WHAT ARE MATHEMATICAL DIAGRAMS?

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ABSTRACT

Although traditionally neglected, mathematical diagrams have recently begun to attract attention from philosophers of mathematics. By now, the literature includes several case studies investigating the role of diagrams both in discovery and justification. Certain preliminary questions have, however, been mostly bypassed. What are diagrams exactly? Are there different types of diagrams? In the scholarly literature, the term "mathematical diagram" is used in diverse ways. I propose a working definition that carves out the phenomena that are of most importance for a taxonomy of diagrams in the context of a practice-based philosophy of mathematics, privileging examples from contemporary mathematics. In doing so, I move away from vague, ordinary notions. I define mathematical diagrams as forming notational systems and as being geometric/topological representations or two-dimensional representations (or both). I also examine the relationship between mathematical diagrams and spatiotemporal intuition. By proposing an explication of diagrams, I explain (away) certain controversies in the existing literature. Moreover, I shed light on why mathematical diagrams are so effective in certain instances, and, at other times, dangerously misleading.

Keywords: Mathematical Diagram, Visualization, Illustration, Mathematical Proof, Heuristic, Anschauung

1. INTRODUCTION

Mathematical diagrams[1] come in many varieties and are used in disparate domains. They aid mathematicians in various phases of their careers: learning, discovering, proving, and teaching. But starting in the 19th century and well into the 20th, the use of diagrams was either ignored or

[1] Where there is no risk of ambiguity, I will simply use "diagram" to refer to "mathematical diagram."

subjected to considerable skepticism. They have a bad reputation, and not without reason. For example, they are thought by philosophers and mathematicians alike to seduce us with their pleasant appearances and draw us away from the inflexible scheme of rigorous mathematical proofs.[2] Diagrams are often thought of as good guides to discoveries but poor companions for proofs. But this is not an accurate representation of mathematical practice. Diagrams are indeed fruitful in guiding us to new results.[3] They are, however, also helpful in proving.[4] There are diagrams, such as, for example, the ones used in category theory, that are akin to algebraic notational systems in two dimensions. Other diagrams, such as knot diagrams, enable us to *tame* our geometric/topological imagination and use it reliably.[5] This means that, even if we restrict our attention to proofs, we can no longer simply disregard diagrams.

But we can also broaden our focus. The *philosophy of mathematical practice* aims at considering previously disregarded aspects of mathematical activity. Mathematical practice in all its colorful forms has become a precious well from which to draw important questions – and yield important answers.[6] In particular, probing the use of diagrams in mathematics leads philosophers to important epistemological issues. The scholarly literature on diagrams now includes a plethora of case studies.[7] These are not, however, linked by a single unifying

[2] It is not necessary, in this context, to work with a specific characterization of mathematical rigor. It will suffice to refer to John Burgess's (2015) broad characterization. Roughly, rigorous proofs are proofs that include enough information to convince (for the right reasons) appropriately trained mathematician(s) that they could be converted into formal proofs. Note that Burgess's characterization of rigor can be tweaked to include pre-19th century mathematics (De Toffoli, 2020).
[3] See, for example, (J. Carter, 2019).
[4] (Halimi, 2012; Mumma, 2010; Shin, 2004).
[5] (De Toffoli & Giardino, 2014; De Toffoli, 2020).
[6] See (Mancosu, 2008; Carter, 2019) for overviews of the philosophy of mathematical practice.
[7] See (Giaquinto, 2015) for a survey. See, for example, (Macbeth, 2010; Manders, 2008; Panza, 2012; Mumma, 2012) for Euclidean geometry, (Chemla, 2018) for ancient Chinese diagrammatic practices, (J. Carter, 2010; De Toffoli, 2017; De Toffoli and Giardino, 2014; Starikova, 2010) for diagrams in contemporary mathematics, and

framework, and the term "diagram" is used in divergent ways.[8] In this article, I sketch the outline of such a unifying framework, privileging examples of diagrams from contemporary mathematics. I gesture towards a *Carnapian explication* for diagrams. Instead of trying to clarify an ordinary, vague concept, I propose another, more precise one in order to identify the phenomena in which scholars working on diagrams are most interested. The framework I offer will help to order the literature on mathematical diagrams and resolve certain controversies.

In ordinary discourse, *diagrams* are often associated with pictures. In mathematics, this has led some scholars to identify diagrams with geometric or topological pieces of notations (see Figure 1(a)). This choice might be appropriate to study diagrams in the history of mathematics. It presents, however, two problems. First, it does not distinguish between diagrams and illustrations (more to come on this). Second, it is not appropriate for contemporary mathematics since it does not include what are perhaps the most ubiquitous diagrams appearing in mathematical journals nowadays: commutative diagrams (see Figure 1(b)).

(a) (b)

Figure 1. A knot diagram and a commutative diagram

(Bellucci and Pietarinen, 2016; Shin, 2002; Schlimm, 2018; Macbeth, 2014) for diagrams in logic. See (Giaquinto, 2007) for a book-length study.

[8] Marcus Giaquinto (2007) provided the most systematic discussion of diagrams, but he does not aim at drawing a general, unified picture.

The proposed account distinguishes between diagrams and illustrations. Only the former are elements of notational systems and thus can be used rigorously in mathematics. Moreover, it treats *diagrams* and what I call *geometric-topological* (GT) notational items separately. Roughly, I define the latter by appealing to their geometric or topological traits, which enable specific visualizations. I define diagrams, instead, as elements of notational systems that are either GT or two-dimensional (or both).

The distinction between GT and non-GT notations roughly traces the "two roads of mathematical comprehension" identified by Hermann Weyl (1932): topology and abstract algebra. Non-GT notations, such as standard algebraic formalism or commutative diagrams, facilitate a broadly algebraic understanding. GT notations enable us instead to think geometrically or topologically about a specific issue, which itself does not necessarily pertain to geometry or topology but can also arise in different branches of mathematics. For example, Venn diagrams are GT diagrams used to reason about logical relations.

Despite the vast amount of new work focusing on diagrams, what has traditionally been skepticism regarding their importance has by no means disappeared.[9] Two assumptions undergird the reluctance to seriously engage with diagrams: 1) Diagrams, like other devices mathematicians use in their reasoning, can be disregarded as philosophically irrelevant *notational variance*. If the same informational content can be conveyed in words (and indeed it can, more or less trivially), then there is no reason to bother with diagrams. 2) There is no principled way to distinguish diagrams that can be used reliably from ones that cannot. I

[9] (Burgess, 2015, p. 98).

challenge both assumptions. The first is based on an excessively narrow restriction of the domain of philosophy of mathematics.[10] The second is based on the failure of distinguishing diagrams from *illustrations*. Just the former are systematic representations that form mathematical notational systems (characterized below) and can be used reliably in proofs.

The plan for the paper is as follows. To get a purchase on the conceptual space occupied by diagrams, I start by discussing mathematical notational systems in general. I motivate their philosophical importance and provide a broad characterization of them. This is the burden of Section 2. In Section 3, I introduce GT notations. I characterize them with the aid of the notion of EMI, *Enhanced Manipulative Imagination*, introduced in (De Toffoli and Giardino, 2014) in the context of a study of knot diagrams. I then turn to mathematical diagrams in their full generality and propose a novel definition. Next, I discuss their typical two-dimensional layout. In Section 4, I show that the proposed taxonomy is indeed fruitful. I explain what differentiates diagrams from illustrations, and I show that my definitions help to resolve controversies on the role of diagrams in proofs in real analysis. I also indicate further applications. In Section 5, I sum up the discussion.

Although I touch on epistemological issues (i.e., by way of neutralizing some of the classical complaints about the use of diagrams in proofs), I leave the analysis of the epistemic role of diagrams aside for future discussion. The aim of this article, in other words, is first and foremost to settle specific foundational issues.

[10] This is in line with the methodological guidelines of the philosophy of mathematical practice, see (Mancosu, 2008).

