

Cantor, Choice, and Paradox^{*}

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Dedicated to the memory of J.T. Chipman

Abstract

I propose a revision of Cantor’s account of set size that understands comparisons of set size fundamentally in terms of surjections rather than injections. This revised account is equivalent to Cantor’s account if the Axiom of Choice is true, but its consequences differ from those of Cantor’s if the Axiom of Choice is false. I argue that the revised account is an intuitive generalization of Cantor’s account, blocks paradoxes—most notably, that a set can be partitioned into a set that is bigger than it—that can arise from Cantor’s account if the Axiom of Choice is false, illuminates the debate over whether the Axiom of Choice is true, is a mathematically fruitful alternative to Cantor’s account, and sheds philosophical light on one of the oldest unsolved problems in set theory.

1 Introduction

Given sets A and B , one or both of which may be infinite, there are three questions we may ask concerning their comparative size:

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1. Is A at least as big as B ?
2. Is A the same size as B ?
3. Is A strictly bigger than B ?

As [Pruss \(2020\)](#) notes, these questions may be interpreted in two main ways, depending on how we understand the notion of set size. If we understand the notion in the “count” sense, then these questions broadly concern *how many* members A has as compared to B . If we understand the notion in the “measure” sense, then these questions broadly concern *how much* of A there is as compared to B . I will be concerned with set size in the former sense. In particular, I will interpret the above questions as follows:

1. Does A have at least as many members as B ?
2. Does A have exactly as many members as B ?
3. Does A have strictly more members than B ?

These questions are widely thought to have been settled by Cantor in the late 1800s.¹ I will propose an alternative answer to them in this paper.

To spell out Cantor’s account of set size, a few preliminary notions are needed. First, say that a function from B to A is an *injection* just in case it maps any two distinct members of B to distinct members of A . Second, say that a function from B to A is a *surjection* just in case every member of A gets mapped from some member of B . Finally, say that function from B to A is a *bijection* just in case it is both injective and surjective.

Cantor’s account of set size says the following:

- (1) A is at least as big as set B if and only if there is an injection from B into A .
- (2) A is the same size as B if and only if there is a bijection from B to A .
- (3) A is strictly bigger than B if and only if there is an injection, but no bijection, from B to A .

¹See [Cantor 1883/1996](#). I will formulate Cantor’s ideas in modern set-theoretic terms in what follows.

While Cantor’s account is widely accepted—indeed, it is the basis of the standard definition of “cardinality” found in all modern set theory textbooks—it is not universally accepted. Perhaps the most prominent objection to Cantor’s account is that it conflicts with the intuitive idea—known as the Part-Whole Principle—that every set is strictly bigger than any of its proper subsets.² This conflict between Cantor’s account and the Part-Whole Principle has led to the development of alternative accounts of set size—most notably, *numerosity theory*—that satisfy the Part-Whole Principle but violate Cantor’s definitions.³

In this paper, I will propose a revision to Cantor’s account that is closer in spirit to Cantor’s account than to numerosity theory. Indeed, I will motivate this revised account via considerations that are unconnected to the Part-Whole Principle. The key difference between Cantor’s account and the revised account lies in how they understand the *at least as big as* relation. According to Cantor’s account, A is at least as big as B just in case there is an injection from B into A . According to the revised account, A is at least as big as B just in case there is a *surjection* from A onto B (or B is empty). In what follows, I will refer to these accounts as the *injective* and *surjective* accounts, respectively. As I will explain later, we can understand the intuitive difference between these accounts as follows:

- Injective account: A is at least as big as B if and only if any two distinct members of B can be paired with distinct *members* of A .
- Surjective account: A is at least as big as B if and only if any two distinct members of B can be paired with disjoint *parts* of A .

Here the “parts” of a set are to be understood as nonempty subsets of that set. So, the members of a set—or, if we prefer, the set’s singleton subsets—can be viewed as the *smallest* parts of that set. In this respect, the surjective

²For example, there is a bijection from the set of even numbers to the set of integers, even though the former is a proper subset of the latter. According to the right-to-left direction of (3), the latter set is *not* strictly bigger than the former. However, by the Part-Whole Principle, the latter set *is* strictly bigger than the former.

³See [Mancosu 2009](#) and [Parker 2009](#) for further discussion of this conflict. Also, strictly speaking, several versions of numerosity theory have been developed since [Benci and Di Nasso \(2003\)](#) first introduced the theory (in English); these versions are largely unified by [Benci and Baglini \(2022\)](#). However, I will speak of “numerosity theory” as a single entity for simplicity in what follows since the points I will make apply to all versions of the theory.

account is a formal generalization of the injective account. I will argue that it is also an *intuitive* generalization of the injective account. Moreover, I will show that the two accounts are equivalent if the Axiom of Choice is true, but their consequences differ if the Axiom of Choice is false.

The Axiom of Choice is the following claim: for any *set* of sets that are nonempty and mutually disjoint, there exists a set that contains exactly one member from each of these sets (and nothing else).⁴ Intuitively, the Axiom of Choice says that it is always possible to arbitrarily *choose* exactly one member from each of these sets—even if there are no distinguishing features among the members of any such set—and thereby create a new set that contains all and only these chosen members. The Axiom of Choice is the most controversial axiom among the standard axioms of set theory, largely because both it and its negation have been thought to lead to paradoxical consequences.

One prominent allegedly paradoxical consequence of the negation of the Axiom of Choice is the possibility known as the *Division Paradox*: a set can be partitioned into a set that is *bigger* than it. The possibility of the Division Paradox is mathematically unassailable if we assume the injective account of set size. However, I will show that the surjective account entails that any set is at least as big as any partition of it—and is never smaller than it—*regardless of whether the Axiom of Choice is true*.⁵ Moreover, since the possibility of the Division Paradox has sometimes been mounted as an argument for the Axiom of Choice, I will argue that the surjective account illuminates the debate over whether the Axiom of Choice is true—namely, by revealing that one prominent argument for the Axiom of Choice can be blocked simply by adopting an intuitively compelling alternative account of set size.

I will also argue that the surjective account of set size is a mathematically fruitful alternative to the injective account. While the two accounts are equivalent if the Axiom of Choice is true, they lead to their own distinctive theories of set size in the absence of the Axiom of Choice: the injective account leads to a theory of “injective” set size; the surjective account leads

⁴See [Rubin and Rubin 1985](#) for many equivalent formulations of the axiom.

⁵So, the surjective account entails that, if the Axiom of Choice is false, A can be at least as big B even if there is no injection from B into A —i.e., the left-to-right direction of (1) can fail if the Axiom of Choice is false. Thus, like [Whittle \(2018\)](#) and [Pruss \(2020\)](#), I will raise a problem for the left-to-right directions of Cantor’s definitions. However, neither author appeals to considerations that distinctively involve the Axiom of Choice.

to a theory of “surjective” set size. As we will see, these two theories have some notable features in common yet differ in interesting ways.

Additionally, I will argue for virtues of a class-theoretic *generalization* of the surjective account. This generalization compares sizes not only among sets but also among *proper classes*—that is, collections that fail to form sets. In particular, I will explain that this generalization ensures that a version of the Division Paradox cannot arise for proper classes, while the analogous class-theoretic generalization of the injective account only ensures this if the class-theoretic generalization of the Axiom of Choice known as *Global Choice* is assumed. Global Choice is the following claim: for any *collection* (set or proper class) of sets that are nonempty and mutually disjoint, there exists a collection that contains exactly one member from each of these sets (and nothing else). I will also show that the class-theoretic generalization of the surjective account leads to a mathematically fruitful line of inquiry regarding various other class-theoretic analogues of the Axiom of Choice.

Finally, I will argue that the surjective account sheds philosophical light on one of the oldest unsolved problems in set theory. This is the question of whether the claim that every set is at least as big as any partition of it entails the Axiom of Choice, where “at least as big as” is understood in the manner of the injective account. As I will explain, this question is straightforwardly answered in the negative when “at least as big as” is understood in the manner of the surjective account.

All of that said, I will not argue that the injective account is “wrong” in some objective sense of the term, as I will neither assume nor deny that there is a uniquely correct conception of “set size.” Indeed, the present paper may happily be read in the spirit of [Parker \(2019\)](#)’s “pragmatic pluralism,” according to which different conceptions of “set size” are suitable for different purposes. Nonetheless, I will argue that the surjective account is an intuitively compelling, mathematically rich, and philosophically illuminating alternative account of set size—and of class size more generally.

While some of this paper is quite technical, I have structured it so that the main thread of philosophical discussion will be accessible to readers who have minimal set theory background. I have placed the more technical aspects of discussion in various footnotes and specific subsections that I will signpost. These parts can be skipped or skimmed without loss of continuity.

2 The Surjective Account of Set Size

Before I spell out the surjective account in full, it will be useful to consider a somewhat simpler, but equivalent, formulation of Cantor’s injective account.

First, define the following relations for arbitrary sets A and B :

- $B \preceq A$ if and only if there is an injection from B into A .
- $A \approx B$ if and only if $B \preceq A$ and $A \preceq B$.
- $B \prec A$ if and only if $B \preceq A$ and $A \not\preceq B$.

Then, the injective account is equivalent to the following collection of claims:

- (1′) A is at least as big as B if and only if $B \preceq A$
- (2′) A is the same size as B if and only if $A \approx B$.
- (3′) A is strictly bigger than B if and only if $B \prec A$.

This equivalence is a straightforward consequence of the Schröder-Bernstein theorem—i.e., if $A \preceq B$ and $B \preceq A$, then there is a bijection from A to B . This theorem holds in all models of Zermelo-Fraenkel set theory (ZF), even if the Axiom of Choice is false.⁶ While the more familiar formulation of Cantor’s account in section 1 appeals to the notions of an injection *and* a bijection, the above formulation only appeals to that of an injection.

Additionally, \approx is defined as the “symmetric” part of \preceq and \prec is defined as the “anti-symmetric” part of \preceq (to use the standard order-theoretic terminology). As such, the above formulation readily allows us to view the injective account as formally analogous to other relational structures that have traditionally figured in measurement theory. For example, in the traditional measurement theory of probability, one begins with the qualitative relation *at least as probable as* and defines *equiprobable to* as its symmetric part and *strictly more probable than* as its anti-symmetric part.⁷ In this manner, the above formulation of the injective account also illustrates the importance of

⁶I will assume ZF throughout the paper.

⁷Indeed, the parallels between measuring set size and measuring probability run further: just as it is customary to define “cardinalities” by appealing to equivalence classes of the *same size as* relation (more on this in section 3.4.2), so one traditional approach to defining numerical probabilities is to appeal to equivalence classes of *equiprobable to*. See [Krantz et al. 1971](#), chap. 5.

considering all three comparative size relations. If we only focus on the *same size as* relation—more widely discussed in the philosophical literature than the other two comparative size relations—then we lose sight of the bigger measurement-theoretic picture.

Now define the following analogous relations:

- $B \preceq^* A$ if and only if there is a surjection from A onto B (or $B = \emptyset$).⁸
- $A \approx^* B$ if and only if $B \preceq^* A$ and $A \preceq^* B$.
- $B \prec^* A$ if and only if $B \preceq^* A$ and $A \not\preceq^* B$.

