Preprint of an article in Journal of Economic Theory 160: 463-493, 2015
Official Publication: https://www.sciencedirect.com/science/article/pii/S0022053115000551

Aggregation theory and the relevance of some issues to others

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March 2015 (first version December 2006)

Abstract

I propose a relevance-based independence axiom on how to aggregate individual yes/no judgments on given propositions into collective judgments: the collective judgment on a proposition depends only on people's judgments on propositions which are *relevant* to that proposition. This axiom contrasts with the classical independence axiom: the collective judgment on a proposition depends only on people's judgments on *the same* proposition. I generalize the premise-based rule and the sequential-priority rule to an arbitrary priority order of the propositions, instead of a *dichotomous* premise/conclusion order resp. a *linear* priority order. I prove four impossibility theorems on relevance-based aggregation. One theorem simultaneously generalizes Arrow's Theorem (in its general and indifference-free versions) and the well-known Arrow-like theorem in judgment aggregation.

Keywords: judgment aggregation, generalized Arrow theorem, generalized premise-based and sequential-priority rules, priority graph, aggregation of non-binary evaluations

JEL Classification Numbers: D70, D71

1 Introduction

The judgment aggregation problem consists in merging many individuals' judgments ('yes' or 'no') on some interconnected propositions into collective judgments on these propositions. Judgment aggregation ('JA') has wide applications. A classic example is decision-making in a jury in court, where the jurors have to merge their judgments on three controversial propositions: (i) the defendant has broken the contract; (ii) the contract is legally valid; (iii) the defendant is guilty (e.g., Kornhauser and Sager [26], List and Pettit [30]). These propositions are interconnected because legal doctrine prescribes that (iii) holds if and only if (i) and (ii) both hold. Another example

[1]The author thanks the referees for very detailed and helpful comments. The paper has been completely rewritten since its 2006 version. It was presented at the Workshop on Logic and Collective Decision Making (Lille, France, 2007), the Social Choice Colloquium (Tilburg University, 2007), the Workshop on Judgment Aggregation (Karlsruhe University, 2007) and the Economics Research Seminar (ETH Zurich, 2007). The author gratefully acknowledges support by the French Agence Nationale de la Recherche (ANR-12-INEG-0006-01) and by the Nuffield Foundation (under its New Career Development Fellowship).

is *preference aggregation*. Here we merge people's judgments on propositions of the kind 'option x is weakly preferable to option y' – in short: xRy – for various pairs of options x and y (where these propositions are interconnected via conditions such as transitivity). In yet another example, we merge people's estimates of some variables (such as GDP, prices and unemployment). In other words, we merge people's judgments on propositions of the sort 'variable k takes value v' for various pairs of a variable k and a possible value v (where these propositions might be interconnected via some macroeconomic equations). Similarly, we might merge grades which people give to some politicians, where the possible grades might be 'good', 'excellent' and 'bad' (as in Balinski and Laraki's [1] voting theory). In other words, we merge people's judgments on propositions of the sort 'politician k is of quality v' for pairs of a politician k and a possible grade v. The last two examples are versions of the *evaluation aggregation* problem, in which we merge people's positions on some matters: people's estimates of variables, people's grades given to politicians, people's degrees of belief in some events, etc. (e.g., Rubinstein and Fishburn [42], Dietrich and List [12], Dokow and Holzman [16]).

Evidently, many 'special' aggregation problems can be stated as JA problems – but does JA theory have to say something interesting about them? JA theory has been particularly successful at generalizing theorems and insights from *preference* aggregation theory, including Arrow's Theorem in its indifference-free version. JA theory has been less successful at addressing some other aggregation problems, including preference aggregation in its general (indifference-permitting) form, the aggregation of (non-binary) evaluations, and the aggregation of judgments on propositions with a more complex priority structure than a dichotomous premise/conclusion structure. Perhaps the main reason is that JA theory draws strongly on the classic but controversial axiom of *proposition-wise independence*: the collective judgment on a proposition should be determined solely by people's judgments on *this* proposition. This axiom denies that other propositions can be relevant. I call a proposition p 'relevant' to another q if people's judgments on p matter for forming the collective judgment on q, so that the latter should draw on the former. Proposition-wise independence implicitly assumes a narrow notion of relevance: each proposition is relevant only to itself. The implausibility of the axiom and its narrow relevance notion becomes evident in our introductory examples:

(a) In the jury example, the popular *premise-based* procedure violates proposition-wise independence and treats the two 'premise propositions' (i) and (ii) as relevant to the 'conclusion proposition' (iii), since the collective judgment on (iii) is derived from jurors' judgments on (i) and (ii). (More precisely, the collective endorses (iii) if and only if each premise proposition is endorsed by majority.) There are many other examples of propositions between which there are relevance connections of a premise-conclusion type, making proposition-wise independence implausible.

(b) Now consider the preference agenda, whose propositions take the form xRy and express betterness comparisons between options. Whether the proposition xRy is collectively endorsed should be sensitive to people's preferences between x and y. Someone's preference between x and y is captured by his judgments on *two* propositions, xRy and yRx (for instance, indifference is captured by 'yes' judgments on both xRy and yRx). So the propositions xRy and yRx

are both relevant to xRy. Yet proposition-wise independence prevents people's judgments on yRx ('yes' or 'no') from affecting the social judgment on xRy – with absurd consequences.[2] This makes the axiom implausible and much stronger than Arrow's axiom of 'independence of irrelevant alternatives'. Both axioms become equivalent only after excluding indifferences.

(c) Finally, consider the evaluation agenda for a group in search of 'positions' on some 'matters' (e.g., estimates of some variables, grades of some politicians, etc.). It is natural to construct the collective position on a matter from people's positions on this matter: the collective estimate of a variable might be an arithmetic average of people's estimates of this variable; the collective grade of a politician might be this politician's median grade; and so on. Such aggregation rules satisfy a *matter-wise* independence axiom: the collective position on a matter depends only on people's positions on this matter. But they violate *proposition-wise* independence: if for instance politician Smith's collective grade is his median grade, then the collective judgment (yes or no) on whether 'he is good' depends not just on people's judgments on this proposition, but on people's judgments on all propositions about Smith ('he is good', 'he is excellent', and so on for other grades). Requiring proposition-wise independence would be utterly implausible.

So far, JA theory faces an all-or-nothing dilemma. Either it accepts proposition-wise independence, which eliminates many plausible aggregation rules and leads into impossibility results. Or it drops the axiom and is left with too many possibilities and no systematic way to prevent irrelevant information from playing a role. In response, I enrich the JA framework by a *'relevance' relation* $\mathcal{R}$ between propositions, where $p\mathcal{R}q$ represents relevance of p to (the collective judgment on) q. I replace proposition-wise independence by *independence of irrelevant propositions* ('IIP'): the collective judgment on a proposition depends solely on people's judgments on propositions relevant to it. There are many interpretations and applications. In the 'classical' case, each proposition is deemed relevant just to itself: $p\mathcal{R}q \Leftrightarrow p = q$. So IIP reduces to proposition-wise independence. For less narrow relevance notions, IIP becomes weaker and more plausible. For instance, relevance of p to q could mean that p is a premise/argument/reason for or against q, as is the case in (a). Such a relevance relation is acyclic. The corresponding axiom IIP represents the condition of *premise-based* aggregation. Alternatively, relevance of p to q could mean that p and q pertain to the same semantic field, topic or matter. Such a relevance relation is an equivalence relation. An example is the relevance relation indicated in (b) for the preference agenda: it reflects Arrow's notion of (ir)relevance and renders IIP equivalent to Arrow's axiom of 'independence of irrelevant alternatives'. Another example is the relevance relation indicated in (c) for the evaluation agenda: here, IIP requires matter-wise independent aggregation, for instance by taking people's average or median position on each matter.

Overview of the findings in the context of the literature. After defining the judgment aggregation framework (in the version of List and Pettit [30] and more

[2] A collective acceptance of xRy can never be reversed if everyone who strictly prefers x to y (i.e., accepts xRy but not yRx) suddenly becomes indifferent (i.e., accepts both xRy and yRx). This is counterintuitive.

precisely Dietrich [7], [8]) and adding the relevance notion (Sections 2 and 3), I explore relevance-based aggregation from a constructive perspective (Section 4) and then an axiomatic perspective (Sections 5-9). In Section 4, the relevance relation is taken to capture premisehood (priority) and to define an (acyclic) 'priority graph' over the propositions. This leads to *priority rules*: aggregation rules which decide on the propositions in an order of priority imposed by the priority graph, where each decision is constrained by the prior decisions. Such rules generalize List's [28] *sequential priority rules*, which are based on a linear priority order rather than a general priority graph. Theorem 1 gives sufficient conditions for priority rules to be 'well-behaved', i.e., to respect our new independence axiom and generate consistent collective judgments.

Later, Sections 5-9 focus on impossibility theorems in an Arrovian tradition. I first introduce a unanimity condition – *unambiguous agreement preservation* – which focuses on 'non-spurious' agreements, in which people agree not just on a judgment, but also on the 'reasons'. The axiom turns out to generalize Arrow's weak Pareto principle. I prove four impossibility theorems that give sufficient conditions under which our new independence and unanimity axioms imply that aggregation is degenerate. The theorems differ in the notion of 'degenerate'. One theorem generalizes Arrow's Theorem in its general version and its indifference-free version, since Arrow's Theorem arises when choosing the *preference* agenda (in its general or indifference-free version). Our theorem also generalizes the known Arrow-like theorem in judgment aggregation (see Dietrich and List [10] and Dokow and Holzman [15], both building on Nehring and Puppe [37]/[39] and strengthening Wilson [43]). The latter theorem arises when choosing the (narrow) classical relevance notion. This known theorem already generalizes Arrow's Theorem in its indifference-free version. Arrow's Theorem in its general version had so far no judgment-aggregation counterpart. It was however derived from a judgment-aggregation theorem (see Dokow and Holzman [17], Corollary 4.4). Ever since Nehring and Puppe [37]/[39] an important goal in the theory has been that theorems be *tight*, i.e., maximally general in their assumptions on the aggregation problem. In particular, the mentioned Arrow-like theorem is tight (see Dokow and Holzman [15]). Our theorems are all tight in the special case of the classical relevance relation (where our new independence axiom reduces to the classical one). But they are not tight in general.

In sum, weakening classical independence opens up new possibilities (such as priority rules), but does not generally free us from impossibility. It is of course well-known that classical independence is very hard to satisfy: besides the cited Arrow-like impossibility theorem, see for instance the impossibility theorems in List and Pettit [30], Pauly and van Hees [40], Dietrich [5], Gärdenfors [23], Mongin [33], Nehring and Puppe [37]/[39], [38], Dietrich and List [9], [11], [13], Dokow and Holzman [17], Nehring [35] and Dietrich and Mongin [14]. The classical independence axiom is often criticised (e.g., Chapman [3], Mongin [33]), but rarely weakened. All weaker independence axioms in the literature – such as independence axioms restricted to premise propositions (Dietrich [5], Dietrich and Mongin [14]) or to atomic propositions (Mongin [33]) – are special cases of our independence axiom: they arise for special choices of the relevance relation. It is also worth mentioning Dokow and Holzman's [16] impossibility theorem on matter-wise independent aggregation of non-binary evaluations. Although matter-wise independence is a special case of our independence axiom, their impossibility theorem is not generalized by our ones (partly because the unanimity axioms do not match). A growing branch of the judgment aggregation literature gives up proposition-wise independence altogether rather than weakening it. This includes the distance-based approach (e.g., Konieczny and Pino-Perez [25], Pigozzi [41], Miller and Osherson [31], Hartmann *et al.* [24], Lang *et al.* [27]), the sequential approach (e.g., List [28], Dietrich and List [9]), Borda-like and scoring-based rules (Zwicker [44], Dietrich [8], Duddy, Piggins and Zwicker [19]), and 'approximately majoritarian' rules (Nehring, Pivato and Puppe [36]).

2 The judgment aggregation framework

A group of $n \geq 2$ individuals, labelled $1, ..., n$, has to form collective judgments on some interconnected propositions.

The agenda. The set of propositions under consideration is the *agenda*. It is subdivided into *issues*, i.e., pairs of opposite proposition, such as 'it will rain' and 'it won't rain'. Writing '$\neg$' for negation, the agenda thus takes the form $X = \{p, \neg p, p', \neg p', ...\}$, with issues $\{p, \neg p\}$, $\{p', \neg p'\}$, ... An individual rationally accepts one proposition from each issue ('completeness') and respects any logical interconnections ('consistency'). Formally:

Definition 1 *An **agenda** is a non-empty set X (of 'propositions') that is endowed with the notions of negation and interconnections, i.e.,*

(a) *to each $p \in X$ corresponds a proposition denoted $\neg p \in X$ ('not p') with $\neg p \neq p = \neg\neg p$ (so X is partitioned into pairs $\{p, \neg p\}$, called 'issues'),*

(b) *certain judgment sets $J \subseteq X$ containing a single member of each issue count as 'rational' (the set of these J is denoted $\mathcal{J} \neq \varnothing$),*

where (in this paper) X is tautology-free, i.e., no $p \in X$ belongs to all $J \in \mathcal{J}$.[3]

Given an agenda X, we fix a subset $X_0 \subseteq X$ containing exactly one member of each pair $p, \neg p$ (no matter which one). So $X = \{p, \neg p : p \in X_0\}$. I often write '$\pm p$' for '$p, \neg p$', and use the term 'issue' for both $\pm p$ and $\{\pm p\}$. In examples, the agenda is often specified *syntactically*, writing propositions as logical sentences and using the logical notions of negation and interconnections. Simple (syntactic) agendas are $X = \{\pm a, \pm b, \pm(a \wedge b)\}$ and $X = \{\pm a, \pm b, \pm c, \pm(c \leftrightarrow (a \wedge b)\}$, where a, b, c are (logically independent) atomic propositions such as, in a jury decision problem, 'the defendant has broken the contract', 'the contract is legally valid' and 'the defendant is liable'.[4]

[3] In algebraic terms, the agenda is the structure $X \equiv (X, \neg, \mathcal{J})$. The negation operator $\neg$ (a function $p \mapsto \neg p$ satisfying $\neg p \neq p = \neg\neg p$) and the set of issues (a partition of X into binary sets) are two interdefinable objects. We could thus equivalently define an agenda as a set endowed with '*issues* and interconnections', and define the negation of p as the unique proposition $\neg p$ such that $\{p, \neg p\}$ is an issue. Algebraically, X would then be the structure $X \equiv (X, \mathcal{I}, \mathcal{J})$, where $\mathcal{I}$ is the set of issues.

[4] Following Dietrich [7], the logic in which propositions are expressed could take many forms: classical or non-classical (e.g., a modal logic), propositional or non-propositional (i.e., a predicate logic).