2. NOTATIONS MATTER

In what follows, I propose an account of mathematical diagrams that puts them on all fours with non-diagrammatic representations forming mathematical notational systems. Following Keith Stenning (2002):

> Systems of representations are sets of objects (sentences, diagrams, films . . .) each of which stands for something else. What makes for system in these sets is that the representations bear relations to each other which correspond in useful ways to the relations between the situations they stand for. (p. 10)

Stenning's analysis focuses on Euler diagrams in which topological relations correspond to logical relations. As I shall explain, it is precisely because they form notational systems that diagrams *can* (but might not) be deployed in proofs;[11] I thus exclude messy and inchoate sketches or other pictures that can be used exclusively for heuristic purposes.[12] This choice might appear to be excessively restrictive. It is not.[13] The majority of studies on diagrams in mathematics consider systems of diagrams, or, in any case, *systematic diagrams*, that is, diagrams that could form systems of representations. In this section, I will discuss mathematical notational systems in general. I will then zero in on diagrams in Section 3.

[11] This does not mean that diagrams are essential in proof – although I make the point in (De Toffoli, 2022) that they can indeed be essential in proofs.
[12] Notice that even pictures and messy sketches could be considered as notations, see (Goodman, 1976; Kulvicki, 2003). However, I am here focusing on genuinely *mathematical* notations and, as we shall see, the bar for what counts as a notation is higher.
[13] This is in line with, for example, (Stenning, 2002, Ch. 2) and (Shin, 2002).

2.1 MATHEMATICAL NOTATIONAL SYSTEMS

As every mathematician experiences in her work, choosing the right notational system can be crucial for solving a problem or proving a claim. In his influential work on heuristics in mathematics George Pólya (1945) writes:

> An important step in solving a problem is to choose the notation. It should be done carefully. The time we spend now on choosing the notation carefully may be repaid by the time we save later by avoiding hesitation and confusion. (p. 136)

Mathematical notational systems are heterogeneous, spanning a wide variety, from algebraic symbolic notations to diagrammatic systems. Examples of notational systems commonly used in mathematics are numerals such as Hindu-Arabic numerals (1, 2, 3, . . .), fractions (e.g., 2/3), decimal expressions (e.g., 3.1415…), letters of the Greek alphabet to refer to particular irrational numbers (e.g., π, or the aura section ϕ), and the classical notations for square roots (e. g., $\sqrt{2}$).[14] More sophisticated notations are used in calculus, and of course notations are crucial in logic as well.[15] Notational systems can also be formed by diagrams, examples of which are Euclidean diagrams, knot diagrams, commutative diagrams, and Venn diagrams.[16]

[14] Florian Cajori's (1928) massive two volume collection of mathematical notations contains many examples of notations for both elementary and advanced mathematics. For a detailed survey on numerical notations see (Ifrah, 1994).
[15] See (Dutilh Novaes, 2012).
[16] See (Brown, 2008, Ch.6) for an analysis of mathematical notations starting from the example of knots.

Mark Colyvan (2012) presents examples in which mathematical notational systems (including diagrammatic ones) are especially important and concludes:

> it might be tempting to suggest that a mathematical object by any other name would be just as useful. But we have seen that this is not so. Sometimes the names encode properties of the objects in question and in such cases other names would be less revealing. [...] In short, properties of the notation are important in mathematics. (p. 170)

Still, in the philosophical jargon, the expression *notational change* is often taken to be equivalent to *superficial* or *non-essential chang*e. And this does track a real phenomenon: changing the font of a mathematical text, its color, or even its language (say from Italian to English) does not lead to any mathematical change. That is why it is crucial to give an account of mathematical notations that, like the one proposed here, individuates notations disregarding changes that are not *mathematically* relevant.

Brown (2008, Ch. 6) and Colyvan (2012, Ch. 8), have pointed out the similarity between mathematics and poetry exactly because of the importance of notation in these two domains. There is, however, a significant dissimilarity with respect to what notation is in poetry and in mathematics. Whereas *poetic notation* is often phonetic natural language also associated with its written counterpart,[17] *mathematical notation* is made of symbols, diagrams, and other inscriptions. Whereas poetic notation is already available (since it is mostly natural language),

[17] Poems in which the shape of the written text matters exist but are not prototypical, at least in the Western tradition.

mathematical notation is made of inscriptions invented specifically for mathematical purposes. In this respect, mathematical notations are much more similar to musical notations, in which the specific notational devices (such as the staff) are inventions with the specific aim of collecting and displaying information in a specific domain.

I consider a mathematical notational system to be a system of items that can be inscribed and that have a specific mathematical interpretation.[18] As explained by Stenning in the quote above, relevant relations between different elements of the same system correspond to relations of what they represent. Moreover, different elements of the same system can be obtained by transforming or combining other elements according to system-specific rules.[19] Other instruments that are relevant in this context are non-notational tools invented for numerical calculation, such as the many kinds of abacus, Napier's bones, and Pascal's calculator.

When analyzing notational systems in mathematical practice, it becomes clear that what matters is not only the carried information, but the possible manipulations that the notational systems support, that is, those manipulations corresponding to mathematical operations (e.g., multiplication, differentiation, group composition, etc.). Consider the case of Arabic and Roman numerals. The two systems support different calculations, and therefore it is often not possible to solve the same numerical problem performing the same calculation in Arabic and Roman numerals. That is, to solve the same problem, we might need to perform different calculations

[18] It is worth noting that although it is common to focus on external inscriptions, notations can also be imagined.
[19] (De Toffoli, 2017). In this context, notational systems are included in what Ferreirós (2016) calls "symbolic frameworks" of a mathematical practice. Notational systems can be used in stable practices in a codified manner.

composed of different steps and a different number of them.[20] In this respect, a new notation might not lead to more results, but it might change the mathematical practice at issue radically.

Limiting my analysis to notational systems does not prevent me from including historical, not entirely precise systems of diagrams such as Euclidean diagrams or diagrams that mathematicians use to find and verify proofs, but are not included in the relative publication.[21] Still, for a representation to be *systematic*, that is, to be (at least potentially) part of a system of representations, it must be in principle possible to spell out the information it carries and the rules to transform it. In contemporary, rigorous mathematical proofs, diagrammatic manipulations correspond to mathematical operations previously defined. They can, therefore, be systematically linked to a discursive format involving clear mathematical concepts. This is not the case for ancient practices. For instance, Euclid, lacking an axiom of continuity, did not have the inferential resources to prove the existence of certain intersection points – such as the one needed in Proposition I 1 – and, at least according Manders's (2008) account, had to rely on diagrams and the implicit knowledge of how to use them without appealing to previously stated inferential rules.

Whereas inscriptions themselves are uninterpreted items, elements of notational systems are interpreted. It is the specific interpretation that assigns the criteria of individuation for elements of notational systems. In order to spell out such identity criteria, I introduce the notion of *enabling* and *constitutive* features.

[20] Contrary to a common opinion, it is indeed possible to calculate with Roman numerals, albeit in ways we are not used to (Detlefsen et al., 1976; Schlimm & Neth, 2008).
[21] An example of such diagrams used in free probability theory is presented in (Carter, 2010).

2.2 ENABLING AND CONSTITUTIVE FEATURES

In the passage quoted above, Colyvan concludes that "properties of the notation are important in mathematics." This is true, but even *non-mathematical* features of notations are important in mathematics. When a formula is printed in an excessively tiny font, it becomes hard to decipher. And being able to perceptually access mathematical symbols is important in mathematics. Nonetheless, enlarging the same symbols merely *enables* the practitioners to access them. The size has nothing to do with the mathematical reasoning supported by the notation. We thus have to single out the mathematically relevant aspects of a notational system. I propose to distinguish between 1) *enabling* and 2) *constitutive* perceptual features and uses of a notational system.[22] Concerning the perceptual features of a given notation, only a subset of them is relevant and can carry mathematical content. For example, intersection points are constitutive features of Euclidean diagrams but not of commutative diagrams (see Figure 1(b)); the specific metric properties in both cases are merely enabling. With respect to the uses of a given notation, constitutive uses correspond to the calculations that can be performed with it, that is, to the supported manipulations that have precise mathematical meaning. For Euclidean diagrams, they are geometric constructions; for linear algebraic notations, they include symbolic manipulations, and, in the case of knot diagrams, such manipulations include ambient isotopies and *Reidemeister moves.*[23] Enabling uses are, for example, writing them on different media such as pieces of paper, blackboards, or computer screens.

[22] More precisely, I should speak of *formal* features instead of *perceptual* features because I hold that notations can also be imagined. However, most of our interactions with notations pass through one of their physical instantiations, so I will keep speaking about perceptual features for simplicity's sake.

[23] These are local diagrammatic moves that do not alter the represented knot. The Reidemeister theorem tells us that they can be used to connect any two diagrams of the same knot (type), see (Adams 1994, Ch. 1).

