Symmetric Relations

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ABSTRACT: There are two ways to characterize symmetric relations. One is intensional: necessarily, *Rxy* iff *Ryx*. In some discussions of relations, however, what is important is whether or not a relation gives rise to the same completion of a given type (fact, state of affairs, or proposition) for each of its possible applications to some fixed relata. Kit Fine calls relations that do 'strictly symmetric'. Is there is a difference between the notions of necessary and strict symmetry that would prevent them from being used interchangeably in such discussions? I show that there is. While the notions coincide assuming an intensional account of relations and their completions, according to which relations/completions are identical if they are necessarily coinstantiated/equivalent, they come apart assuming a hyperintensional account, which individuates relations and completions more finely on the basis of relations' real definitions. I establish this by identifying two definable relations, each of which is necessarily symmetric but nonetheless results in distinct facts when it applies to the same objects in opposite orders. In each case, I argue that these facts are distinct because they have different grounds.

1. Introduction

One of Kit Fine's (2000) arguments against positionalism — the view that the application of a relation to its relata consists in each relatum being assigned to an argument position in the relation — is that it cannot properly handle symmetric relations (2000: 17-18). Following Fine (2000: 4-5), I characterize a *completion* of a relation as a result of that relation applying to (or being saturated by) an appropriate number of objects. Completions potentially include facts, states of affairs, and propositions. Under the assumption that each n-ary relation has n argument positions, each taking at most one argument in a given completion, positionalism yields the wrong number of completions of any symmetric relation by some fixed relata. Consider the binary symmetric relation *being adjacent to*. (A binary relation is a relation which can be completed by at most two objects.) According to positionalism, as just characterized, this relation, as a binary relation, has two argument positions, α and β. Now consider a completion of it by two objects, like Goethe and Charlotte Buff. Given that there are two possible assignments of Goethe and Buff to α and β, there are two ways for them to complete this relation. But, intuitively, there is only one. In Fine's words,

> It seems clear that there are ... relations that are strictly symmetric. For example, the state a's being adjacent to b is surely the same as the state of b's being adjacent to a; and so the ... relation of adjacency is strictly symmetric. (2000: 17)[1]

The notion of symmetry that is operative in Fine's argument is not the typical notion. That notion is intensional, ultimately characterized in terms of relations' extensions across possible worlds.

> **Necessary Symmetry.** A relation R is *necessarily symmetric* $=_{df}$ necessarily, for any x and y, Rxy iff Ryx.[2]

Fine's notion of strict symmetry is different. A binary relation is strictly symmetric "if its completion by the objects a and b is always the same regardless of the argument- places to which they are assigned" (Fine 2000: 17). Strict symmetry can be defined without reference to argument positions, which are features peculiar to only some views about relations, as follows.

> **Strict Symmetry.** A relation R is *strictly symmetric* $=_{df}$ necessarily, for any completions c and c' of a given sort (fact, state of affairs, or proposition) of R by the same objects, $c = c'$.

As Fine puts it, "strict symmetry requires identity of content and not merely identity of extension" (Fine 2000: 17).[3]

[1]Strictly speaking, Fine's argument applies only to *absolute* positionalist views, according to which whether an object is assigned to any position of a relation in a given completion is something which concerns only that object and that position. Such views are defended by Orilia (2011 and 2014), Gilmore (2013 and 2014), and Dixon (2018). It does not apply to Donnelly's (2016) *relative* positionalism, according to which the argument positions of a relation are construed as unary properties that the relation's relata instantiate relative to one another. Donnelly's view results in the correct number of completions of symmetric relations. Absolute positionalist views which allow more than one relatum to be assigned to certain argument positions of certain relations, like those defended by Orilia (2011 and 2014) and Dixon (2018), cannot avoid a generalization of Fine's argument that applies to n-ary relations for any $n \geq 2$, since implementing this strategy does not allow the absolute positionalist to accommodate n-ary relations for $n > 2$ with certain cyclic symmetries (see Fine 2000: 17–18, fn. 10 and Donnelly 2016: sec. 3).

[2]I focus my attention for the time being on binary relations, since the relations that constitute the heart of my case are binary. Henceforth, I will not explicitly qualify binary relations as such until I take the first steps towards generalizing my main claims and arguments to n-ary relations for all $n \geq 2$ in section 4.

