On the Logics with Propositional Quantifiers Extending $\mathsf{S5\Pi}$

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Abstract

Scroggs's theorem on the extensions of $\mathsf{S5}$ is an early landmark in the modern mathematical studies of modal logics. From it, we know that the lattice of normal extensions of $\mathsf{S5}$ is isomorphic to the inverse order of the natural numbers with infinity and that all extensions of $\mathsf{S5}$ are in fact normal. In this paper, we consider extending Scroggs's theorem to modal logics with propositional quantifiers governed by the axioms and rules analogous to the usual ones for ordinary quantifiers. We call them Π-logics. Taking $\mathsf{S5\Pi}$, the smallest normal Π-logic extending $\mathsf{S5}$, as the natural counterpart to $\mathsf{S5}$ in Scroggs's theorem, we show that all normal Π-logics extending $\mathsf{S5\Pi}$ are complete with respect to their complete simple S5 algebras, that they form a lattice that is isomorphic to the lattice of the open sets of the disjoint union of two copies of the one-point compactification of $\mathbb{N}$, that they have arbitrarily high Turing-degrees, and that there are non-normal Π-logics extending $\mathsf{S5\Pi}$.

Keywords: Propositional quantifiers, Scroggs's theorem, lattice of modal logics, algebraic semantics.

1 Introduction

In this paper, we study the modal logics with propositional quantifiers extending the well-studied modal logic $\mathsf{S5}$. Modal logics with propositional quantifiers have been of considerable interest to many modal logicians since their appearances in Fine's dissertation [9] and an early paper by Bull [6]. However, much of the interest is devoted to a few particular systems (e.g., [19,18,4,2,5]) and the expressive power under Kripke semantics (e.g., [7,21,20,16,11,1,3]), and there is an obvious lack of general study of classes of such logics. An exemplary early general study of propositional modal logics is found in Scroggs's famous 1959 paper [22], and it is our intention here to extend it to modal logics with propositional quantifiers.

To this end, we must first define, in general, what is a modal logic with propositional quantifiers. Since we consider here only logics with one modal operator, the language $\mathcal{L}\Pi$ defined below suffices.

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Definition 1.1 Let $\mathcal{L}\Pi$ be the language with the following grammar

$$\varphi ::= p \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \forall p\varphi$$

where $p \in \mathsf{Prop}$, a countably infinite set of propositional *variables.*[2] Other Boolean connectives, $\bot$, and $\Diamond$ are defined as usual.

As is common in the general study of modal logics, we take a modal logic with propositional quantifiers to be a set of formulas satisfying certain closure conditions, which represent the necessary axioms and rules for connectives with fixed meaning. There are many readings of the propositionally quantified sentence $\forall p\varphi$, which result in different axioms and semantics (see [10] for example), but here we take the most straightforward reading: "no matter what proposition p expresses, φ." From a purely logical point of view, this reading should warrant the following widely accepted principles, which we call the Π-*principles*:

- All instances of the universal distribution axiom schema: $\forall p(\varphi \to \psi) \to (\forall p\varphi \to \forall p\psi)$.
- All instances of the universal instantiation axiom schema: $\forall p\varphi \to \varphi^p_\psi$ where ψ is substitutable for p in φ, and φ^p_ψ is the result of this substitution.
- All instances of the vacuous quantification axiom schema: $\varphi \to \forall p\varphi$ where p is not free in φ.
- Universalization rule: if φ is derivable, then $\forall p\varphi$ is derivable.

Then the modal logics with propositional quantifiers, which we call Π-logics in accordance with [6] and most recently [15], can now be defined.

Definition 1.2 A Π-*logic* is a set Λ of formulas in $\mathcal{L}\Pi$ such that Λ contains all instances of propositional tautologies and axioms in the Π-principles, and is closed under *modus ponens* and the only rule, universalization, in the Π-principles.

A *normal* Π-logic Λ is a Π-logic that contains the K axiom and is further closed under necessitation: if $\varphi \in \Lambda$, then $\Box\varphi \in \Lambda$.

For any normal modal logic L in the usual basic modal language, let $\mathsf{L}\Pi$ be the smallest (in terms of inclusion) normal Π-logic containing L.

Then, for example, $\mathsf{S5}\Pi$ is the smallest normal Π-logic extending $\mathsf{S5}$, and $\mathsf{K}\Pi$ is the smallest normal Π-logic extending K, which is just the smallest normal Π-logic.

Following Scroggs, we address the following questions in this paper regarding the Π-logics extending $\mathsf{S5}\Pi$, the set of which we call $\mathsf{Next}\Pi(\mathsf{S5}\Pi)$.

General completeness of logics in $\mathsf{Next}\Pi(\mathsf{S5}\Pi)$ It is well known that $\mathsf{S5}\Pi$ is incomplete with respect to its Kripke frames if propositional quantifiers can vary the valuation of propositional variables to any set of worlds. This was

[2] In contrast, $\top$ is a propositional *constant*. Later we will have another propositional constant.

observed by Fine already in [9] and is in stark contrast to the situation without propositional quantifiers: as is shown by Scroggs, all modal logics in the basic modal language extending $\mathsf{S5}$ are complete with their finite Kripke frames with a totally connected relation. However, Scroggs's proof is algebraic in spirit, and indeed, an algebraic semantics for $\mathcal{L}\Pi$ based on modal algebras is more natural for the normal Π-logics, given our straightforward reading of $\forall p\varphi$. Algebraically, $\forall p\varphi$ is interpreted as the meet (greatest lower bound) of all possible semantic values of φ when we only vary the valuation of p. In short, $\forall p\varphi$ expresses an arbitrary meet. Dually, $\exists p\varphi$ expresses an arbitrary join. For this to work, however, we need the modal algebras to be complete in the sense that for any set of elements in the algebra, the meet and join of this set exist. We will show that all logics in $\mathsf{Next}\Pi(\mathsf{S5}\Pi)$ are complete with respect to their complete simple S5 algebras, to be defined later.

The lattice structure of $\mathsf{Next}\Pi(\mathsf{S5}\Pi)$ From the general completeness for logics in $\mathsf{Next}\Pi(\mathsf{S5}\Pi)$, the lattice structure of $\mathsf{Next}\Pi(\mathsf{S5}\Pi)$ can be reduced to the lattice structure of classes of algebras defined by logics in $\mathsf{Next}\Pi(\mathsf{S5}\Pi)$. We will show from this that the logics in $\mathsf{Next}\Pi(\mathsf{S5}\Pi)$ correspond to the closed sets of a Stone space $\mathcal{S}$, which is homeomorphic to the disjoint union of two copies of the one-point compactification of $\mathbb{N}$ with the natural order topology. Then the lattice $\langle \mathsf{Next}\Pi(\mathsf{S5}\Pi), \subseteq\rangle$ is isomorphic to the lattice of open sets of $\mathcal{S}$ ordered by inclusion.

The computability of logics in $\mathsf{Next}\Pi(\mathsf{S5}\Pi)$ From the correspondence between the logics and the closed sets, we also obtain that there are logics in $\mathsf{Next}\Pi(\mathsf{S5}\Pi)$ of arbitrarily high Turing-degree. While it is known that many natural modal logics with propositional quantifiers are of very high complexity [10,16], this shows that we may still need to face the problem even above $\mathsf{S5}$.

The non-normal Π-logics extending $\mathsf{S5}\Pi$ We will also show that there are many non-normal Π-logics extending $\mathsf{S5}\Pi$, contrary to the situation in the basic modal language, where all modal logics extending $\mathsf{S5}$ are normal. However, we leave a complete study of the non-normal Π-logics extending $\mathsf{S5}\Pi$ to future work.