Then, the surjective account is the following collection of claims:

- (1*) A is at least as big as B if and only if $B \preceq^* A$
- (2*) A is the same size as B if and only if $A \approx^* B$.
- (3*) A is strictly bigger than B if and only if $B \prec^* A$.⁹

In section 1, I claimed that the surjective account possessed several virtues; I will argue for these in the next section. However, for now, I will note some points of comparison with the injective account.

First, both accounts satisfy the following claims:

- (2_{both}) A is the same size as B if and only if A is at least as big as B and B is at least as big as A .
- (3_{both}) A is strictly bigger than B if and only if A is at least as big as B and B is *not* at least as big as A .

⁸The disjunct that B is empty is necessary to ensure that $\emptyset \preceq^* A$ for any nonempty A . Strictly speaking, there is no function—and, thus, no surjection—from any nonempty set to the empty set.

⁹While the above notation for the surjection-based relations is standard in the literature, I am not aware of any previous defense of the claim that the surjective account constitutes an interesting alternative account of set size. For example, [Lindenbaum and Tarski \(1926/1986, 175\)](#)—who introduced the notation for the surjection-based relations—merely say that these relations are “analogues” to the injection-based relations. Similarly, [Banaschewski and Moore \(1990, 375\)](#) simply call \preceq^* a “dual relation” to \preceq .

That is, both accounts take the *same size as* relation to be the symmetric part of *at least as big as*, and both accounts take the *strictly bigger than* relation to be the anti-symmetric part of *at least as big as*. Indeed, the injective and surjective accounts are straightforwardly equivalent to the conjunction of the above two claims with (1) and (1*), respectively. Thus, at an intuitive level, the only substantive difference between the two accounts lies in how they understand *at least as big as*. While the injective account understands this relation in terms of injections, the surjective account understands it in terms of surjections.

Second, the two accounts are equivalent in ZFC—i.e., ZF with the Axiom of Choice. I prove this fact in the Appendix. However, we will see that the two accounts can come apart—indeed, in a striking way—if the Axiom of Choice is false. In particular, while the existence of an injection from B into A is necessary and sufficient for A to be at least as big as B on the injective account, the existence of such an injection is merely a *sufficient* condition for A to be at least as big as B on the surjective account. This is because $B \preceq A$ entails $B \preceq^* A$, but (as we will see) the converse is not the case if the Axiom of Choice is false.¹⁰ Similarly, the existence of a bijection from B to A is merely a *sufficient* condition for A to be the same size as B on the surjective account. Thus, regardless of whether the Axiom of Choice is true, $B \preceq A$ entails $B \preceq^* A$ and $B \approx A$ entails $B \approx^* A$. However, if the Axiom of Choice is false, then $B \preceq^* A$ does not entail $B \preceq A$ and $B \approx^* A$ does not entail $B \approx A$. Moreover, we will see that neither $B \prec A$ nor $B \prec^* A$ entail each other if the Axiom of Choice is false.

Finally, while the Axiom of Choice entails the equivalence of the two accounts, we will see that the question of whether the equivalence of the two accounts entails the Axiom of Choice is itself equivalent to an old open question in set theory—namely, whether the statement ($A \preceq^* B$ implies $A \preceq B$) entails the Axiom of Choice.

A terminological point of caution is worth making before proceeding. Two sets are traditionally called *equinumerous* just in case there is a bijec-

¹⁰To see that $B \preceq A$ entails $B \preceq^* A$, suppose A and B are nonempty. (If B is empty, then we trivially have that $B \preceq^* A$. If A is empty and B is nonempty, then it is not possible that $B \preceq A$.) Let $f : B \rightarrow A$ be injective, and let a be an arbitrary member of A . If $a = f(b)$ for some $b \in B$, then $f^{-1}(a)$ is unique since f is injective. So, there is a function $g : A \rightarrow B$ such that, for some $c \in B$, $g(x) = b$ if $x = f(b)$ for some $b \in B$ and $g(x) = c$ otherwise. Now let b be an arbitrary member of B . Then, $f(b) = a$ for some unique $a \in A$. So, $g(a) = b$. Thus, g is surjective.

tion between them. This terminology suggests that two sets have the *same number of members*—i.e., have exactly as many members as each other—just in case there is a bijection between them. However, we must be careful not to prejudice ourselves against the surjective account solely on the basis of this standard terminology. Here is an example of how such prejudice might mistakenly arise.

The Schröder-Bernstein theorem is often given the following intuitive gloss: if A is at least as big as B and B is at least as big as A , then A and B are equinumerous. If we understand “equinumerous” in terms of bijections, then we might hope that the surjective account entails the following principle: if $A \preceq^* B$ and $B \preceq^* A$, then $A \approx B$. This principle is known as the *Dual Schröder-Bernstein theorem*.¹¹ As we will see later, however, the surjective account does *not* entail this principle in the absence of the Axiom of Choice. While this fact might tempt us to infer that the surjective account is intuitively deficient, such an inference would beg the question against the surjective account. After all, the surjective account provides its own understanding of what makes two sets the same size and, hence, “equinumerous.” Moreover, recall that both accounts satisfy (2_{both}) : A is the same size as B if and only if A is at least as big as B and B is at least as big as A . The right-to-left direction of this claim is simply a pre-theoretic version of the Schröder-Bernstein theorem. Indeed, it is plausibly an intuitive desideratum that our account of set size entails *this* claim—not that it entails the Dual Schröder-Bernstein theorem, which mixes notions of set size from two separate accounts. Thus, to avoid misinterpretation, I will abstain from using the term “equinumerous” in what follows.

3 Virtues of the Surjective Account

Earlier I claimed that the surjective account possessed several virtues:

- (1) It is an intuitive generalization of the injective account.
- (2) It blocks the Division Paradox.
- (3) It illuminates the debate over whether the Axiom of Choice is true.
- (4) It is a mathematically fruitful alternative to the injective account.

¹¹See [Banaschewski and Moore 1990](#).

- (5) It sheds philosophical light on one of the oldest unsolved problems in set theory.

I will now argue for each of these.

3.1 Virtue 1: The Surjective Account Is an Intuitive Generalization of the Injective Account

Let A and B be sets. According to the injective account, A is at least as big as B just in case there is an injection from B into A . Figure 1 depicts an example of such an injection. Intuitively, we can view the members of A and B —or, if we prefer, the singleton subsets of A and B —as being the smallest *parts* of these sets. So, we can view the injective account as saying that A is at least as big as B just in case there is a way of pairing distinct members of B with distinct smallest parts of A .¹²

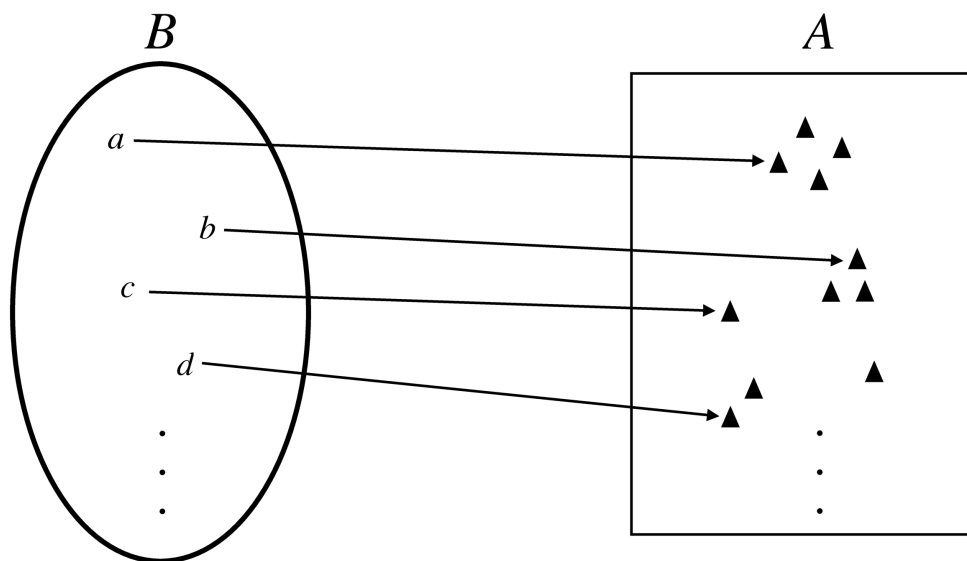


Figure 1: An injection from B into A . This pairs any two distinct members of B with distinct members of A .

More generally, we can view the parts of a set S as being the nonempty *subsets* of S , and we can view a division of S into parts as being a *partition* of

¹²Note that if there is an injection f from B into A , then there is an injection g from B into the set of singleton subsets of A : for any $x \in B$, simply let $g(x) = \{f(x)\}$.

S —that is, a collection of mutually disjoint and jointly exhaustive nonempty subsets of S . So, let us now suppose that there is an injection from B into some partition of A —intuitively, a way of pairing any two distinct members of B with disjoint parts of A . Figure 2 depicts an example of such a mapping.

Note that, in Figure 1, each member of B is only associated with an individual member of A . In contrast, in Figure 2, some members of B are associated with *multiple* members of A . This is because some members of B are associated with non-singleton *subsets* of A . For example, in Figure 2, a is associated with 4 members of A , b is associated with 3 members of A , and d is associated with 2 members of A . (However, as in Figure 1, c is only associated with 1 member of A .) Thus, some members of B are associated with *more* members of A in Figure 2 than they are in Figure 1. Intuitively, this seems to suggest that we have at least as much reason to regard A as being at least as big as B in the scenario depicted in Figure 2 as in the scenario depicted in Figure 1.

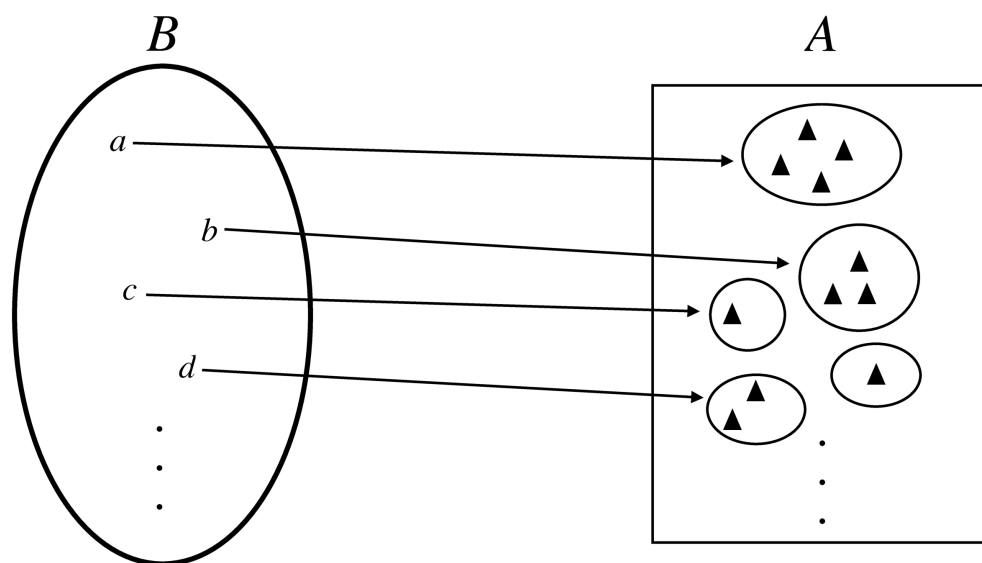


Figure 2: An injection from B into some partition (“division”) of A . This pairs any two distinct members of B with mutually disjoint nonempty subsets (“disjoint parts”) of A .