[3]I presuppose a structured Russellian view of propositions, according to which each singular proposition has a property or relation and its relatum or relata as constituents, arranged in a certain way. On Fregean views,

Are these two notions of symmetry interchangeable? Could Fine have relied, in his argument against positionalism, on the notion of necessary symmetry instead of that of strict symmetry? The answer depends, I will show, on how finely one individuates relations and their completions. If one adopts an intensional account, according to which relations/completions are identical if they are necessarily coinstantiated/equivalent, the two notions of symmetry are coextensional. But if one adopts a hyperintensional account, according to which relations are identical just in case they have the same real definitions (as in Rosen 2015), then strict symmetry is a more discerning notion than necessary symmetry. And given the recent hyperintensional turn from modal to post-modal metaphysics (see Nolan 2014 and Sider 2020: 1–3), the importance of distinguishing between these notions of symmetry has grown considerably. Metaphysicians today are much more likely to individuate entities of various sorts, including properties and relations, hyperintensionally. The various properties these entities may or may not have need to be sensitive to their fine-grained natures. Given the results below, it becomes clear that today's metaphysician will not want to conclude, merely on the basis of a relation's being extensionally symmetric, that all completions of it by the same objects are identical.

I don't expect these results to astound to the reader. I expect many already to take these things to be the case. Fine's use of strict symmetry instead of necessary symmetry in his argument against positionalism is evidence that he does. The name he chose for it is further evidence. Francesco Orilia (2011: 3) and Fraser MacBride (2014: 7) also characterize symmetry strictly rather than in terms of necessity only, suggesting they too recognize a difference between the two notions. But neither Fine nor anyone else, to my knowledge, has provided a specific reason for thinking that strict symmetry is a more discerning notion than necessary symmetry in the context of a hyperintensional account of relations and their

certain propositions which are completions of the same relation by the same objects in opposite orders will be distinct for reasons that appear to have nothing to do with the symmetry of that relation. For example, on such views, the proposition that Hesperus is Hesperus is distinct from the proposition that Hesperus is Phosphorus, not because identity is not strictly symmetric (it presumably *is*), but instead because these propositions have different constituents (senses) on such accounts.

completions. I provide such a reason in what follows, by identifying two relations, each of which is necessarily symmetric but nonetheless results in distinct facts when it applies to the same objects in opposite orders. In each case, I argue that these facts are distinct because they have different grounds.

To show that strict symmetry is a more discerning notion than necessary symmetry, it is enough to establish the following two claims.

(C1) Every strictly symmetric relation is necessarily symmetric.
(C2) There are necessarily symmetric relations that are not strictly symmetric.[4]

It is a relatively straightforward matter to show that (C1) is true. In doing so I rely only on the following plausible existence condition for relational propositions and the following (schematic) identity condition for propositions in general.

> **Proposition Existence.** Necessarily, for any relation R and any x and y, the proposition that Rxy exists.
>
> **Proposition Identity.** If the proposition that p = the proposition that q, then necessarily, p iff q.

Proof. Suppose that an arbitrary relation R is strictly symmetric and consider arbitrary compossible a and b. By Proposition Existence, the proposition that Rab and the proposition that Rba exist. And since R is strictly symmetric, these propositions are identical. By Proposition Identity, necessarily, Rab iff Rba. And since a and b are arbitrary, R is necessarily symmetric. □

Note that this result does not depend on any assumptions about how finely relations and their completions are individuated.

Whether or not (C2) is true, on the other hand, depends on how fine-grained an account of relations and their completions one adopts. It is false if one adopts an intensional account, according to which relations/completions of them are identical when they are necessarily coinstantiated/equivalent.

[4] In his explication of Fine's symmetry-based argument against positionalism, MacBride says, "we arrive at a principle that plausibly governs a significant range of binary [necessarily] symmetric relations: for any such relation, there is only one completion that arises from its saturation by two objects a and b ($aRb = bRa$). Call this principle *Identity*^{sym}." (2007: 36–37) Establishing (C2), then, would suffice to confirm that *Identity*^{sym} governs only a proper subclass of necessarily symmetric binary relations.

Intensional Relation Identity. For any relations R and R', $R = R'$ iff, necessarily, for any x and y, Rxy iff $R'xy$.

Intensional Completion Identity. For any facts/states of affairs/propositions c and c', $c = c'$ iff, necessarily, c exists/obtains/is true iff c' exists/obtains/is true.

Such a view would, for example, identify the relations *being triangular and larger than* and *being trilateral and larger than*, and would identify the following completions of them.

- Alice's being triangular and larger than Bob
- Alice's being trilateral and larger than Bob

According to such a conception of relations, the two notions of symmetry coincide, since, though (C1) is true, (C2) is false. To show that every necessarily symmetric relation is strictly symmetric for each sort of completion, I appeal to the following plausible existence conditions for relational states of affairs and facts in addition to the one for propositions that I stated above.

State of Affairs Existence. Necessarily, for any relation R and any x and y, the state of affairs of x's R-ing y exists.