The plan to address these questions is as follows. In § 2, we present the semantics for $\mathcal{L}\Pi$ and collect the necessary results already appearing in [9] and more recently in [15]. In § 3, we show that, in terms of validity or theoremhood, every formula in $\mathcal{L}\Pi$ is equivalent to a Boolean combination of a few simple formulas. This serves as a good preparation for § 4, where we construct a topological space $\mathcal{S}$ based on all complete simple S5 algebras, which encodes what classes of algebras are definable in terms of validity by $\mathcal{L}\Pi$. Crucially, $\mathcal{S}$ is a Stone space. In § 5, we prove all the main results, which make essential use of the fact that $\mathcal{S}$ is a Stone space and, in particular, that $\mathcal{S}$ is compact. This allows us to prove completeness without using the usual Lindenbaum algebra and quotient construction, though we need to rely on the already proven completeness of $\mathsf{S5}\Pi$. Finally, we conclude with related open problems in § 6.

2 Preliminaries

Recall that a modal algebra is a pair $\langle B, \Box \rangle$ where B is a Boolean algebra and $\Box$ is a unary operator on B satisfying $\Box 1 = 1$ and $\Box(a \wedge b) = \Box a \wedge \Box b$ for any $a, b \in B$. In most cases, we will conflate the notation of an algebra and its carrier set, and we will take $\neg, \wedge, \vee, \Box$ to be the complement, meet, join, and modal operators in modal algebra, despite that they are also in our formal language $\mathcal{L}\Pi$. The usual abbreviations also apply to operations on modal algebras, including $\Diamond a := \neg\Box\neg a$ for all $a \in B$. When confusion may arise, we will use $\neg_B, \wedge_B, \vee_B, \Box_B$ for the operators in a modal algebra B. A modal algebra B is *complete* when its Boolean part is a complete Boolean algebra. Then the semantics for $\mathcal{L}\Pi$ can be defined as follows.

Definition 2.1 For any modal algebra B, a *valuation* V on B is a function from Prop to B. When B is complete, any such valuation can then be extended to a $\mathcal{L}\Pi$-*valuation* $\widehat{V}$ from $\mathcal{L}\Pi$ to B defined recursively by:

(i) $\widehat{V}(p) = V(p)$ for all $p \in \mathsf{Prop}$;

(ii) $\widehat{V}(\top) = 1$; $\widehat{V}(\neg\varphi) = \neg\widehat{V}(\varphi)$; $\widehat{V}(\varphi \wedge \psi) = \widehat{V}(\varphi) \wedge \widehat{V}(\psi)$; $\widehat{V}(\Box\varphi) = \Box\widehat{V}(\varphi)$;

(iii) $\widehat{V}(\forall p\varphi) = \bigwedge\{\widehat{V'}(\varphi) \mid V' : \mathsf{Prop} \to B, V' \sim_p V\}$, where we define $V' \sim_p V$ by $V'(q) = V(q)$ for any $q \in \mathsf{Prop} \setminus \{p\}$.

A formula $\phi \in \mathcal{L}\Pi$ is *valid* on a complete modal algebra B, written as $B \vDash \phi$, if for all valuations V on B, $\widehat{V}(\phi) = 1$.

Since we are only interested in Π-logics extending $\mathsf{S5\Pi}$, we only need modal algebras validating $\mathsf{S5}$. In fact, we only need a very special class of such modal algebras called simple S5 algebras.

Definition 2.2 A *simple S5 algebra* is pair $\langle B, \Box \rangle$ where B is a non-trivial Boolean algebra and $\Box$ is the unary function on B defined for $a \in B$ by

$$\Box a = \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{otherwise.} \end{cases} \quad \text{Then } \Diamond a = \begin{cases} 1 & \text{if } a \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let us denote the class of all simple S5 algebras by $\mathsf{sS5A}$ and the class of all complete simple S5 algebras by $\mathsf{csS5A}$.

modal algebras validating $\mathsf{S5}$ are also known as *monadic algebra* (see [14,13]). However, in the context of monadic algebras, $\Diamond$ and $\Box$ operators are usually denoted by $\exists$ and $\forall$, which we need for propositional quantifiers. We also remark that our simple S5 algebras are indeed simple in its general algebraic sense: they have no non-trivial congruence relation. These algebras are also known as *Henle algebras*.

To formulate completeness with respect to $\mathsf{csS5A}$, it is natural to use the following Galois connection:

Definition 2.3 For any class $\mathsf{C} \subseteq \mathsf{csS5A}$, define $\mathrm{Log}(\mathsf{C}) = \{\varphi \in \mathcal{L}\Pi \mid \forall B \in \mathsf{C}, B \vDash \varphi\}$. We also write $\mathrm{Log}(\{B\})$ as simply $\mathrm{Log}(B)$ for any $B \in \mathsf{csS5A}$.

Conversely, for any set of formulas $\Gamma \subseteq \mathcal{L}\Pi$, define $\mathrm{Alg}(\Gamma) = \{B \in \mathsf{csS5A} \mid \forall\varphi \in \Gamma, B \vDash \varphi\}$. Similarly, $\mathrm{Alg}(\varphi)$ abbreviates $\mathrm{Alg}(\{\varphi\})$.

This finishes the semantics for $\mathcal{L}\Pi$, and now we march into expanding $\mathcal{L}\Pi$, as Fine did in [9], to $\mathcal{L}\Pi\mathrm{Mg}$. This is instrumental for formulating the quantifier elimination on which completeness for $\mathsf{S5}\Pi$ alone in [9,15] depends, and all our new results will also need it. In the following, let $\mathbb{N}_+$ be the set of positive natural numbers, and $\mathbb{N}^\infty$ be the set of natural numbers plus an infinite element ∞. Also, we will use $\mathbb{N}^\infty_+$, which has ∞ but not 0.

Definition 2.4 ([9]) Define $\mathcal{L}\Pi\mathrm{Mg}$ by extending the grammar for $\mathcal{L}\Pi$ with a propositional *constant* g (not in Prop) and countably many new unary operators $\{\mathrm{M}_i \mid i \in \mathbb{N}_+\}$. Then, define $\mathcal{L}\mathrm{Mg}$ as the quantifier free fragment of $\mathcal{L}\Pi\mathrm{Mg}$, which has the following grammar:

$$\varphi ::= p \mid \top \mid \mathrm{g} \mid \Box\varphi \mid \mathrm{M}_i\varphi \mid \neg\varphi \mid (\varphi \wedge \varphi)$$

with $p \in \mathsf{Prop}$.

For future convenience, we refer to the elements in $\mathsf{Prop} \cup \{\top, g\}$ in general as propositional *letters*, and we define $\mathrm{md}(\varphi)$ to be the *modal depth* of φ defined as usual, with M_i's and $\Box$ all treated as modal operators, $\mathrm{free}(\varphi)$ to be the set of free propositional variables in φ, and the *quantificational depth* of φ the maximal length of any chain of nested quantifers in φ, analogous to the usual definition in first-order logics.

Let us also define as in [9] for every $\alpha \in \mathcal{L}\Pi\mathrm{Mg}$ an important formula $\mathrm{atom}(\alpha)$:

$$\mathrm{atom}(\alpha) := \Diamond\alpha \wedge \forall q(\Box(q \to \alpha) \vee \Box(q \to \neg\alpha)) \tag{1}$$

where $q \in \mathsf{Prop}$ does not occur in α. To fix this choice, we assume that there is an enumeration of Prop fixed from the outset. Then whenever we need fresh propositional letters in a definition, the definition picks out the first available propositional variable.

Here g is intended to express the proposition that some atomic proposition is true, and $\mathrm{M}_i\varphi$ the proposition that φ is entailed by at least i many atomic propositions. Hence, g should be evaluated to the join of the atoms in a modal algebra. But this requires that the join exists. Let us call a modal algebra *separable* if the join of its atoms exists. Then we can give the semantics for $\mathcal{L}\mathrm{Mg}$ and $\mathcal{L}\Pi\mathrm{Mg}$ on appropriate modal algebras.

Definition 2.5 For any separable modal algebra B, define g (or g_B when ambiguity arises) as the join of all atoms of B, and M_i an operator on B as follows:

$$M_i a = \begin{cases} 1 & \text{if there are at least } i \text{ distinct atoms below } a \\ 0 & \text{otherwise} \end{cases}$$

for $i \in \mathbb{N}_+$.