Indeed, while Figure 1 depicts an injection from B into A , Figure 2 depicts a kind of *generalized* injection from B into A . In particular, an ordinary

injection from B into A can be viewed as the special case of an injection from B into A 's partition of singletons. So, the intuition that A is at least as big as B by virtue of there existing an injection from B into some partition of A seems to generalize the traditional Cantorian intuition that A is at least as big as B by virtue of there existing an injection from B into A .

More formally, the following seems to be an intuitive generalization of Cantor's (1):

(1**) A is at least as big as B if and only if there is an injection from B into some partition of A .

As I show in the Appendix, (1**) is equivalent to the surjective account's simpler claim (1*)—i.e., A is at least as big as B just in case there is a surjection from A onto B (or B is empty). Moreover, since the injective and surjective accounts satisfy (2_{both}) and (3_{both}), both accounts appear to satisfy the same intuitions about the *same size as* and *strictly bigger than* relations. So, we can succinctly state the key intuitive difference between the accounts as follows:

- Injective account: A is at least as big as B if and only if any two distinct members of B can be paired with distinct *members* of A .
- Surjective account: A is at least as big as B if and only if any two distinct members of B can be paired with disjoint *parts* of A .

Thus, the surjective account appears to be an intuitive generalization of the injective account.

Earlier I said that the surjective and injective accounts are equivalent if the Axiom of Choice is true but that their consequences can differ if the Axiom of Choice is false. We can now intuitively see why the latter is the case—and, moreover, why the intuitive considerations that motivate the surjective account go *beyond* those that motivate the injective account if the Axiom of Choice is false.

Consider the function depicted in Figure 2 again; call this function f . Recall that f is an injection from B into some partition of A . Using f , we might think that we can construct an injection g from B into A simply as follows: let $g(a)$ be an arbitrary member of the set $f(a)$, let $g(b)$ be an arbitrary member of the set $f(b)$, and so on. If a function g could be constructed in this way, then it would clearly be injective, and the two accounts would agree that A is at least as big as B . However, in order to construct the

function g , we need to assume that we can arbitrarily *choose* exactly one member of $f(a)$ to designate as $g(a)$, that we can arbitrarily *choose* exactly one member of $f(b)$ to designate as $g(b)$, and so on. But this is precisely to assume the Axiom of Choice. If the Axiom of Choice is false, then we might *not* be able to make all of these choices. Moreover, if we cannot make all of these choices, then there will not be an injection from B into A , and the injective account will not deem A at least as big as B . In contrast, the surjective account ensures that A at least as big as B , regardless of whether the Axiom of Choice is true, as there is (by assumption) an injection from B into some partition of A . In the next section, I will discuss models of ZF in which the Axiom of Choice fails such that the two accounts indeed differ.

Here is a simple example to illustrate the point. Suppose there are infinitely many farmers, every farmer has at least one dog, and every dog is owned by exactly one farmer. Then, it seems intuitively “obvious” that there are at least as many dogs as farmers. While the surjective account vindicates this intuitive claim, we cannot prove it on the injective account unless we assume the Axiom of Choice. For let B be the set of farmers and A be the set of dogs in Figure 2. Then, there is clearly an injection from B into some partition of A —simply pair each farmer with the set of dogs they own. So, the surjective account entails that A is at least as big as B . In contrast, as we just saw, we need the Axiom of Choice to establish that there is an injection from B into A . So, we cannot prove that there are at least as many dogs as farmers on the injective account unless we assume the Axiom of Choice. Moreover, even if we do accept the Axiom of Choice, the proof of this claim is not immediate on the injective account: we need to employ the Axiom of Choice. In contrast, this claim *does* immediately follow on the surjective account; no appeal to the axiom is needed. The immediacy of this inference suggests that the surjective account is not merely an intuitive generalization of the injective account; it also seems to track our intuitive reasoning about set size more closely than the injective account.¹³

3.2 Virtue 2: The Surjective Account Blocks the Division Paradox

Suppose P is a partition of some set A . Note that there is a surjection from A onto P —simply pair each member x of A with the member of P of which

¹³Thanks to Alex Pruss for this point and the example.

x is a member. Moreover, such a surjection exists regardless of whether the Axiom of Choice is true. So, the surjective account ensures that A is at least as big as P in all models of ZF. That is, the surjective account entails the following principle:

- **Partition Principle (PP)**. Every set is at least as big as any partition of it.

This principle is plausibly one of the various pre-theoretic intuitions we have about “set size”—alongside the transitivity of *at least as big as*, the Part-Whole Principle, among others. Intuitively, **PP** simply says that A can only be divided into as many parts as it has smallest parts. From a pre-theoretic standpoint, this claim seems hard to deny. Other things being equal, then, it seems intuitively desirable to have an account of set size that entails **PP**.

If we assume the injective account, then **PP** is equivalent to the following principle:

- **Partition Principle_{injective} (PP_{injective})**. For every set A and partition P of A , there is an injection from P into A .¹⁴

This technical articulation of **PP** has been held in high esteem by a number of notable thinkers. For example, [Bernstein \(1904, 558\)](#) says, “I regard [**PP_{injective}**] as one of the most important in set theory, and I see no objection to using it.” Similarly, [Moore \(1982, 10\)](#) says that **PP_{injective}** “appears all but self-evident.” [Russell \(1906/2014\)](#) also appears to have accepted the principle.¹⁵

Nonetheless, it turns out that **PP_{injective}** can fail if the Axiom of Choice is false. More precisely, there are models of ZF in which the Axiom of Choice is false (ZF–C) such that **PP_{injective}** fails.¹⁶ So, if the Axiom of Choice is false, then the injective account allows for the counterintuitive possibility that **PP** can fail. In contrast, this possibility is blocked by the surjective account in all models of ZF. So, the surjective account allows us to uphold the intuition that **PP** is necessarily true. Indeed, this fact seems to suggest that, if any principle in the vicinity should be held in high esteem, it is not **PP_{injective}** but rather the pre-theoretic **PP**.

¹⁴This principle is traditionally called *the* Partition Principle in the literature, but it is useful to distinguish it from the pre-theoretic **PP**.

¹⁵I discuss the case of Russell in more detail in section 3.5.

¹⁶See [Taylor and Wagon 2019](#) for a self-contained discussion of how such models arise from the work of Mycielski and Sierpiński.

As a side note, it is well known that the Axiom of Choice entails $\mathbf{PP}_{\text{injective}}$, though whether $\mathbf{PP}_{\text{injective}}$ entails the Axiom of Choice is one of the oldest unsolved problems in set theory.¹⁷ Also, note that $\mathbf{PP}_{\text{injective}}$ is equivalent to the claim that $A \preceq^* B$ entails $A \preceq B$.¹⁸ Since $A \preceq B$ entails $A \preceq^* B$ (cf. section 2), $\mathbf{PP}_{\text{injective}}$ is equivalent to the statement that the injective and surjective accounts are equivalent to each other. Thus, it is also an open question whether the Axiom of Choice is equivalent to the equivalence of the injective and surjective accounts.

It is easy to see that the surjective account also entails the following principle:

- **Weak Partition Principle (WPP).** For every set A and partition P of A , P is *not* strictly bigger than A .¹⁹

If we assume the injective account, then this principle is equivalent to the following:

- **Weak Partition Principle_{injective} (WPP_{injective}).** For every set A and partition P of A , it is *not* the case that there is an injection, but no bijection, from A to P . Equivalently, if $A \preceq^* B$, then $B \not\prec A$.²⁰

As before, the Axiom of Choice entails $\mathbf{WPP}_{\text{injective}}$, but it is not known whether $\mathbf{WPP}_{\text{injective}}$ entails the Axiom of Choice.²¹ Nonetheless, it is known that there are models of $\mathbf{ZF}\neg\mathbf{C}$ in which $\mathbf{WPP}_{\text{injective}}$ fails. For example, there are models of $\mathbf{ZF}\neg\mathbf{C}$ in which the set \mathbb{R} of real numbers can be partitioned into some set P that is *bigger* than \mathbb{R} according to the injective account.²² This sort of possibility seems even more counterintuitive than a

¹⁷Karagila (2013) argues that it is one of the two oldest unsolved problems in set theory (apart from the Continuum Hypothesis, whose status as “unsolved” is controversial). See Moore 1982 and Banaschewski and Moore 1990 for further discussion of the problem.

¹⁸*Proof.* First, suppose $\mathbf{PP}_{\text{injective}}$, so that $P \preceq A$ for any partition P of A . Now suppose $B \preceq^* A$ and let $f : A \rightarrow B$ be surjective. Let B be nonempty. (If B is empty, it trivially follows that $B \preceq A$.) For every $b \in B$, let $A_b = \{x \in A \mid f(x) = b\}$. Note that $P = \{A_b \mid b \in B\}$ is a partition of A . Moreover, clearly $B \preceq P$. Thus, since \preceq is transitive, $B \preceq A$. Now suppose that, if $B \preceq^* A$, then $B \preceq A$. Let P be a partition of A . Define $f : A \rightarrow P$ so that $f(x) = y$ just in case $x \in y$. Clearly, f is surjective, so $P \preceq A$. Hence, $\mathbf{PP}_{\text{injective}}$.

¹⁹The surjective account entails \mathbf{WPP} since \mathbf{PP} and (3_{both}) jointly entail \mathbf{WPP} .

²⁰As before, this principle is traditionally called *the* Weak Partition Principle, but it is useful to distinguish it from the pre-theoretic \mathbf{WPP} .

²¹Again, see Moore 1982 and Banaschewski and Moore 1990 for discussion.

²²See Taylor and Wagon 2019 for discussion of how such models arise.

failure of **PP**. After all, a failure of **PP** is compatible with \mathbb{R} and P merely being *incomparable* in size—that is, with \mathbb{R} not being at least as big as P and P not being at least as big as \mathbb{R} . However, to say that **WPP** fails here means that there is an injection from \mathbb{R} into P but no bijection from \mathbb{R} to P . So, a failure of **WPP** means that the sizes of \mathbb{R} and P *can* be compared, but P is *bigger* than \mathbb{R} —that is, \mathbb{R} can be divided up into more parts than members. As Taylor and Wagon (2019, 307, 310) note, this possibility is as counterintuitive as “a country that has more populated provinces than it has people” or “a sports league having more teams than players”—to wit, *very* counterintuitive.

Thus, if the Axiom of Choice is false, the injective account can lead to the *Division Paradox*: it is possible for a set to be partitioned (“divided”) into a set that is bigger than it.²³ That is, the injective account allows for models of $\text{ZF} \neg \text{C}$ in which **WPP** fails. Indeed, **WPP**_{injective} (and **PP**_{injective}) fail in *all* known models of $\text{ZF} \neg \text{C}$. In contrast, the surjective account satisfies **WPP** (and **PP**) in all models of ZF . So, the surjective account necessarily blocks the Division Paradox.²⁴

Note that, in the above example, $P \preceq^* \mathbb{R}$ and $\mathbb{R} \prec P$, where P is the aforementioned partition of \mathbb{R} . Additionally, $\mathbb{R} \prec P$ entails $\mathbb{R} \preceq^* P$. It follows that $B \prec A$ does *not* entail $B \prec^* A$ if the Axiom of Choice is false. Moreover, since $P \not\approx \mathbb{R}$, it also follows that the Dual Schröder-Bernstein theorem (cf. section 2) can fail if the Axiom of Choice is false. That said, it is an open question whether the Dual Schröder-Bernstein theorem entails the Axiom of Choice.²⁵

Of course, if we should be certain that the Axiom of Choice is true, then we needn’t worry about the Division Paradox. Moreover, although the axiom had a controversial early history, it is now widely (but not universally)

²³The name of this paradox was coined by Taylor and Wagon (2019), though they note that the paradox was known long before then.