Fact Existence. Necessarily, for any relation R and any x and y, the fact that Rxy exists iff Rxy.

I also appeal to the following (again, plausible) truth and obtaining conditions for relational propositions and states of affairs, respectively.

Propositional Truth. Necessarily, for any relation R and any x and y, the proposition that Rxy is true iff Rxy.

State of Affairs Obtaining. Necessarily, for any relation R and any x and y, the state of affairs of x's R-ing y obtains iff Rxy.

I take all of these conditions to be acceptable both to those who individuate relations and their completions intensionally and to those who do so in a more fine-grained way.

Proof. Consider an arbitrary relation R, suppose that R is necessarily symmetric, and consider arbitrary compossible a and b.

Propositions: By Proposition Existence, the proposition that Rab exists, as does the proposition that Rba. By Propositional Truth, necessarily, each of these propositions is true iff Rab and Rba, respectively. But since R is necessarily symmetric, necessarily, Rab iff Rba.

So, necessarily, the proposition that *Rab* is true iff the proposition that *Rba* is true, and so, by Intensional Completion Identity, these propositions are identical.

States of affairs: Mutatis mutandis by State of Affairs Existence and State of Affairs Obtaining.

Facts: Suppose that *Rab*. By Fact Existence, the fact that *Rab* exists. Since R is necessarily symmetric, *Rba*. So, again by Fact Existence, the fact that *Rba* exists. Mutatis mutandis when *Rba*. So, necessarily, the fact that *Rab* exists iff the fact that *Rba* exists, and so, by Intensional Completion Identity, the fact that $Rab =$ the fact that *Rba*. □

When one individuates relations and their completions more finely than this, however, (C2) turns out to be true, and so the notions of necessary and strict symmetry come apart. In particular, there exist necessarily symmetric relations that are not strictly symmetric. I establish this in the next section by identifying a particular such relation. In the section that follows, I consider a potential concern one might have about the first relation, and show that there are other relations available to which the worry does not apply. Afterwards, I discuss a few odds and ends related to my results, including another objection and a discussion about how my results generalize to n-ary relations for any $n \geq 2$.

2. An Extensionally Symmetric Relation that Is Not Strictly Symmetric

In this section, I show that the notions of necessary and strict symmetry come apart when one individuates relations and their completions hyperintensionally. I establish this by identifying a relation that is necessarily symmetric but not strictly symmetric. Completions of it by the same two objects in opposite orders are plausibly distinct.

The hyperintensional account of relations I have in mind is a plausible one, individuating relations according to whether or not they have the same *real definitions*, and requiring of completions that they be no less fine-grained than what that would allow. (I'll make the latter idea more precise soon.) The relational component is a straightforward generalization of Gideon Rosen's real definition-based hyperintensional account of properties.

> **Hyperintensional Property Identity.** F and G are the same property iff
> (a) F and G are definable and for all Φ, $\text{Def}(F, \Phi)$ iff $\text{Def}(G, \Phi)$; or
> (b) F and G are indefinable and $\Box\forall x\,(Fx \leftrightarrow Gx)$. (Rosen 2015: 202)

A real definition, in contrast to a conceptual or lexical definition, provides an analysis of the *thing itself* (i.e., object, kind, property, relation, etc.) (*Ibid.*). In this schema, 'F' and 'G' range over one-place predicates expressing properties. 'Φ' ranges over n-place complex predicates, each expressing a structured complex composed of properties and relations, possibly some objects, but typically featuring some unfilled argument places corresponding to the unfilled argument places in F or G. (As Rosen notes (2015: 190), Φ can be understood as a composite structured Russellian propositional function.) When one of 'F', 'G', or 'Φ' occurs in name position, it abbreviates the corresponding lambda abstraction denoting the property or relation it expresses, e.g., '$\Lambda x\, Fx$', which denotes the property *being an* x_1 *such that* Fx, and '$\Lambda x_1, x_2, \ldots\ \Phi\,(x_1, x_2, \ldots)$', which denotes the property *being* $x_1, x_2, \ldots$ *such that* $\Phi\,(x_1, x_2, \ldots)$. '$\mathrm{Def}\,(F, \Phi)$' says that *being* F is defined by (or consists in, or reduces to) *being* Φ (or to be F is to be Φ).

Generalizing Rosen's account of properties to relations yields the following.

Hyperintensional Relation Identity. R and R' are the same relation iff
(a) R and R' are definable and for all Φ, $\mathrm{Def}\,(R, \Phi)$ iff $\mathrm{Def}\,(R', \Phi)$; or
(b) R and R' are indefinable and $\Box\,\forall x \forall y\,(Rxy \leftrightarrow R'xy)$.

