are different in this respect does not entail that we cannot talk about normative notions and responsibility in epistemology. To be sure, we can still talk about justified and unjustified beliefs. Moreover, there is a substantive body of literature on how we can be considered responsible for our beliefs – and the *ethics of beliefs* is a sub-field in epistemology that has grown exponentially in the last few years.[35]

There is yet another sense in which the subjects of epistemology can be conceived as active. Namely, the status of *being justified* may be connected to the *activity of justifying*. As Ferreirós notes, this can be traced to the tradition of the American pragmatists and goes against what Adam Leite (2004) calls the "Spectatorial Conception," which he characterizes as follows:[36]

> The fundamental idea shared by these views is that being justified is something which happens to you. According to these theories, the justificatory status of a person's belief is determined by certain facts which obtain prior to and independently of the activity of justifying. The activity itself plays no role in determining justificatory status; it is simply a secondary and optional matter of attempting to determine and report, as far as is conversationally necessary, the prior and independent facts which determine the justificatory status of one's belief. Consequently, even if things go badly wrong in the course of the activity, that will not determine one's actual justificatory status. On this conception, one stands in a primarily theoretical or epistemic relation to the justificatory status of one's beliefs: positive justificatory status is something which one finds out about, not something which one brings about. I therefore call this view the Spectatorial Conception. (222)

Leite proposes an alternative account in which the *basing relation*, that is, the relationship that exists between a belief based on a reason and the reason on which it is based, is something that can be described in terms of the activity of justifying. This is another example of how a traditional epistemological debate, the one about the nature of the basing relation, can be applied to a philosophy of mathematical practice – and how in turn considering mathematical practice could bring the debate forward. For instance, in De Toffoli (2021a), I proposed an account of the basing relation for

[32] See, for example, Hieronymi (2006).

[33] Note that such a belief might not be an unqualified belief in the conclusion of the proof but can be the conditional belief that the conclusion is implied by the (explicit or implicit) premises.

[34] The example is adapted from the classic Singer (1972).

[35] See Chignell (2018).

[36] This is also related to John Dewey's critique to traditional epistemology according to which the subject of knowledge, like the classical subject of perception, is passive.

mathematics that is in line with Leite's and that puts the emphasis on our ability to provide reasons for our beliefs rather than on their causal origin.

Let's take stock. I agree with Ferreirós that appropriate subjects of an epistemology of mathematics taking the practice seriously should be considered as embodied and situated in a particular context. I further agree that they should be conceived as active members of their communities. Still, I prefer to keep in line with the tradition and call them epistemic *subjects* merely to emphasize that the focus of our inquiry is belief rather than action – which, to be sure, are interrelated.

Having discussed how epistemic subjects can be characterized, we can turn to the traditional analysis of knowledge and evaluate whether it can be fruitfully applied to mathematical practice.

3 The Analysis of Knowledge

In general epistemology, it is common to deploy formulas beginning with "S knows that p..." We now have a sense of how to conceive of the subject S. Rather than a disembodied, highly idealized subject in a vacuum, our subject will either be a fallible and embodied subject embedded in a social context or be a group of such subjects.

And then? After "S knows that p..." we can satisfy ourselves with an "only if" rather than an "if and only if." Gettier's lesson was precisely that justified true belief is necessary but not sufficient for knowledge. We have: "S knows that p only if p is true, S believes p, and S's belief that p is justified." More schematically:

$$\text{Knowledge} => \text{Justification} + \text{Truth} + \text{Belief}$$

If a subject knows that p, then she believes p. The *belief condition* is the least controversial. Moreover, it is plausible to think that if it presents problems at all, these are not peculiar to mathematics. For this reason, I will set it aside.

The other components require some discussion. The fact that propositional knowledge is knowledge of a true proposition is taken for granted by epistemologists. However, there are reasons to think that this is a non-starter for a philosophy of mathematical practice. I will address this point before turning to an account of justification that allows for Gettier cases in mathematics.

For reasons of space, it is not possible for me to go beyond these necessary conditions and discuss a full analysis of knowledge. I leave the topic for further work. However, let me just briefly list three promising strategies that can be deployed to develop an account of knowledge that fits mathematical practice. One is to tweak modal notions (such as safety or sensitivity) so that they can be applied to necessary truths.[37] A second option is to deploy the virtue epistemology machinery developed by Ernest Sosa (2007).[38] A third approach (which is compatible with the

[37] Some steps in this directions have been taken in (Clarke-Doane 2020).