Individual judgments. The **judgment set** of an individual is the set $J \subseteq X$ of 'accepted' or 'believed' propositions. It is **complete** if it contains a member of each issue $\{p, \neg p\}$, and **consistent** if it is a subset of a rational judgment set. So a judgment set is rational (i.e., in $\mathcal{J}$) just in case it is both consistent and complete. A proposition $p \in X$ (or set $S \subseteq X$) **entails** a proposition $q \in X$ – written $p \vdash q$ (resp. $S \vdash q$) – in case, for every rational judgment set $J \in \mathcal{J}$, if $p \in J$ (resp. $S \subseteq J$), then $q \in \mathcal{J}$.

Aggregation. An *aggregation rule* is a function F that assigns to every profile $(J_1, \ldots, J_n)$ of 'individual' judgment sets (from some domain of admissible profiles) a 'collective' judgment set $F(J_1, \ldots, J_n) \subseteq X$. An example is majority rule, given by

$$F(J_1, ..., J_n) = \{p \in X : |\{i : p \in J_i\}| > n/2\} \text{ for all } (J_1, ..., J_n) \in \mathcal{J}^n.$$

In this paper, the domain of the aggregation rule is always $\mathcal{J}^n$, i.e., any rational input is admissible. If outputs are also rational, F is a function $F : \mathcal{J}^n \to \mathcal{J}$. More generally, F is a function $F : \mathcal{J}^n \to 2^X$ with possibly inconsistent or incomplete outputs. Majority rule notoriously generates inconsistent outputs.

I now give two important examples; other ones will follow in the next section.

Example 1: aggregating strict or general preferences. I now define the preference agenda in two versions: for strict preferences (excluding indifferences) and general preferences (allowing indifferences). Consider a set of two or more alternatives A.

- The **strict preference agenda** for the set of alternatives A is the set of sentences $X = \{xPy : x, y \in A, x \neq y\}$, where xPy reads 'x is (strictly) better than y' and where by definition $\neg xPy = yPx$. The interconnections are defined by the usual conditions on strict preferences. Formally, judgment sets $J \subseteq X$ can be identified with (irreflexive) binary relation $\succ$ on A via the equivalence $x \succ y \Leftrightarrow xPy \in J$ for $x \neq y$; and so we may apply relation-theoretic notions like transitivity to judgment sets. $\mathcal{J}$ is the set of all transitive, anti-symmetric and connected judgment sets $J \subseteq X$. So *rational* judgment sets $J \in \mathcal{J}$ represent strict linear orders on A.
- The **general** (or **weak**) **preference agenda** for the set of alternatives A is the set of sentences $X = \{xRy, \neg xRy : x, y \in A, x \neq y\}$, where xRy reads 'x is weakly better than y'. This agenda has twice the size of the strict preference agenda, as $\neg xRy \neq yRx$, whereas for the strict preference agenda $\neg xPy = yPx$. While $\neg xRy$ cannot be replaced by yRx, I will sometimes write 'yPx' for $\neg xRy$ (reflecting the equivalence between preferring y to x and *not* weakly preferring x to y). The interconnections within the general preference agenda are defined by the usual rationality conditions on weak preferences. Formally, we can apply relation-theoretic notions like transitivity to judgment sets $J \subseteq X$, as each $J \subseteq X$ induces a (reflexive) binary relation $\succsim$ on A via $x \succsim y \Leftrightarrow xRy \in J$ for $x \neq y$. Now $\mathcal{J}$ is the set of all judgment sets (containing exactly one member of each issue $xRy, \neg xRy$) which are transitive and connected. So rational judgment sets represent weak orders on A.

Example 2: aggregating evaluations. Consider the aggregation of people's (possibly non-binary) positions on some matters or issues (e.g., Rubinstein and Fishburn [42], Dietrich and List [12], Dokow and Holzman [16]). As will be seen, this non-binary aggregation problem can be represented in the binary judgment aggregation framework. Given sets K of 'matters' and V of possible 'positions' or 'values' (where $|K| \geq 1$ and $|V| \geq 2$), an *evaluation* is a function $E : K \to V$ assigning a position to each matter, or equivalently a family $(v_k)_{k \in K}$ in V^K. (One might write K and V as $K = \{1, ..., m\}$ and $V = \{0, 1, ..., l\}$ in the finite case.) For instance, K contains political candidates and V possible grades of candidates; or K contains macroeconomic variables (GDP, inflation, etc.) and $V \subseteq \mathbb{R}$; or K contains animal species and V possible sizes of species; or K contains sentences and V truth values in binary or many-valued logic. Not every evaluation counts as coherent, because of interconnections between matters: macroeconomic variables must obey certain equations, animal species cannot all be extinct, etc. Let $\mathcal{E} \subseteq V^K$ be the non-empty set of '*coherent*' evaluations. To study the aggregation of coherent evaluations as a (binary) judgment aggregation problem, consider the agenda

$$X = \{\pm v_k : k \in K, v \in V\} \text{ (the \textbf{evaluation agenda})}$$

where v_k denotes the proposition 'v is the value on matter k'.[5] To each evaluation $E : K \to V$ corresponds a unique judgment set $J_E \subseteq X$, containing those v_k with $E(k) = v$ and those $\neg v_k$ with $E(k) \neq v$. A judgment set is rational just in case it corresponds to a coherent evaluation: $\mathcal{J} = \{J_E : E \in \mathcal{E}\}$.[6]

Evaluation aggregation has so far not been addressed within (binary) judgment aggregation theory. But it has been analysed in other frameworks, and for many kinds of evaluation.[7]

3 Relevance and a new independence axiom

I aim to overcome the following controversial independence axiom, which parallels Arrow's 'independence of irrelevant alternatives' and has led to many impossibility results.

Proposition-wise Independence: For all propositions $p \in X$ and all profiles $(J_1, ..., J_n)$ and $(J'_1, ..., J'_n)$ in the domain, if $p \in J_i \Leftrightarrow p \in J'_i$ for every individual i then $p \in F(J_1, ..., J_n) \Leftrightarrow p \in F(J'_1, ..., J'_n)$.

[5] For simplicity, I use the symbol 'v_k' both for a proposition $v_k \in X$ and a position $v_k = E(k) \in V$.

[6] In a generalized version of Example 2, the set of possible positions is matter-dependent, so that V is replaced by sets V_k ($k \in K$). One matter might consist in estimating a real-valued quantity ($V_k = \mathbb{R}$), another in answering a yes/no question ($V_k = \{\text{yes, no}\}$), and so on.

[7] For instance, the literature on probabilistic opinion pooling deals with aggregating probability functions, i.e., evaluations in which 'matters' are events and 'positions' are subjective probabilities (e.g., Genest and Zidek [22]). Other contributions on (non-binary) evaluation aggregation are made by Rubinstein and Fishburn [42] (who prove a general result on linear aggregation), Claussen and Roisland [4] (who study a non-binary version of the discursive dilemma), Dietrich and List [12] (who seek to unify different aggregation problems), Dokow and Holzman [16] (who prove a general impossibility result), and Pauly and van Hees [40] and Duddy and Piggins [18] (who all study the aggregation of multi-valued logical judgments).

This axiom forbids that the collective judgment on p depends on people's judgments on other propositions. However, often some other propositions are relevant to p, so that people's judgments on them should not be ignored. In our court example, the breach-of-contract proposition and the validity-of-contract proposition both seem relevant to the guilt proposition. Other examples follow shortly. There are many possible interpretations of 'relevance' of q to p: it could for instance mean that q is semantically related to p, or that q is a premise of p. I capture relevance connections by a binary relation $\mathcal{R}$ on the agenda X, where $q\mathcal{R}p$ reads 'q is relevant to p'. The set of propositions relevant to $p \in X$ is denoted

$$\mathcal{R}(p) := \{r \in X : r\mathcal{R}p\}.$$

Often relevance does not distinguish between a proposition and its negation, i.e., is

$$\textit{negation-invariant: } q\mathcal{R}p \Leftrightarrow q'\mathcal{R}p' \text{ if } q' \in \{\pm q\} \text{ and } p' \in \{\pm p\}. \tag{1}$$

Then $\mathcal{R}$ is equivalent to a relation on the set of issues (rather than propositions), or on the set X_0. The informal talk will reflect this. Under negation-invariance, I often write $\mathcal{R}(\pm p)$ to denote both $\mathcal{R}(p)$ and $\mathcal{R}(\neg p)$ (and to imply that $\mathcal{R}(p) = \mathcal{R}(\neg p)$).

The new independence axiom requires collective judgments to depend only on people's judgments on *relevant* propositions:

Independence of Irrelevant Propositions (IIP): For all propositions $p \in X$ and all profiles $(J_1, ..., J_n)$ and $(J'_1, ..., J'_n)$ in the domain, if $J_i \cap \mathcal{R}(p) = J'_i \cap \mathcal{R}(p)$ for every individual i then $p \in F(J_1, ..., J_n) \Leftrightarrow p \in F(J'_1, ..., J'_n)$.

Proposition-wise independence is a special case of IIP with 'classical' relevance given by $\mathcal{R}(p) = \{p\}$. As another example, consider premise-based aggregation for the agenda $X = \{\pm a, \pm b, \pm(a \wedge b)\}$ with premise propositions $\pm a, \pm b$ and conclusion propositions $\pm(a \wedge b)$. The decisions on $\pm a$ and $\pm b$ are made by two separate majority votes, and the decision on $\pm(a \wedge b)$ is deduced logically from the decisions on $\pm a$ and $\pm b$. This rule satisfies IIP if premises are deemed relevant to conclusions, i.e., if

$$\mathcal{R}(\pm a) = \{\pm a\},\ \mathcal{R}(\pm b) = \{\pm b\},\ \mathcal{R}(\pm(a \wedge b)) = \{\pm a, \pm b\}. \tag{2}$$

In full generality, a relevance relation $\mathcal{R}$ need not satisfy any particular relation-theoretic conditions such as transitivity or reflexivity. However I shall assume **non-underdetermination**: every proposition is settled by the judgments on the relevant propositions, i.e., for every $p \in X$ and every consistent set S of the form $\{q^* : q \in \mathcal{R}(p)\}$, where each q^* is q or $\neg q$,

- *either* S entails p (S is then called an ($\mathcal{R}$-)**explanation** of p)
- *or* S entails $\neg p$ (S is then called an ($\mathcal{R}$-)**refutation** of p).

This condition is plausible. It holds automatically if $\mathcal{R}$ is reflexive ('self-relevance'). It also holds for the agenda $\{\pm a, \pm b, \pm(a \wedge b)\}$ with relevance given by (2). Here, each premise proposition $p \in \{\pm a, \pm b\}$ has a single explanation $\{p\}$ (and a single refutation $\{\neg p\}$), $a \wedge b$ has a single explanation $\{a, b\}$, and $\neg(a \wedge b)$ has three explanations $\{a, \neg b\}$, $\{\neg a, b\}$ and $\{\neg a, \neg b\}$.[8] Let me summarize our definitions:

[8] In the paper's unpublished version Dietrich [6], I argue that cases of underdetermination usually stem from having misspecified $\mathcal{R}$; and I show that non-underdetermination is indispensable since otherwise no aggregation rule F on $\mathcal{J}^n$ can satisfy IIP and a mild unanimity condition (requiring $F(J, ..., J) = J$ for all $J \in \mathcal{J}$).

Definition 2 *A* ***relevance relation*** *is a binary relation $\mathcal{R}$ on the agenda X satisfying non-underdetermination (I write $\mathcal{R}(p) := \{q : q\mathcal{R}p\}$). If it is negation-invariant, then it is identified with a relation on X_0 or on the set of issues (and I write $\mathcal{R}(\pm p) := \mathcal{R}(p) = \mathcal{R}(\neg p)$).*

Definition 3 *The* ***classical*** *relevance relation is the one given by $q\mathcal{R}p \Leftrightarrow q = p$, i.e., by $\mathcal{R}(p) = \{p\}$.*

Remark 1: IIP is at most as strong as proposition-wise independence (check this using non-underdetermination), and equivalent to it under classical relevance.

Many informational constraints in social choice theory are instances of IIP relative to 'some' relevance relation. Roughly, the more relevance connections there are, the weaker IIP becomes. IIP is vacuous if everything is relevant to everything, i.e., if $\mathcal{R} = X \times X$. IIP is proposition-wise independence for classical relevance. IIP is Gärdenfors' [23] '*weak*' proposition-wise independence if $\mathcal{R}(p) = \{\pm p\}$ for all $p \in X$. IIP is Dietrich's [5] *independence on premises* – the restriction of proposition-wise independence to a subset $Y \subseteq X$ of 'premises' – if $\mathcal{R}(p) = \{p\}$ for $p \in Y$ and $\mathcal{R}(p) = X$ for $p \in X \backslash Y$. IIP is Mongin's [33] *independence on atomic propositions* if $\mathcal{R}(p) = \{p\}$ for syntactically atomic propositions p and $\mathcal{R}(p) = X$ for syntactically compound propositions p (like $a \wedge b$).

I now give further examples of (negation-invariant) relevance relations.

Example 1 continued. Arrow's condition of *independence of irrelevant alternatives* ('IIA') is equivalent to IIP, where we adopt the '*Arrovian*' relevance relation, which is implicit in IIA and is defined as follows, depending on whether indifferences are allowed:

- In the case of the *strict* preference agenda, Arrovian relevance is defined by

$$\mathcal{R}(\pm xPy) = \{\pm xPy\} \text{ for all } xPy \in X. \tag{3}$$

- In the case of the *general* preference agenda, Arrovian relevance is defined by

$$\mathcal{R}(\pm xRy) = \{\pm xRy, \pm yRx\} \text{ for all } xRy \in X. \tag{4}$$

The asymmetry between (3) and (4) is only apparent, since in (3) we have $\{\pm xPy\} = \{\pm xPy, \pm yPx\}$ (because $\neg xPy = yPx$ and $\neg yPx = xPy$). In (4) it matters that $\mathcal{R}(\pm xRy)$ contains not just $\pm xRy$, but also $\pm yRx$, since an individual's judgments on both of these issues are needed to capture how he ranks x relative to y, i.e., whether he prefers x, prefers y, or is indifferent.

Example 2 continued. For the evaluation agenda $X = \{\pm v_k : k \in K, v \in V\}$, where v_k represents position v on matter k, one might view v_k as relevant to each proposition v'_k concerning the same matter k, but irrelevant to any proposition $v'_{k'}$ concerning another matter $k' \neq k$. Formally:

$$\mathcal{R}(\pm v_k) = \{\pm v'_k : v' \in V\}. \tag{5}$$

Example 3: relevance as an equivalence relation (of sameness in topic). In many cases including Examples 1 and 2, relevance is an equivalence relation: $\mathcal{R}$ is reflexive ('self-relevance'), symmetric, and transitive. So the agenda X is partitioned into equivalence classes of inter-relevant propositions. Interpreting each equivalence class as a *topic*, IIP requires *topic-wise* (not proposition-wise) aggregation. One topic might deal with weather, another with the economy, and so on. A topic can be as small as a single issue $\{\pm p\}$ or much larger. The general preference agenda in Example 1 has topics of the form $\{\pm xRy, \pm yRx\}$ (the topic of comparing x and y). In Example 2, topics correspond to matters in K.