The distinction between enabling and constitutive features applies directly to both non-diagrammatic and diagrammatic notational systems. By way of example, let us consider the notational system constituted by Euclidean diagrams. Colors are merely enabling features of standard Euclidean diagrams, just as they are enabling features of most symbolic notations, and, for this reason, should be disregarded. If Figure 2 (a), representing a diagram used to prove the Pythagorean theorem,[24] were red instead of black, it would be correctly interpreted as not only representing the same geometric configuration but also the very same diagram. There are, however, notational systems in which color is a constitutive feature. One of them is deployed in the beautiful edition of the first six books of the *Elements* by Oliver Byrne (1847), Figure 2 (b). The constitutive uses are going to be, in both cases, geometric constructions.

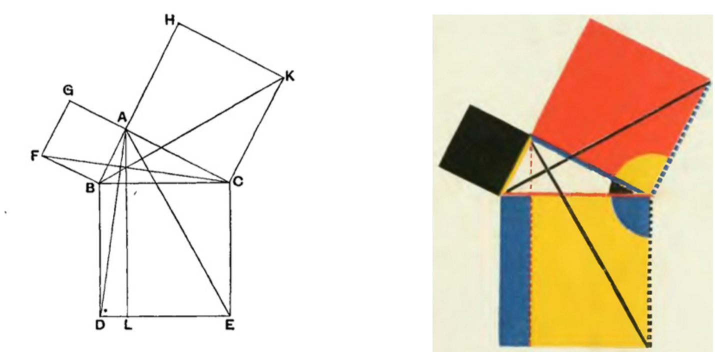

Figure 2. Diagrams belonging to different notations used to prove the Pythagorean Theorem

Crucially, the interpretation of a notational system includes information about which perceptual features are constitutive. Generally, due to our familiarity with them, it is trivial for

[24] Proposition I 47 of the *Elements.*

us to identify the constitutive features of linear symbolic notational systems.[25] However, as I shall discuss below, the situation is not so simple with diagrams – for example, are the exact angles in the triangle ABC in Figure 2 (a) constitutive? The answer is No, but not obviously so. Moreover, it is even less obvious that they could not be. I suggest that this is because both constitutive perceptual features and constitutive uses of the elements of a notational system must satisfy some general, basic constraints:

(1) The constitutive perceptual features are clearly identifiable.
(2) The constitutive perceptual features are reproducible.
(3) The constitutive uses are cognitively manageable.

These conditions admit degrees. Yet, there are minimal thresholds that must be met by any notational system. This is because of our human cognitive and perceptual limitations.[26] In Section 4, I will explain how these conditions are modulated in the particular case of diagrammatic notational systems.

These three constraints can be fleshed out as follows. (1) That the constitutive features should be clearly identifiable means that the informational content (that is, the information we can extract from it) of an expression in a notational system can in principle be reliably carried by a finite number of propositions. This is uncontroversial for linguistic expressions as well as for expressions in non-diagrammatic mathematical notations. However, it holds in general. This is

[25] For these types of representations, we can use the usual type/token distinction.
[26] With respect to perceptual limitations and mathematical notations, see (Azzouni, 2017). See also (Goodman, 1976).

because there can only be a finite number of constitutive features, and they can be related in a finite number of ways. Notational systems, diagrammatic or not, are, in this sense, *discrete* – otherwise we could not use them reliably. For example, in the case of Euclidean diagrams, the information that is carried by the diagrams can be codified in few propositions, as has been done in (Avigad, Dean, and Mumma, 2009). In the case of knot diagrams, the informational content can be carried by specific numerical codes.[27] Notice that to successfully use a notational system and recognize it as such, a mathematical community need not possess the theoretical resources required to convert its expressions into propositions.[28] To be sure, Euclidean diagrams were recognized and used well before the work of Avigad, Dean, and Mumma. It must, however, be *in principle* possible to extract the informational content of expressions in a notational system in propositional form.[29] This means that while I adopt modern tools to characterize notational systems, this is not the only way to do so. In the case of Euclidean diagrams, the fact that they were integrated into a stable mathematical practice might have been sufficient to identify them as genuine notational systems. Here it is important to note that, as I will discuss below, this does not necessarily make the particular format of diagrams dispensable.

(2) In order for the constitutive perceptual features to be reproducible, they have to be stable, and it must be possible for an average practitioner with the *appropriate* tools (such as computer software) to copy an expression in a given notational system. The tools have to be appropriate because they must enable the practitioner not only to reproduce specific expressions but also to create new ones. For example, taking a photograph of a drawn knot

[27] (Adams, 1994, Ch.2).
[28] Thanks to one of the anonymous referees to raising this issue.
[29] See (Feferman, 2012, p. 381) for a discussion of this point.

diagram will not be an appropriate way of reproducing it.[30] Of course, the difficulty involved in reproducing different notations varies greatly.

(3) The manipulations with mathematical meaning are relatively easily applied by an average practitioner with the relevant training. Because of the first condition, diagrams must be convertible into propositional form. This third condition implies that the manipulations of the notational system must also admit a description in terms of propositions. This does not mean, however, that the manipulation of a notational system and its conversion are necessarily on par from a cognitive and epistemological perspective. In fact, this is generally not the case. Moreover, observing historical practices we realize that mathematicians had *practical* knowledge of how to manipulate notations (knowledge-how) without necessarily being aware of explicit rules (knowledge-that).[31]

Notational systems satisfy these three constraints. Single notational items that do not belong to an actual notational system can satisfy them as well. These are notational items that can be integrated into a mathematical notational system but might not (yet) be. I include these into the category of *systematic* notational items.

3. DIAGRAMS: A TAXONOMY

Before starting to sketch out a framework for diagrams, let me address a preliminary concern, namely, whether such a general framework is needed at all. Doubts might arise from the fact

[30] Printing different copies of the same book including diagrams is perfectly fine. The issue here is not with reproducibility in general, but with reproducibility in the context of a mathematical use.
[31] This point is emphasized in (Netz, 1999) for the case of Euclidean geometry.

that it is only by focusing on specific case studies that we can appreciate subtle aspects of the use of different notational formats: diagrams are so heterogenous that a unified analysis might seem utterly uninteresting and lacking in adequate specificity. But this is not the case, or so I claim. Notwithstanding their differences, mathematical diagrams share important features.[32] My aim is to provide an explication of diagrams that is both faithful to actual mathematical practice and helpful in explaining why diagrams can be especially effective reasoning tools, both within and beyond the context of proofs. The definitions I will propose will serve to both create order in the existing literature and resolve certain controversies.

To bring clarity to debates surrounding the nature and role of diagrams, I follow Carnap's methods of *explication*. This is based on the idea of substituting a vague concept with a clearer one. My aim is to put forward a concept that, if not perfectly exact, is at least more precise than the pre-theoretical one. I do not draw a sharp line between diagrams and non-diagrams. My definition will allow for borderline cases. This is not a problem – there will be room for further explications.[33] Moreover, as Timothy Williamson observed, "showing that a distinction has borderline cases does not show that it is unhelpful for theoretical purposes" (2013, p. 294). Crucially, however, the divisions I propose can be exemplified by way of distinct prototypical cases.

In common jargon, the term *diagram* tends to pick out multiple properties. Marcus Giaquinto suggests that "there are several independent features whose presence or absence

[32] I restrict my attention to mathematics. My account, however, generalizes without any modification to certain diagrams used in theoretical physics, such as Feynman diagrams. See (Brown, 2018) for a friendly introduction to the topic.
[33] The process of explication can be, in fact, iterated multiple times (Dutilh Novaes and Reck, 2017).

unconsciously affects our inclination to classify thinking as symbolic or diagrammatic" (2007, p. 248). This is plausible. But not all of such independent features carry the same weight. I submit that two features are predominant in influencing our judgments of what a mathematical diagram is: 1) the presence of geometrical or topological elements that trigger a special form of spatio-temporal imagination and 2) the fact that they are *two-dimensional*.

My plan is to start by discussing the concept fitting the first point and isolate the sub-category of *GT diagrams*. To do this, I will invoke a special form of imagination that is enabled by GT diagrams and is akin to what Felix Klein labeled "refined intuition" (1893). In Section 3.3, I will turn to the second point and define mathematical diagrams in their full generality.