Then, any valuation V on B can be recursively extended to an $\mathcal{L}$Mg-*valuation* $\widehat{V}$ from $\mathcal{L}$Mg to B by the same clauses for Boolean connectives and $\Box$ as in Definition 2.1, plus the following two clauses:

(i) $\widehat{V}(\mathrm{g}) = g_B$

(ii) $\widehat{V}(\mathrm{M}_i\varphi) = M_i\widehat{V}(\varphi)$.

If B is actually complete, define the $\mathcal{L}\Pi$Mg-*valuation* extending $\widehat{V}$ by combining the clauses above and in Definition 2.1. It is not hard to see that the $\mathcal{L}$Mg-valuation, $\mathcal{L}\Pi$-valuation, and $\mathcal{L}\Pi$Mg-valuation extending V are compatible. Hence by $\widehat{V}$, we always mean the defined valuation with the maximal domain. This will be either an $\mathcal{L}$Mg-valuation or an $\mathcal{L}\Pi$Mg-valuation, depending on whether the codomain of V is merely separable or is complete. We also extend the definition of validity and also the Alg operator to formulas in $\mathcal{L}\Pi$Mg in the obvious way.

Regarding atom(α), it is intended to express the proposition that α expresses an atomic proposition. Its definition does not always achieve this intended meaning, but assuming that it is interpreted on complete simple S5 algebras, this definition indeed singles out atoms in simple S5 algebras. The proof of this can be found in [15].

Lemma 2.6 *For any $\alpha \in \mathcal{L}\Pi$Mg, any complete simple S5 algebra B, and any valuation V on B, we have*

$$\widehat{V}(\mathrm{atom}(\alpha)) = \begin{cases} 1 & \textit{if } \widehat{V}(\alpha) \textit{ is an atom in } B \\ 0 & \textit{otherwise.} \end{cases}$$

Now to the logics in the language $\mathcal{L}\Pi$Mg. They are obtained by adding two axiom schemata to define the new operators M_i's and g by formulas in $\mathcal{L}\Pi$.

Definition 2.7 For any normal Π-logic Λ, define $\Lambda\mathsf{Mg}$ as the smallest normal Π-logic (with formula variables in the schemata and rules of the definition now ranging over $\mathcal{L}\Pi$Mg) that contains the following two axiom schemata for each $n \in \mathbb{N}_+$:

$$\mathrm{M}_n\phi \leftrightarrow \exists q_1 \cdots \exists q_n \Big(\bigwedge_{1\leqslant i<j\leqslant n} \Box(q_i \to \neg q_j) \wedge \bigwedge_{1\leqslant i\leqslant n} (\mathrm{atom}(q_i) \wedge \Box(q_i \to \varphi))\Big) \quad (\mathsf{M})$$

$$\mathrm{g} \leftrightarrow \exists q(q \wedge \mathrm{atom}(q)) \quad (\mathsf{g})$$

where $q_1, \cdots q_n \in \mathsf{Prop}$ do not occur in φ, and $q \in \mathsf{Prop}$.

With the help of Lemma 2.6, it is not hard to directly observe that both (M) and (g) are sound. In fact, we have the following theorem, mostly by Fine and Holliday, on which our new results depend.

Lemma 2.8 $\mathsf{S5\Pi Mg}$ *is a conservative extension of* $\mathsf{S5\Pi}$. *Namely,* $\mathsf{S5\Pi Mg} \cap \mathcal{L}\Pi = \mathsf{S5\Pi}$. *Also,* $\mathsf{S5\Pi Mg}$ *is sound and complete with respect to* $\mathsf{csS5A}$.

Proof. For any $\varphi \in \mathcal{L}\Pi$, if $\varphi \in \mathsf{S5\Pi Mg}$, then we can replace all M_i's and g in its derivation by their definitions in Definition 2.7. The resulting derivation is in $\mathsf{S5\Pi}$. Hence $\varphi \in \mathsf{S5\Pi}$. This shows that $\mathsf{S5\Pi Mg} \cap \mathcal{L}\Pi \subseteq \mathsf{S5\Pi}$. The other direction is trivial. This is observed and first used by Fine in [9].

It is first shown algebraically in [15] that $\mathsf{S5\Pi Mg}$ is sound and that $\mathsf{S5\Pi}$ is sound and complete with respect to $\mathsf{csS5A}$. Now for any $\varphi \in \mathcal{L}\Pi\mathrm{Mg}$ that is valid in $\mathsf{csS5A}$, we can first replace all M_i's and g in φ by their definitions to obtain ψ. Then $\varphi \leftrightarrow \psi \in \mathsf{S5\Pi Mg}$. We know that $\mathsf{S5\Pi Mg}$ is sound on $\mathsf{csS5A}$. So ψ is valid. Then, since $\psi \in \mathcal{L}\Pi$, $\psi \in \mathsf{S5\Pi}$, which means $\psi \in \mathsf{S5\Pi Mg}$. By *modus ponens*, $\varphi \in \mathsf{S5\Pi Mg}$. □

The proof of the completeness of $\mathsf{S5\Pi}$ in [15] relies on a fairly intricate quantifier elimination in $\mathsf{S5\Pi Mg}$ found first by Fine in [9], which says that for any $\varphi \in \mathcal{L}\Pi$, there is a formula $\psi \in \mathcal{L}\mathrm{Mg}$ such that $\phi \leftrightarrow \psi \in \mathsf{S5\Pi Mg}$. We will also make use of this technical result. In fact, ψ can be chosen from a much smaller fragment of $\mathcal{L}\mathrm{Mg}$. Following Fine, we call them model descriptions and define them now.

Definition 2.9 For any $\varphi \in \mathcal{L}\Pi\mathrm{Mg}$, first define the following abbreviations:

$$\mathrm{Q}_0 := \neg \mathrm{M}_1\varphi; \quad \mathrm{Q}_i\varphi := \mathrm{M}_i\varphi \wedge \mathrm{M}_{i+1}\varphi, i \in \mathbb{N}_+; \quad \mathrm{N}\varphi := \Diamond(\neg \mathrm{g} \wedge \varphi).$$

For any finite subset $P \subseteq \mathsf{Prop}$, a *state description* s over P is a conjunction of literals from P in which every $p \in P$ occurs. We follow the convention that an empty conjunction is $\top$ and an empty disjunction is $\bot$. Let 2^P be the set of all state descriptions over P. Then, a *model description of degree n over P* is a conjunction of:

(i) either g or ¬g;

(ii) a state description $a \in 2^P$;

(iii) for each $s \in 2^P$, either $\mathrm{M}_n s$ or some $\mathrm{Q}_n s$ for some $i < n$;

(iv) for each $s \in 2^P$, either $\mathrm{N}s$ or $\mathrm{N}\neg s$.

Lemma 2.10 ([9], § 4.2) *For any $\varphi \in \mathcal{L}\Pi$, there exists a $qf(\varphi) \in \mathcal{L}\mathrm{Mg}$ such that $\varphi \leftrightarrow qf(\varphi) \in \mathsf{S5\Pi Mg}$. Moreover, $qf(\varphi)$ is a disjunction of model descriptions over* $\mathrm{free}(\varphi)$ *of degree 2^n where n is the quantification degree of φ.*

For the construction of qf, the reader can also see the appendix of [15].

3 Semantical and syntactical reduction

In this section, we show that for any $\varphi \in \mathcal{L}\Pi$, any complete simple S5 algebra A, and any $\Lambda \in \mathsf{Next\Pi(S5\Pi)}$, we can construct a formula, which we call $basic(\varphi)$, such that:

- $\varphi \in \Lambda$ iff $basic(\varphi) \in \Lambda\mathsf{Mg}$;
- $A \vDash \varphi$ iff $A \vDash basic(\varphi)$;
- $basic(\varphi)$ is a Boolean combination of $\Diamond\neg g$ and $\mathrm{M}_i\top$ for $i \in \mathbb{N}_+$.

To facilitate the proof, let us first define a number of useful fragments of $\mathcal{L}\Pi$.