Example 4: relevance as an acyclic relation (of priority/premisehood). I now interpret '$q\mathcal{R}p$' as 'q is a premise/reason/argument for (or against) p'. To make sense of this interpretation, I exclude priority cycles. Formally, $\mathcal{R}$ is a negation-invariant relevance relation without cycles of issues; it is referred to as a 'priority graph' (see Definition 4). IIP then represents the condition of *premise-based* aggregation: the collective judgment on any proposition $p \in X$ is determined by people's reasons for or against p. This generalizes classical premise-based aggregation, which

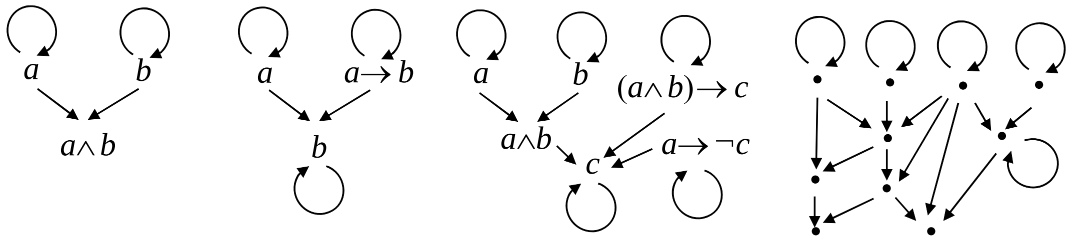

Figure 1: Priority graphs on four agendas. Arrows indicate relevance (priority). Agenda 1: $X_0 = \{a, b, a \wedge b\}$. Agenda 2: $X_0 = \{a, a \rightarrow b, b\}$. Agenda 3: $X_0 = \{a, b, c, a \wedge b, (a \wedge b) \rightarrow c, a \rightarrow \neg c\}$. Agenda 4: X_0 contains 10 propositions indicated by "$\cdot$".

has only two levels of priority, 'premises' and 'conclusions', as in the first and second agenda of Figure 1. By allowing an arbitrary priority structure, I permit 'premises of premises' and many other interesting constellations. A premise-based – that is, IIP – aggregation rule can be thought of as being a sequential procedure, which first decides on the roots of the priority graph (the 'basic premises'), and then works itself forward along each branch of the graph. Let me be more concrete. I call $p \in X$ a **root proposition** and $\pm p$ a **root issue** if p has no external premise, i.e., $\mathcal{R}(\pm p) = \{\pm p\}$. For instance, a and b are root propositions in the first priority graph of Figure 1. By IIP, root issues $\pm p$ are settled by a vote on the issue, ignoring other issues. In *this* sense, root issues must be decided first. Decisions on non-root issues $\pm p$ must come later because – to ensure collective rationality – they must respect the decisions on root issues and other prior issues. If the prior decisions impose no logical constraint on the current issue $\pm p$ – for instance, if for the second agenda in Figure 1 the decisions on the two root issues are $\neg a$ and $a \rightarrow b$, which have no logical implication for the issue $\pm b$ – then there is some freedom in how to settle the current issue. All the premise-based approach (i.e., IIP) requires here is that the current issue $\pm p$ be decided based on people's judgments on the premises $\mathcal{R}(\pm p)$. This can be done in many ways. *One* route is to base the decision on $\pm p$ on people's judgments on $\pm p$ (and thereby indirectly on their judgments on $\mathcal{R}(\pm p)$[9]), for instance by taking a majority vote on $\pm p$. This route is taken by 'priority rules', studied in the next section.

4 Priority rules relative to a priority graph

This section follows Example 4's interpretation of relevance as priority/premisehood, so that IIP requires premise-based aggregation. I again assume that relevance defines a 'priority graph'. This notion is now defined formally:

Definition 4 *A **priority graph** is a negation-invariant relevance relation $\mathcal{R}$ which is acyclic as a relation over issues, i.e., there are no $p_1, ..., p_m \in X$ ($m \geq 2$) from distinct issues such that $p_1 \mathcal{R} p_2 \mathcal{R} \cdots p_m \mathcal{R} p_1$. Given a priority graph, the axiom IIP is also called **(generalized) premise-basedness**.*

I now introduce *priority rules* relative to a priority graph, where X is finite for simplicity.[10] They generalize List's [28] *sequential priority rules*, which are defined relative to a linear order of issues – a 'linear priority graph' – and decide the issues one by one in order of diminishing priority: the decision on any issue is *either* deduced from past decisions *or* made by voting on the current issue, depending on whether past decisions logically constrain the current decision. Linear priority graphs are of course a very special case. In applications, priority is often non-linear. For instance, two issues can be on a par, so that neither has priority over the other (as for the two 'premise issues' $\pm a$ and $\pm b$ in our example agenda $\{\pm a, \pm b, \pm(a \wedge b)\}$).

A priority rule relative to an arbitrary priority graph begins by a vote on every root issue (of maximal priority). Next, one considers each issue of second-highest priority, to which only root issues (and possibly the present issue) are relevant: if the past decisions on relevant root issues imply some decision on the current issue, then this decision is adopted mechanically; otherwise a local vote is taken on the present issue, neglecting other issues. And so on for other issues. When taking a local vote on an issue $\pm p$, the group uses a **local decision method for** p, i.e., an aggregation rule for the one-issue agenda $\{\pm p\}$ given by a function $D_p : \mathcal{J}'^n \to \mathcal{J}'$ where $\mathcal{J}' = \{\{p\}, \{\neg p\}\}$. D_p could for instance be majority voting. In sum, the group judgment set $J \subseteq X$ is constructed step-by-step by forming group judgment sets $J(p)$ for the various one-issue agendas $\{\pm p\} \subseteq X$, and then taking their union. To state the definition formally, recall that $X_0 \subseteq X$ contains one proposition from each issue.

Definition 5 *A **priority rule** (relative to a priority graph $\mathcal{R}$ on a finite agenda X) is an aggregation rule $F = F_{(D_p)_{p \in X_0}}$ on $\mathcal{J}^n$ which is given by some local decision methods $(D_p)_{p \in X_0}$ (one per proposition in X_0, i.e., per issue) as follows. Fix a profile $(J_1, ..., J_n) \in \mathcal{J}^n$. Form 'local judgment sets' $J(p) \subseteq \{\pm p\}$, $p \in X_0$, by a recursive procedure: for each $p \in X_0$, after having formed the local judgment sets $J(q)$ for the propositions $q \in X_0$ prior to p (i.e., in $\mathcal{R}(p) \backslash \{p\}$), take the set of all prior judgments*

[9] The reason is that these judgments are determined by those on $\mathcal{R}(\pm p)$ by non-underdetermination.

[10] See the paper's unpublished version Dietrich [6] for the infinite case.

$J_<(p) := \cup_{q \in X_0 : q \in \mathcal{R}(p) \backslash \{p\}} J(q)$ *and put*

$$J(p) := \left\{ \begin{array}{ll} \{\tilde{p} \in \{\pm p\} : J_<(p) \text{ entails } \tilde{p}\} & \text{if } J_<(p) \text{ entails a } \tilde{p} \in \{\pm p\} \\ D_p(J_1 \cap \{\pm p\}, ..., J_n \cap \{\pm p\}) & \text{if } J_<(p) \text{ entails no } \tilde{p} \in \{\pm p\}. \end{array} \right. \quad (6)$$

The full judgment set is the union $F(J_1, ..., J_n) := \cup_{p \in X_0} J(p)$.[11]

How are the sets $J(p)$ constructed in practice? The (initial) decision on each root proposition $p \in X_0$ is always made by voting on $\pm p$, as $J_<(p) = \varnothing$. Later decisions on non-root propositions $p \in X_0$ are made *either* by entailment from the past judgments $J_<(p)$ *or* by a vote on $\pm p$, depending on whether $J_<(p)$ settles p. The local rules D_p may all be the same rule, e.g., majority voting. Alternatively, D_p could depend on who has expertise on the present proposition p (physicists might have expertise on physical propositions), or on who is personally affected by the decision on p (citizens of Brighton are affected by decisions on urban planning for Brighton).[12]

When are priority rules well-behaved, i.e., satisfy IIP and generate logically consistent outputs? The following theorem gives sufficient conditions. Let me motivate them first.

- *Do priority rules satisfy IIP?* This question is related to whether $\mathcal{R}$ is transitive. To see why, note that if $p\mathcal{R}q$ and $q\mathcal{R}r$, then people's judgments on p could affect the group decision on r (via the group decisions on p and q), assuming these three propositions are logically interconnected. By Theorem 1, transitivity of $\mathcal{R}$ is sufficient for IIP. (This is true even in the absence of reflexivity, although one might at first think that violations of IIP can occur when deciding on a self-*irrelevant* proposition using a *local* vote.)
- *Do priority rules generate consistent decisions?* This depends on two factors. First, transitivity of $\mathcal{R}$ once again matters. To get an intuition, note that if $p\mathcal{R}q$ and $q\mathcal{R}r$ but not $p\mathcal{R}r$, then, after the group has decided on p and q, it decides on r by respecting the decision on q (as $q \in \mathcal{R}(r)$), but without giving any attention to the decision on p (as $p \notin \mathcal{R}(r)$) – which threatens collective consistency. Second, collective consistency is also threatened by logical connections between different (mutually irrelevant) branches of the priority graph, intuitively because the decisions in one branch ignore those in other branches (by mutual irrelevance), even if there are logical connections. Theorem 1 shows that collective consistency is guaranteed if $\mathcal{R}$ is transitive and certain kinds of logical connections are excluded.

Before stating the theorem, recall that negation-closed sets $S_1, ..., S_m \subseteq X$ are *logically independent* if any consistent subsets $J_1 \subseteq S_1, ..., J_m \subseteq S_m$ have a consistent union.

[11] The sets $J(p)$ are recursively well-defined, as the priority graph is an acyclic and finite, and thus well-founded relation on issues (see the well-founded recursion theorem, e.g., Fenstad [20]). The construction of $F(J_1, ..., J_n)$ in Definition 5 can be restated without introducing the sets $J(p)$: $F(J_1, ..., J_n)$ is the unique set $J \subseteq X$ such that for all $p \in X_0$

$$J \cap \{\pm p\} = \left\{ \begin{array}{ll} \{\tilde{p} \in \{\pm p\} : J \cap \mathcal{R}(p) \backslash \{\pm p\} \text{ entails } \tilde{p}\} & \text{if this set is non-empty} \\ D_p(J_1 \cap \{\pm p\}, ..., J_n \cap \{\pm p\}) & \text{otherwise.} \end{array} \right.$$

[12] D_p might be majority voting among a particular subgroup of experts on p, as in a *distributed premise-based procedure* (see List [29]).

This condition fails trivially if two of the sets S_i overlap, as one can then pick consistent subsets whose union contains a pair $p, \neg p$. To exclude such trivial cases, I call $S_1, ..., S_m$ *logically quasi-independent* if any consistent subsets $J_1 \subseteq S_1, ..., J_m \subseteq S_m$ have a consistent union as long as this union contains no pair $p, \neg p$ (equivalently, if any set $S \subseteq S_1 \cup \cdots \cup S_m$ is consistent whenever each restriction $S \cap S_i$ is consistent).

Theorem 1 *All priority rules (relative to a priority graph $\mathcal{R}$ on a finite agenda X)*
(a) *satisfy IIP (i.e., are premise-based rules) if $\mathcal{R}$ is transitive;*
(b) *generate rational outcomes if $\mathcal{R}$ is transitive and for all pairwise irrelevant propositions $p_1, ..., p_m \in X$ the sets $\mathcal{R}(p_1), ..., \mathcal{R}(p_m)$ are logically quasi-independent.*

It is worth considering Theorem 1 for two special priority graphs:

- If the priority graph defines a *linear* order over issues, Theorem 1's conditions (of transitivity and logical quasi-independence) hold trivially. So Theorem 1 implies that List's [28] sequential priority rules satisfy IIP and (as is known) generate rational outcomes. To see why the logical quasi-independence condition holds, note that linearity implies that for any $p_1, ..., p_m \in X$ we may assume without loss of generality that $p_1 \mathcal{R} p_2 \mathcal{R} \cdots \mathcal{R} p_m$, so that $\mathcal{R}(p_1) \subseteq \mathcal{R}(p_2) \subseteq \cdots \subseteq \mathcal{R}(p_m)$.
- Now assume the (degenerate) priority graph in which each issue is only relevant to itself ($\mathcal{R}(\pm p) = \{\pm p\}$). Then Theorem 1's transitivity assumption holds trivially, and the logical quasi-independence condition reduces to the condition that all issues in X are mutually independent, i.e., that there are no logical interconnections whatsoever between issues. For this priority graph, Theorem 1 is tight, i.e., minimal in its assumptions, as an anonymous referee kindly pointed out.[13] But tightness fails for some other priority graphs. For instance, in the absence of any logical interconnections between issues, transitivity is not needed in (b), since rationality of outcomes is guaranteed.

Finally, Theorem 1's logical quasi-independence condition reduces to a logical independence condition under a simple structural condition on the priority graph: no proposition is relevant to two mutually irrelevant propositions (so that the sets $\mathcal{R}(p_1), ..., \mathcal{R}(p_m)$ in Theorem 1 must be pairwise disjoint). This condition holds for the first three graphs of Figure 1.

5 A new unanimity axiom restricted to unambiguous agreements

We now turn to the axiomatic analysis of relevance-based aggregation. IIP cannot be our only axiom: it fails to exclude *constant* rules, which totally neglect people's judgments. The usual strategy is to impose a unanimity condition, typically by requiring preservation of all unanimous judgments:

Unanimity Principle: For every profile $(J_1, ..., J_n)$ in the domain and every proposition $p \in X$, if $p \in J_i$ for all individuals i then $p \in F(J_1, ..., J_n)$.

[13] To see why the absence of logical interconnections is *necessary* for part (b)'s consistency conclusion, note that only the second case in (6) ever applies as $J_<(p)$ is always empty.

This axiom is not very natural under the relevance-based approach: why should people's judgments on the propositions relevant to p suddenly not matter? Even if everyone agrees on p, there can be much disagreement on relevant propositions. Such 'spurious agreements' – agreements with disagreements on the 'reasons' – are often believed to lack normative force (e.g., Mongin [32], Nehring [35], Bradley [2]). Note however that spurious agreements are impossible on those propositions which can be justified (explained) in only one way. I call such propositions 'unambiguous':

Definition 6 *Given the relevance relation $\mathcal{R}$, a proposition in X is ($\mathcal{R}$-)**unambiguous** if it has only one explanation, and ($\mathcal{R}$-)**ambiguous** otherwise. The set of unambiguous propositions is denoted $U_{\mathcal{R}}$.*

In our example agenda $X = \{\pm a, \pm b, \pm(a \wedge b)\}$ with relevance given by (2), $U_{\mathcal{R}} = X \backslash \{\neg(a \wedge b)\}$. Proposition $\neg(a \wedge b)$ is ambiguous as it has three explanations: $\{\neg a, b\}$, $\{a, \neg b\}$ and $\{\neg a, \neg b\}$. So an agreement on $\neg(a \wedge b)$ can be spurious. The new unanimity axiom is restricted to unambiguous propositions, hence, to non-spurious agreements:

Unambiguous Agreement Preservation (UAP): For every profile $(J_1, ..., J_n)$ in the domain and every unambiguous proposition $p \in U_{\mathcal{R}}$, if $p \in J_i$ for all individuals i then $p \in F(J_1, ..., J_n)$.[14]

Remark 2: UAP is equivalent to the classical (global) unanimity principle under classical relevance, as $U_{\mathcal{R}} = X$. In sum, both of our axioms – IIP and UAP – reduce to their classical counterparts under classical relevance.