²⁴Additionally, as Taylor and Wagon (2019, 311) explain, the injective account (which they implicitly assume) can lead to a *Double* Division Paradox if the Axiom of Choice is false: a set A can be partitioned into a bigger set B , which in turn can be partitioned into a bigger set C . Still more strikingly, the injective account can lead to an *Infinite* Division Paradox if the Axiom of Choice is false—e.g., the possibility of “more teams than players, more conferences than teams, more leagues than conferences, more sports than leagues, and so on.” (Taylor and Wagon attribute the latter paradox to Asaf Karagila.) Since the surjective account entails **WPP**, it blocks the Double and Infinite Division Paradoxes as well.

²⁵See Banaschewski and Moore 1990 for discussion.

accepted by mathematicians and philosophers.²⁶ However, even among those who accept the axiom, very few are *completely* certain of its truth. So, even the vast majority of those who accept the Axiom of Choice can still consider the epistemic *possibility* that it is false and ask what might happen if we assume it is false. In particular, we can ask if it is epistemically possible for the Division Paradox to arise if the Axiom of Choice is false. If we accept the injective account, then the answer to this question appears to be “yes.” This appears to be an intuitive cost of the injective account.

For my part, the mere epistemic possibility of something as counterintuitive as the Division Paradox seems reason enough to pursue an independently motivated means of blocking it without assuming the Axiom of Choice. However, some who accept the Axiom of Choice may be unperturbed by this possibility and thus unmotivated to pursue an alternative account of set size. After all, we might think that the Axiom of Choice seems pretty obviously true, so it is not terribly bad if our account of set size needs to rely on it to get the intuitively correct results.²⁷

Nonetheless, while the Axiom of Choice is widely accepted, there is substantially less agreement about its *class-theoretic* analogues (e.g., Global Choice).²⁸ So, those who accept the Axiom of Choice and are unperturbed by the epistemic possibility of the Division Paradox may worry yet about a version of the paradox arising for *proper classes*—that is, collections that fail to form sets and (unlike sets) cannot be members of other collections. As I will explain, if we accept the natural class-theoretic generalization of the injective account, then a class-theoretic version of the Division Paradox *necessarily* arises if the Axiom of Choice holds yet any of its class-theoretic analogues fails. However, if we accept the natural class-theoretic generalization of the surjective account, then the class-theoretic Division Paradox is necessarily *blocked*—regardless of the truth of any class-theoretic analogue of the Axiom of Choice.

²⁶See Moore 1982 for discussion of the early controversies. As McCarty et al. (2023) note, those working in intuitionistic or constructive mathematics usually (but not universally) reject the Axiom of Choice. Its truth has also been doubted by other contemporary mathematicians and philosophers who (to my knowledge) are neither self-avowed intuitionists nor constructivists, including Potter (2004, sect. 14), Penrose (2005, 14–15, 366), and Herrlich (2006, 7).

²⁷Thanks to an anonymous referee for raising this thought.

²⁸Thanks to a separate anonymous referee for raising this point.

3.2.1 The Surjective Account Blocks the Class-Theoretic Division Paradox

Note. This is a more technical subsection and can be skipped or skimmed without loss of continuity.

Strictly speaking, the injective and surjective accounts are accounts of *set* size. As such, neither account permits us to compare the sizes of proper classes. Nonetheless, it is straightforward to generalize these accounts to be of class size by understanding functions in a class-theoretic way. The key idea is simply to treat functions as classes of ordered pairs—hereafter, *binary relational classes*—rather than (as is standard in set theory) sets of ordered pairs.²⁹ We can then formulate class-theoretic generalizations of the injective and surjective accounts by understanding injections and surjections class-theoretically.³⁰ We can also retain the notation ‘ $A \preceq B$ ’, ‘ $A \preceq^* B$ ’, etc., to indicate the existence of the relevant functions between classes. Moreover, as in the set-theoretic case, it straightforwardly follows that $A \preceq B$ implies $A \preceq^* B$ and that $A \approx B$ implies $A \approx^* B$.

Axiomatic theories that involve both sets and classes have been developed by various authors, the two most prominent theories being Gödel-Bernays set theory (**GB**) and Morse-Kelley set theory (**MK**). Each of these theories is an extension of **ZF**, and each is often (though not always) formulated in a way that includes its own class-theoretic analogue of the Axiom of Choice. Because **GB** is the weaker theory, I will work in this theory in this section. Also, I will not assume that **GB** includes either the set-theoretic Axiom of Choice—hereafter, **AC**—or any class-theoretic analogue thereof.

Let us now consider how we might formulate class-theoretic versions of **PP** and **WPP**. At first blush, we might think there are no class-theoretic analogues of these principles since they appeal to the notion of a “partition”—a particular kind of set of sets—yet it is not sensible to speak a class *of*

²⁹Following Uzquiano (2015), call a *binary relational class* R a class of ordered pairs. The *domain* of R , $Dom(R)$, is the class $\{x : \exists y(\langle x, y \rangle \in R)\}$. The *range* of R , $Rng(R)$, is the class $\{y : \exists x(\langle x, y \rangle \in R)\}$. A binary relational class F is a *functional class*—for short, a *function*—just in case, for all x, y, z , if $\langle x, y \rangle \in F$ and $\langle x, z \rangle \in F$, then $y = z$. In such a case, let $F(x)$ denote the unique y such that $\langle x, y \rangle \in F$.

³⁰Let F be a functional class. Say that F is an *injection* from A into B just in case $Dom(F) = A$, $Rng(F) \subseteq B$, and for all $x, y \in Dom(F)$, $F(x) = F(y)$ implies $x = y$. Additionally, say that F is a *surjection* from A onto B just in case $Dom(F) = A$, $Rng(F) = B$, and for every $y \in B$, there is some $x \in A$ such that $F(x) = y$. Finally, say that F is a *bijection* just in case it is both injective and surjective.

proper classes. However, it turns out that we can *encode* class-theoretic generalizations of these principles. We can do this by drawing on the connection between surjections and partitions as well as an old trick due to Bernays (1942) that enables us to simulate “class-valued” functions.

First, let f be a surjection from a (possibly proper) class A onto another class B . This surjection will associate distinct members of B with nonempty, mutually disjoint subclasses of A , and the union of all such subclasses will simply be A . So, there is a sense in which f *simulates* a “partition” of A into those subclasses.

Second, we can use Bernays (1942)’ trick to simulate a *function*—specifically, a bijection—from B to these subclasses of A . Here is Uzquiano (2015, 9)’s description of Bernays’ trick (which I have slightly generalized):

Bernays simulates a “class-valued function” from B to subclasses of A by means of a binary relational class R : in particular, we take R to map a member b of B to the class of members of A to which b is related, i.e., $\{x \in A : \langle b, x \rangle \in R\}$, which is itself a subclass of A . We may even write $R(b) = S$ to abbreviate: $\forall x(\langle b, x \rangle \in R \leftrightarrow x \in S)$.

More specifically, say that R simulates an *injection* from B into subclasses of A just in case: $R(x) = R(y)$ only if $x = y$. Additionally, say that R simulates a *surjection* from B onto subclasses of A just in case: for every subclass S of A , there is some $b \in B$ such that $R(b) = S$. Finally, say that R simulates a *bijection* from B to subclasses of A just in case R simulates a function that is both injective and surjective. Now consider the surjection f from A onto B again. Clearly, f induces a binary relational class R that simulates a bijection from B to nonempty, mutually disjoint and jointly exhaustive subclasses of A : for any $b \in B$, simply let $R(b) = \{x \in A : f(x) = b\}$. R then encodes a bijection from B to a “partition” of A .

Next, note that on the class-theoretic versions of both the injective and surjective accounts, the existence of a bijection from one class to another is sufficient for each to be at least as big as the other. Let us analogously assume that the existence of the aforementioned simulated bijection is sufficient for there to be at least as many members of B as there are aforementioned subclasses of A and vice versa.³¹ Let us also assume that the *at least as many*

³¹Uzquiano (2015, 9) makes a similar assumption about simulated “class-valued” functions—namely, that a necessary condition for a class S to have more subclasses than

as relation is transitive, where this relation compares how many members or subclasses a given class has to how many members or subclasses another class has. Then, there are at least as many members of A as there are the aforementioned nonempty, mutually disjoint, and jointly exhaustive subclasses of A just in case there are at least as many members of A as there are B . Thus, the following principle encodes the claim that any class A —whether it is a set or proper class—is at least as big as any “partition” of A :

- **Class-theoretic Partition Principle (CPP)**. For any classes A and B , if there is a surjection from A onto B , then A is at least as big as B .

If we additionally assume that *strictly bigger than* is the antisymmetric part of *at least as big as*—as is also the case on the class-theoretic versions of both the injective and surjective accounts—then the following principle encodes the claim that no class can be “partitioned” into a bigger class:

- **Class-theoretic Weak Partition Principle (CWPP)**. For any classes A and B , if there is a surjection from A onto B , then B is *not* strictly bigger than A .

Note that, if we assume the class-theoretic version of the injective account, then **CPP** and **CWPP** are equivalent to the following:

- **CPP_{injective}**. For any classes A and B , if $B \preceq^* A$, then $B \preceq A$.
- **CWPP_{injective}**. For any classes A and B , if $B \preceq^* A$, then $B \not\prec A$.

Let us now turn to class-theoretic analogues of **AC**. [Uzquiano \(2015\)](#) discusses five such principles:

- **Global Choice (GC)**. For any class of sets that are nonempty and mutually disjoint, there exists a class that contains exactly one member from each of these sets (and nothing else).³²

members is that it is impossible to simulate a surjective class-valued function from S onto the subclasses of S .

³²This is a different, but equivalent, formulation of **GC** to that stated by Uzquiano. I have stated [Rubin and Rubin \(1985, 191\)](#)’s “CAC 2” formulation of **GC**, whereas Uzquiano states their “E” formulation.

- **Global Well-Ordering (GWO)**. There is a well-ordering of V , where V is the class of all sets.³³
- **Maximality (Max)**. A is a proper class only if $A \approx V$.
- **Cardinal Comparability (CC)**. Given two classes A and B , $A \preceq B$ or $B \preceq A$.
- **Projection (Proj)**. A is a proper class only if $\text{Ord} \preceq A$, where Ord is the class of all ordinals.

The first four principles, as well as the conjunction of **Proj** and **AC**, are equivalent to each other in **GB**.³⁴ Moreover, each of the first four principles is strictly stronger than **AC**—that is, all models of **GB** that satisfy the first four principles also satisfy **AC** but not all models of **GB** that satisfy **AC** also satisfy the first four principles³⁵—and indeed distinctive concerns have been raised for them that do not apply to **AC**.³⁶ Thus, there is considerably less agreement about the truth of these class-theoretic principles than there is about the truth of **AC**.