Definition 3.1 Recall that $\mathcal{L}$Mg is the quantifier free fragment of $\mathcal{L}\Pi$Mg. Now, Define the following propositional-variable-free fragments of $\mathcal{L}$Mg where i ranges in $\mathbb{N}_+$:

$$\begin{aligned}\mathcal{S}\mathrm{Mg} \ni \varphi &::= \top \mid \mathrm{g} \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \mathrm{M}_i\varphi \\ \mathcal{S}_{\leqslant 1}\mathrm{Mg} \ni \varphi &::= \top \mid \mathrm{g} \mid \Diamond\neg\mathrm{g} \mid \mathrm{M}_i\top \mid \neg\varphi \mid (\varphi \wedge \varphi) \\ \mathcal{S}\mathrm{Basic} \ni \varphi &::= \top \mid \Diamond\neg\mathrm{g} \mid \mathrm{M}_i\top \mid \neg\varphi \mid (\varphi \wedge \varphi)\ .\end{aligned}$$

The $\mathcal{S}$ instead of $\mathcal{L}$ in their names means "$\mathcal{S}$entence." It is not hard to see that $\mathcal{S}$Mg collects all propositional-variable-free formulas in $\mathcal{L}$Mg and that $\mathcal{S}_{\leqslant 1}$Mg collects some formulas with modal depth at most 1 in $\mathcal{S}$Mg, which are enough for our purposes.

For any $\varphi \in \mathcal{L}\Pi$, we will construct $basic(\varphi)$ as the following with u and $comp$ to be defined:

$$basic(\varphi) = comp(\Box qf(u(\varphi))).$$

Here, $u(\varphi)$ is the universal closure of φ, which is defined as $\forall p_1 \forall p_2 \cdots \forall p_n \varphi$ where $p_1, p_2, \cdots, p_n$ enumerate the free propositional letters in φ. And recall that qf returns the result of quantifier elimination. Since $u(\varphi)$ has no free propositional variable, according to Lemma 2.10, $qf(u(\varphi)) \in \mathcal{S}$Mg is a disjunction of model descriptions of some finite degree over $\varnothing$. From Definition 2.9, we can see that all model descriptions of degree n over $\varnothing$ are of the form

$$\pm g \wedge \mathrm{M}_i\top \wedge \neg\mathrm{M}_{i+1}\top \wedge \pm\Diamond\neg\mathrm{g} \quad \text{or} \quad \pm\, g \wedge \mathrm{M}_n\top \wedge \pm\Diamond\neg\mathrm{g}$$

where $i < n$, $\pm$ stands for $\neg$ or nothing, and $\mathrm{M}_0\top$ for $\top$. In short, $qf(u(\varphi))$ is a Boolean combination of g, $\mathrm{M}_i\top$ for $i \in \mathbb{N}_+$, and $\Diamond\neg g$, and hence is in $\mathcal{S}_{\leqslant 1}$Mg.

Now we construct $comp$ as a function that simplifies a boxed modal description over $\varnothing$ to a formula in $\mathcal{S}$Basic in a provably equivalent way.

Lemma 3.2 *For any ψ a disjunction of model descriptions of degree n over $\varnothing$, there is a formula $comp(\Box\psi) \in \mathcal{S}$Basic such that $\Box\psi \leftrightarrow comp(\Box\psi) \in \mathsf{S5\Pi Mg}$.*

Proof. Let pos be the number of model descriptions in ψ where g appears positively and neg the number of model descriptions in ψ where g appears negatively. For any $1 \leqslant i \leqslant pos$, let α_i be the result of deleting the conjunct g in the ith model description in ψ where g appears positively, and similarly define β_i for $1 \leqslant i \leqslant neg$, where we need to delete the $\neg$g conjunct.

Let $\alpha = \bigvee_{1 \leqslant i \leqslant pos} \alpha_i$ and $\beta = \bigvee_{1 \leqslant i \leqslant neg} \beta_i$, which are now Boolean combinations of $\mathrm{M}_i\top$'s and $\Diamond\neg$g. Then obviously $\psi \leftrightarrow ((\mathrm{g} \wedge \alpha) \vee (\neg\mathrm{g} \wedge \beta))$ and $\Box\psi \leftrightarrow \Box((\mathrm{g} \wedge \alpha) \vee (\neg\mathrm{g} \wedge \beta))$ are in $\mathsf{S5\Pi Mg}$ using propositional tautologies and normality. Let us write for any $\varphi_1, \varphi_2 \in \mathcal{L}\Pi$Mg, $\varphi_1 \equiv_{\mathsf{S5\Pi Mg}} \varphi_2$ iff

$\varphi_1 \leftrightarrow \varphi_2 \in \mathsf{S5\Pi Mg}$. Then we have

$$\begin{aligned}
\Box\psi &\equiv_{\mathsf{S5\Pi Mg}} \Box((\mathrm{g} \wedge \alpha) \vee (\neg\mathrm{g} \wedge \beta)) && (2)\\
&\equiv_{\mathsf{S5\Pi Mg}} \Box((\mathrm{g} \vee \neg\mathrm{g}) \wedge (\mathrm{g} \vee \beta) \wedge (\neg\mathrm{g} \vee \alpha) \wedge (\alpha \vee \beta)) && (3)\\
&\equiv_{\mathsf{S5\Pi Mg}} \Box(\mathrm{g} \vee \neg\mathrm{g}) \wedge \Box(\mathrm{g} \vee \beta) \wedge \Box(\neg\mathrm{g} \vee \alpha) \wedge \Box(\alpha \vee \beta) && (4)\\
&\equiv_{\mathsf{S5\Pi Mg}} \Box(\mathrm{g} \vee \Box\beta) \wedge \Box(\neg\mathrm{g} \vee \Box\alpha) \wedge \Box(\Box\alpha \vee \Box\beta) && (5)\\
&\equiv_{\mathsf{S5\Pi Mg}} (\Box\mathrm{g} \vee \Box\beta) \wedge (\Box\neg\mathrm{g} \vee \Box\alpha) \wedge (\Box\alpha \vee \Box\beta) && (6)\\
&\equiv_{\mathsf{S5\Pi Mg}} (\Box\mathrm{g} \vee \beta) \wedge (\Box\neg\mathrm{g} \vee \alpha) \wedge (\alpha \vee \beta) && (7)\\
&\equiv_{\mathsf{S5\Pi Mg}} (\neg\Diamond\neg\mathrm{g} \vee \beta) \wedge (\neg\mathrm{M}_1\top \vee \alpha) \wedge (\alpha \vee \beta). && (8)
\end{aligned}$$

In the above chain of provable equivalences, (5), (7), and (8) require more explanation. Note that in $\mathsf{S5\Pi Mg}$, we have $\Box\forall p\varphi \equiv_{\mathsf{S5\Pi Mg}} \forall p\Box\varphi$, and dually, $\Diamond\exists p\varphi \equiv_{\mathsf{S5\Pi Mg}} \exists p\Diamond\varphi$. With all the axioms in $\mathsf{S5}$, and also the axiom (M) defined in Definition 2.7, $\mathrm{M}_i\top \equiv_{\mathsf{S5\Pi Mg}} \Box\mathrm{M}_i\top$. Then, $\alpha \equiv_{\mathsf{S5\Pi Mg}} \Box\alpha$ and $\beta \equiv_{\mathsf{S5\Pi Mg}} \Box\beta$, since α and β are Boolean combinations of $\mathrm{M}_i\top$ and $\Diamond\neg g$. Thus we have (5) and (7). For (8), some manipulation of axioms (g) and (M) gives us $\Box\neg g \equiv_{\mathsf{S5\Pi Mg}} \neg\mathrm{M}_1\top$. The rest of the equivalences are standard $\mathsf{S5}$ reasoning. We can then define $basic(\Box\psi)$ to be the right hand side in (8), which is provably equivalent to $\Box\psi$ and is in $\mathcal{S}$Basic. □

Now we show that for any $\varphi \in \mathcal{L}\Pi$, $basic(\varphi)$ has the required properties.