Example 1 continued. For the preference agenda in its strict or general version (with Arrovian relevance), UAP is equivalent to the weak Pareto principle, which requires preserving unanimous *strict* betterness judgments. This is because the unambiguous propositions are precisely the propositions expressing *strict* betterness comparisons:

- For the strict preference agenda, *all* propositions in X express strict betterness comparisons, and indeed $U_{\mathcal{R}} = X$ since each proposition $xPy \in X$ has a single explanation, $\{xPy\}$.
- For the general preference agenda, only the propositions in X of the form $yPx := \neg xRy$ express strict betterness comparisons, and indeed

$$U_{\mathcal{R}} = \{\neg xRy : x, y \in A, x \neq y\} = \{yPx : x, y \in A, x \neq y\} \qquad (7)$$

  since each $\neg xRy \in X$ has a single explanation, $\{\neg xRy, yRx\}$, while each $xRy \in X$ has two explanations, $\{xRy, yRx\}$ and $\{xRy, \neg yRx\}$.

Example 2 continued. Consider the evaluation agenda $X = \{\pm v_k : k \in K, v \in V\}$ with the 'matter-wise' relevance relation (5). Each $v_k \in X$ has only one explanation ($\{v_k\} \cup \{\neg v'_k : v' \in V \backslash \{v\}\}$) and each $\neg v_k \in X$ has $|V| - 1$ explanations (of the form

[14] In a generalization of UAP, p ranges not over $U_{\mathcal{R}}$ but over a given subset $\mathcal{P} \subseteq U_{\mathcal{R}}$ of 'privileged' propositions. All following theorems survive this generalization: see the paper's unpublished version Dietrich [6].

$\{w_k\} \cup \{\neg w'_k : w' \in V \backslash \{w\}\}$ with $w \in V \backslash \{v\}$). So, as long as $|V| > 2$, the set of unambiguous propositions is

$$U_{\mathcal{R}} = \{v_k : k \in K, v \in V\}. \tag{8}$$

Here UAP is far more plausible than the (global) unanimity principle: requiring to preserve a unanimously endorsed proposition $\neg v_k \in X \backslash U_{\mathcal{R}}$ strikes as implausible, because the position v could be a good compromise although *no-one* holds it.

6 Three impossibility theorems and Arrow's Theorem in both versions as special cases

Are there appealing aggregation rules satisfying our two axioms, IIP and UAP? General answers to this question are harder to give than for classical axioms, because we have to address not just logical links, but also relevance links. Indeed, the interplay between both kinds of links matters. Theorem 1 above is a *possibility* result: it gives sufficient conditions for the existence of well-behaved (priority) rules. I now turn to *impossibility* results, which give sufficient conditions for the inexistence of any non-degenerate rules $F : \mathcal{J}^n \to \mathcal{J}$ satisfying both axioms.

First, what is a 'degenerate' aggregation rule? I shall draw on various familiar versions of dictatorship. In preference aggregation theory, (i) a 'strong dictator' can impose his entire preference relation, (ii) a '(weak) dictator' can impose his *strict* preferences (not his weak preferences which can be indifferences), and (iii) a 'veto-dictator' can prevent ('veto') any *strict* preference. All of this can be rephrased in relevance-based terminology, drawing on the fact that *strict* preferences are expressed by *unambiguous* propositions in the preference agenda (see Example 1). Indeed, for the preference agenda: (i) a strong dictator can impose his entire judgment set, (ii) a (weak) dictator can impose any unambiguous proposition, and (iii) a vetodictator can prevent any unambiguous proposition. I now generalize these three classical notions (and two other ones, namely semi-dictatorship and semi-vetodictatorship) to arbitrary judgment aggregation problems (agendas):

Definition 7 *Under an aggregation rule $F : \mathcal{J}^n \to \mathcal{J}$, an individual i is*

- *a **strong dictator** if $F(J_1, ..., J_n) = J_i$ for all $(J_1, ..., J_n) \in \mathcal{J}^n$;*
- *a **dictator** (respectively, **semi-dictator**) if, for every unambiguous proposition $p \in U_{\mathcal{R}}$, we have $p \in F(J_1, ..., J_n)$ for all $(J_1, ..., J_n) \in \mathcal{J}^n$ such that $p \in J_i$ (respectively, such that $p \in J_i$ and $p \notin J_j$, $j \neq i$);*
- *a **vetodictator** (respectively, **semi-vetodictator**) if, for every unambiguous proposition $p \in U_{\mathcal{R}}$, i has a* veto *(respectively,* semi-veto*) on p, i.e., a judgment set $J_i \in \mathcal{J}$ not containing p such that $p \notin F(J_1, ..., J_n)$ for all $J_j \in \mathcal{J}$, $j \neq i$ (respectively, for all $J_j \in \mathcal{J}$, $j \neq i$, containing p).*

*F is called **strongly dictatorial** (respectively **(semi-)dictatorial**, **(semi-)vetodictatorial**) if some individual is a strong dictator (respectively (semi-)dictator, (semi-)vetodictator).*

Remark 3: Under classical relevance, dictatorship and strong dictatorship are equivalent (as $U_{\mathcal{R}} = X$).

Note that the difference between (veto)dictatorship and *semi*-(veto)dictatorship only arises if F is not proposition-wise monotonic, i.e., if additional support for a proposition can reverse a collective acceptance of that proposition.

Standard impossibility theorems on judgment aggregation are usually driven by *conditional entailments* between propositions (first used by Nehring and Puppe [37]). A conditional entailment is an entailment that is conditional on some other propositions (with a non-triviality condition on the choice of these other propositions):

Definition 8 *Proposition $p \in X$* ***conditionally entails*** *$q \in X$ if $\{p\} \cup Y \vdash q$ for some (possibly empty) set $Y \subseteq X$ that is consistent with p and with $\neg q$. The conditional entailment is* ***proper*** *if $p \nvdash q$, i.e., p is consistent with $\neg q$.*

We need a stronger variant of conditional entailment that is sensitive to relevance links. I call a set $Y \subseteq X$ *strongly consistent* with $p \in X$ if it is consistent with every ($\mathcal{R}$-)explanation of p (hence also with p itself). Loosely speaking, this means that Y is consistent with any reasons that could underlie p.

Definition 9 *Proposition $p \in X$* ***constrainedly entails*** *$q \in X$ (written '$p \vdash_{\mathcal{R}} q$') if $\{p\} \cup Y \vdash q$ for some (possibly empty) set $Y \subseteq U_{\mathcal{R}}$ that is strongly consistent with p and with $\neg q$ (i.e., consistent with all explanations of p and all ones of $\neg q$). In this case, p* ***constrainedly entails*** *q* ***in virtue of*** *Y (written '$p \vdash_{\mathcal{R},Y} q$').*

Remark 4: Constrained entailment implies conditional entailment, and is equivalent to it under classical relevance (as then each proposition p has the only explanation $\{p\}$).

Examples are due. First, every *unconditional* entailment is a constrained entailment: just take $Y = \varnothing$. Next, the general preference agenda X of Example 1 (with Arrovian relevance) contains many constrained entailments (and this is indeed a source of impossibility). For instance, for pairwise distinct options x, y, z, we have $xRy \vdash_{\mathcal{R},\{yPz\}} xPz$, because $\{xRy, yPz\} \vdash xPz$, where yPz belongs to $U_{\mathcal{R}}$ and is consistent with each explanation of xRy ($\{xRy, yRx\}$ and $\{xRy, \neg yRx\}$) and with the only explanation of $\neg xPz = zRx$ ($\{zRx\}$). By contrast, *no* constrained entailments (besides the trivial self-entailments) exist in our example agenda $X = \{\pm a, \pm b, \pm(a \wedge b)\}$ with relevance given by (2). For instance, it is neither the case that $a \vdash_{\mathcal{R},\{\neg(a\wedge b)\}} \neg b$ (as $\neg(a \wedge b) \notin U_{\mathcal{R}}$), nor the case that $a \vdash_{\mathcal{R},\{b\}} a \wedge b$ (as $\{b\}$ is inconsistent with the explanation $\{a, \neg b\}$ of $\neg(a \wedge b)$). As a result, our impossibility results will not apply to this agenda – and indeed this agenda allows for plenty of well-behaved (premise-based) aggregation rules. In general, the more relevance connections there are, the fewer constrained entailments there are.[15]

Verifying whether $p \vdash_{\mathcal{R}} q$ requires checking whether $p \vdash_{\mathcal{R},Y} q$ for any set $Y \subseteq U_{\mathcal{R}}$. Fortunately, one can restrict this test to sets $Y \subseteq U_{\mathcal{R}} \backslash (\mathcal{R}(p) \cup \mathcal{R}(\neg q))$, and as long as $\mathcal{R}$ is negation-invariant even to sets $Y \subseteq U_{\mathcal{R}} \backslash (\mathcal{R}(\pm p) \cup \mathcal{R}(\pm q))$. This is shown by Lemma 3 below. Loosely speaking, one can thus restrict attention to sets Y of *unambiguous and irrelevant* propositions.

[15] Formally, if $\mathcal{R}, \mathcal{R}'$ are relevance relations on X with corresponding constrained entailment relations $\vdash_{\mathcal{R}}, \vdash_{\mathcal{R}'}$ ($\subseteq 2^X \times X$), then $\mathcal{R} \subseteq \mathcal{R}' \Rightarrow \vdash_{\mathcal{R}'} \subseteq \vdash_{\mathcal{R}}$. Indeed, if $\mathcal{R}$ is refined, then $U_{\mathcal{R}}$ shrinks and explanations increase in size and number, so that the requirements on Y get stronger.

Recall that a conditional entailment from p to q is 'proper' if p is consistent with $\neg q$ ('no unconditional entailment'). For a *constrained* entailment to be 'proper', something subtly stronger than consistency of p with $\neg q$ is required:

Definition 10 *A proposition $p \in X$* ***properly constrainedly entails*** *another $q \in X$ if $p \vdash_{\mathcal{R}} q$ and every explanation of p is consistent with every explanation of $\neg q$.*

Remark 5: Under classical relevance, proper constrained entailment is equivalent to proper conditional entailment (as each $p \in X$ has the only explanation $\{p\}$).

For the strict or general preference agenda (Example 1), all constrained entailments without unconditional entailment are proper; for instance, the constrained entailment $xRy \vdash_{\mathcal{R}} xPz$ is proper because each explanation of xRy ($\{xRy, yRx\}$ and $\{xRy, \neg yRx\}$) is consistent with each explanation of $\neg xPz$ ($= zRx$). There are many other examples.[16]

Our impossibility results draw on paths of constrained entailments.

Definition 11 (a) *For propositions $p, q \in X$, if X contains propositions $p_1, ..., p_m$ ($m \geq 2$) with $p = p_1 \vdash_{\mathcal{R}} p_2 \vdash_{\mathcal{R}} ... \vdash_{\mathcal{R}} p_m = q$, I write $p \vdash\vdash_{\mathcal{R}} q$; if moreover one of these constrained entailments is proper, I write $p \vdash\vdash_{\mathcal{R}}^{\text{proper}} q$.*

(b) *A set $Z \subseteq X$ is* ***pathlinked*** *if $p \vdash\vdash_{\mathcal{R}} q$ for all $p, q \in Z$, and* ***properly pathlinked*** *if moreover $p \vdash\vdash_{\mathcal{R}}^{\text{proper}} q$ for some (hence all) $p, q \in Z$.*

Pathlinkedness of a set Z leads to a limited form of *neutral* aggregation within Z: the same coalitions are 'semi-decisive' (in a technical sense) for each proposition in Z. Such a neutrality argument is the first step to establish our impossibility theorems; the next step consists in proving that only singleton coalitions $\{i\}$ can be 'semi-decisive'.

Theorem 2 *If the set $U_{\mathcal{R}}$ of unambiguous propositions is properly pathlinked and inconsistent, every aggregation rule $F : \mathcal{J}^n \to \mathcal{J}$ satisfying IIP and UAP is semi-vetodictatorial.*

Theorem 3 *If the set $\{p, \neg p : p \in U_{\mathcal{R}}\}$ of unambiguous or negated unambiguous propositions is properly pathlinked, every aggregation rule $F : \mathcal{J}^n \to \mathcal{J}$ satisfying IIP and UAP is semi-dictatorial.*

In both theorems, there may exist several semi-(veto)dictators, and there need not exist any (veto)dictator. Like in all our theorems, the assumptions are not generally tight, but become tight for classical relevance. All this will become clear in Section 8, where I apply the theorems to classical relevance.

To be able to strengthen 'semi-dictatorial' to 'dictatorial' in Theorem 3, it suffices to add a small extra condition on the paths in Theorem 3. I call a constrained entailment $p \vdash_{\mathcal{R}} q$ 'irreversible' if it is not a 'constrained equivalence', i.e., if p constrainedly entails q in virtue of a set Y without it being the case that q entails p given Y:

[16] Every (non-unconditional) constrained entailment between root propositions is proper (see Example 4), again because a root proposition p has only explanation $\{p\}$. If relevance is an equivalence relation (as in Examples 1-3) which moreover partitions X into *pairwise logically independent* subagendas ('topics'), then all constrained entailments across equivalence classes are proper. (Two subagendas X_1, X_2 are logically independent if the union of consistent subsets $A \subseteq X_1, B \subseteq X_2$ is consistent.)

Definition 12 *For $p, q \in X$, p* ***irreversibly constrainedly entails*** *q if $p \vdash_{\mathcal{R},Y} q$ for a set Y for which $\{q\} \cup Y \nvdash p$.*

In the strict or general preference agenda (Example 1), *all* constrained entailments between distinct propositions are irreversible. For instance, xRy irreversibly entails xRz (where x, y, z are distinct options), as $xRy \vdash_{\mathcal{R},\{yPz\}} xRz$ where $\{xRz, yPz\} \nvdash xRy$. By the next result, the semi-dictatorship of Theorem 3 becomes a dictatorship if *at least one* constrained entailment is irreversible.