3.2 GEOMETRIC-TOPOLOGICAL DIAGRAMS

Let us start by considering geometric or topological notational items such as Euclidean diagrams and M.C. Escher illustrations of tiling. These usually trigger spatiotemporal imagination, or intuition. The term *intuition* has been used in many different ways and has a long philosophical history. Relevant to the current discussion is intuition intended as the translation of the German term *Anschauung*.[34] This is also a widely discussed notion that admits various interpretations – especially in relation to the Kantian tradition.[35] In this context, however, I wish to use this term as it is commonly used in the context of mathematics. In English, this term is at times translated as *intuition* and other times as *imagination* – I prefer to keep the German in order to avoid confusion. A popular book aimed at developing Anschauung in mathematics is *Anschauliche*

[34](Burgess, 2015, p. 27).
[35] See (Friedman, 2000; 2012) for an analysis of the nature of intuition as including a kinematic component in Immanuel Kant and his successors Hermann von Helmholtz and Henri Poincaré.

Geometrie (1932) by David Hilbert and Stephan Cohn-Vossen. The title is commonly translated in English as *Geometry and the Imagination* (1990) – but it could also have been translated as *Intuitive Geometry*. Hans Hahn's popular essay "Die Krise der Anschauung,"[36] is translated into English as "Crisis in Intuition" (1980) and deals with the pitfalls of Anschauung in dealing with infinite processes that became clear at the end of the 19th century with the rigorization of analysis.

Diagrams and illustrations have been traditionally associated with Anschauung – and Hilbert and Cohn-Vossen's volume contains plenty. Much more recently, John Conway and his co-authors (2018) write:

> Imagination, an essential part of mathematics, means not only the facility which is imaginative, but also the facility which calls to mind and manipulates mental images. (p. 9)

We are now ready to propose our first definition:

> *Definition 1. A notational item is* ***geometric-topological (GT)*** *if geometric or topological perceptual features are relevant and enable the use of Anschauung.*

[36] This is the title of Hans Hahn's (1980) influential article that originated from a lecture he gave in Vienna in the 1920s.

The domain of GT notational items includes all sorts of geometric illustrations, from very clear ones to messy sketches. They tend to trigger Anschauung in the viewer. It is important to consider that Anschauung does not pertain only to a mathematical field but designates a way of reasoning. Roughly, GT representations lead us to the road of topology identified by Weyl (1932).

Felix Klein (1892) suggested that a university's mathematics department should have at least one full professor for each of the following three types of mathematician:

1) The philosopher, who thinks with concepts,
2) The analyst, who essentially manipulates formulas,
3) The geometer, whose starting point is "Anschauung."[37]

Of course, a single mathematician should be proficient in all three modes of reasoning, but she will generally privilege one in particular. In order to pursue her research, the geometer might start with a naïve form of Anschauung but will have to refine it in order to prove results.

Klein (1897) proposed to distinguish between "naïve Anschauung" and "refined Anschauung." [38] Naïve Anschauung is innate but develops in our experiences:

[37] "1) Der Philosoph, der von den Begriffen construirt,
2) Der Analytiker, der wesentlich mit der Formel operirt,
3) Der Geometer, der von der Anschauung ausgeht."

[38] (Torretti, 1978, p. 147). See also (Bråting and Pejlare, 2008, p. 347).

> Mechanical experiences, such as the one we have in the manipulation of solid bodies, contribute to forming our ordinary metric intuitions, while optical experiences with light rays and shadows are responsible for the development of a 'projective' intuition. (p. 593)

Klein recognized both the importance and the limits of naïve Anschauung. With mathematical training, such limits can be pushed, and a "refined Anschauung" can be developed. This new type of Anschauung is shaped by our knowledge of precise mathematical definitions. And it is at play when manipulating diagrams rigorously. Klein (1893) proceeds by explaining that

> It is [refined intuition] that we find in Euclid; he carefully develops his system on the basis of well-formulated axioms, is fully conscious of the necessity of exact proofs, clearly distinguishes between the commensurable and incommensurable, and so forth. (p. 41)

Klein's suggestions are insightful and my proposal is inspired by them. However, I do not aim here to do exegetical work but only to contextualize and motivate my use of the concept "Anschauung."

By definition, GT notational items enable Anschauung. More precisely, being systematic representations, GT *diagrams* enable the use of a form of refined Anschauung. In a previous work on knot diagrams in collaboration with Valeria Giardino (2014), we called this form of Anschauung *enhanced manipulative imagination, EMI* in short. Like Klein's naïve intuition, EMI

> derives from our interaction with concrete objects and our familiarity with manipulating them. (pp. 830-831)

However, it is enhanced insofar as it is developed with training and has to be applied according to specific rules:

> In topology, which is informally referred to as 'rubber-band geometry', a practitioner develops the ability to imagine continuous deformations. Manipulations of topological objects are guided by the consideration of concrete manipulations that would be performed on rubber or other deformable material. (*Ibid.*, p. 831)

The manipulations we perform using EMI are tightly connected to mathematical concepts. For example, in knot theory we can use topological moves to show the equivalence of different knot diagrams. We can deploy EMI in knot theory because specific conventions on knot diagrams make it easy for us to interpret them as representing curves *in space* and thus trigger our visualization of three-dimensional space. For example, we can convince ourselves that the two diagrams in Figure 3 represent the same knot by imagining pulling the middle strand in Figure 3 (a) down.

Figure 3. Two equivalent knot diagrams (De Toffoli and Giardino, 2014, p. 833)

These diagrams can actually be shown to be representing the trivial knot, that is, to be unknotted:

> This move alone only allows us to conclude that both diagrams represent equivalent knots; to see that they actually represent the unknot we would have to apply further similar moves. In the following, we present a formalization for these moves. (*Ibid*., p. 834)

While these moves are supported by EMI, they admit of formalization because they are subjected to specific rules.[39]

We are now ready to define GT diagrams:

> *Definition 2. A **GT diagram** is a systematic notational item whose constitutive perceptual features include geometric or topological elements that enable the use of EMI.*

[39] See also (De Toffoli, 2020).

GT diagrams are the most prototypical type of diagrams. They include knot diagrams and Euclidean diagrams.[40]

While Euclidean diagrams are manipulated according to geometric constructions and rigid motions, topological diagrams can be manipulated in ways that track homeomorphisms (that is, continuous maps with a continuous inverse). In both cases, EMI is involved. Crucially, the use of EMI is subjected to precise rules (not necessarily explicit) – this is because in the present proposal diagrams are always systematic. This is not the case for the type of Anschauung at play with the use of GT notational items in general. I will return to the distinction between diagrams and illustrations in Section 4.

EMI refers to our intuition of space and accounts for the possibility of imagining precise topological and geometric transformations on GT diagrams. It is enhanced by training (and therefore, it is not innate), and it involves kinesthetic sense as well, and thus cannot be reduced to visual imagination alone. Moreover, it does not require visual perception since it can be activated by three-dimensional models or diagrams in bas-relief.[41] EMI is not only at play in geometry and topology. For example, using Venn diagrams or other logical diagrams, we reason

[40] One anonymous referee pointed at the threat of circularity in this definition. This danger arises from the use of geometric-topological notions to explicate diagrams while at the same time the concept of diagram that we form by observing mathematical practice and its history is itself tied to geometry and topology. I do not think there is a vicious circularity here. I take it that we know what geometric-topological notions are and instead of associating them with all diagrams, I use them to characterize a special kind of diagrams (and representations more generally). The issue, however, is a complex one and I cannot discuss it in detail in this context.

[41] Consider this example involving a blind mathematician. George Francis and Richard Morin (1979) were the first to spell out an explicit sphere eversion. As Silvio Levy (1995) observes:

> Morin, incidentally, is blind, and the fact that he was one of the first people to understand how a sphere can turn inside out is both a tribute to his ability and a convincing proof that "visualization" goes far beyond the physical sense of sight.

Here the term "visualization" is another term that can be traced back to Anschauung.

with the help of systems exploiting enclosure relations that can be interpreted and manipulated with the aid of EMI.

The stage has now been set to discuss mathematical diagrams in all of their generality.

3.3 DIAGRAMS

Before the middle of the 20th century, the vast majority of mathematical diagrams were GT.[42] However, we should not be tempted into thinking that all diagrams are GT. Examples of non-GT diagrams are shown in Figure 1(b) and in Figure 4.