'R' and 'R'' range over two-place predicates expressing binary relations. This account of relations distinguishes between *being triangular and larger than* and *being trilateral and larger than*, since, presumably, they have different real definitions. The former is presumably defined in terms of the property *being a side*, while the latter is presumably defined in terms of *being an angle*, viz.,

x is triangular and larger than $y =_{df}$ (i) x is triangular and (ii) x is larger than y
x is trilateral and larger than $y =_{df}$ (i) x is trilateral and (ii) x is larger than y,

where

x is triangular $=_{df}$ (i) x is polygonal and (ii) x has exactly three angles
x is trilateral $=_{df}$ (i) x is polygonal and (ii) x has exactly three sides.

Recall that the intensional account identifies these relations, since they are necessarily coinstantiated.

The hyperintensional account of completions I have in mind is one according to which, roughly speaking, completions are no less fine-grained than are the relations of which they are completions. A first pass at capturing this idea more precisely is as the claim that completions of distinct relations are always distinct.[5] But this formulation ignores certain views about the nature of relations. According to directionalism, for example, every relation applies to its relata in an order, and every (binary) relation has a converse, which (necessarily) applies to some objects in the opposite order that it applies to them whenever it applies to them.[6] Some directionalists might be inclined to distinguish between a completion of a relation, like *loving* by, say, Goethe and Buff in that order (the state of affairs of Goethe's loving Buff) and a completion of its distinct converse, *being loved by*, by them in the opposite order (the state of affairs of Buff's loving Goethe), regarding the former as having Goethe, Buff, and *loving* as constituents but the latter as having Goethe, Buff, and *being loved by* as constituents instead. Such a directionalist can endorse the idea that completions of distinct relations are always distinct. But other directionalists may prefer to identify completions like these, in this case taking each to have Goethe, Buff, and *both* of *loving* and *being loved by* as constituents. If possible, we should not adopt identity conditions of completions that would rule out views like the latter form of directionalism by fiat.

The following identity condition for completions does not do so.

> **Hyperintensional Completion Identity.** For any completions c and c' of a given sort of relations R and R', respectively, $c = c'$ only if either (a) $R = R'$ or (b) R' and R are converses of one another.

Those who distinguish a completion of a relation from every completion of its converse need only ever invoke the falsity of clause (a) to show that completions c and c' are distinct, since,

[5] This amounts to Fine's (2000: 5) principle Uniqueness: no completion is a completion of more than one relation.

[6] For statements and endorsements of directionalism, see Russell 1903: secs 94–95 and secs 218–19 and Dixon forthcoming. Directionalism is to be contrasted with *neutral* views of relations, according to which relations are not inherently directional, and that the manner in which a relation applies to its relata is not to be ultimately understood in terms of the order in which it applies to them, but in some other way. (Absolute positionalists, for example, believe that each manner in which a relation R can apply consists in a possible assignment of its relata to its argument positions.) Neutral view theorists believe either that every relation is its own (only) converse (as in Williamson 1985), or that there *is* no meaningful notion of a converse of a relation (as in Fine 2000).

on each of their views, completions of distinct relations will presumably always be distinct. But clause (b) enables one to allow for completion identities even when $R \neq R'$ if they wish, as long as R and R' are converses.

It may come as no surprise to the reader that the necessarily but not strictly symmetric relations I have in mind are definable. The hyperintensional account of relations I operate with, after all (see above), still individuates *in*definable relations intensionally, and the proof I gave at the end of section 1 can easily be modified to show that every necessarily symmetric indefinable relation is strictly symmetric on the account. One of the relations I have in mind is defined as follows.

> $R_1xy =_{df}$ (i) x is the same diameter as y, (ii) y is the same diameter as x, and (iii) either y is more massive than x or y isn't more massive than x.

or, symbolically, where $\ulcorner Dxy \urcorner$ abbreviates $\ulcorner x$ is the same diameter as $y \urcorner$ and $\ulcorner Mxy \urcorner$ abbreviates $\ulcorner x$ is more massive than $y \urcorner$,

$$R_1xy =_{df} \text{(i) } Dxy \,\&\, \text{(ii) } Dyx \,\&\, \text{(iii) } Myx \vee \neg\, Myx.$$

I first establish that R_1 is necessarily symmetric.