[38] See Tanswell (2016).

second) is to go knowledge-first. Following Timothy Williamson (2000), the idea here is to take knowledge as a primitive notion and explain justification in terms of it rather than the other way around – note that although Williamson and other proponents of the knowledge-fist approach are infallibilist about justification (and thus in my view their theories can hardly be applied to analyze mathematical practice), fallibilist knowledge-first theories of justification have been developed as well.[39]

Let us then turn to the truth and justification conditions for knowledge.

3.1 True Propositions

If knowledge requires justified true belief, it must be knowledge of true propositions. I have already discussed in the introduction that to analyze a scientific practice, it might not be enough to consider beliefs exclusively. Beliefs play, however, a central role in any practice that has a theoretical orientation.

In the tradition of Robert Stalnaker (1976), propositions have been coarsely identified using the semantics of possible worlds. A proposition can be conceived as the set of possible worlds where it is true. But this implies that there is just one necessary proposition. David Lewis (1996) is explicit about this point:

> What I choose to call 'propositions' are individuated coarsely, by necessary equivalence. For instance, there is only one necessary proposition. (551)

But plausibly mathematical propositions are necessary propositions. And if there is just one mathematical true proposition, then propositional knowledge in mathematics will be quite uninteresting since it won't even be possible to differentiate our knowledge of different mathematical propositions. Lewis continues:

> So the necessary proposition is known always and everywhere. Yet this known proposition may go unrecognised when presented in impenetrable linguistic disguise, say as the proposition that every even number is the sum of two primes.[40] Likewise, the known proposition that I have two hands may go unrecognised when presented as the proposition that the number of my hands is the least number n such that every even number is the sum of n primes. [. . .] These problems of disguise shall not concern us here. Our topic is modal, not hyperintensional, epistemology. (*ibid.*, 552)

It is clear, however, that the problem of disguise must concern us *here*. And in fact, we need to individuate propositional content in a finer way to talk about mathematics at all! This problem should not be, however, a roadblock but a challenge to overcome. Several methods are available. One option is to index beliefs

[39] See, for example, Silva (2022).

[40] Actually, the Goldbach conjecture says that every even number *larger than 2* is the sum of two primes.

to sentences rather than to propositions.[41] Another is to adopt one of the strategies that have been developed under the label of *hyperintensionality*.[42]

Besides problems arising from the very conception of propositions, another issue concerns talking about *truth* in mathematics. Here is Ferreirós (2016) on the matter:

> From the time of the ancient Greeks, with their definition of knowledge as "justified true belief," to the twentieth century under the influence of denotational semantics, it has been common to consider mathematics as a body of truths (otherwise, it is felt, there couldn't be knowledge of them). Math has been regarded as the discipline that provides justifications of the strongest kind, namely deductive proofs. But what are mathematical truths true of? (6–7)

Let us bracket the claim about justification (to which I will return shortly) and focus on truth. To dodge the question "what are mathematical truths true of?" and avoid talking about truth altogether, Ferreirós completely abandons the classical analysis of knowledge. He prefers to focus on objectivity instead of truth.[43] I agree with him that concentrating on objectivity rather than truth is better suited to studying mathematical practice. "The problem," as Kreisel famously put it, "is not the existence of mathematical objects, but the objectivity of mathematical statements."[44]

However, I want to indicate here that we are not forced to abandon the traditional analysis of knowledge (that has at its center the notion of true propositions) altogether. The issue is massive; in this context, I just want to suggest that talking about true mathematical propositions could be applied even to Ferreirós's own account of mathematics, or something in line with it, anyway.

Ferreirós asks, "what are mathematical truths true of?" It is plausible to think that mathematical truths are truths about mathematical objects. This reflects Benacerraf's (1973) desideratum (shared by many philosophers of mathematics) of having a homogeneous semantics between mathematical and non-mathematical discourse. How we conceive of these objects is another matter, however. Moreover, there are alternatives to this simple answer. As a way of example, consider Geoffrey Hellman's (1989) modal structuralism. In his view, mathematical truths are about possible systems of objects that exemplify structures. In his words:

> [O]rdinary mathematical statements are construed as elliptical for hypothetical statements as to what would hold in any structure of the appropriate type [. . .] Absolute reference to mathematical objects is eliminated entirely. (53)

Hellman's position is inspired by Hilary Putnam's (1967) seminal (and controversial) paper "Mathematics Without Foundations," in which the author claims that

[41] This is done in Pettigrew (2021).

[42] See, for example, Berto and Nolan (2021).

[43] A similar approach can be (and has been) adopted with respect to science. See for example the discussion in Bird (2010).

[44] Reported in Dummett (1978, xxviii).

it is possible to give two different interpretations of the same mathematical fact: platonistic or modal.