Lemma 3.3 *For any $\varphi \in \mathcal{L}\Pi$ and $\Lambda \in \mathsf{Next\Pi(S5\Pi)}$, $\varphi \in \Lambda$ iff $basic(\varphi) \in \Lambda\mathsf{Mg}$.*

Proof. Since $\Lambda\mathsf{Mg}$ is a conservative extension of Λ (by Lemma 2.8), $\varphi \in \Lambda$ iff $\varphi \in \Lambda\mathsf{Mg}$. Using universalization and also universal instantiation, $\varphi \in \Lambda\mathsf{Mg}$ iff $u(\varphi) \in \Lambda\mathsf{Mg}$. Since $\Lambda \in \mathsf{Next\Pi(S5\Pi)}$, $\Lambda\mathsf{Mg}$ extends $\mathsf{S5\Pi Mg}$. Together with the quantifier elimination result in Lemma 2.10, $qf(u(\varphi)) \leftrightarrow u(\varphi) \in \Lambda\mathsf{Mg}$. Thus $u(\varphi) \in \Lambda\mathsf{Mg}$ iff $qf(u(\varphi)) \in \Lambda\mathsf{Mg}$. By necessitation and also the T axiom derivable in $\mathsf{S5}$, $qf(u(\varphi)) \in \Lambda\mathsf{Mg}$ iff $\Box qf(u(\varphi)) \in \Lambda\mathsf{Mg}$. Finally, due to Lemma 3.2 and the fact that $qf(u(\varphi))$ is indeed a disjunction of model descriptions of some finite degree over $\varnothing$, we have $basic(\varphi) = comp(\Box qf(u(\varphi))) \leftrightarrow \Box qf(u(\varphi)) \in \Lambda\mathsf{Mg}$. Thus $basic(\varphi) \in \Lambda\mathsf{Mg}$ iff $\Box qf(u(\varphi)) \in \Lambda\mathsf{Mg}$. Connecting all the equivalences, $\varphi \in \Lambda$ iff $basic(\varphi) \in \Lambda\mathsf{Mg}$. □

On the semantical side, we first make an easy observation.

Lemma 3.4 *For any complete (resp. separable) modal algebra B and any $\varphi \in \mathcal{L}\Pi\mathrm{Mg}$ (resp. $\mathcal{L}\mathrm{Mg}$) such that $\mathrm{free}(\varphi) = \varnothing$, for any two valuations V and V' on B, $\widehat{V}(\varphi) = \widehat{V'}(\varphi)$.*

Due to this observation, we define for any separable modal algebra B, a fixed trivial valuation V_B which maps every $p \in \mathsf{Prop}$ to 1_B. Then $B \vDash \varphi$ iff $\widehat{V_B}(\varphi) = 1_B$ for any $\varphi \in \mathcal{L}\mathrm{Mg}$ (or $\mathcal{L}\Pi\mathrm{Mg}$ when B is complete) such that $\mathrm{free}(\varphi) = \varnothing$.

Then we can prove the semantical requirement for *basic*.

Lemma 3.5 *For any $\varphi \in \mathcal{L}\Pi$ and any complete simple S5 algebra B, $B \vDash \varphi$ iff $B \vDash basic(\varphi)$.*

Proof. First consider the following chain of equivalences:

$B \vDash \varphi$ iff $B \vDash u(\varphi)$	by the definition of validity
iff $\widehat{V_B}(u(\varphi)) = 1$	by the definition of validity
iff $\widehat{V_B}(qf(u(\varphi))) = 1$	B validates S5ΠMg, and $u(\varphi) \leftrightarrow qf(u(\varphi)) \in$ S5ΠMg
iff $\widehat{V_B}(\Box(qf(u(\varphi)))) = 1$	$\Box 1 = 1$ and $\Box a \neq 1$ if $a \neq 1$
iff $\widehat{V_B}(comp(\Box(qf(u(\varphi))))) = 1$	B validates S5ΠMg, and Lemma 3.2

Note that $basic(\varphi)$ must have no free propositional variable, because $basic(\varphi) = comp(\Box qf(u(\varphi)))$, $\text{free}(u(\varphi)) = \varnothing$, and neither qf nor $comp$ introduces new free variables. Then by the observation in the previous lemma, $B \vDash basic(\varphi)$ iff $\widehat{V_B}(basic(\varphi)) = 1$. Connecting all the equivalences, $B \vDash \varphi$ iff $B \vDash basic(\varphi)$. □

4 Types and type space

In the last section, we have shown that many formulas are equivalent in terms of validity or theoremhood. In this section, we do the same to the algebras: many algebras are equivalent in terms of the formulas in $\mathcal{L}\Pi$ they validate. This equivalence relation can in fact bring csS5A from a class to a countable set, which we will call the type space. Then, to study the classes of algebras definable by formulas in $\mathcal{L}\Pi$, we can just study the sets of types of the algebras in those classes. This in turn gives us a topology on the type space. Now we start with the definition of the types, which is in fact a much simplified version of the famous Tarski invariant for Boolean algebras (see § 5.5 in [8]), due to the completenss of the algebras we are interested in.

Definition 4.1 For any complete simple S5 algebra B, its *type* $t(B)$ is a pair $\langle t_0(B), t_1(B)\rangle$ where

$$t_0(B) = \begin{cases} 1 & \text{if } g \neq 1 \\ 0 & \text{if } g = 1, \end{cases} \quad t_1(B) = \begin{cases} i \in \mathbb{N} & \text{if } B \text{ has exactly } i \text{ atoms} \\ \infty & \text{if } B \text{ has infinitely many atoms.} \end{cases} \tag{9}$$

Recall that g of B is the join of its atoms. Hence, t_0 says whether this algebra contains an atomless part, and t_1 counts the atoms it has. Let S be the set of all types of complete simple S5 algebras, the type space.

Proposition 4.2 $S = (\{0,1\} \times \mathbb{N}^\infty) \backslash \{\langle 0,0\rangle\}$.

Proof. Apparently, $S \subseteq \{0,1\} \times \mathbb{N}^\infty$ as any type is a pair $\langle t_0, t_1\rangle$ where the first component can only be 0 or 1 and the second component can only be a natural number of ∞. Also, if the type of a complete simple S5 algebra A is $\langle 0,0\rangle$, then A has no atom and also no atomless part, which means A is trivial and thus not a complete simple S5 algebra in our Definition 2.5. So we have shown the inclusion of left to right. Now to the other direction. The right-hand-side can be decomposed into three parts:

- $\langle 0, n\rangle$ for $n \in \mathbb{N}_+$. Types of this form can be realized by the Boolean algebra B_n of the powerset of a set of n elements, with $\Box$ defined as in Definition 2.5.
- $\langle 1, 0\rangle$. To realize this type, take the countable free Boolean algebra B. It is well known that B is atomless, but not complete. However, we can take the MacNeille completion B^+ of B, a complete Boolean algebra (unique up to isomorphism) such that B embeds into and that every element of B^+ is a join of images of elements of B. For a construction of this B^+, see [12], Chap. 25. Then B^+ is complete and atomless, as if there is an atom, it must be the image of an atom of B, but B is atomless. Now turn B^+ to a simple S5 algebra by defining $\Box$ as in Definition 2.5. Then B^+ has type $\langle 1, 0\rangle$.
- $\langle 1, n\rangle$ for $n \in \mathbb{N}_+$. Consider the product of B^+ and B_n with $\Box$ again defined as above. It is not hard to see that it has an atomless part: g of this algebra is $\langle 1, 0\rangle$. It also has n many atoms, listed by $\langle 0, a\rangle$ where a range over atoms in B_n. Thus this simple S5 algebra has type $\langle 1, n\rangle$.

Hence we realized all types in the right-hand-side. So the inclusion from right to left is also shown. □

Now we define the equivalences between algebras. Then we will show that types capture this equivalence relation.

Definition 4.3 For any two complete simple S5 algebras A, B, and any $\mathcal{L} \in \{\mathcal{L}\Pi, \mathcal{S}\text{Basic}\}$ we say $A \equiv_{\mathcal{L}} B$ if for any sentence $\phi \in \mathcal{L}$, $A \vDash \varphi$ iff $B \vDash \varphi$.