It is straightforward to show that, if **GC** is true, then **CPP**_{injective}—and, thus, **CWPP**_{injective}—holds in **GB**.³⁷ So, if **GC** holds and we adopt

³³That is, there is a binary relational class, with domain and range V , that is a linear order—i.e., is reflexive, transitive, antisymmetric, and total—for which every nonempty subclass of V has a smallest member by the lights of this order.

³⁴See Hamkins 2014 and the appendices of Linnebo 2010 and Uzquiano 2015.

³⁵See Felgner 1976, sect. 3.

³⁶One general concern stems from the traditional view—as, e.g., articulated by Maddy (1983)—that sets instantiate the *combinatorial* notion of a collection, while classes instantiate the *logical* notion of a collection. In particular, Schindler (2019, 408) writes:

As [Bernays (1983)] remarks, the axiom of choice “is an immediate application of the combinatorial concepts in question.” On the logical notion of [collection], on the other hand, it is doubtful whether a class satisfying the requirements set out in the axiom of choice can always be found.

Other concerns about these class-theoretic principles are more technical. For example, **Max** is an essential premise of an argument Uzquiano (2015) presents for a “recombination” paradox. Additionally, Barton and Williams (2023) argue that **GC** raises problems for an independently-motivated view they call “class-theoretic potentialism.”

³⁷*Proof.* Given classes A and B , let f be a surjection from A onto B . Since **GC** and **GWO** are equivalent, there is a well-ordering of V . For any $b \in B$, let $g(b)$ be the smallest member a of A , relative to this well-ordering, for which $f(a) = b$. Clearly, g is an injection from B into A .

the class-theoretic version of the injective account, then **CPP** and **CWPP** hold. Additionally, as I show in the Appendix, the class-theoretic versions of the injective and surjective accounts are equivalent if **GC** holds. However, as I will now explain, if **GC** fails, then the class-theoretic versions of these accounts are not equivalent.

In particular, Hamkins (2023) shows that **CWPP**_{injective} fails in *all* models of **GB** in which **AC** is true yet **GC** fails. So, **CWPP**_{injective} fails in all models of **GB** in which **AC** is true yet any of its above class-theoretic analogues fails. In other words, if we accept the class-theoretic version of the injective account, then a class-theoretic Division Paradox *necessarily* arises if **AC** is true yet any of its class-theoretic analogues fails.³⁸

Now let us turn to the class-theoretic version of the surjective account. It is obvious that this account entails both **CPP** and **CWPP** and thus necessarily blocks the class-theoretic Division Paradox. Since there is substantially less agreement about the truth of **AC**'s class-theoretic analogues than about the truth of **AC**, it seems we now have an even stronger intuitive point in favor of the surjective account—at least, for proper classes.

While it is possible to adopt the surjective account for proper classes but the injective account for sets, it seems more intuitively natural to adopt a unified account of size for all classes. It would seem quite odd, for example, if whether one class had *more* members than another depended partly on whether these classes were sets or proper classes. Intuitively, one would have thought that such a comparison depended only on what the *members* of the two classes were. Thus, the foregoing class-theoretic considerations seem to constitute a strong intuitive point in favor of the surjective account for classes generally—sets and proper classes alike.

3.3 Virtue 3: The Surjective Account Illuminates the Debate over the Axiom of Choice

The Axiom of Choice is the most controversial axiom of ZFC, largely because both it and its negation have been thought to lead to paradoxical consequences. Perhaps the most famous allegedly paradoxical consequence

³⁸Note that this situation is somewhat different from the set-theoretic situation. All that is known about the latter is: if we accept the set-theoretic version of the injective account, then the set-theoretic Division Paradox *can* arise if **AC** fails—that is, there are *some* models of ZF−C such that **WPP**_{injective} fails. However, **WPP**_{injective} does fail in all *known* models of ZF−C.

of *assuming* the Axiom of Choice is the Banach-Tarski Paradox. This is Banach and Tarski (1924)'s theorem that any solid sphere can be decomposed into a finite number of parts and then—paradoxically—reassembled to form two solid spheres that have the same volume as the original. Since the Banach-Tarski Paradox can be shown to arise only if the Axiom of Choice is true, it is sometimes mounted as an argument against the Axiom of Choice.³⁹ However, it has also been thought that the *negation* of the Axiom of Choice leads to paradoxical consequences—perhaps most prominently, the possibility of the Division Paradox.⁴⁰ The possibility of the Division Paradox is thus sometimes mounted as an argument *for* the Axiom of Choice.⁴¹

We can now see that the possibility of the Division Paradox should *not* be regarded as an argument for the Axiom of Choice—at least, not a direct argument for it. This is because the possibility of the paradox is not a consequence of the negation of the Axiom of Choice but rather of the *conjunction* of the negation of the Axiom of Choice and the injective account. However, as we have seen, if we assume the *surjective* account, then the negation of the Axiom of Choice can never lead to the Division Paradox. Thus, contrary to what is sometimes claimed—at least, in the set-theoretic folklore—the negation of the Axiom of Choice does *not*, by itself, lead to the possibility of the Division Paradox. Indeed, the Division Paradox can be blocked simply by adopting an intuitively compelling alternative account of set size—namely, the surjective account.

3.4 Virtue 4: The Surjective Account Is a Mathematically Fruitful Alternative to the Injective Account

Note. This is a more technical subsection and can be skipped or skimmed without loss of continuity.

The injective account of set size has been enormously mathematically

³⁹See Tomkowicz and Wagon 2017 for discussion.

⁴⁰I say the *possibility* of the Division Paradox because, assuming the injective account, the Division Paradox *can* arise if the Axiom of Choice is false—i.e., $\mathbf{WPP}_{\text{injective}}$ fails in *some* models of $\text{ZF}\neg\text{C}$. Again, it is not known if $\mathbf{WPP}_{\text{injective}}$ fails in all models of $\text{ZF}\neg\text{C}$, though it does fail in all *known* models of $\text{ZF}\neg\text{C}$.

⁴¹See Taylor and Wagon 2019. Curiously, the Division Paradox is much less discussed than the Banach-Tarski Paradox among philosophers. Nonetheless, the paradox is well known among set theorists. For example, Halbeisen and Shelah (2001) discuss it at the beginning of their paper in connection with the vexed status of the Axiom of Choice.

fruitful. Since the injective and surjective accounts are equivalent if the Axiom of Choice is true, they are equally mathematically fruitful if we assume the axiom. While the consequences of the surjective account in the absence of the Axiom of Choice have not been explored in as much depth as those of the injective account in the absence of the Axiom of Choice, in this section I will argue that the surjective account is nonetheless a mathematically fruitful alternative in this context.

Note that the injective and surjective accounts are, strictly speaking, only accounts of *comparative* set size. To turn the injective account into a fully fledged theory of set size, it is customary to define a monadic notion of set size—namely, “cardinality”—that is fundamentally based on injections (or, what amounts to the same thing, bijections). When ZF is supplemented with the injective account, what emerges is a rich theory of set size—even in the absence of the Axiom of Choice. I will now explain how a similarly rich theory of *surjective* set size plausibly emerges from the surjective account in the absence of the Axiom of Choice.

I will begin by noting some interesting facts about the relations of comparative set size that figure in the surjective account—that is, the relations \preceq^* , \approx^* , and \prec^* . I will refer to these as the “surjective” relations and the comparative set size relations associated with the injective account—that is, \preceq , \approx , and \prec —as the “injective” relations. Then, I will turn to nontrivial questions raised by the surjective analogue of “cardinality” to which the surjective relations lead.

3.4.1 The Surjective Comparative Set Size Relations

We have already seen two distinctive features of the surjective relations: they satisfy **PP** and **WPP** in all models of ZF, whereas there are models of ZF–C in which the injective relations violate these principles. I will now note some nontrivial features that the surjective relations have in *common* with the injective relations in the absence of the Axiom of Choice. Then, I will note some distinctive consequences regarding the size of the continuum to which they can lead if the Axiom of Choice is false.

First, the surjective relations share several prominent order-theoretic properties with the injective relations. In particular, \preceq^* and \approx^* are clearly reflexive—as \preceq and \approx are—and \prec^* is clearly irreflexive—as \prec is. Also, each of the surjective relations is transitive, just as each of the injective re-

lations is.⁴² Additionally, \approx^* is clearly symmetric, as \approx is. Thus, \approx and \approx^* are both equivalence relations on the class of all sets. Moreover, on both accounts, Trichotomy—i.e., A is strictly bigger than B , B is strictly bigger than A , or A is the same size as B —is equivalent to the Axiom of Choice.⁴³ Finally, we saw earlier that both accounts entail a pre-theoretic version of the Schröder-Bernstein theorem.

The surjective relations also lead to analogues of Cantor’s most prominent results concerning comparative sizes of infinite sets. In particular, since the injective account entails that the set of natural numbers is the same size as the set of rational numbers, so does the surjective account. The surjective account also satisfies the property that any set, including any infinite set, has strictly more subsets than members. On the injective account, this is simply the fact that, for any set A , $A \prec \mathcal{P}(A)$, where $\mathcal{P}(A)$ is the set of A ’s subsets. On the surjective account, this is the claim that, for any set A , $A \prec^* \mathcal{P}(A)$. The latter claim holds since (if A is nonempty) there is a surjection from $\mathcal{P}(A)$ onto A —simply map every singleton $\{x\}$ to x and every other subset of A to an arbitrary member of A —yet Cantor’s theorem ensures that there is no surjection from A onto $\mathcal{P}(A)$. Thus, both the injective and surjective relations lead to a never-ending hierarchy of infinite set sizes: the set \mathbb{N} of natural numbers is smaller than $\mathcal{P}(\mathbb{N})$, which is smaller than $\mathcal{P}(\mathcal{P}(\mathbb{N}))$, and so on.

That said, the surjective relations can lead to distinctive consequences regarding the size of the continuum—that is, the size of \mathbb{R} —if the Axiom of Choice is false. One way for this to happen is if the Axiom of Determinacy—which is inconsistent with the Axiom of Choice—is true.⁴⁴ If the Axiom of Determinacy is true, then there are striking differences between how the

⁴²It is obvious that \preceq^* and \approx^* are transitive. To see that \prec^* is transitive, suppose that $A \prec^* B$ and $B \prec^* C$. Clearly, $A \preceq^* C$. To show that $C \not\prec^* A$, suppose for *reductio* that $C \preceq^* A$, i.e., that there is a surjection f from A onto C . Since $B \preceq^* C$, there is a surjection g from C onto B . Then, the function $g \circ f$ is a surjection from A onto B , contradicting the assumption that $A \prec^* B$.

⁴³It is well known that Trichotomy on the injective account is equivalent to the Axiom of Choice. On the surjective account, Trichotomy amounts to the claim that $A \prec^* B$, $B \prec^* A$, or $A \approx^* B$. To see that the latter is also equivalent to the Axiom of Choice, first note that the Axiom of Choice is equivalent to the claim that, for any A and B , $A \preceq^* B$ or $B \preceq^* A$. (See Rubin and Rubin 1985, sect. 3.) Simple logic shows that the latter is equivalent to the claim that, for any A and B , $(B \preceq^* A \text{ and } A \not\prec^* B)$ or $(A \preceq^* B \text{ and } B \not\prec^* A)$ or $(B \preceq^* A \text{ and } A \preceq^* B)$ —i.e., $A \prec^* B$, $B \prec^* A$, or $A \approx^* B$.