Proof. Consider arbitrary compossible a and b and suppose that R_1ab. It follows by the definition of R_1 that (i) a is the same diameter as b, (ii) b is the same diameter as a, and (iii) either b is more massive than a or b isn't more massive than a. So (i) b is the same diameter as a, and (ii) a is the same diameter as b. And, of course, it is logically true that either a is more massive than b or a is not more massive than b, and so it is true whenever R_1ab. So (i) b is the same diameter as a, (ii) a is the same diameter as b, and either a is more massive than b or a is not more massive than b. Hence R_1ba. And since a and b are arbitrary, necessarily, for any x and y, R_1xy iff R_1yx. □

R_1 is not, however, *strictly* symmetric. There are distinct facts which are completions of it by the same objects. This can be seen by noting that certain completions of a single type (facts, states of affairs, or propositions) of R_1 by two objects in opposite orders have different grounds. Narvi and Tarqeq (two moons of Saturn) are both 7 km in diameter, and so we get the following two facts involving R_1. (Let n be Narvi, t be Tarqeq, and $[p]$ be the fact that p.)

f_1: $[R_1nt]$

f_2: $[R_1tn]$

(i) Narvi is the same diameter as Tarqeq, and (ii) Tarqeq is the same diameter as Narvi. Also, it is a logical truth that (iii) either Tarqeq is more massive than Narvi or Tarqeq isn't more massive than Narvi. And (i) Tarqeq is the same diameter as Narvi, and (ii) Narvi is the same diameter as Tarqeq. And again, it is a logical truth that (iii) either Narvi is more massive than Tarqeq or Narvi isn't more massive than Tarqeq. So R_1nt and R_1tn, and therefore facts f_1 and f_2 exist.[7]

Due to the conjunctive nature of R_1, each of the facts f_1 and f_2 is naturally treated as a conjunctive fact of the form $[p \,\&\, q \,\&\, r]$, viz.,

f_1: $[Dnt \,\&\, Dtn \,\&\, (Mtn \vee \neg Mtn)]$

f_2: $[Dtn \,\&\, Dnt \,\&\, (Mnt \vee \neg Mtn)]$.[8]

It is typically supposed that a conjunctive fact is fully grounded in its conjuncts taken together, where, roughly, a fact (or some facts) fully ground a fact x when it provides (or they provide) a complete metaphysical explanation of x.

(&I) If p and q, then $[p \,\&\, q]$ is fully grounded in $[p], [q]$.[9]

Given (&I), each of f_1 and f_2 is fully grounded by the following facts.[10]

full grounds of f_1	full grounds of f_2
f_3: $[Dnt]$	f_4: $[Dtn]$
f_4: $[Dtn]$	f_3: $[Dnt]$
f_5: $[Mtn \vee \neg Mtn]$	f_6: $[Mnt \vee \neg Mnt]$

[7] Here and elsewhere in the following discussion, I rely on Fact Existence.

[8] Alternatively, one might take $[R_1nt]$ and $[R_1tn]$ to be *grounded* by these conjunctive facts, as in Rosen 2010: sec. 10 and 2015: sec. 6, rather than being *identical* to them. $[R_1nt]$ and $[R_1tn]$ have different grounds in either case. Indeed, each has the *same* grounds in either case, save for the relevant conjunctive fact itself.

[9] See Correia 2010: 267-68, Schnieder 2011: 449, and Fine 2012a: 58. The conditions that $p \neq [p \,\&\, q]$ and $q \neq [p \,\&\, q]$ are sometimes added. But these conditions are met in the case under consideration.

[10] Though (&I) and ($\vee$I) (see below) specify the *full* grounds of conjunctive and disjunctive facts, respectively, it suffices to show that f_1 and f_2 are distinct that they have different *partial* grounds, where something partially grounds x exactly when it is among some things that fully ground x. Henceforth I will drop any explicit qualification when the grounds I am discussing are partial. For more discussion of the distinction between full and partial grounding, including the definition of the latter in terms of the former that I have invoked, see Rosen 2010: 115 and Fine 2012a: 50. It has been questioned whether every partial ground must be among some full grounds (see, for example, Leuenberger 2020 and Trodgdon and Witmer 2021). Fortunately, I need rely only on the less controversial assumption that if some things fully ground x then every individual thing among those things partially grounds x.

It is true that f_1 and f_2 have *some* grounds in common, viz., f_3 and f_4. But all that I must show to establish that f_1 and f_2 are distinct is that they have *some* different grounds. And to show this, it suffices to establish just that one has a ground that the other doesn't. And each has a ground the other doesn't, since f_5 and f_6 are presumably distinct facts.

The distinctness of f_5 and f_6 can be corroborated by attending to their grounds. While these facts are facts concerning only logical truths, there is nevertheless a fact of the matter about the relative masses of Narvi and Tarqeq. Currently we don't have precise measurements of their masses. And if they turn out to be exactly equal in mass, then the example may need to be traded with another. But it is at least possible for two things with the same diameter to have different masses, and so there are possible worlds where one of Narvi and Tarqeq is more massive than the other, even if this world is not one of them. For simplicity, let's pretend that this world is one in which Narvi is more massive than Tarqeq. Then f_5 and f_6 have different grounds, and so must be distinct.