Definition 13 (a) *For propositions $p, q \in X$, I write $p \vdash\vdash_{\mathcal{R}}^{\text{irrev}} q$ if X contains propositions $p_1, ..., p_m$ $(m \geq 2)$ with $p = p_1 \vdash_{\mathcal{R}} p_2 \vdash_{\mathcal{R}} ... \vdash_{\mathcal{R}} p_m = q$, where at least one of these constrained entailments is irreversible.*

(b) *A pathlinked set $Z \subseteq X$ is* ***irreversibly pathlinked*** *if $p \vdash\vdash_{\mathcal{R}}^{\text{irrev}} q$ for some (hence all) $p, q \in Z$.*

Theorem 4 *If the set $\{p, \neg p : p \in U_{\mathcal{R}}\}$ of unambiguous or negated unambiguous propositions is properly and irreversibly pathlinked, every aggregation rule $F : \mathcal{J}^n \to \mathcal{J}$ satisfying IIP and UAP is dictatorial.*

This theorem generalizes Arrow's Theorem in its general and indifference-free versions. To see why, note the following fact (shown in the appendix):

Remark 6: The strict or general preference agenda for a set of at least three alternatives (with Arrovian relevance) satisfies the assumptions of Theorem 4, i.e., the set $\{p, \neg p : p \in U_{\mathcal{R}}\}$ (which equals X) is properly and irreversibly pathlinked.[17]

By this observation, Theorem 4 has Arrow's Theorem as a special case:

Corollary 1 *(Arrow's Theorem in both versions) Given the strict or general preference agenda for a set of at least three alternatives (with Arrovian relevance), every aggregation rule $F : \mathcal{J}^n \to \mathcal{J}$ satisfying IIP (equivalent to Arrow's independence of irrelevant alternatives) and UAP (equivalent to the weak Pareto principle) is dictatorial.*

I now apply our impossibility theorems to the classical relevance relation (Section 7) and then to a concrete example of evaluation aggregation (Section 8).[18]

7 The Arrow-like theorem in judgment aggregation as a special case

I now state the special cases of Theorems 2-4 for classical relevance.[19] Here these theorems become tight, i.e., minimal in their assumptions (as long as the agenda is

[17] In the case of the strict preference agenda, the pathlinkedness of X follows directly from the well-known pathconnectedness of X (Nehring [34], Dietrich and List [10], Dokow and Holzman [15]).

[18] For brevity, I do not also apply the results to the case where relevance represents priority/premisehood (Example 4). In this case the sets U_R and $\{\pm p : p \in U_R\}$ contain all root propositions of the priority graph (and perhaps other propositions), so that our theorems assume certain paths between root propositions (and perhaps other propositions). Whether these assumptions hold – i.e., whether such paths can be constructed – depends very much on the specific case, i.e., on the interplay between the priority graph and logical connections.

[19] I thank an anonymous referee for asking me to establish this important link.

finite).

Theorem 4 becomes the known Arrow-like impossibility theorem, i.e., the counterpart for judgment aggregation of Arrow's Theorem (Dietrich and List [10], Dokow and Holzman [15], both building on Nehring and Puppe [37] and strengthening Wilson [43]). Indeed, the assumptions and axioms of Theorem 4 reduce to those of the Arrow-like theorem. The Arrow-like theorem assumes, firstly, that the agenda is *pathconnected*. Pathconnectedness is defined like pathlinkedness, except that one uses conditional rather than constrained entailment (it is introduced by Nehring and Puppe [37] under the label 'total blockedness'). The Arrow-like theorem assumes, secondly, that the agenda is *pair-negatable*. Recall that a set $Y \subseteq X$ is *minimal inconsistent* if it is inconsistent and its proper subsets are consistent. Pair-negatability means that X has a minimal inconsistent subset Y which can be made consistent by negating some pair of propositions, i.e., $(Y \backslash \{p, q\}) \cup \{\neg p, \neg q\}$ is consistent for some pair of distinct propositions $p, q \in Y$. For instance, the strict and general preference agendas are pair-negatable once there are three distinct alternatives x, y, z, as the subset $Y = \{xPy, yPz, zPx\}$ is minimal inconsistent and becomes consistent if we (for instance) replace xPy and yPz by yPx and zPy.[20] To be precise, Theorem 4's assumption (of proper irreversible pathlinkedness) reduces to pathconnectedness and *a slightly generalized version of* pair-negatability. Pair-negatability in this generalized version means that X has an inconsistent subset Y such that the sets $(Y \backslash \{p, q\}) \cup \{\neg p, \neg q\}$, $(Y \backslash \{p\}) \cup \{\neg p\}$ and $(Y \backslash \{q\}) \cup \{\neg q\}$ are each consistent for some pair of distinct propositions $p, q \in Y$. This version implies the standard one since it does not require Y to be *minimal* inconsistent and since $(Y \backslash \{p\}) \cup \{\neg p\}$ and $(Y \backslash \{q\}) \cup \{\neg q\}\}$ are automatically consistent if Y is *minimal* inconsistent. For finite X, both versions are equivalent since a finite inconsistent set has a minimal inconsistent subset. I henceforth understand 'pair-negatability' in the generalized sense.

The following observation (proved in the appendix) shows that Theorem 4's assumptions indeed reduce to pathconnectedness and pair-negatability. The observation in fact boils down to well-known facts, given the equivalence (for classical relevance) of constrained and conditional entailment (see the citations in the proof).

Remark 7: For classical relevance, the agenda X $(= U_{\mathcal{R}} = \{p, \neg p : p \in U_{\mathcal{R}}\})$ is
- pathlinked if and only if it is pathconnected,
- irreversibly pathlinked if and only if it is pathconnected and pair-negatable;

moreover pathlinkedness of X is equivalent to proper pathlinkedness of X.

So, Theorem 4 reduces to the Arrow-like theorem in the case of classical relevance:

Corollary 2 *(the Arrow-like theorem in judgment aggregation) If the agenda is pathconnected and pair-negatable, every aggregation rule $F : \mathcal{J}^n \to \mathcal{J}$ satisfying propositionwise independence and the unanimity principle is dictatorial.*[21]

[20] Pair-negatability can be defined equivalently in terms of negating an *even number* (rather than a *pair*) of propositions. Another equivalent statement is Dokow and Holzman's [15] 'non-affineness' condition.

[21] Here and in Corollary 4, 'dictatorial' can be read in the weak or strong sense, as both are equivalent for classical relevance (see Remark 3).

If X is finite, this result is tight, as it has been proved with an 'if *and only if*' by Dokow and Holzman [15].

Now we turn to Theorems 2 and 3. Under classical relevance (for which $X = U_{\mathcal{R}} = \{p, \neg p : p \in U_{\mathcal{R}}\}$), these two results collapse into a single result, stated in the next corollary. The reason is that each theorem's assumption becomes equivalent to pathconnectedness of X (by Remark 7), and (semi-)dictatorship becomes equivalent to (semi-)vetodictatorship given proposition-wise independence.

Corollary 3 *If the agenda is pathconnected, every aggregation rule $F : \mathcal{J}^n \to \mathcal{J}$ satisfying proposition-wise independence and the unanimity principle is semi-dictatorial.*[22]

If X is finite, this result is also derivable by combining two known results, namely the Arrow-like theorem (Corollary 2) and a theorem by Dokow and Holzman [15].[23] Again, the result is tight, i.e., would hold with an 'if *and only if*', for finite X. In Corollary 3, 'semi-dictatorial' cannot be strengthened to 'dictatorial' and there can be multiple semi-dictators (as is clear from footnote 23).

Finally, we can also deduce a seminal theorem by Nehring and Puppe [39], as an anonymous referee kindly remarked. A rule $F : \mathcal{J}^n \to \mathcal{J}$ is *monotonic* if additional support for the collectively endorsed judgment set never reverses its collective endorsement (i.e., for all $J_1, ..., J_n, J \in \mathcal{J}$ and individuals i, $F(J_1, ..., J_n) = J \Rightarrow F(J_1, ..., J_{i-1}, J, J_{i+1}, ..., J_n) = J$). Since semi-dictatorship implies dictatorship once we assume monotonicity, Corollary 3 implies the following result:

Corollary 4 *If the agenda is pathconnected, every aggregation rule $F : \mathcal{J}^n \to \mathcal{J}$ satisfying proposition-wise independence, monotonicity and the unanimity principle is dictatorial.*

Here again, the assumptions are tight (Nehring and Puppe [39]).

While Corollary 1 (Arrow's Theorem), Corollary 2 (the Arrow-like theorem in judgment aggregation) and Corollary 3 are special cases of our results – for particular agendas and/or relevance relations – Corollary 4 is not a special case, but 'only' a consequence of our results. Thus Nehring and Puppe's theorem has not been generalized, but re-derived.

8 A concrete illustration

I now reconsider the evaluation agenda of Example 2 and work out a concrete case in which Theorem 2's semi-vetodictatorship or even Theorem 4's dictatorship applies.

[22] 'Semi-dictatorial' can again be read in the weak or strong sense, as both are equivalent for the classic relevance relation. The strong sense is defined like the weak sense, except that p ranges over the entire agenda X, not $U_{\mathcal{R}}$. Given proposition-wise independence, strong semi-dictatorship by individual i means that $p \in F(J_1, ..., J_n)$ whenever $p \in J_i$ but $p \notin J_j$ for $j \neq i$.

[23] To see why, let X be pathconnected. If X is also pair-negatable, it is dictatorial by Corollary 2, hence in particular semi-dictatorial. If X is *not* also pair-negatable (and X is finite), F must be a 'parity rule' (see Dokow and Holzman's [15] Proposition 4.3), hence in particular a semi-dictatorship. Under a parity rule, the collective endorses those propositions which are endorsed by an *odd* number of individuals from M, where $M \subseteq N$ is a fixed subgroup of odd size. If X is not pair-negatable, parity rules turn out to map into $\mathcal{J}$.

Suppose a country's inflation rate y (in $\mathbb{R}$) depends on two economic quantities a and b (in $\mathbb{R}$). I consider two cases.

Case 1: exogenously given theory of inflation. Assume that it is uncontroversially known that inflation obeys the equation '$y = f(a, b)$' for a given function $f : \mathbb{R}^2 \to \mathbb{R}$, where the equation is uniquely solvable in a (for any fixed b and y) and in b (for any fixed a and y). The group needs to estimate a, b and y (matters 1, 2 and 3, respectively), subject to the equation. Formally, the set of matters is $K = \{1, 2, 3\}$, and the set of possible positions on a matter ('estimates') is $V = \mathbb{R}$. An evaluation is a function $E : K \to \mathbb{R}$, or equivalently a triple of estimates $(a, b, y) \in \mathbb{R}^3$. It is coherent if and only if $y = f(a, b)$. So the agenda is $X = \{\pm v_k : k \in K, v \in \mathbb{R}\}$, where v_k is the proposition 'v is the value of variable k'. To an evaluation E corresponds the judgment set $J \subseteq X$ containing those v_k with $E(k) = v$ and those $\neg v_k$ with $E(k) \neq v$. Rational judgment sets are judgment sets corresponding to coherent evaluations. A proposition is relevant to another if and only if it pertains to the same variable (matter). Formally, $\mathcal{R}(\pm v_k) = \{\pm v'_k : v' \in \mathbb{R}\}$. The unambiguous propositions are

$$U_{\mathcal{R}} = \{v_k : k \in K, v \in \mathbb{R}\} = \{v_1, v_2, v_3 : v \in \mathbb{R}\}.$$

Theorem 2 applies, so that only vetodictatorships obey our conditions on aggregation. To show this, I prove that $U_{\mathcal{R}}$ is properly pathlinked and inconsistent. Inconsistency is obvious, since a variable cannot have many values. Proper pathlinkedness follows from two observations:

(a) Between any pair $v_k, v'_{k'} \in U_{\mathcal{R}}$ with $k \neq k'$ there is a proper constrained entailment $v_k \vdash_{\mathcal{R}} v'_{k'}$, i.e., a 'one-step path'. For instance, for any $a, y \in \mathbb{R}$, $a_1 \vdash_{\mathcal{R},Y} y_3$ with $Y = \{b_2\}$, where b is chosen such that $y = f(a, b)$. To see that this is a well-defined constrained entailment, note that Y is consistent with the only explanation of a_1, i.e., $\{a_1\} \cup \{\neg a_1 : a' \neq a\}$, and with each explanation of $\neg y_3$, i.e., each $\{\bar{y}_3\} \cup \{\neg y_3 : y' \neq \bar{y}\}$ for $\bar{y} \neq y$. The constrained entailment is proper because the explanation of a_1 is consistent with each explanation of $\neg y_3$.

(b) Between any pair $v_k, v'_k \in U_{\mathcal{R}}$ involving the same variable k there is a 'two-step path': choosing any $w_{k'} \in U_{\mathcal{R}}$ with $k' \neq k$, we have $v_k \vdash_{\mathcal{R}} w_{k'} \vdash_{\mathcal{R}} v'_k$ by (a).

Case 2: controversial theory of inflation. Now assume there are two rival theories: one claims the equation '$y = f(a, b)$', the other claims the equation '$y = g(a, b)$', where $f, g : \mathbb{R}^2 \to \mathbb{R}$ are distinct functions and each equation is again uniquely solvable in a (for any fixed b, y) and in b (for any fixed a, y). Consider a refined decision problem in which the group needs not only to estimate a, b and y (matters 1, 2 and 3), but also to choose one of the two theories (matter 4). This leads to another special case of Example 2, yet in the generalized version of footnote 6 in which the set of possible positions on a matter is matter-dependent ('V_k' instead of 'V'). Formally, the set of matters (issues) is now $K = \{1, 2, 3, 4\}$, and the set of possible positions on a matter k is $V_k = \mathbb{R}$ if $k \in \{1, 2.3\}$ and $V_k = \{f, g\}$ if $k = 4$. An evaluation E assigns a position in V_k to each matter k; equivalently, it is a tuple $(a, b, y, h) \in \mathbb{R}^3 \times \{f, g\}$ containing the three estimates and the chosen theory (function). it is coherent if and only if the estimates respect the chosen theory, i.e., $y = h(a, b)$. The agenda is thus $X = \{\pm v_k : k \in K, v \in V_k\}$, where proposition v_k means that v is the value on matter

k. As usual, to an evaluation E corresponds the judgment set $J \subseteq X$ containing those v_k with $E(k) = v$ and those $\neg v_k$ with $E(k) \neq v$; rational judgment sets are judgment sets corresponding to coherent evaluations. The relevance relation is given by $\mathcal{R}(\pm v_k) = \{\pm v'_k : v' \in V_k\}$. The unambiguous propositions are

$$U_{\mathcal{R}} = \{v_k : k \in K, v \in V_k\} = \{v_1, v_2, v_3 : v \in \mathbb{R}\} \cup \{\pm f_4, \pm g_4\}.$$

This time Theorem 4 applies, so that dictatorships are the only aggregation rules satisfying our conditions. I now show this. Note that the closure under negation of $U_{\mathcal{R}}$ is the entire agenda X. So I need to show that X is properly and irreversibly pathlinked. First, the set $U^* := \{v_k : k \in \{1,2,3\}, v \in \mathbb{R}\}$ is properly pathlinked; the argument resembles that in Case 1, as U^* corresponds to the set of unambiguous propositions in Case 1.[24] I now establish pathlinkedness of X by proving that, for all $p \in X$,

$$p \vdash\!\vdash_{\mathcal{R}} h_4 \text{ and } h_4 \vdash\!\vdash_{\mathcal{R}} p \text{ for each } h \in \{f, g\}. \tag{9}$$

I distinguish between different cases (in cases (a) and (b) the constructed paths have only one step, so that '$\vdash\!\vdash_{\mathcal{R}}$' can be replaced by '$\vdash_{\mathcal{R}}$'):

(a) *Case $p \in U^*$*: As U^* is pathlinked, it suffices to show (9) for *some* $p \in U^*$. Firstly, for any $a_1 \in U^*$ and $h \in \{f, g\}$ we have $h_4 \vdash_{\mathcal{R}} a_1$ in virtue of $Y = \{b_2, y_3\}$ with any $b \in \mathbb{R}$ and $y := h(a, b)$ (because a is the only solution of the equation $y = h(a, b)$). Secondly, as long as a and b have been chosen such that $f(a, b) \neq g(a, b)$, we also have $a_1 \vdash_{\mathcal{R},Y} h_4$.