⁴⁴The Axiom of Determinacy was introduced by Mycielski and Steinhaus (1962).

“injective” size of \mathbb{R} and the “surjective” size of \mathbb{R} compare to the aleph numbers. First, the injective size of \mathbb{R} is *incomparable* to \aleph_1 —that is, $\mathbb{R} \not\preceq \aleph_1$ and $\aleph_1 \not\preceq \mathbb{R}$.⁴⁵ However, the surjective size of \mathbb{R} is comparable to \aleph_1 , as $\aleph_1 \prec^* \mathbb{R}$. In fact, \mathbb{R} is surjectively much bigger than \aleph_1 . In particular, for any natural number m , $\aleph_m \prec^* \mathbb{R}$. Moreover, let Θ be the least ordinal α for which $\alpha \not\preceq^* \mathbb{R}$. Then, for every ordinal α such that $\aleph_\alpha \prec^* \aleph_\Theta$, $\aleph_\alpha \preceq^* \mathbb{R}$. So, the surjective relations lead to a distinctive line of inquiry regarding how the size of the continuum compares to the aleph numbers if the Axiom of Determinacy is true. Indeed, this line of inquiry is considerably richer than that to which the injective relations lead, which all but stops at the incomparability between \mathbb{R} and \aleph_1 .⁴⁶

Class-theoretic analogues of the surjective relations also lead to a distinctive line of inquiry involving class-theoretic analogues of the Axiom of Choice. The following are surjective analogues of three such principles:

- **Maximality*** (*Max**). A is a proper class only if $A \approx^* V$, where V is the class of all sets.
- **Projection*** (*Proj**). A is a proper class only if $\text{Ord} \preceq^* A$, where Ord is the class of all ordinals.
- **Cardinal Comparability*** (*CC**). Given two classes A and B , $A \preceq^* B$ or $B \preceq^* A$.

As in the injective case, **GC** is equivalent to *Max** in **GB**.⁴⁷ However, unlike its injective counterpart, *Proj** holds in all models of **GB**, regardless of whether **AC** or **GC** is true.⁴⁸ Thus, on the surjective account, there is necessarily a smallest proper class—namely, Ord —but whether all proper classes are the same size depends on whether **GC** is true. Further, **GC** trivially entails the conjunction of *Proj** and **AC**. Also, unlike the injective case, the

⁴⁵The results I describe in this paragraph are discussed, or are straightforward consequences of the results discussed, by Kanamori (2003, chap. 6).

⁴⁶Note that the above example also shows that $B \prec^* A$ does not entail $B \prec A$ if the Axiom of Choice is false, as $\aleph_1 \prec^* \mathbb{R}$ yet $\aleph_1 \not\prec \mathbb{R}$ if the Axiom of Determinacy is true.

⁴⁷Uzquiano (2015, Appendix) shows that **GC** is equivalent to *Max*. Since $A \approx V$ entails $A \approx^* V$, **GC** entails *Max**. Now suppose **GC** fails. Hamkins (2014) shows that **GC** is equivalent to the claim that, for any class A , $A \preceq^* \text{Ord}$. Since **GC** fails, there is some class A such that $A \not\preceq^* \text{Ord}$. Moreover, since A is a subclass of V , $V \not\preceq^* \text{Ord}$. Thus, $\text{Ord} \not\approx^* V$. So, *Max** fails for the proper class Ord . Hence, *Max** entails **GC**.

⁴⁸See Hamkins 2014.

conjunction of **Proj**^{*} and **AC** does *not* entail **GC**. This is simply because there are models of **GB** in which **AC** holds but **GC** fails. Moreover, **GC** entails **CC**^{*},⁴⁹ but it is an interesting and (to my knowledge) open question whether **CC**^{*} entails **GC**.

3.4.2 Surjective Cardinalities

Let us now turn to the question of how we may develop a surjective analogue of “cardinality” in the absence of the Axiom of Choice. The standard way to define cardinality in the absence of the Axiom of Choice is to apply “Scott’s trick” to equivalence classes of \approx .⁵⁰ For example, Jech (1973, chap. 11) defines the cardinality $|A|$ of a set A as $|A| = \{B : B \approx A \text{ and } B \text{ is of least rank}\}$.⁵¹ Let us call this the *injective* cardinality of a set, and let us call any set that is such a cardinality an injective *cardinal*. By analogy, we may define the *surjective* cardinality $|A|^*$ of a set A as $|A|^* = \{B : B \approx^* A \text{ and } B \text{ is of least rank}\}$, and we may understand surjective cardinals similarly.⁵² We may also define arithmetical operations on surjective cardinals by analogy with the standard approach for injective cardinals: $|A|^* + |B|^* = |A \cup B|^*$ for disjoint A and B ,⁵³ $|A|^* \cdot |B|^* = |A \times B|^*$, $|A|^{|B|^*} = |{}^B A|^*$, and $2^{|A|^*} = |\mathcal{P}(A)|^*$.⁵⁴ The mathematically interesting question regarding surjective cardinals, then, is what properties they necessarily satisfy—or can fail to satisfy—in the absence of the Axiom of Choice.

It is well known that many properties of arithmetic that hold for infinite

⁴⁹Recall that $A \preceq B$ implies $A \preceq^* B$ and $B \preceq A$ implies $B \preceq^* A$. Since **GC** entails **CC**, it readily follows that **GC** entails **CC**^{*}.

⁵⁰See Scott 1955.

⁵¹Let $V_\alpha = \bigcup_{\beta < \alpha} \mathcal{P}(V_\beta)$ for each ordinal α . The *rank* of a set A is the smallest ordinal α such that $A \subseteq V_\alpha$.

⁵²This terminology is due to Truss (1984).

⁵³If A and B are not disjoint, then we may take $|A|^* + |B|^* = |(A \times \{0\}) \cup (B \times \{1\})|^*$.

⁵⁴Note that these arithmetical operations satisfy many of the same basic properties as their injective counterparts. For example, like injective cardinal addition, surjective cardinal addition is commutative and associative. Surjective cardinal multiplication is also commutative, associative, and distributive over surjective cardinal addition. Additionally, $|A|^{|B|^* + |C|^*} = |A|^{|B|^*} \cdot |A|^{|C|^*}$ and $(|A|^{|B|^*})^{|C|^*} = |A|^{|B|^* \cdot |C|^*}$. The proof of each of these facts can be constructed simply considering its injective counterpart and then employing the fact that $A \approx B$ implies $A \approx^* B$. For example, since $(A \times B) \approx (B \times A)$, we also have that $(A \times B) \approx^* (B \times A)$ and thus that $|A|^* \cdot |B|^* = |B|^* \cdot |A|^*$. Additionally, suppose $(A \cap C) = \emptyset$, $(B \cap D) = \emptyset$, $|A|^* \leq |B|^*$, and $|C|^* \leq |D|^*$. (The latter two conditions mean that $A \preceq^* B$ and $C \preceq^* D$.) Then, it is straightforward that $|A|^* + |C|^* \leq |B|^* + |D|^*$.

injective cardinals in the presence of the Axiom of Choice can fail in its absence. For example, it is a theorem of ZFC that the sum and product of two infinite injective cardinals—or, equivalently in ZFC, two infinite surjective cardinals—is simply their maximum. However, this statement is not provable in ZF alone. Moreover, the question of what arithmetical properties of infinite injective cardinals do hold in all models in ZF and what properties only hold in some models has led to a nontrivial and fruitful area of research.⁵⁵ The analogous question has also been explored to some extent for infinite surjective cardinals. For example, Truss (1984) shows, along with other “cancellation” laws, that $k \cdot \mathfrak{m}^* \approx^* k \cdot \mathfrak{n}^*$ implies $\mathfrak{m}^* \approx^* \mathfrak{n}^*$ in ZF for arbitrary positive integer k and surjective cardinals $\mathfrak{m}^*, \mathfrak{n}^*$. Also, it is known that $2^{\mathfrak{m}} \preceq \mathfrak{m}^2$ implies $\mathfrak{m} \leq 4$ if \mathfrak{m} is an injective cardinal. However, as Halbeisen (2017, 133) notes, it remains an open question whether $2^{\mathfrak{m}} \preceq^* \mathfrak{m}^2$ implies $\mathfrak{m} \leq 4$. This is a question about how the *surjective* sizes of two injective cardinals—namely, $2^{\mathfrak{m}}$ and \mathfrak{m}^2 —compare. It also appears to be an open question whether $2^{\mathfrak{m}^*} \preceq^* \mathfrak{m}^{*2}$ implies $\mathfrak{m}^* \leq 4$ if \mathfrak{m}^* is a surjective cardinal. This is a question about how the surjective sizes of two *surjective* cardinals—namely, $2^{\mathfrak{m}^*}$ and \mathfrak{m}^{*2} —compare. So, the study of cardinal arithmetic for infinite surjective cardinals in the absence of the Axiom of Choice appears to be an area of comparable richness to that involving infinite injective cardinals.

It is worth noting that Truss (1984) also shows that the *strong* cancellation law that $k \cdot \mathfrak{m}^* \preceq^* k \cdot \mathfrak{n}^*$ implies $\mathfrak{m}^* \preceq^* \mathfrak{n}^*$ is *unprovable* in ZF. In contrast, the analogous strong cancellation law involving injective cardinals is provable in ZF. In this respect, surjective cardinal arithmetic is somewhat awkward compared to injective cardinal arithmetic. Nonetheless, as Truss’ work shows, to see how this awkwardness arises is nontrivial and mathematically interesting. Compare: it is an awkward but mathematically interesting theorem of ZFC that $2 \cdot \mathfrak{m} = \mathfrak{m}$ if \mathfrak{m} is an infinite injective cardinal.

Finally, let us turn to *large cardinal* axioms. These are axioms that state the existence of infinite cardinals (“large cardinals”) with certain properties whose existence cannot be proven in ZFC.⁵⁶ Such axioms are standardly understood in terms of the injective conception of cardinals. However, once we understand cardinals surjectively, we can consider surjective versions of large cardinal axioms in the absence of the Axiom of Choice. For example, an injective cardinal κ is standardly taken to be *strongly inaccessible* just

⁵⁵See Jech 1973, chap. 11 for notable results in this area.

⁵⁶See Kanamori 2003 for a survey of large cardinal axioms.

in case (i) κ is uncountable, (ii) κ is not a sum of fewer than κ -many injective cardinals smaller than κ (where “smaller” and “fewer” are understood injectively), and (iii) $\alpha \prec \kappa$ implies $2^\alpha \prec \kappa$. By analogy, we may take a surjective cardinal κ^* to be *surjectively* strongly inaccessible just in case (i^{*}) κ^* is uncountable, (ii^{*}) κ^* is not a sum of fewer than κ^* -many surjective cardinals smaller than κ^* (where “smaller” and “fewer” are now understood surjectively), and (iii^{*}) $\alpha^* \prec^* \kappa^*$ implies $2^{\alpha^*} \prec^* \kappa^*$. So, it would appear interesting to ask of these large surjective cardinal axioms questions analogous to those that are standardly asked of the usual large cardinal axioms in the context of ZFC. For example, how do their consistency strengths compare to one another in ZF alone?⁵⁷ And how do large surjective cardinal axioms bear on the Continuum Hypothesis?⁵⁸ We may also ask: how do the consistency strengths of the large surjective cardinal axioms compare to those of their injective counterparts? To my knowledge, these questions are unexplored.