We can endorse our favorite position in the ontology of mathematics – we can be platonists or nominalists, or neither of them.[45] We can adopt a structuralist strategy and think of mathematical objects as positions in abstract structures. This should not, however, force upon us a specific epistemology of mathematics. However, it is true that some metaphysical views may rule out certain epistemologies. For example, some versions of fictionalism take mathematical statements like "2 is a prime number" at face value to be false because they are about objects that do not exist.

Another relevant consideration is that, looking at the practice, it is often natural to focus on *conditional* mathematical propositions. Although traditional inquiries in the epistemology of mathematics have been focused on our knowledge of axioms, from the perspective of the working mathematician, the truth of axioms is seldom addressed.[46] What matters the most from a practice-based perspective is whether certain premises entail certain conclusions.[47]

When we consider mathematical knowledge in practice, we think about mathematicians producing correct proofs for some mathematical proposition. We have a pretty good logical story to understand what *correct* means[48] – although this story might not be readily applied to the practice.[49] It is then plausible to think that mathematical knowledge does not require a commitment to the absolute truth of the axioms we use to prove our theorems, but only a commitment to the correctness of our proofs.

In contemporary mathematics, it is often accepted that mathematical proofs are supposed to establish that their conclusions are true *if* their premises are true. Moreover, when investigating certain domains, it is common practice to discharge certain premises. In Euclidean geometry, when we talk about a theorem T, such as the Angle Sum Theorem, our real conclusion is not the stated conclusion of T, but rather: *in any Euclidean plane*, T.[50] Obviously, the same stated conclusion would not be a truth of hyperbolic geometry. Nowadays, everybody accepts that both Euclidean and non-Euclidean geometry are legitimate mathematical domains, and that we can know geometrical truths.

Thinking of mathematical propositions (or at least some of them) as conditional on the axioms is indeed Ferreirós' preferred approach. He offers a *hypothetical*

[45] See, for example, Burgess (1983).

[46] There are exceptions, however. Sometimes the truth of certain axioms (such as the axiom of choice) is debated by practicing mathematicians.

[47] This is in line with Peirce's view of mathematics. According to him, in deductive reasoning we start from an "hypothetical state of things and are led to conclude that, however it may be with the universe in other respects, wherever and whenever the hypothesis may be realized, something else not explicitly supposed in that hypothesis will be true invariably" (1998, 212).

[48] Even though this story is not universally accepted. For example, it is criticized by constructivist mathematicians. This unveils other instances (in this case linked to intuitionism) in which metaphysical positions can indeed lead one to endorse particular epistemological views.

[49] See Avigad (2021).

[50] To be sure, this holds nowadays, not of the historical practice.

conception of advanced mathematics[51] in which the axioms are only conditionally accepted. In so doing, he indexes truth to specific theories:

> My use of the word 'truth' at this point must be relativized by implicit or explicit reference to a mathematical theory. (Ferreirós 2016, 8)

It is because of this relativization that Ferreirós claims that it is advisable to talk of objectivity rather than truth – but this is no roadblock to understanding even mathematical knowledge in terms of true propositions.[52,53] And indeed Ferreirós himself does not renounce to such talk:

> I shall try to provide a strong account of how we arrive at–and share–mathematical truths, without invoking mathematical objects in any strong ontological sense. (*ibid.*)

Crucially, endorsing Ferreirós's hypothetical view, we do not have to postulate the existence of any mathematical object in a strong ontological sense. This is because rather than *describing* a given mathematical domain, axioms would *constitute* it. A final note on the hypothetical view proposed by Ferreirós is in order. It should not be conflated with a simple form of *if-thenism* in mathematics because, in his view, not all hypotheses are on a par – in short, this is explained by Ferreirós appealing to the internal relationships of different mathematical practices as well as to the relationships of mathematical practices with scientific and technical ones. It is by observing the close connection of mathematics with science and even technical practices (from the elementary ones of counting and measuring) that we can explain the non-arbitrariness of mathematics.

Let me also note *en passant* that talking about truth in mathematics does not need to parallel talking about truth in other contexts. Alethic pluralists contend that there are different ways of being true, which are appropriate to different domains of discourse.[54] For our concerns, it suffices to note that what truth amounts to in mathematics might not be the same compared to what truth amounts to when we talk about ordinary objects such as chairs and tables. For instance, while a correspondence theory of truth might be attractive for the latter, for the former, a coherence theory of truth might be more appropriate.[55] Moreover, there could be ways of modulating the same theory in different domains. For example, in her

[51] It is worth noting that he proposes a different account for elementary mathematics.