To see why f_5 and f_6 have different grounds, note that it is typically supposed that a disjunction is fully grounded in each of its true disjuncts.

($\vee$I) If p, then $[p \vee q]$ is fully grounded in $[p]$.
If q, then $[p \vee q]$ is fully grounded in $[q]$.[11]

Given ($\vee$I), f_5 is fully grounded in $[\neg\, Mtn]$, while f_6 is fully grounded in $[Mnt]$. And *these* facts are clearly distinct. $[\neg\, Mtn]$, after all, can exist whether or not $[Mnt]$ exists. (Narvi might be greater in mass than Tarqeq, making 'Mnt' true; or they might have the same mass, making 'Mnt' false.) This is so even if these two facts have the same grounds, which they might; it is plausible that each is fully grounded in $[m(n) = m_n]$, $[m(t) = m_t]$, and $[m_n > m_t]$, taken together, where $\ulcorner m(x) \urcorner$ abbreviates $\ulcorner$the mass of $x\urcorner$. Figure 1 depicts the facts and grounding connections I have identified so far.

f_5 and f_6, then, are distinct facts, and so each of f_1 and f_2 has a ground that the other

[11] See Correia 2010: 267-68, Schnieder 2011: 449, and Fine 2012a: 58. As in the case of (&I), the conditions that $p \neq [p \vee q]$ in the first case and that $q \neq [p \vee q]$ in the second are sometimes added. But as before, these conditions are met in the case under consideration. It should also be acknowledged that ($\vee$I), as stated above, has been called into question by Jon Litland (2015).

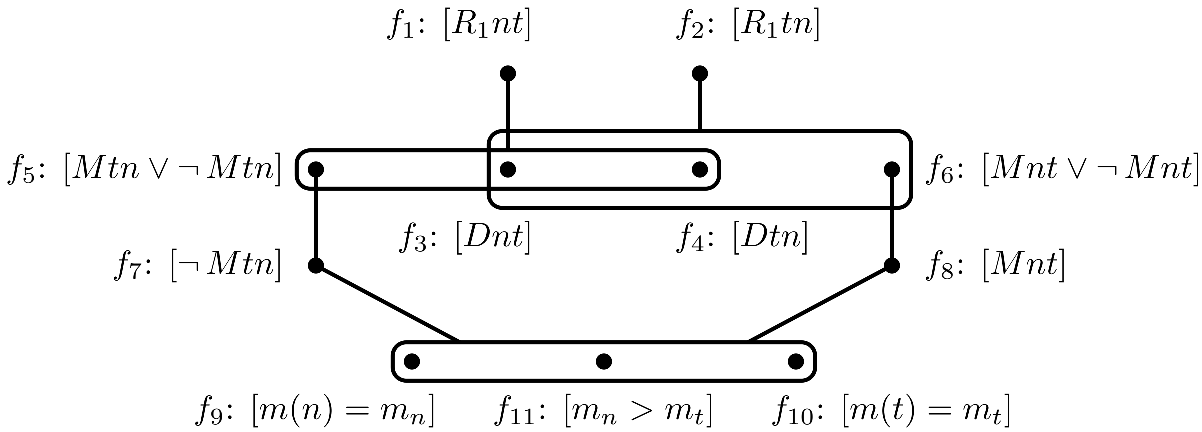

Figure 1: The grounds of f_1 and f_2. A solid line running in a downward direction from a node x to another node y, which may run through one or more other nodes, indicates that x is fully grounded in y. A solid line connecting a node x to a solid box enclosing nodes $y_1, y_2, \ldots$ indicates that x is fully grounded in $y_1, y_2, \ldots$

lacks. As a result, f_1 and f_2 have different grounds, and therefore must be distinct. This shows that R_1 is not strictly symmetric; there are distinct facts which are completions of it by the same objects, Narvi and Tarqeq. And because R_1 is necessarily symmetric, there are necessarily symmetric relations that are not strictly symmetric. The notions of necessary symmetry and strict symmetry are, therefore, not coextensive on a plausible hyperintensional account of relations and their completions.