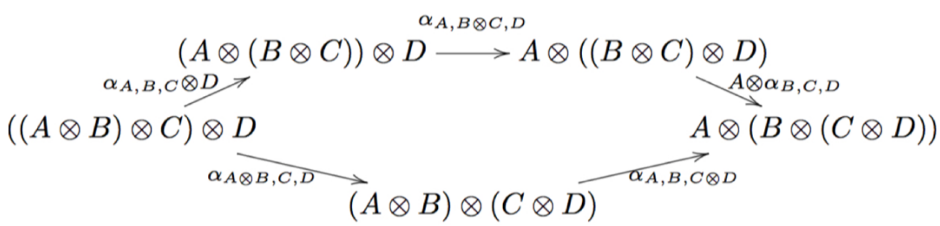

Figure 4. A commutative diagram, the "pentagon axiom" of monoidal categories, see Sec. 3.1 of (Selinger, 2009).

It seems plausible that diagrams can be defined in terms of their two-dimensionality:

*Tentative Definition. A **mathematical diagram** is a two-dimensional systematic notational item.*

As previously mentioned, by *systematic*, I do not mean that it necessarily belongs to a full-fledged mathematical notational system, but only that it could since it satisfies the three criteria listed in Section 2.1.

[42] (Johansen, Misfeldt, and Pallavicini, 2018).

As we shall see in Section 4.2, systematicity comes in degrees. The same is true for two-dimensionality. A two-dimensional representation exploits in a non-trivial way the space of the page, i.e., it is not only constituted by sequences of symbols. Of course, also sequences of symbols exploit two-dimensionality, but they do it in a somewhat trivial fashion: 1) the symbols themselves are two-dimensional, and 2) the sequence is divided into different lines (or columns). In a classic paper, Jill Larkin and Herbert Simon (1987) also use two-dimensionality to define diagrams. However, their work concerns not only mathematical diagrams but also diagrams used in other intellectual endeavors:

> we have provided a simple distinction between sentential representations, in which data structure is indexed by a position in a list, with each element 'adjacent' only to the next element in the list, and diagrammatic representations, in which information is indexed by location in a plane, many elements may share the same location, and each element might be 'adjacent' to any number of other elements. (p. 98)

The definition above includes non-GT diagrams but leaves all non-systematic representations out. Non-GT notational items also trigger a particular type of cognitive ability linked to our perceptual system, but it is not, like EMI, an ability that is essentially activated by topological or geometrical features of the representations. According to Giardino (2018), diagrams in general can be viewed as *multirecruiting systems*. That is,

> they constitute an 'interface' with which to integrate information coming from perception or action, already functioning in pragmatic contexts, and other more cognitive resources such as conceptual knowledge. (p. 173).

Although non-diagrammatic notations can also function as multirecruiting systems, diagrams are particularly effective due to their two-dimensionality or/and the presence of geometric or topological constitutive features. Moreover, two-dimensionality is conducive to phenomena that often increase the effectiveness of notations: multiple-readability, aspect shifting, and the presence of free rides.[43]

Alternative definitions of diagrams in the literature exclude non-GT diagrams.[44] This choice might be appropriate to study geometric diagrams in the history of mathematics but is too restrictive to investigate diagrams used in mainstream contemporary mathematics.[45] While GT diagrams pose deep challenges to the traditional ideal of mathematical proof, non-GT diagrams are generally accepted in proof.

In the proposed definition, two-dimensional representations are contrasted with one-dimensional representations, or linear representations. However, once again, the distinction is

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[43] See, for example, (Shin, 2015). Discussing these three phenomena is beyond the scope of the current article.
[44]See (Barwise & Etchemendy, 1996; Shimojima, 1999; Stenning, 2002). Keith Stenning's terminological choices diverge from mine partially because of our different goals and the different sets of examples. My primary aim is to track mathematical practice rather than to provide a definition targeted to psychologists and mathematics education scholars. I mainly focus on successful mathematical diagrammatic notational systems – some of which, such as node-and-link diagrams or commutative diagrams, are not diagrammatic in Stenning's sense – and primarily on epistemic issues. I thus prioritize correctly characterizing those examples. For reasons of space, I cannot discuss Stenning's proposal here in detail.
[45] It is true that there is a substantive difference here, but symbolic diagrams cannot be ignored. For example, the mathematician John Sullivan (2012) feels the need to distinguish commutative diagrams from knot diagrams and calls the former "formal diagrams" and the latter "informal" – still, both are *diagrams*.

not as clear-cut as it can seem at first. Written natural language is mostly linear.[46] Linear, in this case, means that we canonically read written text from beginning to end.[47] But in special contexts we can read a world in a different direction, such as when we appreciate the phenomenon of palindromes. Here is one: *Amore, Roma*.

Moreover, there are a number of mathematical notations that present two-dimensional arrangements, but nevertheless remain mostly linear insofar as a canonical reading direction is usually adopted. Common examples are the standard notations for functions, (multi) exponential, continued fractions, and integrals:

$$\frac{x_1+x_2+x_3+x_4}{y_1y_2y_3},\ x^{2^{2^{2^{\{\dots\}}}}},\ a_1+\frac{1}{a_2+\cdots\frac{1}{a_n+\cdots}},\ \int_a^b x^2dx.$$

Equation 1. Formulae presenting two-dimensional elements

Linear expressions can be arranged in two dimensions. Sometimes we write equations in different lines to read them both line by line and vertically. Calculating with Arabic numerals, we exploit the two-dimensionality of the page to create cognitively useful vertical alignments. Derivations in sequent caluculs also have a two-dimensional structure. These, however, are not single notational items, like, say, a commutative diagram, but groups of representations temporarily linked together only for a specific purpose.

Going further into typical two-dimensional representations, we encounter displays made of symbols arranged in two dimensions, such as tables and matrices. Reading them line by line,

[46] It is plausible that this derives from the phonetic origin of language and the linearity of time (Krämer, 2003).
[47] I will gloss over the fact that written natural languages include devices, such as footnotes, that interrupt their linearity.

they resemble linear displays, but at the same time they allow us to define operations on them that their linear counterparts do not support, such as the calculation of the determinant or the multiplication of matrices, which works by reading the first matrix to multiply line by line and the second one column by column. The fact that a matrix is not just a sequence of symbols allows experts to identify specific patterns, such as symmetries. Matrices therefore share important similarities with diagrams. However, in order to respect our pre-theoretical distinction between diagrams on the one hand and table and matrices on the other, I will exclude them. I therefore stipulate that tables and matrices are not two-dimensional in the relevant sense.[48] One rationale behind this choice is that genuine two-dimensional notational items can be *read* with similar cognitive effort in more than one or even two directions.[49]

But there is a lingering problem with the tentative definition proposed. Although most diagrams exploit the two-dimensionality of the plane in a non-trivial way, there are exceptions. Atsushi Shimojima (1996) gives the example of a linear map representing relative distances from different cities to Indianapolis:

Indianapolis——Bloomington————————Louisville—Michigan City

Figure 5. A linear map

[48] Azzouni (2020, p. 73) suggests that diagrams can be characterized through their two-dimensionality. More specifically, they "rely on distinctively two-dimensional inference packages." The dimensionality of inferential packages is, in turn, characterized in terms of the visual capacities they enable. According to Azzouni's proposal, matrices and notations such as the ones in Equation 1 rely only on adjacency relations and are therefore not diagrammatic. This is in line with the current proposal.

[49] Thanks to Michael Stuart for this suggestion. Note that this is compatible with the fact that certain non-diagrammatic notational items admit multiple readings (Bellucci & Pietarinen, 2016).