[52] Interesting similarities could perhaps be drawn from Putnam's *internal realism*, but Ferreirós does not discuss them.

[53] Ironically, Justin Clarke-Done (2020) uses examples similar to the ones by Ferreirós to argue for mathematical realism and *against* mathematical objectivity. In his view, an objective question admits a single answer. He uses the case of non-Euclidean geometry precisely to highlight the non-objective character of mathematics. Although there is much more to be said about this, for reasons of space I have to postpone the discussion to another time.

[54] To be sure, this claim can be interpreted in many ways – see Pedersen and Wright (2018).

[55] A coherence notion of truth for mathematics is developed by Crispin Wright (1992, 1999). For a detailed discussion on how to apply Wright's ideas to mathematics, see (S. Shapiro 2007).

investigations on logic, Gila Sher (2016) has argued in favor of the correspondence theory of truth. In her view, however, what this view amounts to changes in different contexts – she explicitly considers the case of mathematics.

How we should understand truth in mathematics raises monumental questions that I cannot even start to discuss. But I do not need to do so in this context. My aim was just to indicate that Ferrerirós's own proposal contains the resources needed to apply the traditional analysis of knowledge to mathematics. More generally, the suggestion here is that the epistemology of mathematics – especially an epistemology that aims at understanding mathematics as it is actually practiced – can be (mostly) separated from the metaphysics of mathematics.

Let us then turn to the notion of justification.

3.2 Justification

"Math has been regarded as the discipline that provides justifications of the strongest kind, namely deductive proofs," says Ferreirós. It is for this reason that Gettier cases are generally thought to be impossible in mathematics:

> [I]n general epistemology Gettier cases typically involve instances in which S's justification for p is understood to either *proceed through false assumptions* (as in Gettier's original examples) or be such that it is only "by luck" that it delivers a true belief (as in Goldman's Barn County example[56]). *The former issue is ruled out in the mathematical setting in virtue of the fact that a correct mathematical proof cannot proceed through false lemmas* (and that the presence of such fallacious steps can be decided by examining the structure of PROOF itself – a feature which is not true in the case, e.g., of perceptual evidence). And the latter is ruled out in the mathematical setting because the counterfactual dimension of the relevant notion of "luck" (e.g. that S happened to be looking at the one genuine barn in the field as opposed to a fake one in a nearby world) has no clear mathematical analog.[57] (Dean and Kurokawa 2014, 207, *emphasis added*)

In previous work (2021a), focusing on mathematical propositions conditional on axioms, I argued that there is a plausible account of mathematical justification according to which correct mathematical proofs are not the only arguments that provide mathematical justification – even excluding non-deductive methods. In my view, flawed arguments that look like proofs to the relevant practitioners may confer justification as well. Endorsing a fallibilist account of mathematical justification opens the possibility of Gettier cases arising in mathematics, at least those cases involving false assumptions.

To be sure, in the literature, different forms of mathematical fallibilism have been proposed – many authors have focused on how we justify axioms and urged us to leave behind the idea that they are self-evident. For one, Kurt Gödel (1964) himself

[56] This example is actually due to Carl Ginet.

[57] I changed the notation for the subject and the proposition to make it coherent with the one adopted in the rest of the chapter.

was explicit in claiming that some axioms could be justified only *extrinsically* in terms of their consequences. But as I mentioned in the previous section, it is natural for the philosophers of mathematical practice to focus on mathematical propositions other than axioms.

Imre Lakatos (1976) has linked his fallibilism to the quasi-empirical nature of mathematics and to the fluidity of mathematical concepts. However, the type of fallibilism proposed here is different than Lakatos' because it applies straightforwardly to contemporary mathematics as well, not only to the history of mathematics. Rather than deriving from the open-ended nature of mathematical concepts, it originates from our human inability to get things right 100% of the time.

If the subjects at the center of our inquiry are fallible and have limited resources, then it is reasonable to think that justification in general is not factive, that is, that a subject can be justified in holding a false belief or in holding a true belief for wrong reasons. Mathematical justification should be no exception.

Roughly, in my account, whether a subject is justified is not just a matter of logic but hangs on social considerations as well: arguments that are in line with a community's good inferential practices but that contain subtle mistakes may confer justification. In my terminology, these are *simil-proofs*:

> (SIMIL-PROOF) An argument is a Simil-Proof (SP) when it is shareable, and some agents who have judged all its parts to be correct as a result of checking accept it as a proof. Moreover, the argument broadly satisfies the standards of acceptability of the mathematical community to which it is addressed. (De Toffoli 2021a, 835)

Of course, this is a general proposal that needs to be specified in a case-by-case manner. For instance, we should be able to determine what the standards of acceptability of a mathematical community are. However, it is a start toward a fallibilist account of mathematical justification.