3. Are Logical Truths Too Cheap?

One might be concerned that R_1, as defined in the previous section, is not a genuine relation. The definition of R_1 includes a logical truth, and, one might worry, it is "tacked onto" a substantive relational claim (that involving the relation *having the same diameter as*) in the definition of R_1 in an unsatisfying kind of way. One might think that plausible existence conditions for definable relations would not include provisions to allow for logical truths (and falsities) to play a role in defining up genuine relations. Readers who are not concerned with this matter can skip ahead to section 4. For those who are, I'll begin by noting that it is easy to voice this concern, but it is much more difficult to motivate plausible existence conditions that do not allow such conditions to define up genuine relations. Moreover, it's not obvious that we'll *never* want to allow logical truths and falsities from making appearances in the definitions of definable relations.[12] Still it would be nice if my argument did not hang on this technicality. And fortunately, it does not. Other necessarily symmetric but not strictly symmetric relations can be defined that avoid appealing to logical truths in this potentially unsatisfying way.

One such relation is defined disjunctively, but includes a different sort of condition in the second clause of its disjunctive definition that *non-trivially* follows from the truth of the first conjunct of the clause.

> $R_2xy =_{df}$ (i) x is an element of y or (ii) y is an element of x and the cardinality of the union of the elements of x is greater than or equal to the cardinality of y.

Formally,

$$R_2xy =_{df} x \in y \vee (y \in x \,\&\, |\bigcup_{z \in x} z| \geq |y|).$$

($|x|$ is the cardinality, i.e., the number of elements, of x.) The domain of R_2 can be any collection of at least one non-set (such as a material object) and the sets that can be constructed from it or them.[13] Like R_1, R_2 is necessarily symmetric.

Proof. Consider arbitrary compossible a and b and suppose that R_2ab. It follows by the definition of R_2 that either (i) $a \in b$ or (ii) $b \in a$ and the cardinality of the union of the elements of a is greater than or equal to the cardinality of b. Suppose first that $a \in b$. For it to be the case that R_2ba, it must be the case that either (i) $b \in a$ or (ii) $a \in b$ and the cardinality of the union of the elements of b is greater than or equal to the cardinality of a. We have supposed that $a \in b$. And the cardinality of the union of the elements of any set x can't be less than the cardinality of any set $y \in x$, since the union of the elements of x must contain everything that y contains.[14] Since, by our supposition, b has an element, we know it is a set, and so the cardinality of the union of the elements of b is greater than or equal to the cardinality of a. It now follows by clause (ii) of the definition of R_2 that R_2ba. Now suppose that $b \in a$ and the cardinality of the union of the elements of a is greater than or equal to the cardinality of b. Since $b \in a$, it follows by clause (i) of the definition of R_2 that R_2ba. Either way, then, R_2ba. And since a and b are arbitrary, necessarily, for any x and y, R_2xy iff R_2yx. □

[12] Thanks to Eileen Nutting for a helpful discussion about these points.

[13] One can stipulate the domain to be the von Neumann set-theoretic hierarchy, and let $\varnothing$ replace any non-set in the originally envisaged domain. But I expect most readers to be committed to at least one material object, and my case is a bit easier to appreciate when it is stated in terms of a material object rather than $\varnothing$.

[14] I assume that the cardinality of any non-set is defined but is 0, since it has no elements.

But, again like R_1, R_2 is not strictly symmetric. The following facts are distinct yet are completions of R_2 by the same objects.

f_1: [{Socrates} R_2s {{Socrates}, {{Socrates}}}]
f_2: [{{Socrates}, {{Socrates}}} R_2s {Socrates}][15]

{Socrates} $\in$ {{Socrates}, {{Socrates}}}, and so, via clause (i) of the definition of R_2, {Socrates} R_2s {{Socrates}, {{Socrates}}}. And since R_2 is necessarily symmetric, {{Socrates}, {{Socrates}}} R_2s {Socrates}. So facts f_1 and f_2 exist.

R_2 is defined disjunctively, and so each of f_1 and f_2 is naturally treated as a disjunctive fact. Let s be Socrates.

f_1: [$\{s\} \in \{\{s\}, \{\{s\}\}\}$ or ($\{\{s\}, \{\{s\}\}\} \in \{s\}$ and the cardinality of the union of the elements of $\{s\}$ $\geq$ the cardinality of $\{\{s\}, \{\{s\}\}\}$)]
f_2: [$\{\{s\}, \{\{s\}\}\} \in \{s\}$ or ($\{s\} \in \{\{s\}, \{\{s\}\}\}$ and the cardinality of the union of the elements of $\{\{s\}, \{\{s\}\}\}$ $\geq$ the cardinality of $\{s\}$)]

Solving for the unions yields:

f_1: [$\{s\} \in \{\{s\}, \{\{s\}\}\}$ or ($\{\{s\}, \{\{s\}\}\} \in \{s\}$ and the cardinality of $\varnothing \geq$ the cardinality of $\{\{s\}, \{\{s\}\}\}$)]
f_2: [$\{\{s\}, \{\{s\}\}\} \in \{s\}$ or ($\{s\} \in \{\{s\}, \{\{s\}\}\}$ and the cardinality of $\{s, \{s\}\}$ $\geq$ the cardinality of $\{s\}$)]

($\vee$I), together with certain other plausible assumptions, implies that f_1 and f_2 have different grounds.