Although this notational item is linear, it is both systematic and GT and it is plausible to considered it to be a diagram.   Therefore, the previous definition needs to be revised:

> *Definition 3. A* ***mathematical diagram*** *is a systematic notational item that is either GT or two-dimensional, or both.*

In what follows, I will use the term *diagram* in this technical sense.[50]

I added an exception to the two-dimensionality of diagrams exclusively for GT representations.[51] Other linear representations are, in fact, commonly conceived as not being diagrammatic. For example, although an exact sequence (such as the one in Figure 6) can be considered to be a limit case of a commutative diagram, following standard terminology, I do not consider it to be a diagram.

$$\cdots \to X_{i-2} \to X_{i-1} \to X_i \to X_{i+1} \to \cdots$$

Figure 6. An exact sequence

It is precisely to rule out exact sequences that Mikkel Johansen et al. (2018) decided to limit the domain of diagrams to two-dimensional representations:

[50] Another proposed way to distinguish diagrams from other notational items is to consider the perspectival nature of pictures. Roberto Casati and Valeria Giardino (2013) distinguish pictures from diagrams by the fact that the former are *perspectival* (i.e., perspective or viewpoint is indispensable for their interpretation) and the latter are not. This distinction seems to track a cognitive difference, since there is evidence that our visual system operates in and through perspectival representations. Moreover, it distinguishes between illustrations of three-dimensional mathematical models from diagrammatic notations for logic such as the one formed by Venn diagrams. Nonetheless, it radically departs from mathematical practice since it excludes that knot diagrams and other geometric or topological diagrams are, in fact, diagrams.

[51] There are other linear GT diagrams. For example, the one constituted by Leibniz's lines is a system that is equivalent to Euler diagrams, but it is constituted by lines of different length rather than by circles. Thanks to Javier Legris for pointing me to this example.

> It is our goal in this paper to create an instrument that makes it possible to track and understand aspects of mathematical practice, so we have to make pragmatic compromises. One such compromise is to adopt the criterion that a diagram has to be two dimensional, even if this criterion may seem arbitrary from a theoretical perspective [.] (p. 112)

In the proposed definition, two-dimensionality is not a necessary condition for diagrams, but it *is* a necessary condition for non-GT diagrams.

The new proposed definitions cut the space of notational items used in mathematics in three orthogonal ways:

1) systematic vs. non-systematic and
2) GT vs. non-GT
3) Two-dimensional vs. non-two-dimensional

These distinctions are not sharp. Above I have shown this with respect to two-dimensionality. In Section 4 I will show that systematicity also comes in degrees. Moreover, some diagrams are non-GT but can be modified to include GT elements. For instance, in commutative diagrams, we can integrate other expressions in different ways, such as in Figure 7 —this example comes from higher category theory, and the curved lines are best interpreted as belonging to GT notations.

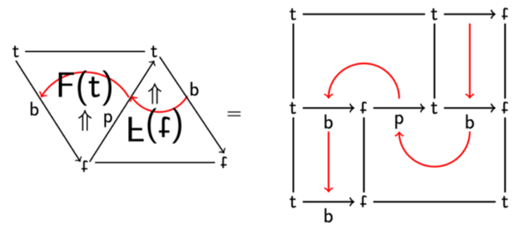

Figure 7. Commutative diagrams of 1-arrows with additional 2-arrows in red, from (Carter and Kamada, n.d., 136)

Figure 8 illustrates the proposed taxonomy.

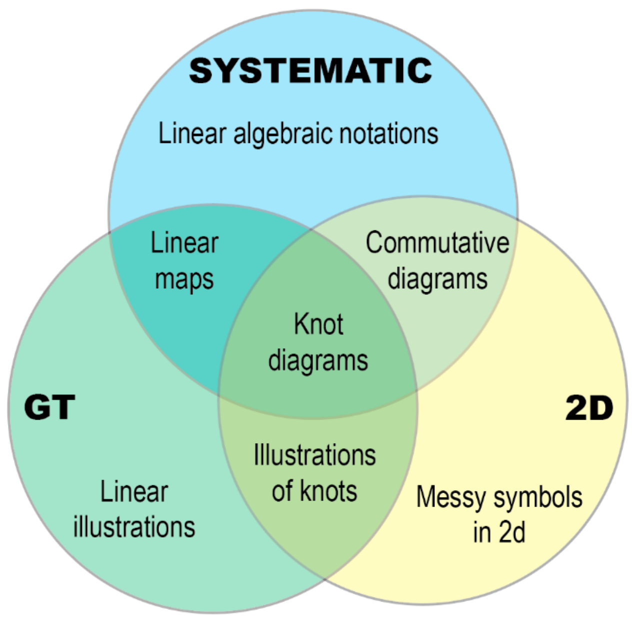

Figure 8. Taxonomy

Diagrams are at the intersection of the systematic representations with either GT figures or with two-dimensional figures. Knot diagrams (as well as Euclidean diagrams) are at the very center of the Venn diagram. This was to be expected, as they are, in fact, prototypical mathematical diagrams.

4. FRUITFULNESS OF THE TAXONOMY

Some advantages of this new taxonomy are that: 1) it explains why and in virtue of what features diagrams can play a non-trivial role in proofs, and 2) it allows us to create order in the existing literature by resolving certain controversies. Moreover, it lends itself as a basic framework to study diagrams from the perspective of cognitive psychology. In 4.1 I discuss the features of diagrams that make them adequate to enter into the inferential structure of proofs. I start by considering Brendan Larvor's proposal (2019) and put forward a revised version of it. In 4.2, I explain that systematicity is a graded notion and that, in some cases, the same notational item can be interpreted either as a diagram or as an illustration. Afterward, in 4.3, I show how we can use the distinctions I introduced to resolve a controversy about the use of diagrams in proofs in real analysis.

4.1 NECESSARY CONDITIONS TO BE A DIAGRAM

In the proposed taxonomy, GT diagrams are systematic. This is what distinguishes them from illustrations.[52] Salvador Dalí's *Crucifixion (Corpus Hypercubicus)* and M.C. Escher's *Möbius Strip II* are not diagrams. They can, however, be appealed to in mathematics in order to illustrate geometrical and topological concepts. Illustrations commonly drawn by geometers are generally less aesthetically pleasing but serve a similar purpose. Such illustrations range from simple

[52] This approach is also endorsed in (De Toffoli and Giardino, 2014) with respect to knot diagrams.

sketches to meticulous drawings, such as the one in Figure 9, which represents a modification of a three-manifold by a process called *Dehn surgery*.

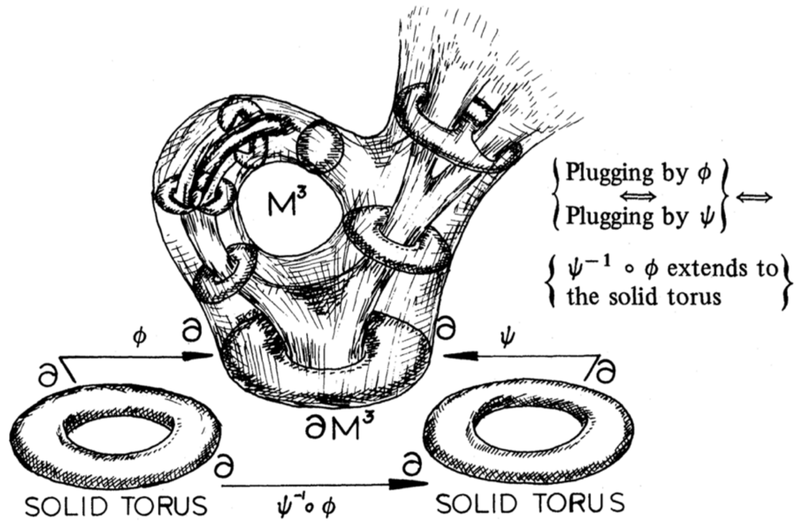

Figure 9. Modifying a 3-manifold by Dehn surgery, (Thurston, 1982, p. 361).

Beyond their importance as didactic tools, illustrations play crucial roles in research contexts as well. For instance, they include computer-generated images that allow geometers to visualize intricate structures. They can even be a *sine qua non* condition for new discoveries.[53] Nonetheless, illustrations are not systematic representations. Their informational content is often unclear. For example, is the particular pattern of ramifications part of the information conveyed in Figure 9 or is it an accidental feature? As we have already seen, similar questions also arise for Euclidean geometry. In the case of diagrams (such as Euclidean diagrams), they are solved by appealing to the distinction between constitutive and merely enabling perceptual

[53] Paolo Mancosu (2005, p. 19) reports a case in which computer-generated images guide researchers in finding a proof.

features. When we use a geometric diagram in order to obtain a general result, we have to be careful not to rely on those features of our diagram that are not shared with all other diagrams of the same geometrical object.[54] It is for this reason that it is of paramount importance to ensure the possibility of distinguishing between features that convey information and ones that do not, that is, between constitutive and enabling perceptual features. Since this distinction is not generally clear for illustrations, these are typically used as heuristic devices that enhance understanding – but cannot play a more important role in proofs.