(b) *Case $\neg p \in U^*$, i.e., $p = \neg v_k$ with $v_k \in U^*$*: Consider $h = f$ (the proof is analogous for $h = g$). By (a), $v_k \vdash_{\mathcal{R}} g_4$ and $g_4 \vdash_{\mathcal{R}} v_k$. So $\neg g_4 \vdash_{\mathcal{R}} \neg v_k$ and $\neg v_k \vdash_{\mathcal{R}} \neg g_4$ using contraposition (see Lemma 1). In other words, $f_4 \vdash_{\mathcal{R}} \neg v_k$ and $\neg v_k \vdash_{\mathcal{R}} f_4$ (as desired), because the constrained entailment relation does not distinguish between f_4 and $\neg g_4$ (they are logically equivalent).

(c) *Case $p, \neg p \notin U^*$, i.e., $p \in \{\pm f_4, \pm g_4\}$*: Consider any $h \in \{f, g\}$. I may assume without loss of generality that $p \in \{f_4, g_4\}$, as $\neg f_4$ is interchangeable with g_4, and $\neg g_4$ with f_4 (see (b)). Pick any $q \in U^*$. By (a) $p \vdash\!\vdash_{\mathcal{R}} q$ and $q \vdash\!\vdash_{\mathcal{R}} h_4$, so that $p \vdash\!\vdash_{\mathcal{R}} h_4$; similarly, $h_4 \vdash\!\vdash_{\mathcal{R}} q$ and $q \vdash\!\vdash_{\mathcal{R}} p$, so that $h_4 \vdash\!\vdash_{\mathcal{R}} p$.

Finally, the pathlinkedness of X is proper and irreversible. It is proper since constrained entailments $v_k \vdash_{\mathcal{R}} v'_{k'}$ with $k \neq k'$ are proper. It is irreversible because, for all $a, b \in \mathbb{R}$ and all $y \neq f(a, b)$, there is an irreversible constrained entailment $a_1 \vdash_{\mathcal{R}} \neg y_3$ (take $Y = \{b_2, f_4\}$).

9 An impossibility theorem with strong dictatorship

When do our conditions on aggregation even imply *strong* dictatorship? For the general preference agenda, this cannot be the case: it is indeed well-known that Arrow's axioms allow for non-strong dictatorships (in the form of 'lexicographic dictatorships', in which a 'second-order dictator' acts as tie-breaker wherever the 'first-order dictator' is indifferent).

[24] To adapt the argument, simply add proposition f_4 to each set Y used in a constrained entailment. This enforces the equation '$y = f(a, b)$' which used to be exogenous in Case 1.

A first observation is that if *all* propositions are unambiguous, i.e., if $U_{\mathcal{R}} = X$, as is for instance true for classical relevance, then a dictatorship is automatically strong, so that Theorem 4 becomes a strong dictatorship result:

Corollary 5 *If X is properly and irreversibly pathlinked and $U_{\mathcal{R}} = X$, every aggregation rule $F : \mathcal{J}^n \to \mathcal{J}$ satisfying IIP and UAP is strongly dictatorial.*

The condition $U_{\mathcal{R}} = X$ can in fact be weakened to the condition that all propositions in X are disjunctions of unambiguous propositions. We call $p \in X$ the *disjunction* of the set of propositions $S \subseteq X$ if accepting p is (rationally) equivalent to accepting at least one member of S (i.e., for any rational judgment set $J \in \mathcal{J}$, $p \in J \Leftrightarrow S \cap J \neq \varnothing$). For instance, the proposition 'it rains or snows' is presumably the disjunction of {'it rains', 'it snows'}.[25] Finally, by a disjunction *of unambiguous propositions* I of course mean a disjunction *of some set $S \subseteq U_{\mathcal{R}}$*.

Theorem 5 *If $\{p, \neg p : p \in U_{\mathcal{R}}\}$ is properly and irreversibly pathlinked and all ambiguous propositions are disjunctions of unambiguous propositions, then every aggregation rule $F : \mathcal{J}^n \to \mathcal{J}$ satisfying IIP and UAP is strongly dictatorial.*

I give two examples where the additional condition holds, and one where it fails:

- The condition holds trivially if $U_{\mathcal{R}} = X$ (no ambiguous propositions), hence in particular if relevance is classical. So, for classical relevance Theorem 5 reduces, just like Theorem 4, to the Arrow-like theorem in judgment aggregation stated in Corollary 2.
- The condition also holds for the evaluation agenda of Example 4 (and thus for the illustrations of Section 8). Here, an ambiguous propositions is of type $\neg v_k$, saying that v is *not* the value holding on matter k; this is the disjunction of the unambiguous propositions of type v'_k, $v' \neq v$, saying that some other value v' holds on matter k. In the illustration under Case 2 of Section 8 all premises of Theorem 5 hold, so that strong dictatorships are the only 'solutions'.
- The condition fails for the general preference agenda: no ambiguous proposition (of type xRy) is a disjunction of unambiguous propositions (of type $x'Py' = \neg y'Rx'$). This is why Arrow's theorem in its general version is not a strong dictatorship result.

10 Conclusion

The relevance-based approach to judgment aggregation hopefully opens up new perspectives, by overcoming proposition-wise independence without allowing for arbitrariness. On the constructive side, I have generalized sequential-priority and premise-based aggregation towards an arbitrary priority structure, captured by a 'priority graph' over the propositions. On the axiomatic side, I have introduced more general, relevance-based axioms of independence and unanimity-preservation, and shown various impossibility theorems based on these axioms. In the special case of the classical relevance notion, the two axioms reduce to their classical counterparts, and

[25] Given a set $S \subseteq X$, X need of course not contain a proposition that is its disjunction. The disjunction is unique as long as X contains no two equivalent propositions.

the theorems reduce to familiar results such as the Arrow-like theorem in judgment aggregation.

This paper is a first step. Future research could focus on other relevance-based aggregation rules, axioms and theorems. It could also address a more normative question: how should the relevance relation be designed in the first place? For instance, when should relevance be transitive? Reflexive? Acyclic? Should relevance connections and logical connections be related in any systematic way? Under the priority interpretation of relevance, which propositions should have priority? Such questions are obviously difficult. Yet the relevance-based approach needs systematic criteria for designing the relevance relation.

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A Proofs

I now prove all results. Throughout, N is the set of individuals $\{1, ..., n\}$.

A.1 Theorem 1 on priority rules

To prove this theorem, let $F \equiv F_{(D_p)_{p \in X_0}}$ be a priority rule relative to a priority graph $\mathcal{R}$ on a finite agenda X. The set of $\mathcal{R}$-maximal (resp. $\mathcal{R}$-minimal) elements of a set $S \subseteq X$ is denoted $\max_{\mathcal{R}}(S)$ (resp. $\min_{\mathcal{R}}(S)$) and contains those $s \in S$ for which there is no $r \in S \backslash \{s\}$ such that $s\mathcal{R}r$ (resp. $r\mathcal{R}s$). As $\mathcal{R}$ is acyclic on X_0 and as X_0 is finite,

$$\max_{\mathcal{R}} S \neq \varnothing \text{ and } \min_{\mathcal{R}} S \neq \varnothing \text{ for all } \varnothing \neq S \subseteq X_0. \tag{10}$$

(a) Let $\mathcal{R}$ be transitive. Suppose for a contradiction that IIP is violated. Then not all $p \in X_0$ have the property that, for all $(J_1, ..., J_n), (J'_1, ..., J'_n) \in \mathcal{J}^n$, if $J_i \cap \mathcal{R}(p) = J'_i \cap \mathcal{R}(p)$ for all i then

$$F(J_1, ..., J_n) \cap \{\pm p\} = F(J'_1, ..., J'_n) \cap \{\pm p\}. \tag{11}$$

Let $p \in X_0$ be $\mathcal{R}$-minimal such that this property fails. Pick $(J_1, ..., J_n), (J'_1, ..., J'_n) \in \mathcal{J}^n$ such that $J_i \cap \mathcal{R}(p) = J'_i \cap \mathcal{R}(p)$ for all i and (11) is violated. Choose any $q \in \mathcal{R}(p) \backslash \{\pm p\}$. By $\mathcal{R}$'s transitivity $\mathcal{R}(q) \subseteq \mathcal{R}(p)$, and so $J_i \cap \mathcal{R}(q) = J'_i \cap \mathcal{R}(q)$ for all i. By p's minimality property, (11) holds for q instead of p. As this is so for all $q \in \mathcal{R}(p) \backslash \{\pm p\}$,

$$F(J_1, ..., J_n) \cap \mathcal{R}(p) \backslash \{\pm p\} = F(J'_1, ..., J'_n) \cap \mathcal{R}(p) \backslash \{\pm p\}. \tag{12}$$

Now let $Y := \{\tilde{p} \in \{\pm p\}$: the set (12) entails $\tilde{p}\}$.

Case 1: $Y \neq \varnothing$. Then, by definition of the priority rule, we have $F(J_1, ..., J_n) \cap \{\pm p\} = Y$ and $F(J'_1, ..., J'_n) \cap \{\pm p\} = Y$. This implies (11), contradicting the choice of p.

Case 2: $Y = \varnothing$. Then, again by definition of the priority rule, $F(J_1, ..., J_n) \cap \{\pm p\} = D_p(J_1 \cap \{\pm p\}, ..., J_n \cap \{\pm p\})$ and $F(J'_1, ..., J'_n) \cap \{\pm p\} = D_p(J'_1 \cap \{\pm p\}, ..., J'_n \cap \{\pm p\})$. These two sets are distinct as (11) is violated. So there is an i such that $J_i \cap \{\pm p\} \neq J'_i \cap \{\pm p\}$. So, as $J_i \cap \mathcal{R}(p) = J'_i \cap \mathcal{R}(p)$, $\mathcal{R}(p)$ cannot contain both of $\pm p$, hence contains none of $\pm p$ by negation-invariance. In other words, $\mathcal{R}(p) = \mathcal{R}(p) \backslash \{\pm p\}$. So the set (12) equals $F(J_1, ..., J_n) \cap \mathcal{R}(p)$, which contains a member of each pair $\pm r \in \mathcal{R}(p)$ and thus entails p or $\neg p$ by non-underdetermination. This contradicts that $Y = \varnothing$.

(b) Assume the transitivity and quasi-independence conditions. For all $p \in X$, put $\mathcal{R}^p := \mathcal{R}(p) \cup \{\pm p\}$ and $\mathcal{R}_p := \mathcal{R}(p) \backslash \{\pm p\}$. Let $(J_1, ..., J_n) \in \mathcal{J}^n$. The consistency of $J := F(J_1, ..., J_n)$ follows from three claims:

Claim 1: $X = \cup_{p \in \max_{\mathcal{R}} X_0} \mathcal{R}^p$; hence, $J = \cup_{p \in \max_{\mathcal{R}} X_0} (J \cap \mathcal{R}^p)$.

Claim 2: for any pairwise irrelevant propositions $(p_i)_{i \in I}$, the sets $(\mathcal{R}^{p_i})_{i \in I}$ are logically quasi-independent; hence, the sets $(\mathcal{R}^p)_{p \in \max_{\mathcal{R}} X_0}$ are logically quasi-independent.

Claim 3: $J \cap \mathcal{R}^p$ is consistent for each $p \in X_0$, hence also for each $p \in \max_{\mathcal{R}} X_0$.

Proof of Claim 1. For a contradiction, suppose $X \backslash \cup_{p \in \max_{\mathcal{R}} X_0} \mathcal{R}^p \neq \varnothing$. Then, by negation-invariance, $X_0 \backslash \cup_{p \in \max_{\mathcal{R}} X_0} \mathcal{R}^p \neq \varnothing$. Hence by (10) there is a $q \in \max_{\mathcal{R}} (X_0 \backslash \cup_{p \in \max_{\mathcal{R}} X_0} \mathcal{R}^p)$. As $q \notin \cup_{p \in \max_{\mathcal{R}} X_0} \mathcal{R}^p$, we have $q \notin \max_{\mathcal{R}} X_0$. So q is relevant to some $q' \in X_0 \backslash \{q\}$. As q is maximal in $X_0 \backslash \cup_{p \in \max_{\mathcal{R}} X_0} \mathcal{R}^p$ and relevant to q', it does not belong to $X_0 \backslash \cup_{p \in \max_{\mathcal{R}} X_0} \mathcal{R}^p$. So $q' \in \cup_{p \in \max_{\mathcal{R}} X_0} \mathcal{R}^p$. Hence, as $\mathcal{R}$ is transitive, q is relevant to some $p \in \max_{\mathcal{R}} X_0$, a contradiction as $q \notin \cup_{p \in \max_{\mathcal{R}} X_0} \mathcal{R}^p$.

Proof of Claim 2. Consider pairwise irrelevant propositions $(p_i)_{i \in I}$ and consistent sets $B_i \subseteq \mathcal{R}^{p_i}$ $(i \in I)$ whose union contains no pair $\pm p$. I show that $\cup_{i \in I} B_i$ is consistent. Without loss of generality let each B_i contain a member of each pair $\pm p \in \mathcal{R}^{p_i}$ (otherwise extend the B_i's to consistent sets $\bar{B}_i \subseteq \mathcal{R}^{p_i}$ with this property; the present proof shows the consistency of $\cup_{i \in I} \bar{B}_i$, hence that of $\cup_{i \in I} B_i$). As the sets $\mathcal{R}(p_i)$ $(i \in I)$ are logically quasi-independent, (*) $\cup_{i \in I}(B_i \cap \mathcal{R}(p_i))$ is consistent. By non-underdetermination, (**) each $B_i \cap \mathcal{R}(p_i)$ entails a $\tilde{p}_i \in \{\pm p_i\}$. Since each B_i entails $\tilde{p}_i$ $(\in \{\pm p\})$ and by definition equals $(B_i \cap \mathcal{R}(p_i)) \cup \{p_i\}$ or $(B_i \cap \mathcal{R}(\tilde{p}_i)) \cup \{\neg p_i\}$, it must by consistency equal $(B_i \cap \mathcal{R}(p_i)) \cup \{\tilde{p}_i\}$. So, taking the union, $\cup_{i \in I} B_i = \cup_{i \in I}((B_i \cap \mathcal{R}(p_i)) \cup \{\tilde{p}_i\})$. This set is consistent by (*) and (**).