3.4.3 Summary

The above is just a sampling of the rich line of mathematical inquiry that is opened by the surjective account. However, the fact that all of the above issues *can* be investigated seems to show that the surjective account is indeed a mathematically fruitful alternative to the injective account.

3.5 Virtue 5: The Surjective Account Sheds Philosophical Light on One of the Oldest Unsolved Problems in Set Theory

In section 3.2, I noted that whether $\mathbf{PP}_{\text{injective}}$ entails the Axiom of Choice—and, thus, whether $\mathbf{PP}_{\text{injective}}$ is equivalent to the Axiom of Choice—is one of the oldest unsolved problems in set theory. This fact alone seems to show that the question is a *mathematically* interesting one. If we accept the injective account—for which $\mathbf{PP}_{\text{injective}}$ is the technical articulation of \mathbf{PP} —then this is plausibly a philosophically interesting question as well. In an unpublished manuscript, Russell (1906/2014, 301–2) suggests that $\mathbf{PP}_{\text{injective}}$ might be

⁵⁷As Koellner (2010, footnote 10) notes, it is also an open question how the consistency strengths of large injective cardinal axioms compare to one another in ZF.

⁵⁸As Karagila (2017) notes, in the absence of the Axiom of Choice, there are a number of inequivalent versions of the Continuum Hypothesis—for example, the claim that $\aleph_1 \prec \aleph_2$ implies $\aleph_1 \approx \aleph_2$, the claim that $\aleph_1 \approx \aleph_2$, and the claim that $\aleph_2 \not\approx \aleph_1$.

“worthy to be an axiom” and asserts—without proof—that it is indeed equivalent to the Axiom of Choice.⁵⁹ Moreover, he uses this alleged equivalence to elucidate the philosophical content of the Axiom of Choice and thereby motivate the foundational significance of the axiom to set theory.

Nonetheless, if we accept the surjective account, then the broader question of whether **PP** entails the Axiom of Choice loses much of its philosophical significance since it is straightforwardly answered in the *negative*. This is because the surjective account ensures that **PP** is true in all models of **ZF**. So, on the surjective account, it is possible for the Axiom of Choice to hold while **PP** holds, and it is possible for the Axiom of choice to fail while **PP** holds. Thus, the Axiom of Choice is *logically independent* of **PP** on the surjective account (assuming **ZF** is consistent).⁶⁰

4 Taking Intuitive Stock

I have discussed three accounts of set size—the injective account, the surjective account, and numerosity theory (cf. section 1)—as well as how they fare with respect to various alleged intuitive desiderata. Here I will expand upon that discussion in order to obtain a more comprehensive picture of their relative intuitive merits.

In what follows, when discussing whether the injective or surjective account entails a particular claim, I mean whether it entails that claim *without* assuming the Axiom of Choice. Also, though I have not spelled out the mathematical details of numerosity theory, those details will largely be irrelevant in my discussion.⁶¹

4.1 A Size for Every Set

The injective and surjective accounts entail that every set has a size—an injective and surjective cardinality, respectively—regardless of whether any axioms beyond **ZF** are assumed. In contrast, as [Benci et al. \(2006, 52\)](#) explain, numerosity theory requires Global Choice (**GC**) in order to entail that every

⁵⁹More precisely, Russell claims that **PP**_{injective} is equivalent to his “Multiplicative Axiom,” which [Russell \(1908/2014\)](#) later proved to be equivalent to the Axiom of Choice.

⁶⁰For similar reasons, the Axiom of Choice is logically independent of **WPP** on the surjective account.

⁶¹I will discuss the relevant such details in footnotes.

set has a size. Recall that this principle is more controversial than the Axiom of Choice. It seems intuitively problematic that numerosity theory requires it to ensure that every set has a size.

Because numerosity theory requires **GC** to ensure that every set has a size, it cannot entail any principle that ascribes a size-related property to all sets—e.g., Trichotomy, the Part-Whole Principle, the Partition Principle, or the Weak Partition Principle—without assuming **GC**.

4.2 Trichotomy

According to Trichotomy, for any sets A and B , A is bigger than B , B is bigger than A , or A is the same size as B . Recall that the injective and surjective accounts entail Trichotomy just in case the Axiom of Choice is true. This fact appears to be an intuitive deficiency of both accounts. Similarly, numerosity theory entails Trichotomy just in case **GC** is true.⁶²

4.3 The Part-Whole Principle

According to the Part-Whole Principle, every set is strictly bigger than any of its proper subsets. Like the injective account, the surjective account entails that this principle is false.⁶³ This fact is an intuitive deficiency of the two accounts. Although numerosity theory is often said to entail the Part-Whole Principle, in fact it only entails the Part-Whole Principle if **GC** is assumed. This fact seems to lessen the intuitive advantage numerosity theory might have appeared to have with respect to this principle.⁶⁴

⁶²This follows from the fact that “numerosities”—i.e., set sizes in numerosity theory—are part of a totally ordered field; see [Benci and Baglini 2022](#).

⁶³Since there is a bijection from the set of even numbers to the set of integers, the surjective account also entails that the latter is *not* strictly bigger than the former.

⁶⁴That said, numerosity theory does not require **GC** to entail *restricted* versions of the Part-Whole Principle—e.g., one that only applies to countably infinite sets ([Benci and Di Nasso 2003](#)) or one that only applies to sets of ordinals ([Benci et al. 2006](#)). Similarly for restricted versions of Trichotomy and the claim that every set has a size.

4.4 The Partition Principle and Weak Partition Principle

The surjective account entails both the Partition Principle and the Weak Partition Principle (and class-theoretic generalizations thereof). In contrast, the injective account does not entail these principles if the Axiom of Choice (or **GC**, in the class-theoretic case) is false. This fact is an intuitive deficiency of the injective account. While numerosity theory requires **GC** to entail these principles, the question of whether numerosity theory does indeed entail them when **GC** is assumed has not (to my knowledge) been pursued.

4.5 Hume's Principle and the Dual Hume's Principle

According to *Hume's Principle*,⁶⁵ two sets are the same size if and only if there is a bijection between them. Obviously, the injective account entails this principle. While the surjective account entails the right-to-left direction of this principle, it does not entail the left-to-right direction.⁶⁶ Additionally, while numerosity theory entails the left-to-right direction of this principle,⁶⁷ it does not entail the right-to-left direction.⁶⁸ As a number of philosophers have regarded Hume's Principle as central to the concept of set size, many would regard the fact that neither the surjective account nor numerosity theory entails this principle as intuitive defects of these accounts.

That said, the injective account does not entail the claim that if there is a surjection from A onto B , then A is at least as big as B . As I argued in section 3.1, this claim is at *least* as intuitively plausible as the injective account's claim that if there is an injection from B into A , then A is at least as big as B . However, the injective account does entail that if A is at least as big as B , then there is a surjection from A onto B . Of course, according to the surjective account, A is at least as big as B if and only if there is a surjection from A onto B . Moreover, this biconditional—in conjunction with the claim, entailed by both accounts, that A and B are the same size if and only if A is at least as big as B and B is at least as big as A —entails the

⁶⁵So named by Boolos (1990).

⁶⁶This follows from the fact that surjective account does not entail the Dual Schröder-Bernstein Theorem if the Axiom of Choice is false (cf. section 3.2).

⁶⁷The left-to-direct direction is what Benci et al. (2006) call the *Half Cantor Principle*.

⁶⁸For example, there is a bijection between the set of even numbers and the set of integers, but the latter set is bigger than the former on numerosity theory.

following:

- Dual Hume’s Principle. A and B are the same size if and only if there is a surjection from A onto B and a surjection from B onto A .

Thus, the Dual Hume’s Principle has a strong claim to being an intuitively plausible alternative to Hume’s Principle. While the Dual Hume’s Principle is entailed by the surjective account, only its left-to-right direction is entailed by the injective account and numerosity theory.⁶⁹ I will not take a stand here on which principle (if either) is *more* intuitively plausible, but we must take care not to let the historical prominence of Hume’s Principle by itself prejudice us against its dual. After all, the dual principle has not (to my knowledge) even been discussed—much less, defended—in the literature previously.⁷⁰

4.6 Subtraction and Division

In section 3.4.2, I noted that the injective and surjective accounts have the property that, in ZFC, the sum and product of two infinite cardinals is simply their maximum. For example, $|\mathbb{N}| + |\mathbb{R}| = |\mathbb{R}| + |\mathbb{R}| = |\mathbb{N}| \cdot |\mathbb{R}| = |\mathbb{R}| \cdot |\mathbb{R}| = |\mathbb{R}|$. So, there are no unique cardinals \mathfrak{m} and \mathfrak{n} such that $\mathfrak{m} + |\mathbb{R}| = |\mathbb{R}|$ and $\mathfrak{n} \cdot |\mathbb{R}| = |\mathbb{R}|$. As a result, while addition and multiplication are well-defined for the two accounts, subtraction and division are not. This fact seems to be an intuitive deficiency of the accounts. In contrast, subtraction and division are well-defined in numerosity theory (as are addition and multiplication).⁷¹

4.7 Strong Cancellation Law

In section 3.4.2, I noted that, on the surjective account, the strong cancellation law—i.e., $k \cdot \mathfrak{m}^* \preceq^* k \cdot \mathfrak{n}^*$ implies $\mathfrak{m}^* \preceq^* \mathfrak{n}^*$, for arbitrary positive integer k —can fail in ZF–C. While this fact is mathematically interesting, it seems intuitively problematic.⁷² In contrast, on the injective account, the

⁶⁹Recall that $A \approx B$ entails $A \approx^* B$ but $A \approx^* B$ does not entail $A \approx B$.

⁷⁰Apart from its intrinsic intuitive plausibility, the Dual Hume’s Principle might also admit of an “extrinsic” justification similar to that which Hume’s Principle is often given. In particular, Hume’s Principle is often extrinsically justified by pointing to the fact that it can serve as a neo-logicist foundation for Peano arithmetic; see Tennant 2017 for an overview. It would be interesting to explore whether such a justification can also be provided for the dual principle.

⁷¹This is because numerosities are part of a field; see Benci and Baglini 2022.

⁷²Thanks to Joshua Thong for pressing me on this point.

analogous strong cancellation law involving injective cardinals is provable in ZF. Numerosity theory also entails an analogous strong cancellation law.⁷³

4.8 Comparative Size Principle

It is straightforward to show that the injective account entails the following intuitively plausible principle:

- Comparative Size Principle. A is at least as big as B if and only if B is the same size as some subset of A .⁷⁴

Indeed, the injective account is often formulated by taking the *same size as* relation as primitive—treating two sets as the same size just in case there is a bijection between them—and defining *at least as big as* via the above principle.⁷⁵ Nonetheless, I do not know whether the surjective account entails the Comparative Size Principle. If it does not, then that would seem to be an intuitive deficiency of the surjective account. To my knowledge, it is an open question whether numerosity theory entails the Comparative Size Principle—and, moreover, whether numerosity theory is even consistent with the principle when uncountably infinite sets are assigned sizes.⁷⁶

4.9 Arbitrariness

Here is another apparent intuitive desideratum: our account of set size determines, by itself, how (and whether) any two sets compare in size. Note that both the injective and surjective accounts readily satisfy this desideratum. For example, whether A is at least as big as B is determined entirely by whether there is an injection from B into A or a surjection from A onto B , respectively.