It is worth stressing that claiming that mathematical justification is not merely a matter of logic but depends on social and cognitive factors does not force us into some sort of relativism. There is a fact of the matter whether a simil-proof is a genuine proof or not, but we might not be able to establish this fact at a given time – and a notion of mathematical justification that is faithful to actual mathematical practice should be sensitive to it.

That justification is fallible means that there is a gap between justification and truth. Linda Zagzebiski (1994) showed that when this is the case, we can always construct Gettier cases. Analyzing such cases can give us special insight into mathematical practice because it leads us to the consideration of putative proofs that were accepted by the mathematical community for a time and were later deemed to be inadequate. One such example is Kempe's original proof of the 4-color conjecture – it was first published in 1879 and only found wanting 11 years later.[58] Another is Dehn's Lemma:

[58] See the discussion in De Toffoli (2022).

> (DEHN) In 1910 Max Dehn published an argument for what is now called Dehn's Lemma. In 1929 Hellmuth Kneser discovered that Dehn's argument contained a significant gap and thus that the lemma has not been proved (although it had been used as a premise to prove many other results). It was only in 1957 that Christos Papakyriakopoulos published a proof of it. (De Toffoli 2021a, 837)

It is my contention that Dehn was initially justified in believing his result. He had a justified true belief that did not amount to knowledge. Moreover, in the years in which Dehn's simil-proof was accepted by the mathematical community, it would have been perfectly fine in some contexts to use his lemma to establish other results. And this could lead to additional Gettier cases in which a subject is justified in believing a result, the result is true, but the justification is somewhat disconnected from its truth. In these cases, the subject's justification would proceed through false assumptions.

Building on these ideas, Neil Barton (manuscript) has argued that there are different kinds of Gettier cases in mathematics. Besides cases analogous to the one mentioned, Barton also discusses how Gettier cases arise with respect to our beliefs in axioms. In so doing, he builds on a more general conception of mathematical justification, one that does not apply only to conditional mathematical propositions but to axioms as well.

In light of these considerations, it is fair to say that the possibility of Gettier cases in mathematics was dismissed too quickly. And this in turn shows that even traditional concerns of epistemology, which seem so remote from mathematical practice, can be relevant to it.

4 Conclusion

Although the interests of epistemologists and of philosophers of mathematical practice might seem to be disjoint, they are not. I discussed the issue at a very general level.

To sum up, in order to fruitfully apply epistemology to the philosophy of mathematical practice, we have to reconceive the epistemic subjects along lines that have already been traced by social and feminist epistemologists.[59] The subjects at the center of our epistemological theorizing should be embodied, fallible, and embedded in a specific mathematical context. Bringing down the level of idealization at play for the knowing subjects leads us naturally to a fallibilist view of mathematical justification. This, in turn, opens the door to Gettier cases, which, contrary to what one might expect, can be of interest to philosophers of mathematical practice. This is a signal that other unexpected points of contact between the two disciplines could be found.

Moreover, I also stressed that epistemological concerns are better kept separate from metaphysical ones. It is my contention that we can pursue an analysis of

[59] Other relevant approaches in epistemology are ones inspired by the embodied cognition research program, broadly understood to include embedded, enactive, and extended cognition (Shapiro and Spaulding 2021).

mathematical knowledge, even within the traditional epistemological framework, without having to endorse any specific position in the metaphysics of mathematics.

Let me end by listing some specific themes in epistemology that are directly relevant to the philosophy of mathematical practice – themes that I could not discuss for reasons of space. I mentioned that the traditional focus on individual propositional knowledge could be broadened to include different forms of social knowledge as well as an analysis of knowledge how. Moreover, I alluded to the fact that the case of mathematics could be used to inquire about the nature of the basing relation – the relationship that holds between a belief and the reasons on which it is based. There are many other themes in epistemology that have already been applied, or that could be applied to the study of mathematical practice. Here are a few examples: epistemic defeaters (Easwaran 2015), themes in virtue epistemology (Aberdein et al. 2021), epistemic injustice (Rittberg et al. 2020), and disagreement (Aberdein 2023; De Toffoli and Fontanari 2023).

My goal was to lay the ground for more such specific applications and therefore to indirectly encourage them. More generally, I hope that clarifying how we should reconceive the epistemological subject(s) if we want to pursue an inquiry about justification and knowledge in mathematics that is faithful to the practice will dispel some skepticism with respect to the application of epistemology to the philosophy of mathematical practice.

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