Proof of Claim 3. Suppose the contrary: there is a $p \in X_0$ for which $J \cap \mathcal{R}^p$ is inconsistent. By (10), we can pick a $p \in X_0$ that is $\mathcal{R}$-maximal subject to $J \cap \mathcal{R}^p$ being inconsistent. By an argument similar to that made for Claim 1,

$$\mathcal{R}_p = \cup_{q \in \max_{\mathcal{R}}(X_0 \cap \mathcal{R}_p)} \mathcal{R}^q; \text{ hence } J \cap \mathcal{R}_p = \cup_{q \in \max_{\mathcal{R}}(X_0 \cap \mathcal{R}_p)} (J \cap \mathcal{R}^q). \tag{13}$$

By Claim 2, the sets $\mathcal{R}^q, q \in \max_{\mathcal{R}}(X_0 \cap \mathcal{R}_p)$, are logically quasi-independent. Hence, as each $J \cap \mathcal{R}^q$ in (13) is consistent (by the maximality of p), the set $J \cap \mathcal{R}_p = J \cap \mathcal{R}(p) \backslash \{\pm p\}$ is consistent. By construction of priority rules, the consistency is inherited to the augmented set $(J \cap \mathcal{R}(p) \backslash \{\pm p\}) \cup (J \cap \{\pm p\})$ (see the restated definition in footnote 11). This set equals $J \cap \mathcal{R}^p$. The consistency of $J \cap \mathcal{R}^p$ contradicts the choice of p. ■

A.2 Constrained entailment and (semi-)decisive coalitions: preparatory lemmas

Before proving the impossibility theorems, I show some lemmas that help us understand constrained entailment and its effect on (semi-)winning coalitions.

First, as this definition of constrained entailment is symmetric in p and $\neg q$, constrained entailment satisfies contraposition, as the reader checks easily:

Lemma 1 *(contraposition) For all $p, q \in X$ and all $Y \subseteq U_{\mathcal{R}}$, $p \vdash_{\mathcal{R},Y} q \Leftrightarrow \neg q \vdash_{\mathcal{R},Y} \neg p$.*

I now give a sufficient condition for when a constrained entailment reduces to an unconditional entailment.

Lemma 2 *For all $p, q \in X$ with $\mathcal{R}(p) \subseteq \mathcal{R}(\neg q)$ or $\mathcal{R}(\neg q) \subseteq \mathcal{R}(p)$, $p \vdash_{\mathcal{R}} q$ if and only if $p \vdash q$.*

Proof. Let p, q be as specified. Obviously, $p \vdash q$ implies $p \vdash_{\mathcal{R},\varnothing} q$. Suppose for a contradiction that $p \vdash_{\mathcal{R}} q$, say $p \vdash_{\mathcal{R},Y} q$, but $p \nvdash q$. Then $\{p, \neg q\}$ is consistent. So there is a $B \in \mathcal{J}$ containing p and $\neg q$. Then

- the set $B \cap \{r, \neg r : r\mathcal{R}p\}$ is an explanation of p;
- the set $B \cap \{r, \neg r : r\mathcal{R}\neg q\}$ is an explanation of $\neg q$.

One of these two sets is a superset of the other one, as $\mathcal{R}(p) \subseteq \mathcal{R}(\neg q)$ or $\mathcal{R}(\neg q) \subseteq \mathcal{R}(p)$; call this superset J. As $p \vdash_{\mathcal{R},Y} q$, $Y \cup J$ is consistent. So, as J entails both p and $\neg q$, also $Y \cup J \cup \{p, \neg q\}$ is consistent. In particular, $Y \cup \{p, \neg q\}$ is consistent, in contradiction to the fact that $p \vdash_{\mathcal{R},Y} q$. ■

The following fact helps in choosing the set Y in a constrained entailment.

Lemma 3 *For all $p, q \in X$, if $p \vdash_{\mathcal{R}} q$, then $p \vdash_{\mathcal{R},Y} q$ for some set $Y \subseteq U_{\mathcal{R}} \backslash (\mathcal{R}(p) \cup \mathcal{R}(\neg q))$.*

Proof. Let $p, q \in X$, and assume $p \vdash_{\mathcal{R}} q$, say $p \vdash_{\mathcal{R},Y} q$. The proof is done by showing that $p \vdash_{\mathcal{R},Y \backslash (\mathcal{R}(p) \cup \mathcal{R}(\neg q))} q$. Suppose for a contradiction that not $p \vdash_{\mathcal{R},Y \backslash (\mathcal{R}(p) \cup \mathcal{R}(\neg q))} q$. Then

(*) $\{p, \neg q\} \cup Y \backslash (\mathcal{R}(p) \cup \mathcal{R}(\neg q))$ is consistent.

I show that

(**) $p \vdash p'$ for all $p' \in Y \cap \mathcal{R}(p)$ and $\neg q \vdash q'$ for all $q' \in Y \cap \mathcal{R}(\neg q)$,

which together with (*) implies that $\{p, \neg q\} \cup Y$ is consistent, a contradiction since $p \vdash_{\mathcal{R},Y} q$. Let $p' \in Y \cap \mathcal{R}(p)$. For a contradiction suppose $p \nvdash p'$. Then there is a $B \in \mathcal{J}$ containing p and $\neg p'$. The set $J := B \cap \{r, \neg r : r\mathcal{R}p\}$ does not entail $\neg p$, hence is an explanation of p (by definition of a relevance relation). So $J \cup Y$ is consistent (as $p \vdash_{\mathcal{R},Y} q$), a contradiction since $J \cup Y$ contains both p' and $\neg p'$. This shows that $p \vdash p'$. For analogous reasons, $\neg q \nvdash q'$ for all $q' \in Y \cap X^l$. ■

Now I introduce notions of decisive and semi-decisive coalitions, and I show that semi-decisiveness is preserved along paths of constrained entailments.

Definition 14 *A possibly empty coalition $C \subseteq N$ is* decisive *(respectively,* semi-decisive*) for $p \in X$ if its members have judgment sets $J_i \in \mathcal{J}$, $i \in C$, containing p, such that $p \in F(J_1, ..., J_n)$ for all $J_i \in \mathcal{J}$, $i \in N \backslash C$ (respectively, for all $J_i \in \mathcal{J}$, $i \in N \backslash C$, not containing p).*

While a decisive coalition for p can always enforce p (by using appropriate judgment sets), a semi-decisive coalition can enforce p provided all other individuals reject p. Let $\mathcal{W}(p)$ and $\mathcal{C}(p)$ be the sets of decisive and semi-decisive coalitions for $p \in X$, respectively. The next lemma shows that semi-decisiveness is 'contagious' along constrained entailments. The lemma parallels many other 'contagion lemmas' in social choice theory; indeed most standard proofs of Arrow's Theorem use a contagion lemma (see, e.g., Gaertner's [21] textbook).

Lemma 4 *(contagion lemma) For all $p, q \in X$, if $p \vdash_{\mathcal{R}} q$ then $\mathcal{C}(p) \subseteq \mathcal{C}(q)$. In particular, if $Z \subseteq X$ is pathlinked, all $p \in Z$ have the same semi-decisive coalitions.*[26]

Proof. Suppose $p, q \in X$, and $p \vdash_{\mathcal{R}} q$, say $p \vdash_{\mathcal{R},Y} q$, where by Lemma 3 without loss of generality $Y \cap \mathcal{R}(p) = Y \cap \mathcal{R}(\neg q) = \varnothing$. Let $C \in \mathcal{C}(p)$. So there are sets $J_i \in \mathcal{J}$, $i \in C$, containing p, such that $p \in F(J_1, ..., J_n)$ for all $J_i \in \mathcal{J}$, $i \in N \backslash C$, containing $\neg p$. By Y's consistency with every explanation of p, it is possible to change each J_i, $i \in C$, into a set (still in $\mathcal{J}$) that contains every $y \in Y$ and has the same intersection with $\mathcal{R}(p)$ as J_i; this change preserves the required properties, i.e., it preserves that $p \in J_i$ for all $i \in C$ (as $\mathcal{R}$ is a relevance relation), and preserves that $p \in F(J_1, ..., J_n)$ for all $J_i \in \mathcal{J}$, $i \in N \backslash C$, containing $\neg p$ (by $Y \cap \mathcal{R}(p) = \varnothing$ and IIP). So we may assume without loss of generality that $Y \subseteq J_i$ for all $i \in C$. Hence, by $\{p\} \cup Y \vdash q$, all J_i, $i \in C$, contain q.

To establish that $C \in \mathcal{C}(q)$, I consider any sets $J_i \in \mathcal{J}$, $i \in N \backslash C$, all containing $\neg q$, and I show that $q \in F(J_1, ..., J_n)$. We may assume without loss of generality that $Y \subseteq J_i$ for all $i \in N \backslash C$, by an argument like the one above (using that Y is consistent with any explanation of $\neg q$, $\mathcal{R}$ is a relevance relation, $Y \cap \mathcal{R}(\neg q) = \varnothing$, and IIP). As $\{\neg q\} \cup Y \vdash \neg p$, all J_i, $i \in N \backslash C$, contain $\neg p$. Hence $p \in F(J_1, ..., J_n)$. Moreover, $Y \subseteq F(J_1, ..., J_n)$ by $Y \subseteq U_{\mathcal{R}}$. So, as $\{p\} \cup Y \vdash q$, $q \in F(J_1, ..., J_n)$, as desired. ■

For any set $\mathcal{S}$ of coalitions $C \subseteq N$, let $\overline{\mathcal{S}} := \{C' \subseteq N : C \subseteq C' \text{ for some } C \in \mathcal{S}\}$.

[26] Constrained entailments preserve semi-decisiveness but usually not decisiveness.

Lemma 5 *For all $p, q \in X$,*
(a) *$p \vdash_{\mathcal{R}} q$ irreversibly if and only if $\neg q \vdash_{\mathcal{R}} \neg p$ irreversibly;*
(b) *if $p \vdash_{\mathcal{R}} q$ irreversibly then $\overline{\mathcal{C}(p)} \subseteq \mathcal{C}(q)$.*

Proof. Let $p, q \in X$. Part (a) follows from Lemma 1 and the fact that, for all $Y \subseteq U_{\mathcal{R}}$, $\{q\} \cup Y \nvdash p$ if and only if $\{\neg p\} \cup Y \nvdash \neg q$.

Regarding (b), suppose $p \vdash_{\mathcal{R}} q$ irreversibly, say $p \vdash_{\mathcal{R},Y} q$ with $\{q\} \cup Y \nvdash p$. We can assume without loss of generality that $Y \cap \mathcal{R}(p) = Y \cap \mathcal{R}(\neg q) = \varnothing$, since otherwise we could replace Y by $Y' := Y \backslash (\mathcal{R}(p) \cup \mathcal{R}(\neg q))$, for which still $p \vdash_{\mathcal{R},Y} q$ (by the proof of Lemma 3) and $\{q\} \cup Y' \nvdash p$. To show $\overline{\mathcal{C}(p)} \subseteq \mathcal{C}(q)$, consider any $C' \in \overline{\mathcal{C}(p)}$. So there is a $C \in \mathcal{C}(p)$ with $C \subseteq C'$. Hence there are $J_i \in \mathcal{J}$, $i \in C$, containing p, such that $p \in F(J_1, ..., J_n)$ for all $J_i \in \mathcal{J}$, $i \in N \backslash C$, containing $\neg p$. Like in earlier proofs, I may suppose without loss of generality that, for all $i \in C$, $Y \subseteq J_i$ (using that Y is consistent with all explanations of p, $\mathcal{R}$ is a relevance relation, IIP, and $Y \cap \mathcal{R}(p) = \varnothing$); hence, by $\{p\} \cup Y \vdash q$, $q \in J_i$ for all $i \in C$. Further, as $\{\neg p, q\} \cup Y$ is consistent (by $\{q\} \cup Y \nvdash p$), there are sets $J_i \in \mathcal{J}$, $i \in C' \backslash C$, such that $\{\neg p, q\} \cup Y \subseteq J_i$ for all $i \in C' \backslash C$.

I have to show that $q \in F(J_1, ..., J_n)$ for all $J_i \in \mathcal{J}$, $i \in N \backslash C'$, containing $\neg q$. Consider such sets J_i, $i \in N \backslash C'$. Again, we may assume without loss of generality that for all $i \in N \backslash C'$, $Y \subseteq J_i$ (as Y is consistent with all explanations of $\neg q$, $\mathcal{R}$ is a relevance relation, IIP, and $Y \cap \mathcal{R}(\neg q) = \varnothing$), which by $\{\neg q\} \cup Y \vdash \neg p$ implies that $\neg p \in J_i$ for all $i \in N \backslash C'$. In summary then,

$$J_i \supseteq \begin{cases} \{p, q\} \cup Y & \text{for all } i \in C \\ \{\neg p, q\} \cup Y & \text{for all } i \in C' \backslash C \\ \{\neg p, \neg q\} \cup Y & \text{for all } i \in N \backslash C'. \end{cases}$$

So $p \in F(J_1, ..., J_n)$ (by the choice of the sets J_i, $i \in C$) and $Y \subseteq F(J_1, ..., J_n)$ (by $Y \subseteq U_{\mathcal{R}}$). Hence, as $\{p\} \cup Y \vdash q$, $q \in F(J_1, ..., J_n)$, as desired. ■

In the following characterisation of decisive coalitions it is crucial that $p \in U_{\mathcal{R}}$.

Lemma 6 *If $p \in U_{\mathcal{R}}$, $\mathcal{W}(p) = \{C \subseteq N$: all coalitions $C' \supseteq C$ are in $\mathcal{C}(p)\}$.*

Proof. Let $p \in U_{\mathcal{R}}$ and $C \subseteq N$. If $C \in \mathcal{W}(p)$ then clearly all coalitions $C' \supseteq C$ are in $\mathcal{C}(p)$. Conversely, suppose all coalitions $C' \supseteq C$ are in $\mathcal{C}(p)$. As $C \in \mathcal{C}(p)$, there are sets J_i, $i \in C$, containing p, such that $p \in F(J_1, ..., J_n)$ for all sets J_i, $i \in N \backslash C$, not containing p. To show that $C \in \mathcal{W}(p)$, consider any sets J_i, $i \in N \backslash C$ (containing or not containing p); I show that $p \in F(J_1, ..., J_n)$. Let $C' := C \cup \{i \in N \backslash C : p \in J_i\}$. By $C \subseteq C'$, $C' \in \mathcal{C}(p)$. So there are sets B_i, $i \in C'$, containing p, such that $p \in F(B_1, ..., B_n)$ for all sets B_i, $i \in N \backslash C'$, not containing p. As p has a single explanation, we have for all $i \in C'$ $J_i \cap \{r, \neg r : r \in \mathcal{R}(p)\} = B_i \cap \{r, \neg r : r \in \mathcal{R}(p)\}$, hence $J_i \cap \mathcal{R}(p) = B_i \cap \mathcal{R}(p)$. So, by IIP and the definition of the sets B_i, $i \in C'$, and since $p \notin J_i$ for all $i \in N \backslash C'$, $p \in F(J_1, ..., J_n)$, as desired. ■

A.3 Theorems 2 and 3 on (semi-)vetodictatorship

Proof of Theorem 2. Let $U_{\mathcal{R}}$ be inconsistent and properly pathlinked. I first prepare the proof by establishing three simple claims.