In contrast, as Parker (2013) explains, numerosity theory does not satisfy this desideratum. For example, let ODD be the set of odd numbers, $EVEN$ the set of even numbers, and $EVEN + 2$ the set of even numbers greater than

⁷³Let $\mathfrak{n}(A)$ be the numerosity of a set A . In the context of numerosity theory, the analogous cancellation law is: if $k \cdot \mathfrak{n}(A) \leq k \cdot \mathfrak{n}(B)$, then $\mathfrak{n}(A) \leq \mathfrak{n}(B)$. This readily follows from the fact that numerosities belong to the nonnegative part of an ordered field that includes all positive integers; see Benci and Baglini 2022.

⁷⁴See Hamkins 2021, sect. 3.8.

⁷⁵See, for example, Gödel 1947, 515.

⁷⁶See Benci et al. 2006, sect. 4.1.

2. That is, let $ODD = \{1, 3, 5, \dots\}$, $EVEN = \{2, 4, 6, \dots\}$, and $EVEN + 2 = \{4, 6, 8, \dots\}$. Numerosity theory is compatible with assigning the same sizes to ODD and $EVEN$, but it is also compatible with assigning the same sizes to ODD and $EVEN + 2$ (which, by the Part-Whole Principle, must both be smaller than $EVEN$). Numerosity theory does not, by itself, tell us how these sets compare in size. Rather, it is compatible with infinitely many different assignments of size to them, each one of which intuitively seems to be “arbitrary.”⁷⁷

4.10 Summary

The following table summarizes the foregoing discussion. I have tried to discuss the most prominent alleged intuitive desiderata among which the three accounts differ, though I cannot adjudicate which are *genuine* intuitive desiderata. Note that none of the accounts satisfy *all* of the desiderata—even when the Axiom of Choice or Global Choice is assumed—and no set of desiderata satisfied by one account is a subset of those satisfied by another. This makes it difficult to assess which account, if any, is the *most* intuitive of the bunch. For my part, I am simply inclined to view each account as having its own distinctive intuitive virtues.

At this point, one might ask why we should care about how intuitive an account of set size (or class size) is—and, moreover, why we should care about how its intuitiveness compares to that of the injective account. After all, the injective account has led to some beautiful mathematics—its relation to intuition be damned. Nonetheless, it is worth noting that the consequences of adopting the injective account—i.e., the basis of the textbook definition of “cardinality”—have received a disproportionate amount of attention from philosophers, and it is worth asking why this is the case.

Why, for example, have philosophers taken greater interest in the consequences of adopting the textbook definition of “cardinality” than in the consequences of adopting the textbook definition of “matrix”? Considerations of mathematical fruit alone cannot explain this disparity, as matrix theory does not seem *less* rich than the set theory that results from adopting Cantor’s definition of “cardinality” (along with standard axioms of sets). Instead, it seems more plausible that, unlike the textbook definition of “matrix,” Cantor’s definition of “cardinality” tracks intuitions we have about

⁷⁷See [Parker 2013](#) for further discussion.

some pre-theoretic concept—namely, “set size”—that has immediate relevance to philosophically significant questions—namely, those about the infinite. (Is there more than one size of infinity? Is there a biggest infinity?) As Carnap might have put the point, Cantorian “cardinality” *explicates* a philosophically significant pre-theoretic concept.⁷⁸ The concept of “matrix,” while mathematically and scientifically fruitful, does not. So, if we can devise an explication of “set size” that has intuitive virtues not possessed by the injective account, there would appear to be philosophical value in doing so. The surjective account and numerosity theory appear to be precisely such explications.

⁷⁸See Carnap 1950, chap. 1. That said, to my knowledge, Carnap did not specifically discuss the question of how “set size” may be explicated.

Intuitive Desideratum	Injective	Surjective	Numerosity
	Without AC With AC	Without AC With AC	Without GC With GC
Account entails. . .			
Every set has a size	✓ ✓	✓ ✓	× ✓
Trichotomy	× ✓	× ✓	× ✓
Part-Whole Principle	× ×	× ×	× ✓
Partition Principle	× ✓	✓ ✓	× ?
Weak Partition Principle	× ✓	✓ ✓	× ?
Hume's Principle: Left-to-right direction	✓ ✓	× ✓	✓ ✓
Hume's Principle: Right-to-left direction	✓ ✓	✓ ✓	× ×
Dual Hume's Principle: Left-to-right direction	✓ ✓	✓ ✓	✓ ✓
Dual Hume's Principle: Right-to-left direction	× ✓	✓ ✓	× ×
Strong Cancellation Law	✓ ✓	× ✓	✓ ✓
Subtraction and division are well-defined	× ×	× ×	✓ ✓
Comparative Size Principle	✓ ✓	? ✓	? ?
No arbitrary comparisons of set size	✓ ✓	✓ ✓	× ×

Table 1: Differences in Accounts of Size. AC = Axiom of Choice. GC = Global Choice. Note that the claims numerosity theory is known to entail when GC is assumed—but not when GC is absent—are not entailed by numerosity theory when AC is assumed but GC is absent. Additionally, neither the injective nor surjective account entail the Part-Whole Principle or the well-definedness of subtraction and division when GC is assumed. Further, the class-theoretic version of the surjective account entails class-theoretic versions of the Partition Principle and Weak Partition Principle without assuming GC, but the class-theoretic version of the injective account only entails them if GC is assumed.

5 Conclusion

The surjective account of set size is a seemingly minor tweak to Cantor's injective account: simply understand comparisons of size fundamentally in terms of surjections rather than injections. Nonetheless, this tweak yields significant benefits: we arrive at an intuitively compelling and mathematically rich alternative account of set size (and of class size more generally), gain insight into the debate concerning the Axiom of Choice, and shed philosophical light on one of the oldest unsolved problems in set theory. I will close with some final thoughts.

First, one might ask whether we can devise an account of size that satisfies *all* of the intuitive desiderata we have considered. This is not possible, however, since the Part-Whole Principle is inconsistent with both Hume's Principle and the Dual Hume's Principle. For my part, I am inclined to regard numerosity theory simply as constituting a fundamentally different approach to set size from the broadly Cantorian approaches of the injective and surjective accounts.

Second, we might be tempted to regard the foregoing discussion as constituting an argument *for* the Axiom of Choice. As the above table shows, each of the three accounts satisfies *more* intuitive desiderata when the Axiom of Choice or Global Choice is assumed. So, whether we opt for a broadly Cantorian approach or not, we might think that we need to assume at least the Axiom of Choice to arrive at a maximally intuitively plausible account of size.

This inference strikes me as too quick, however. For one, it is not merely an intuitive desideratum that we satisfy intuitive principles like the Partition Principle; it is also an intuitive desideratum that we satisfy such principles *in an intuitive way*. For example, recall the case of farmers and dogs from section 3.1. Intuitively, it seems that the existence of a surjection from the set of dogs onto the set of farmers should *by itself* entail that there are at least as many dogs as farmers. We shouldn't *need* to make the additional inferences licensed by the Axiom of Choice to conclude this. Indeed, it seems intuitively *problematic* that the injective account requires the axiom to deliver this verdict—and this is an intuitive cost that goes beyond the mere epistemic possibility that the axiom is false. Of course, it might turn out to be impossible for any account to entail a larger number of the aforementioned principles in an intuitive way without the Axiom of Choice, but I think it is too early to write off the prospects for such an account.

Indeed, it seems to me that we have merely scratched the surface of the space of intuitively compelling alternative accounts of size. First, I have not yet mentioned modal variations of the above accounts, on which (for example) the merely *possible* existence of a bijection from one set to another suffices for them to be the same size.⁷⁹ Second, recall that the intuitive motivation for the surjective account is but a simple generalization of those considerations that motivate the injective account's understanding of *at least as big as*. It may be that the motivation for the surjective account itself admits of a simple generalization that enables us to satisfy nearly all of our intuitive desiderata without assuming the Axiom of Choice. And it may be that any account that results from such a generalization bears substantial mathematical fruit in even less explored mathematical terrain. Regardless of where this particular inquiry leads, the recent proliferation of approaches to set size suggests we may be entering a rich new era in our exploration of the infinite.

6 Appendix

Theorem 1. The injective and surjective accounts are equivalent in ZFC.

Proof. In what follows, (1)–(3) and (1*)–(3*) refer to the statements of the injective and surjective accounts, respectively.

First, assume (1) and that A is at least as big as B . Since $B \preceq A$, we have that $A \preceq^* B$ —i.e., (1*) is true. The further assumption of (2) and (3) then immediately entails (2*) and (3*). Now assume (1*), the Axiom of Choice, and that A is at least as big as B . Since the Axiom of Choice is true, $\mathbf{PP}_{\text{injective}}$ is true (cf. section 3.2). So, since $A \preceq^* B$, we have that $B \preceq A$ —i.e., (1) is true. The further assumption of (2*) and (3*) then immediately entails (2) and (3). \square

Theorem 2. (1*) and (1**) are equivalent in ZF.

Proof. If A or B is empty, it is obvious that (1*) and (1**) are equivalent. So, suppose A and B are nonempty.

To show that (1*) entails (1**), let f be a surjective function from A onto B . For every $b \in B$, let $A_b = \{x \in A \mid f(x) = b\}$. Note that $P = \{A_b \mid b \in B\}$

⁷⁹Different modal extensions of the injective account have recently been explored by Pruss (2020), Scambler (2021), and Builes and Wilson (2022).

is a partition of A . Now define $g : B \rightarrow P$ such that $g(x) = A_x$. Clearly, if $x \neq y$, then $g(x) \neq g(y)$. So, g is injective.

To show that (1^{**}) entails (1^*) , let f be an injection from B into some partition of A . Let a be an arbitrary member of A . Note that, if $a \in f(c)$ for some $c \in B$, then c is unique since $f(x) \cap f(y) = \emptyset$ for any distinct $x, y \in B$. So, there is a function $g : A \rightarrow B$ such that, for some $b \in B$, $g(x) = c$ if $x \in f(c)$ for some $c \in B$ and $g(x) = b$ otherwise. Moreover, g is surjective: for any $x \in B$, there is some $y \in A$ such that $y \in f(x)$ and therefore $g(y) = x$. \square

Theorem 3. The class-theoretic versions of the injective and surjective accounts are equivalent in **GB** if **GC** holds.

Proof. Here (1)–(3) and (1^*) – (3^*) will refer to the class-theoretic generalizations of these claims.

The proof that (1)–(3) entail (1^*) – (3^*) is exactly analogous to the set-theoretic case. Now assume (1^*) , **GC**, and that A is at least as big as B . Since **GC** is true, **CPP**_{injective} is true (cf. section 3.2.1). Since $A \preceq^* B$, it follows that $B \preceq A$ —i.e., (1) is true. The further assumption of (2^*) and (3^*) then immediately entails (2) and (3). \square

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