Lemma 4.4 *For any two complete simple S5 algebras A, B, $A \equiv_{\mathcal{L}\Pi} B$ iff $A \equiv_{\mathcal{S}\text{Basic}} B$.*

Proof. Immediate from Lemma 3.5. □

Lemma 4.5 *For any two complete simple S5 algebras A, B, $t(A) = t(B)$ iff $A \equiv_{\mathcal{S}\text{Basic}} B$. Hence, together with Lemma 4.4, $t(A) = t(B)$ iff $A \equiv_{\mathcal{L}\Pi} B$.*

Proof. Recall that for all $\varphi \in \mathcal{S}$Basic and any complete simple S5 algebra A, $A \vDash \varphi$ iff $\widehat{V_A}(\varphi) = 1$, because free$(\varphi) = \varnothing$. Also notice that for $\varphi = \Diamond\neg g$ or $\mathrm{M}_i\top$ for any $i \in \mathbb{N}_+$, $\widehat{V_A}(\varphi)$ is either 0 or 1. Since $\mathcal{S}$Basic consists of all and only the Boolean combinations of these formulas, in fact for any $\varphi \in \mathcal{S}$Basic, $\widehat{V_A}(\varphi) \in \{0, 1\}$, and in other words, $A \vDash \varphi$ or $A \vDash \neg\varphi$.

Now suppose $t(A) = t(B)$. Then, conflating 1_A and 1_B, and also 0_A and 0_B, we can easily verify that $\widehat{V_A}(\Diamond\neg \mathrm{g}) = \widehat{V_B}(\Diamond\neg \mathrm{g})$ and that for all $i \in \mathbb{N}_+$, $\widehat{V_A}(\mathrm{M}_i\top) = \widehat{V_B}(\mathrm{M}_i\top)$. Then a simple induction propagates these equalities to all $\varphi \in \mathcal{S}$Basic. Thus we see that if $t(A) = t(B)$, then $A \equiv_{\mathcal{S}\text{Basic}} B$.

On the other hand, if $t(A) \neq t(B)$, then there are two cases:

- $t_1(A) \neq t_1(B)$. In this case, $\Diamond\neg \mathrm{g}$ distinguishes the two algebras.
- $t_2(A) \neq t_2(B)$. Let n be the smaller number among them. Then $n \in \mathbb{N}$, $n + 1 \in \mathbb{N}_+$, and $\neg\mathrm{M}_{n+1}$ distinguishes the two algebras.

Hence, if $t(A) \neq t(B)$, then $A \not\equiv_{\mathcal{S}\text{Basic}} B$. □

Due to this lemma, the function t can be seen as the quotient map from $\mathsf{csS5A}$ to $\mathsf{csS5A}/\equiv_{\mathcal{L}\Pi}$. This means that the Galois connection between $\mathsf{csS5A}$ and $\mathcal{L}\Pi$ by Alg and Log can be reduced to the following Galois connection between S and $\mathcal{L}\Pi$.

Definition 4.6 For any type $s \in S$ and any $\varphi \in \mathcal{L}\Pi$, let us write $s \vDash \varphi$ just in case for any $A \in \mathsf{csS5A}$ such that $t(A) = s$, $A \vDash \varphi$.

Then define $\mathrm{Type}(\Gamma)$ for every $\Gamma \subseteq \mathcal{L}\Pi$ as $\{s \in S \mid \forall \varphi \in \Gamma, s \vDash \varphi\}$, with $\mathrm{Type}(\varphi)$ again abbreviating $\mathrm{Type}(\{\varphi\})$. On the other direction, define $\mathrm{Log}(T)$ for any subset T of S as $\{\varphi \in \mathcal{L}\Pi \mid \forall s \in T, s \vDash \varphi\}$, with $\mathrm{Log}(s)$ abbreviating $\mathrm{Log}(\{s\})$ for any $s \in S$ as well.

Then, we can collect the following easy but useful observations.

Lemma 4.7 *For any* $\Gamma \subseteq \mathcal{L}\Pi$*,* $\varphi, \psi \in \mathcal{S}\mathrm{Basic}$*:*

- $\mathrm{Alg}(\Gamma) = t^{-1}(\mathrm{Type}(\Gamma))$*,* $t(\mathrm{Alg}(\Gamma)) = \mathrm{Type}(\Gamma)$*, and then* $\mathrm{Log}(\mathrm{Type}(\Gamma)) = \mathrm{Log}(\mathrm{Alg}(\Gamma))$*;*
- $\mathrm{Type}(\Gamma) = \bigcap\{\mathrm{Type}(\varphi) \mid \varphi \in \Gamma\}$*;*
- $\mathrm{Type}(\neg\varphi) = S \setminus \mathrm{Type}(\varphi)$*,* $\mathrm{Type}(\varphi \wedge \psi) = \mathrm{Type}(\varphi) \cap \mathrm{Type}(\psi)$*.*

Using this lemma, we can study the following topology that will be important to us for both general completeness and the lattice structure of $\mathsf{Next}\Pi(\mathsf{S5}\Pi)$.

Definition 4.8 Let $\mathcal{S}$ be the topological space with the type space S as the underlying set and $\{\mathrm{Type}(\varphi) \mid \varphi \in \mathcal{L}\Pi\}$ as basic opens.

Lemma 4.9 *$\mathcal{S}$ is a Stone space, homeomorphic to the disjoint union of two copies of the one-point compactification of $\mathbb{N}$ with the usual order topology.*

Proof. From Lemma 3.5, the basic opens of $\mathcal{S}$ are just sets in $\{\mathrm{Type}(\varphi) \mid \varphi \in \mathcal{S}\mathrm{Basic}\}$. By the third bullet in Lemma 4.7, we know that $\{\mathrm{Type}(\varphi) \mid \varphi \in \mathcal{S}\mathrm{Basic}\}$ is a field of sets on S. Thus $\mathcal{S}$ is zero-dimensional. To see that $\mathcal{S}$ is Hausdorff, take two different $s_1, s_2 \in S$. Recall that S is $t(\mathsf{csS5A})$. So we can find two complete simple S5 algebras B_1 and B_2 such that $t(B_1) = s_1$ and $t(B_2) = s_2$. Then, by Lemma 3.5, $B_1 \not\equiv_{\mathcal{S}\mathrm{Basic}} B_2$. So we can find a formula $\varphi \in \mathcal{S}\mathrm{Basic}$ such $\mathrm{Type}(\varphi)$ that separates B_1 and B_2. So $\mathcal{S}$ is Hausdorff.

To show that $\mathcal{S}$ is compact, we need a more detailed analysis of $\{\mathrm{Type}(\varphi) \mid \varphi \in \mathcal{S}\mathrm{Basic}\}$. First, note that $\mathrm{Type}(\Diamond\neg g) = \{1\} \times \mathbb{N}^{\infty}$, $\mathrm{Type}(\neg\Diamond\neg g) = \{0\} \times \mathbb{N}_{+}^{\infty}$, and they partition S into two parts. Let us name them by S_1 and S_0 respectively. Hence $\mathcal{S}$ is the disjoint union of $\mathcal{S}_1$ and $\mathcal{S}_0$ defined as the subspaces of S on S_1 and S_0 respectively. So we only need to show that they are both compact. On $\mathcal{S}_1$, the basic clopens are now Boolean combinations of $\mathrm{Type}(\mathrm{M}_i)$ for $i \in \mathbb{N}_+$ and $\mathrm{Type}(\Diamond\neg g)$, all restricted to S_1. But $\mathrm{Type}(\Diamond\neg g) = S_1$. Then the clopens are actually the field of sets on S_1 generated by $\{\mathrm{Type}(\mathrm{M}_n) \cap S_1 \mid n \in \mathbb{N}\} = \{\{\langle 1, i\rangle \mid n \leqslant i \leqslant \infty\} \mid n \in \mathbb{N}\}$. Hence it is not hard to see that $\mathcal{S}_1$ is just (homeomorphic to) the one-point compactification of the order topology on $\mathbb{N}$. The situation for $\mathcal{S}_0$ is almost the same, except that the space is on $\mathbb{N}_{+}^{\infty}$. But it is still homeomorphic to the one-point compactification of $\mathbb{N}$. □

5 Main results

Now we are prepared to prove the main results regarding Π-logics extending S5Π. Let us start with the general completeness.