Diagrams, however, can lift heavier epistemological weight because they can form mathematical notational systems, and thus their interpretation and use are subject to precise constraints (as we have seen, more or less explicit).[55] That diagrams are systematic implies that they must satisfy the conditions for notational systems listed in Section 2.2:

(1) The constitutive perceptual features are clearly identifiable.

(2) The constitutive perceptual features are reproducible.

(3) The constitutive uses are cognitively manageable.

Working specifically with the case of diagrams, Larvor (2019) proposed a list of conditions for diagrams to be used systematically, that is, the kind of diagrams that can appear in proofs (and thus, for me, to count as diagrams *tout court*):

[54] That is, we have to avoid over-generalizations.

[55] Diagrams, as characterized here, share core similarities with what Azzouni calls "diagrammatic algorithmic systems" (Azzouni 2020, p. 31).

(A) They are easy to draw.

(B) They are not metrical.

(C) It is “possible to put the inferences into systematic mathematical relation with other mathematical inferential practices.” (*Ibid.* p. 2735)

I share what I take to be the gist of Larvor’s proposal. In point of fact, I want to show that the conditions (1), (2), and (3) I proposed for notations in general can be seen as generalizations of conditions (A), (B), and (C). The latter are designed by Larvor with the case of Euclidean diagrams in mind. They apply very well to that case, as well as for other cases of diagrams. *As stated*, however, conditions (A), (B), and (C) do not apply to all diagrams I have been discussing.

First, diagrams are not always easy to draw. Even the small knot in Figure 1(a) is not especially easy to draw; larger knot diagrams can be extremely difficult to draw (both by hand and with the aid of technological tools). Flipping through Rolfsen’s (1976) *Knots and Links,* it quickly becomes evident that diagrams are not always easy to draw. However, I take the idea behind this condition to be that *it must be possible for an average agent in normal circumstances with the appropriate training and the appropriate tools to reliably reproduce a diagram*.[56] This is a specification of (2).

Second, the idea that diagrams should not be metrical is inspired by the influential work done by Manders (2008) on Euclidean geometry. To give a brief account of a longer and more complicated story, Manders’ insight consists of dividing information that can be read off from

[56] I added the clause: “in normal circumstances” to avoid counterexamples involving specific impediments such as tiredness or sickness.

diagrams from information that has to be carried by the text. On the one hand, what he called *exact* attributes are exact metric properties (e.g., lengths and angles) that can only be carried by the text. On the other hand, *co-exact* attributes roughly correspond to topological features (e.g., intersection points) and are invariant under small perturbations – thus, they can reliably be read-off diagrams.[57] Larvor's condition is thus intended to exclude the appeal to exact attributes of a diagram in a proof. In the proposed terminology, Larvor is right that exact metric features of Euclidean diagrams cannot be constitutive.[58] However, (non-exact) metrical features *can* be constitutive in certain diagrams.[59] For example, the fact that the distance between Indianapolis and Louisville is much greater than the one between Indianapolis and Bloomington is a metric constitutive feature of the diagram in Figure 5.[60] Nevertheless, it is a stable and easily identifiable feature. Certain computer visualizations can also offer precise metric details in a reliable way.[61] In light of these considerations, we can formulate a condition in the vicinity of the

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[57] A well-known hurdle for Manders' account is that after certain construction steps, diagrams that differed only with regards to metric properties give rise to diagrams that have different topological features (Mumma, 2010). For an alternative analysis of Euclidean geometry, see (Beere & Morison, n.d.). In this context, I can remain neural with respect to the correctness of Manders' analysis for the specific case of Euclidean geometry. In particular, the analysis of specific historical mathematical practices is not relevant to the current discussion.
[58] To be precise, the problem is not solely with metric properties, but with any property that is not invariant under small perturbation. For example, if Byrne used similar shades of red instead of red, blue, black, and yellow, his diagrams would have brought confusion and would have been hard to interpret correctly.
[59] Actually, this holds also for Euclidean diagrams themselves. For example, we can reliably talk of a segment being shorter than another if it is included in the former – this type of metric feature is in fact co-exact in Manders' terminology.
[60] In certain passages, it seems Larvor (2019, p. 2721, p. 2724, p. 2733) uses the term "metric" simply to refer to exact properties in the context of Euclidean diagrams. If that is true, then we agree for the case of Euclidean geometry and my contribution consists in extending his conditions to other cases.
[61] For example, graphs of functions obtained with visualization software are *sensitive*: small alterations change the identity of the graph. They are not, however, unreliable representations, (Kulvicki, 2003, p. 328). Moreover, in other geometric practices, like the one of ancient Chinese mathematics, diagrams are supposed to represent certain metric properties, and they do so reliably; see, for example, (Chemla, 2018). Thanks to an anonymous referee for this point.

one proposed by Larvor: *Diagrams' constitutive perceptual features are easily identifiable and carry mathematical content reliably*. This can be traced to condition (1).

Larvor's last condition can be interpreted, using the terminology I introduced, as: *diagrams' constitutive uses correspond to well-defined mathematical operations.* This means that the manipulations supported by diagrams correspond to precise mathematical operations and are connected to the foundational apparatus of the field in question. This should hold if the manipulations on diagrams can be put into "systematic mathematical relation with other mathematical inferential practices". Of course, these manipulations must be defined exclusively on the constitutive perceptual features of the diagrams. As we saw, thanks to condition (1), diagrams admit of specific codifications. This third condition tells us that it must be possible to describe the constitutive uses of diagrams with such codifications and is thus equivalent to the general condition (3). As we have seen, this last condition is critical if we want to use diagrams rigorously. However, it is key to remark that it does not exclude that it is possible to appeal to refined Anschauung, or, more precisely, to EMI, in a rigorous way.[62]

It is crucial to bear in mind that this does not mean that diagrams and their codifications are mutually translatable since they might support different manipulations corresponding to mathematical operations and different ways of extracting information. The same exact information can be conveyed in two different formats, and this makes a difference in the way we can access and use it, see (De Toffoli, 2022). This is clearly seen with GT diagrams that allow us to exploit EMI, which is not true of their corresponding codifications. However, this holds also for non-GT diagrams. In certain cases, non-GT diagrams also support a specific type of reasoning

[62] (De Toffoli, 2021).

that is not supported in their linear codification. One example is given by proof by *diagram chasing*.[63]

To summarize, here is the new list of conditions that have to be satisfied by diagrams:

(A) It is possible for an agent with the appropriate training and the appropriate tools to reliably reproduce a diagram.

(B) Diagrams' constitutive perceptual features are easily identifiable and carry mathematical content reliably.

(C) Diagrams' constitutive uses correspond to well-defined mathematical operations.

4.2 DEGREES OF SYSTEMATICITY

I now turn to the possibility of diagrams entering the inferential structure of mathematical proofs. Skepticism *vis-à-vis* the use of diagrams in proofs derives, at least in part, from the fact that it is not always easy to distinguish them from illustrations. Both diagrams and illustrations are *interpreted* figures. In some cases, the same *figure* (i.e., the physical token on the paper or the mental image) can not only be interpreted as different diagrams, but also as an illustration.

To complicate things, the diagram/illustration distinction actually represents a spectrum rather than a dichotomy. I characterized diagrams as systematic representations. However, I glossed over the fact that systematicity comes in degrees. Interpreted figures can be ordered

[63] (De Toffoli, 2017).

with respect to their degree of systematicity.[64] Commutative diagrams and knot diagrams are certainly systematic, and thus diagrams, but things are not always so clear. There are representations that do not clearly fall on either side of the divide between diagrams and illustrations. In Figure 10 are three topological sketches representing orientable closed connected surfaces of genus one (i.e., a torus, or doughnut), of genus two, and of genus three (i.e., a pretzel).[65] These sketches are common among topologists. Moreover, they form a generalizable system: it is easy to draw an orientable closed surface of genus *n* following the same pattern.

Figure 10. Closed connected orientable surfaces of genus one, two, and three

These sketches can be interpreted both as diagrams and as illustrations. If they are interpreted as diagrams, then they have to satisfy all three conditions listed in the previous section. This implies, in particular, that they can be given a precise interpretation and rules of manipulation. For example, we can use them to define operations such as the *connected sums* (with this operation, we can, for example, attach together two tori and obtain a surface of genus two). As illustrations, they can be used informally to evoke topological objects.