Claim 1. (i) The set $\mathcal{C}(p)$ is the same for all $p \in U_{\mathcal{R}}$; call it $\mathcal{C}_0$. (ii) The set $\mathcal{C}(\neg p)$ is the same for all $p \in U_{\mathcal{R}}$.

Part (i) follows from Lemma 4. Part (ii) follows from it too because, by Lemma 1, $\{\neg p : p \in U_{\mathcal{R}}\}$ is like $U_{\mathcal{R}}$ pathlinked, q.e.d.

Claim 2. $\varnothing \notin \mathcal{C}_0$ and $N \in \mathcal{C}_0$.

By UAP, $N \in \mathcal{C}_0$. Suppose for a contradiction that $\varnothing \in \mathcal{C}_0$. Consider any judgment set $J \in \mathcal{J}$. Then $F(J, ..., J)$ contains all $p \in U_{\mathcal{R}}$, by $N \in \mathcal{C}_0$ if $p \in J$, and by $\varnothing \in \mathcal{C}_0$ if $p \notin J$. Hence $F(J, ..., J)$ is inconsistent, a contradiction, q.e.d.

By Claim 2, there is a minimal coalition C in $\mathcal{C}_0$ (with respect to inclusion), and $C \neq \varnothing$. By $C \neq \varnothing$, there is a $j \in C$. Write $C_{-j} := C \backslash \{j\}$. As $U_{\mathcal{R}}$ is properly pathlinked, there exist $p \in U_{\mathcal{R}}$ and $r, s \in X$ such that $p \vdash\vdash_{\mathcal{R}} r$, $r \vdash_{\mathcal{R}} s$ properly, and $s \vdash\vdash_{\mathcal{R}} p$.

Claim 3. $\mathcal{C}(r) = \mathcal{C}(s) = \mathcal{C}_0$; hence $C \in \mathcal{C}(r)$ and $C_{-j} \notin \mathcal{C}(s)$.

By Lemma 4, $\mathcal{C}(p) \subseteq \mathcal{C}(r) \subseteq \mathcal{C}(s) \subseteq \mathcal{C}(p)$. So $\mathcal{C}(r) = \mathcal{C}(s) = \mathcal{C}(p) = \mathcal{C}_0$, q.e.d.

Now let Y be such that $r \vdash_{\mathcal{R},Y} s$, where by Lemma 3 without loss of generality $Y \cap \mathcal{R}(r) = Y \cap \mathcal{R}(\neg s) = \varnothing$. By $C \in \mathcal{C}(r)$, there are judgment sets $J_i \in \mathcal{J}$, $i \in C$, containing r, such that $r \in F(J_1, ..., J_n)$ for all $J_i \in \mathcal{J}$, $i \in N \backslash C$, not containing r. I assume without loss of generality that

$$\text{for all } i \in C_{-j},\ Y \subseteq J_i, \text{ hence (by } \{r\} \cup Y \vdash s)\ s \in J_i, \tag{14}$$

which I may do by an argument like that in the proof of Lemma 4 (using that Y is consistent with any explanation of q, $\mathcal{R}$ is a relevance relation, $Y \cap \mathcal{R}(r) = \varnothing$, and IIP). By (14) and as $C_{-j} \notin \mathcal{C}(s)$ (see Claim 3), there are sets $B_i \in \mathcal{J}$, $i \in N \backslash C_{-j}$, containing $\neg s$, such that, writing $B_i := J_i$ for all $i \in C_{-j}$,

$$\neg s \in F(B_1, ..., B_n). \tag{15}$$

I may without loss of generality modify the sets B_i, $i \in N \backslash C_{-j}$, into new sets in $\mathcal{J}$ as long as their intersections with $\mathcal{R}(\neg s)$ stays the same (because the new sets then still contain $\neg s$ as $\mathcal{R}$ is a relevance relation, and still satisfy (15) by IIP). First, I modify the set B_i for $i = j$: as $r \vdash_{\mathcal{R}} s$ properly, $B_j \cap \{t, \neg t : t \in \mathcal{R}(\neg s)\}$ (an explanation of $\neg s$) is consistent with any explanation of r, hence with $J_j \cap \{t, \neg t : t \in \mathcal{R}(r)\}$, so that I may assume that $J_j \cap \{t, \neg t : t \in \mathcal{R}(r)\} \subseteq B_j$; which implies that

$$B_i \cap \mathcal{R}(r) = J_i \cap \mathcal{R}(r) \text{ for all } i \in C. \tag{16}$$

Second, I modify the sets B_i, $i \in N \backslash C$: I assume (using that $Y \cap \mathcal{R}(\neg s) = \varnothing$ and Y's consistency with any explanation of $\neg s$) that

$$\text{for all } i \in N \backslash C,\ Y \subseteq B_i, \text{ hence (as } \{\neg s\} \cup Y \vdash \neg r)\ \neg r \in B_i. \tag{17}$$

The definition of the sets J_i, $i \in C$, and (17) imply, via (16) and IIP, that

$$r \in F(B_1, ..., B_n). \tag{18}$$

By (15), (18), and the inconsistency of $\{r, \neg s\} \cup Y$, the set Y is not a subset of $F(B_1, ..., B_n)$. So there is a $y \in Y$ with $y \notin F(B_1, ..., B_n)$. We have $\{j\} \in \mathcal{C}(\neg y\}$ for the following two reasons.

- B_j contains $\neg y$; otherwise $y \in B_i$ for all $i \in N$, so that $y \in F(B_1, ..., B_n)$ by $y \in U_{\mathcal{R}}$.
- Consider any sets $C_i \in \mathcal{J}$, $i \neq j$, not containing $\neg y$, i.e., containing y. I show that $\neg y \in J := F(C_1, ..., C_{j-1}, B_j, C_{j+1}, ..., C_n)$. For all $i \neq j$, $C_i \cap \{t, \neg t : t \in \mathcal{R}(y)\}$ is consistent with y, hence is an explanation of y (as $\mathcal{R}$ satisfies non-underdetermination); for analogous reasons, $B_i \cap \{t, \neg t : t \in \mathcal{R}(y)\}$ is an explanation of y. These two explanations must be identical by $y \in U_{\mathcal{R}}$. So $C_i \cap \mathcal{R}(y) = B_i \cap \mathcal{R}(y)$. Hence, by $y \notin F(B_1, ..., B_n)$ and IIP, $y \notin J$. So $\neg y \in J$, as desired.

By $\{j\} \in \mathcal{C}(\neg y)$ and Claim 1, $\{j\} \in \mathcal{C}(\neg q)$ for all $q \in U_{\mathcal{R}}$. So j is a semi-vetodictator. ■

Proof of Theorem 3. Let $\{\pm p : p \in U_{\mathcal{R}}\}$ be properly pathlinked. I will reduce the proof to that of Theorem 2. I start again with two simple claims.

Claim 1. The set $\mathcal{C}(q)$ is the same for all $q \in \{\pm p : p \in U_{\mathcal{R}}\}$; call it $\mathcal{C}_0$.

This follows immediately from Lemma 4, q.e.d.

Claim 2. $\varnothing \notin \mathcal{C}_0$ and $N \in \mathcal{C}_0$.

By UAP, $N \in \mathcal{C}(p)$ for all $p \in U_{\mathcal{R}}$; hence $N \in \mathcal{C}_0$. This implies, for all $p \in U_{\mathcal{R}}$, that $\varnothing \notin \mathcal{C}(\neg p)$; hence $\varnothing \notin \mathcal{C}_0$, q.e.d.

Now by an analogous argument to that in the proof of Theorem 2, but based this time on the present Claims 1 and 2 rather than on the two first claims in Theorem 2's proof, one can show that there exists an individual j such that $\{j\} \in \mathcal{C}(\neg q)$ for all $q \in U_{\mathcal{R}}$. So, by the present Claim 1 (which is stronger than the first claim in Theorem 2's proof),

$$\{j\} \in \mathcal{C}(q) \text{ for all } q \in U_{\mathcal{R}}. \tag{19}$$

So j is a semi-dictator, for the following reason. Let $q \in U_{\mathcal{R}}$ and let $(J_1, ..., J_n) \in \mathcal{J}^n$ be such that $q \in J_j$ and $q \notin J_i$, $i \neq j$. By (19) there is a set $B_j \in \mathcal{J}$ containing q such that $q \in F(B_1, ..., B_n)$ for all $B_i \in \mathcal{J}$, $i \neq j$, not containing q. Since q has only one explanation (by $q \in U_{\mathcal{R}}$), the two explanations $J_j \cap \{t, \neg t : t \in \mathcal{R}(q)\}$ and $B_j \cap \{t, \neg t : t \in \mathcal{R}(q)\}$ are identical. So $J_j \cap \mathcal{R}(q) = B_j \cap \mathcal{R}(q)$. Hence, using IIP and the definition of B_j, $q \in F(J_1, ..., J_n)$, as desired. ■

A.4 Theorems 4 and 5 on weak or strong dictatorship and related results

Proof of Theorem 4. Let $\{\pm p : p \in U_{\mathcal{R}}\}$ be properly and irreversibly pathlinked. By Theorem 3, there is a semi-dictator i. I show that i is a dictator.

Claim. For all $q \in \{\pm p : p \in U_{\mathcal{R}}\}$, $\mathcal{C}(q)$ contains all coalitions containing i.

Consider any $q \in \{\pm q : q \in U_{\mathcal{R}}\}$ and any coalition $C \subseteq N$ containing i. By proper pathlinkedness there exist $p \in U_{\mathcal{R}}$ and $r, s \in X$ such that $p \vdash\vdash_{\mathcal{R}} r \vdash_{\mathcal{R}} s \vdash\vdash_{\mathcal{R}} q$, where $r \vdash_{\mathcal{R}} s$ is a properly constrained entailment. By $\{i\} \in \mathcal{C}(p)$ and Lemma 4, $\{i\} \in \mathcal{C}(r)$. So, by Lemma 5(b), $C \in \mathcal{C}(s)$. Hence, by Lemma 4, $C \in \mathcal{C}(q)$, q.e.d.

By this claim and Lemma 6, $\{i\} \in \mathcal{W}(p)$ for all $p \in U_{\mathcal{R}}$. This implies that i is a dictator, by an argument similar to the one that completed the proof of Theorem 3. ■

Proof of Remark 6. Let X be the general preference agenda with Arrovian $\mathcal{R}$ (the proof for the strict preference agenda is left to the reader). Recall that $U_{\mathcal{R}} = \{xPy : x, y \in A, x \neq y\}$ where $xPy := \neg yRx$. I show that (i) $U_{\mathcal{R}}$ is pathlinked, and (ii) there are $r, s \in U_{\mathcal{R}}$ with proper and irreversible constrained entailments $r \vdash_{\mathcal{R}} \neg s \vdash_{\mathcal{R}} r$. Then, by (i) and Lemma 1, $\{\neg p : p \in U_{\mathcal{R}}\}$ is (like $U_{\mathcal{R}}$) pathlinked, which together with (ii) implies that $\{\pm p : p \in U_{\mathcal{R}}\}$ $(= X)$ is properly and irreversibly pathlinked, completing the proof.

Proof of (i): Consider any $xPy, x'Py' \in U_{\mathcal{R}}$. I show that $xPy \vdash\vdash_{\mathcal{R}} x'Py'$. The paths to be constructed depend on whether or not $x \in \{x', y'\}$ and whether or not $y \in \{x', y'\}$. I consider the following list of cases (which is exhaustive since $x \neq y$ and $x' \neq y'$):

- Case $x \neq x', y' \& y \neq x', y'$: Here $xPy \vdash_{\mathcal{R},\{x'Px, yPy'\}} x'Py'$.
- Case $y = y' \& x \neq x', y'$: Here $xPy \vdash_{\mathcal{R},\{x'Px\}} x'Py = x'Py'$.
- Case $y = x' \& x \neq x', y'$: Here $xPy \vdash_{\mathcal{R},\{yPy'\}} xPy' \vdash_{\mathcal{R},\{x'Px\}} x'Py'$.
- Case $x = x' \& y \neq y', x'$: Here $xPy \vdash_{\mathcal{R},\{yPy'\}} xPy'$.
- Case $x = y' \& y \neq x', y'$: Here $xPy \vdash_{\mathcal{R},\{x'Px\}} x'Py \vdash_{\mathcal{R},\{yPx\}} x'Px$.
- Case $x = x' \& y = y'$: Here $xPy \vdash_{\mathcal{R},\varnothing} xPy$.
- Case $x = y' \& y = x'$: Here, for any $z \in A \backslash \{x, y\}$, $xPy \vdash_{\mathcal{R},\{yPz\}} xPz \vdash_{\mathcal{R},\{yPx\}} yPz \vdash_{\mathcal{R},\{zPx\}} yPx$.

Proof of (ii): For any pairwise distinct options $x, y, z \in A$, we have $xPy \vdash_{\mathcal{R},\{yPz\}} xRz$ $(= \neg zRx)$, and $xRz \vdash_{\mathcal{R},\{zPy\}} xPy$, in each case properly and irreversibly. ■

Proof of Remark 7. Assume classical relevance. Constrained and conditional entailment coincide by Remark 4. This implies the first bullet point. The second bullet point follows from the additional fact that, for any $p, q \in X$, $Z \subseteq X$, the following are equivalent (see Dokow and Holzman [15] for a parallel argument): (i) p irreversibly constrainedly (= conditionally) entails q in virtue of Z, i.e., $p \vdash_{\mathcal{R},Z} q$ while $\{q\} \cup Z \nvdash p$; (ii) there is an instance of pair-negatability, i.e., the set $Y := \{p, \neg q\} \cup Z$ is inconsistent and becomes consistent if one negates p and/or $\neg q$. Finally, pathlinkedness implies proper pathlinkedness, because any path of conditional entailments from a proposition to its negation must contain at least one *properly conditional* entailment (as is well-known since Nehring and Puppe [37]), and because 'conditional' is equivalent to 'constrained'. ■

Proof of Theorem 5. Let the assumptions hold. By Theorem 4, there is a dictator i. To show that i is a strong dictator, I consider any $(J_1, ..., J_n) \in \mathcal{J}^n$, and show that $J_i = F(J_1, ..., J_n)$. It suffices to show that $J_i \subseteq F(J_1, ..., J_n)$. Suppose $q \in J_i$. By assumption, q is the disjunction of some set $S \subseteq U_{\mathcal{R}}$. So, as $q \in J_i$, we can pick a $p \in J_i \cap S$. As $p \in J_i \cap U_{\mathcal{R}}$ and as i is a (weak) dictator, $p \in F(J_1, ..., J_n)$. So, as q is the disjunction of a set containing p, we have $q \in F(J_1, ..., J_n)$, as desired. ■