Theorem 5.1 *For any* $\Lambda \in \mathsf{Next}\Pi(\mathsf{S5}\Pi)$*,* $\Lambda = \mathrm{Log}(\mathrm{Alg}(\Lambda))$*.*

Proof. As is shown in Lemma 4.7, $\mathrm{Log}(\mathrm{Alg}(\Lambda)) = \mathrm{Log}(\mathrm{Type}(\Lambda))$. Also, it is trivial that $\Lambda \subseteq \mathrm{Log}(\mathrm{Type}(\Lambda))$. Hence we just need to show that, for any $\varphi \in \mathcal{L}\Pi$, if $\varphi \in \mathrm{Log}(\mathrm{Type}(\Lambda))$, then $\varphi \in \Lambda$.

Let us assume the antecedent. Then for any $s \in \mathrm{Type}(\Lambda)$, $s \vDash \varphi$. In other words, $\mathrm{Type}(\varphi) \supseteq \mathrm{Type}(\Lambda)$. As we observed in Lemma 4.7, $\mathrm{Type}(\Lambda) = \bigcap\{\mathrm{Type}(\psi) \mid \psi \in \Lambda\}$. By Lemma 3.5, $\mathrm{Type}(\psi) = \mathrm{Type}(basic(\psi))$. Note also that $\psi \in \Lambda$ iff $basic(\psi) \in \Lambda\mathsf{Mg}$, which is shown in Lemma 3.3. Thus the set $\{\mathrm{Type}(\psi) \mid \psi \in \Lambda\} \subseteq \{\mathrm{Type}(\psi) \mid \psi \in \Lambda\mathsf{Mg} \cap \mathcal{S}\mathrm{Basic}\}$. On the other hand, for any $\psi \in \Lambda\mathsf{Mg} \cap \mathcal{S}\mathrm{Basic}$, using the axioms defining M_i and g, there is a $\psi' \in \mathcal{L}\Pi$ such that $\psi \leftrightarrow \psi' \in \Lambda\mathsf{Mg}$. This means that ψ' is in $\Lambda\mathsf{Mg}$, hence also in Λ, and that $\mathrm{Type}(\psi) = \mathrm{Type}(\psi')$, using Lemma 2.8. Hence $\{\mathrm{Type}(\psi) \mid \psi \in \Lambda\} = \{\mathrm{Type}(\psi) \mid \psi \in \Lambda\mathsf{Mg} \cap \mathcal{S}\mathrm{Basic}\}$, which we now call F.

Now this is a filter of basic clopens in $\mathcal{S}$ for the following reasons.

- For any $X, Y \in F$, we can find $\alpha, \beta \in \Lambda\mathsf{Mg} \cap \mathcal{S}\mathrm{Basic}$ such that $X = \mathrm{Type}(\alpha)$ and $Y = \mathrm{Type}(\beta)$. Now $\alpha \wedge \beta \in \Lambda\mathsf{Mg} \cap \mathcal{S}\mathrm{Basic}$, since $\Lambda\mathsf{Mg}$ has all propositional tautologies and *modus ponens*. Hence $X \cap Y = \mathrm{Type}(\alpha) \cap \mathrm{Type}(\beta) = \mathrm{Type}(\alpha \wedge \beta) \in F$.
- Recall that the basic clopens in $\mathcal{S}$ are just $\{\mathrm{Type}(\beta) \mid \beta \in \mathcal{S}\mathrm{Basic}\}$. For any $X \in F$ and any basic clopen Y such that $X \subseteq Y$, we first find $\alpha \in \Lambda\Pi\mathsf{Mg} \cap \mathcal{S}\mathrm{Basic}$ and $\beta \in \mathcal{S}\mathrm{Basic}$ such that $X = \mathrm{Type}(\alpha)$ and $Y = \mathrm{Type}(\beta)$. Then note that $\mathrm{Type}(\alpha \to \beta) = (S \setminus X) \cup Y = S$, since $X \subseteq Y$. Then by the completeness of $\mathsf{S5}\Pi\mathsf{Mg}$ (Lemma 2.8), $\alpha \to \beta \in \mathsf{S5}\Pi\mathsf{Mg}$. Then by *modus ponens* in $\Lambda\mathsf{Mg}$, which extends $\mathsf{S5}\Pi\mathsf{Mg}$ as Λ extends $\mathsf{S5}$, $\beta \in \Lambda\mathsf{Mg}$. Remember that $\beta \in \mathcal{S}\mathrm{Basic}$. Hence $\beta \in \Lambda\mathsf{Mg} \cap \mathcal{S}\mathrm{Basic}$ and $Y = \mathrm{Type}(\beta) \in F$.

Thus we have $\mathrm{Type}(\Lambda) = \cap F$, a filter of basic clopens in $\mathcal{S}$, and we assumed that $\mathrm{Type}(\varphi) \supseteq \mathrm{Type}(\Lambda)$. Take $basic(\varphi)$. We have $\mathrm{Type}(basic(\varphi)) = \mathrm{Type}(\varphi)$ and that it is basic clopen in $\mathcal{S}$. We have shown that $\mathcal{S}$ is a Stone space in Lemma 4.9. Hence by compactness, there is actually an element $Z \in F$ such that $\mathrm{Type}(\varphi) \subseteq Z$. By the definition of F, we can find a $\psi \in \Lambda\mathsf{Mg} \cap \mathcal{S}\mathrm{Basic}$ such that $Z = \mathrm{Type}(\psi)$. Then $\mathrm{Type}(\psi \to basic(\varphi)) = S$. By completeness again, we have $\psi \to basic(\varphi) \in \mathsf{S5}\Pi\mathsf{Mg}$ and thus also $\Lambda\mathsf{Mg}$. Then, since ψ is taken in $\Lambda\mathsf{Mg}$, $basic(\varphi) \in \Lambda\mathsf{Mg}$. By Lemma 3.3, $\varphi \in \Lambda$. This finishes the completeness of Λ. □

Then we describe the lattice structure of $\mathsf{Next}\Pi(\mathsf{S5}\Pi)$.

Theorem 5.2 *The lattice* $\langle \mathsf{Next}\Pi(\mathsf{S5}\Pi), \subseteq \rangle$ *is isomorphic to the lattice of the open sets of* $\mathcal{S}$*. The isomorphism is* $\Lambda \mapsto S \setminus \mathrm{Type}(\Lambda)$*, or in the other direction,* $X \mapsto \mathrm{Log}(S \setminus X)$*.*

Proof. It is shown in the proof of Theorem 5.1 that for any $\Lambda \in \mathsf{Next\Pi(S5\Pi)}$, Type($\Lambda$) is the intersection of a filter of the basic opens of $\mathcal{S}$. By the basic theory of Stone spaces, this means that Type(Λ) is always a closed set in $\mathcal{S}$. Also, for any $\Lambda_1, \Lambda_2 \in \mathsf{Next\Pi(S5\Pi)}$, obviously $\Lambda_1 \subseteq \Lambda_2$ iff $\text{Type}(\Lambda_1) \supseteq \text{Type}(\Lambda_2)$. This means that, if we can establish that for every closed set $X \in S$, there is a $\Lambda \in \mathsf{Next\Pi(S5\Pi)}$ such that $X = \text{Type}(\Lambda)$, then the lattice structure of $\mathsf{Next\Pi(S5\Pi)}$ is precisely the inverse lattice of the closed sets of $\mathcal{S}$, or just the lattice of the open sets of $\mathcal{S}$, and the isomorphism will be given by $\Lambda \mapsto S \setminus \text{Type}(\Lambda)$.