[64] (De Toffoli and Giardino, 2015, p. 328).
[65] The *genus* corresponds to the number of holes.

In order to overcome this ambiguity, we can set a systematicity threshold for diagrams, one that increases and decreases depending on the context. It can be set at the level of the least systematic representations that are generally accepted to play a non-redundant role in proofs in a certain mathematical practice. Since the standards of what counts as a proof are different in different fields and have become more stringent since the introduction of modern logic at the end of the 19th century, the threshold will not be fixed but movable.[66] While it is reasonable for a scholar of ancient mathematics to consider Euclidean diagrams as fully systematic notational systems, this is not necessarily the case for scholars interested in contemporary mathematics.

4.3 A CONTROVERSY IN ANALYSIS

It is true, however, that it is not always easy to determine what is acceptable in proofs. A famous point of contention is the role of diagrams in real analysis. One much-discussed example is the graph usually accompanying proofs of the Intermediate Value Theorem (IVT). This theorem states that any continuous curve f on an interval [a,b] takes any value between f(a) and f(b). This means that for all c such that c is in [f(a), f(b)] there exists x such that f(x) = c, see Figure 11.

[66] See (De Toffoli, 2020) for a discussion on the standards of acceptability in mathematics.

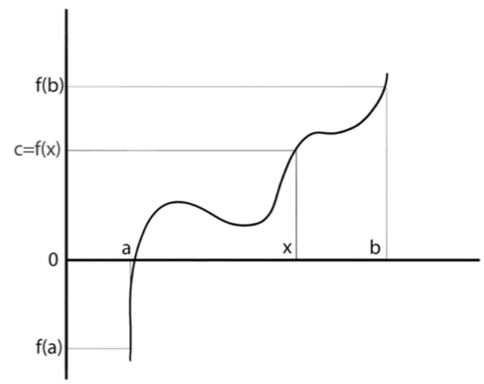

Figure 11. Intermediate Value Theorem

Marcus Giaquinto (2011) claims that one is not warranted to infer the existence of an intersection point (the one between the line parallel to the x-axis going through c intersects the curve) by looking at the graph in Figure 11. James R. Brown (2008, p. 29) and Jody Azzouni (2013, pp. 327-329) disagree. According to the received view, the IVT was first established by Bolzano in the early 19th century. Brown argues, however, that "using the picture alone, we can be certain of this result – if we can be certain of anything" (*Ibid.*, p. 29) and thus that the result was established well before Bolzano's contribution to what Klein called the "arithmetization of analysis." It is crucial to notice that Brown considers the figure to be an illustration: "the way the picture works is much like a direct perception; it is not some sort of encoded argument" (*Ibid.*, p. 30). Giaquinto takes issue with this approach. He argues that the property relevant to the theorem (i.e., ε-δ continuity) cannot be visualized.[67] As he correctly observes, an inference supported by the figure would be unreliable if it was used to reason about functions on a domain

[67] Giaquinto (2011) contrasts this case to the one of Euclidean geometry. In short, according to him, in this latter case it is permissible to infer the existence of an intersecting point from a diagram because of the perceptual origin of our geometric concepts.

that is not complete, such as that of the rational numbers. The IVT is a theorem about the real numbers, and the key property at play is their *completeness*,[68] which must, therefore, be (directly or indirectly) appealed to in proving the theorem. Crucially, however, this is not a property that a graph of a function can externalize.

Azzouni (*Ibid.*) has a different take on this example.[69] He argues that the domain of application of the diagrammatic inference is only conventionally assigned to the figure and can be understood to include only a subclass of all ε-δ continuous functions.

> For we can certainly precisely delineate an application class of curves if we want to: everywhere smooth curves, for example. It's important to realize […] that the class of mathematical objects that a diagrammatic proof can be supposed to apply to is not determined on the basis of the kinds of things the diagram appears to apply to (according to what mathematical objects its figures resemble). (p. 328)

As we saw, the same figure can be interpreted either as a diagram or as an illustration. And even when interpreted as a diagram, we can still interpret it as a diagram of a curve in the class of continuous curve, or a diagram of a curve in some restricted class of curves. Restricting the class of curves to which the theorem applies, we lose generality but not rigor, according to Azzouni.

With the aid of the proposed framework, the divergence between Giaquinto and Azzouni can be explained (away) in terms of their different interpretations of the same figure. On the

[68] That the reals are complete means that any non-empty set of real numbers which is bounded above has a least upper bound.
[69] See also (Azzouni, 2020, p. 65).

one hand, criticizing Brown, Giaquinto interprets it as an illustration, helpful to give us a rough idea of the theorem but inadequate to be appealed to in a proof. On the other hand, Azzouni interprets the figure as a diagram that has a specific interpretation and thus can be used in a proof. That is, pointing at the same figure, Giaquinto and Azzouni are not referring to the same notational item. This is not an isolated case.[70] Often disputes of this type can be resolved by giving an explicit account of what is intended with the word *diagram*.

Another issue worth exploring is the relative reliability of diagrammatic and non-diagrammatic proofs. Even granting the legitimacy of diagrammatic proofs, are diagram-free proofs always to be preferred? There is evidence that this is not the case. As David Corfield (2003) remarks for the case of monoidal categories, the probability of introducing an error in an attempt to prove a theorem would be higher if we were to adopt the linear algebraic notation:

> having followed a representation of the proof written in standard linear notation, one has no reason to be more confident in its correctness than having followed the pictures. (p. 255)

Note that Corfield's "pictures" are diagrams in my sense. In Jeremy Avigad's (2019) analysis, visualization and diagrammatic reasoning are listed among the strategies that might increase the reliability of informal proofs. However, this is a matter for further research.

[70] See, for example, (Burgess 2015, p. 98).

5. CONCLUSION

The use of diagrams and illustrations in mathematics is puzzling. At times they are effective cognitive aids of the mathematician while other times, they are avoided because they are deemed to be dangerous. In order to explain these different features, I distinguished diagrams from illustrations. I also offered a definition of diagram in the spirit of Carnap's explication. Although my definition does not perfectly track how we use the word *diagram* in ordinary discourse, it captures the phenomena of most importance for a practice-based taxonomy of diagrams in mathematics. I started by discussing mathematical notational systems in general. I then introduced the category of GT notational items, which can be illustrations or diagrams.

I characterized diagrams as being systematic, that is, forming mathematical notational systems, and as being either GT or two-dimensional, or both. While GT diagrams invite a geometric-topological mode of reasoning, non-GT diagrams are more akin to algebraic expressions. I then explained that all notational systems have to satisfy specific constraints. These are often trivially met in the case of non-diagrammatic notational systems, but not for diagrammatic ones. The fact that these constraints are usually not explicitly stated might have contributed to the reluctance to accept diagrams in proofs.

In order to spell out such constraints, I distinguished between *enabling* and *constitutive* features of a notational system, and I used such distinction to sharpen the constraints proposed by Larvor (2019) that diagrams must be subject to in order to be used rigorously. I articulated a second list, intended to unpack the conditions that genuine diagrams must meet:

(A) It is possible for an agent with the appropriate training and the appropriate tools to reliably reproduce a diagram.

(B) Diagrams' constitutive perceptual features are easily identifiable and carry mathematical content reliably.

(C) Diagrams' constitutive uses correspond to well-defined mathematical operations.

These conditions guarantee the systematicity of diagrams and explain why we can use them in a proof without incurring the risk of compromising the proof's reliability.

I traced the efficacy of GT diagrams in the fact that they enable the use of EMI (enhanced manipulative imagination), which is a powerful cognitive ability connected with Anschauung. In particular, by using EMI we manage to exploit spatiotemporal and kinematic imagination in a regimented way, that is, in a way that tracks precise mathematical operations. I also suggested that it is plausible to think that the effectiveness of diagrams in mathematics can be connected to the fact that they typically exhibit a two-dimensional layout. This is because two-dimensional representations are conducive to three phenomena that increase their efficiency as mathematical notational systems: multiple readability, the possibility of aspect shift, and the presence of free rides.

With the proposed taxonomy, I shed new light on the intriguing phenomena involved in mathematical diagrams. I showed that getting clear on the terminology can help order the existing literature and resolve specific points of contention. Moreover, I hope that this work can guide further research on diagrams, both properly philosophical, on the epistemology of mathematical diagrams, and empirical, on the cognitive abilities triggered by diagrams and their usefulness in various contexts, such as teaching and research.

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