Now take an arbitrary closed set X in $\mathcal{S}$. Then $\text{Log}(X) \in \mathsf{Next\Pi(S5\Pi)}$ as it is the set of formulas in $\mathcal{L}\Pi$ valid on a class of complete simple S5 algebras. Then what remains to be shown is that $\text{Type}(\text{Log}(X)) = X$. Again, the direction $X \subseteq \text{Type}(\text{Log}(X))$ is trivial. Now take an arbitrary type $s \in S \setminus X$. Then we just need to show that $s \notin \text{Type}(\text{Log}(X))$. Since $\mathcal{S}$ is a Stone space, X is closed, and $s \notin X$, we know that s and X can be separated by a basic clopen. Then, we can find a $\varphi \in \mathcal{S}\text{Basic}$ such that $X \subseteq \text{Type}(\varphi)$ but $s \notin \text{Type}(\varphi)$. But then, $\varphi \in \text{Log}(X)$. Since $\text{Type}(\text{Log}(X)) = \bigcap\{\text{Type}(\psi) \mid \psi \in \text{Log}(X)\}$, we see that $s \notin \text{Type}(\text{Log}(X))$. This finishes the proof. □

Since we have shown in the process of proving Theorem 4.9 that $\mathcal{S}$ is isomorphic to the disjoint union of two copies of the one-point compactification of $\mathbb{N}$, we have the following corollary.

Corollary 5.3 *The lattice* $\langle \mathsf{Next\Pi(S5\Pi)}, \subseteq \rangle$ *is isomorphic to the lattice of open sets of the disjoint union of two copies of the one-point compactification of* $\mathbb{N}$*, which is further isomorphic to the lattice of filters of the direct product of two copies of the field of finite and cofinite sets in* $\mathbb{N}$*.*

Another corollary of this characterization of all logics in $\mathsf{Next\Pi(S5\Pi)}$ is that, in terms of computability, there are arbitrarily complex logics (coded as sets of natural numbers in some natural way). More precisely, for any $X \subseteq \mathbb{N}$, there is a $\Lambda \in \mathsf{Next\Pi(S5\Pi)}$ such that X and Λ are Turing-equivalent.

Theorem 5.4 *For any* $X \subseteq \mathbb{N}$*,* $\text{Log}(\{1\} \times (X \cup \{\infty\}))$ *is Turing-equivalent to* X*.*

Proof. It is not hard to see that $\{1\} \times (X \cup \{\infty\})$ is a closed set in $\mathcal{S}$. Let us name $\{1\} \times (X \cup \{\infty\})$ by C, and $\{1\} \times \mathbb{N}^\infty$ by S_1 as we did before. Then $C = \text{Type}(\text{Log}(C))$, and thus for any $\varphi \in \mathcal{L}\Pi$, $\varphi \in \text{Log}(C)$ iff $\text{Type}(\varphi) \supseteq C$.

To reduce $\text{Log}(C)$ to X, the core idea is that to decide $\text{Type}(\varphi) \supseteq C$, we only need to see whether $\text{Type}(\varphi) \cap S_1 \supseteq C$, as $C \subseteq S_1$. Also $\text{Type}(\varphi) \cap S_1 = \text{Type}(basic(\varphi)) \cap S_1$, which is a finite union of intervals with finite left end points in S_1, with these end points readily computable from φ. Then, when X, and hence C, is given by an oracle, to decide whether $\varphi \in \text{Log}(C)$, we just need to do the following:

- If there is no cofinite interval, then return “no”. This is correct because only a cofinite interval contains $\langle 1, \infty \rangle$, which is in C by definition.
- Otherwise, for each $s \in S_1$ that is not in $\text{Type}(\varphi)$, of which there are only finitely many, check whether $s \in C$. If the oracle for C ever returns “yes”,

then return "no", as now Type(φ) is not a superset of C. Otherwise, $S_1 \cap \text{Type}(\varphi) \supseteq C$, and $\varphi \in \Lambda$.

When X, and hence C, is finite, then we can have an algorithm that directly checks whether for each $s \in C$, s is also in Type(φ). In sum, $Log(C)$ can be reduced to X.

On the other hand, suppose Log(C) is given by an oracle. Then to compute whether $n \in X$ for some $n \in \mathbb{N}$, we only need to use the formula $\varphi_n = \neg\mathrm{M}_{n-1} \vee \mathrm{M}_{n+1}$, with M_{-1} and M_0 here defined as $\top$. Note that $\text{Type}(\varphi_n) = S_1 \setminus \{\langle 1, n\rangle\}$. This means that:

- If $\varphi_n \in \text{Log}(C)$, then $S_1 \setminus \{\langle 1, n\rangle\} \supseteq C$, and hence $\langle 1, n\rangle \notin C$ and $n \notin X$.
- If $\varphi_n \notin \text{Log}(C)$, then $S_1 \setminus \{\langle 1, n\rangle\} \not\supseteq C$. Then $\langle 1, n\rangle \in C$ and $n \in X$.

Thus we only need to use the oracle to decide whether $\varphi_n \in \text{Log}(C)$ and then return the opposite answer. □

Regarding the non-normal Π-logics extending $\mathsf{S5}\Pi$, we limit ourselves in this paper to merely point out that there are many such logics. Algebraically, non-normal modal logics come from *matrices* (see §1.5 of [17]), which are algebras of propositions with a set of designated truth values. To exhibit a non-normal Π-logic extending $\mathsf{S5}\Pi$, we can use just one particular structure. Let B be the complete simple S5 algebra whose Boolean part is the direct product of the powerset algebra of $\mathbb{N}$ and the MacNeille completion of the free Boolean algebra with countably many generators. Note that $t(B) = \langle 1, \infty\rangle$. Now consider the following set:

$$\Lambda = \{\varphi \in \mathcal{L}\Pi \mid \forall V : \mathsf{Prop} \to B, \widehat{V}(\varphi) \geqslant g\}.$$

It is not hard to see that $\Lambda \supseteq \text{Log}(B)$, as the latter collects formulas whose valuation stay at 1, hence necessarily above g. Also, Λ is a Π-logic. In particular, universalization is valid because if φ only evaluates to elements above g, then $\forall\varphi$ evaluates to the meet of those elements above g, which must stay above g. Moreover, $\exists q(q \wedge \text{atom}(q)) \in \Lambda$, as this formula evaluates precisely to g. However, $\Box\exists q(q \wedge \text{atom}(q))$ is not in Λ, since $\Box g$ is $\bot$, because $g \neq 1$ in B: there is an atomless part in B. This means we obtained a non-normal Π-logic extending a normal Π-logic Log(B) which has no proper while consistent normal extension: the only closed proper subset of $\{B\}$ in $\mathcal{S}$ is $\varnothing$. Obviously, for any complete simple S5 algebra B that has both a non-trivial atomless part and a non-trivial atomic part, we can obtain a non-normal Π-logic in the same fashion. We could also use the requirement that $\widehat{V}(\varphi) \geqslant \neg g$, which will result in non-normal Π-logics including $\neg g$ but not $\Box\neg g$.

6 Conclusion

In this paper, we investigated Π-logics extending $\mathsf{S5}\Pi$. In particular, we see that complete simple S5 algebras are semantically adequate for all normal Π-logics extending $\mathsf{S5}\Pi$, that the lattice of these normal Π-logics are isomorphic to the lattice of the open sets of the type space $\mathcal{S}$ that is homeomorphic to the disjoint union of two copies of the one-point compactification of $\mathbb{N}$, that they can have arbitrarily high Turing-degree, and that they do not exhaust all the Π-logics extending $\mathsf{S5\Pi}$ as there are non-normal ones.

A major unresolved problem though, is the characterization of all Π-logics, instead of only the normal ones, extending $\mathsf{S5\Pi}$. We conjecture that a similar strategy can be used, though we need to be more careful about the choice of types. With an informative characterization, we may also be able to find a simple syntactical condition for a Π-logic extending $\mathsf{S5\Pi}$ to be normal and describe how the normal ones are distributed in the lattice of all Π-logics extending $\mathsf{S5\Pi}$.

Finally, we ask whether there is a way to prove all the results, especially the completeness of $\mathsf{S5\Pi}$ and stronger Π-logics, without using explicitly quantifier elimination. That this is important is because for many modal logics L, there is little hope that one can obtain a manageable quantifier elimination for $\mathsf{L\Pi}$. Hence, we need some technique that can be more easily generalized.

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