Consider first the grounds of f_1. $\{s\} \in \{\{s\}, \{\{s\}\}\}$, and so, by ($\vee$I), f_1 is grounded in

f_3: [$\{s\} \in \{\{s\}, \{\{s\}\}\}$]

But $\{\{s\}, \{\{s\}\}\} \notin \{s\}$, and so the right disjunct of f_1 is false. Thus there is no fact corresponding to its truth. Hence no such fact grounds f_1. Now consider the grounds of f_2. $\{\{s\}, \{\{s\}\}\} \notin \{s\}$, so f_2 is not grounded in [$\{\{s\}, \{\{s\}\}\} \in \{s\}$]. As I just pointed out, this fact does not exist. However, the cardinality of $\{s, \{s\}\}$ *is* greater than or equal to the cardinality of $\{s\}$. And since $\{s\} \in \{\{s\}, \{\{s\}\}\}$, the following conjunctive fact exists.

[15] By choosing these facts, I have endeavored to maximize the strength of my case while keeping it as simple as possible. Simpler completions of R_2 by the same objects, such as [Socrates R_2s {Socrates}] and [{Socrates} R_2s Socrates], don't as obviously have different grounds as do f_1 and f_2.

f_4: $[\{s\} \in \{\{s\}, \{\{s\}\}\}$ and the cardinality of $\{s, \{s\}\} \geq$ the cardinality of $\{s\}]$

(Replace 'y' in R_2's definition with $\{s\}$ and 'x' with $\{\{s\}, \{\{s\}\}\}$.) By ($\vee$I), f_4 grounds f_2.

f_4, however, does not ground f_1. By (&I), f_4 is grounded in f_3 together with

f_5: [the cardinality of $\{s, \{s\}\} \geq$ the cardinality of $\{s\}$].

Figure 1 depicts the facts and grounding connections I have identified so far. But f_5, while

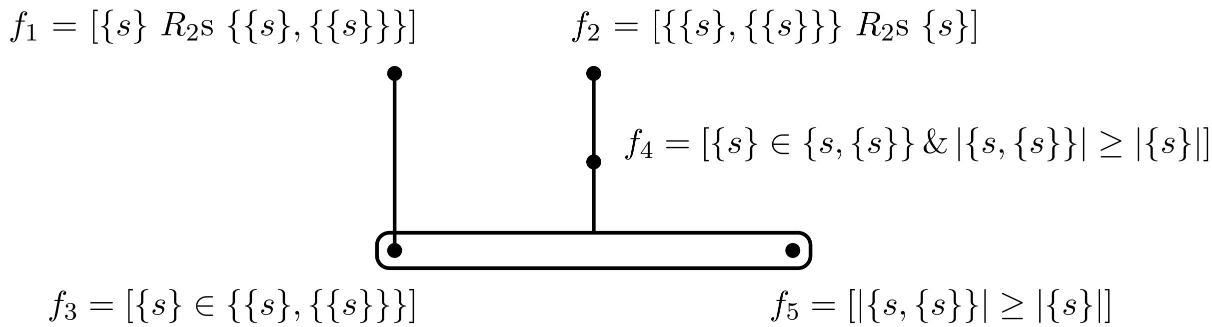

Figure 2: The grounds of f_1 and f_2 (for R_2)

a ground of f_2, does not plausibly ground f_1. f_3 is presumably f_1's sole *immediate* ground. ($\vee$I) and (&I) plausibly specify the immediate grounds of disjunctive facts and conjunctive facts, respectively. While it is not straightforward to define immediate grounding in terms of grounding (which is neutral with respect to the immediate/mediate distinction), it can be taken as primitive and grounding can be defined as its transitive closure (cf. Fine 2012a: 50–51).[16] f_3 is a full ground of f_1, and so suffices to explain it. And the only other potential immediate ground of f_1 specified by ($\vee$I), viz., $[\{\{s\}, \{\{s\}\}\} \in \{s\}$ and the cardinality of the union of elements of $\{s\}$ is $\geq$ the cardinality of $\{\{s\}, \{\{s\}\}\} \in \{s\}]$, does not exist. So if any other fact grounds f_1, it must presumably do so by transitivity via f_3. Now for a fact x to ground a fact y, x must be relevant to y (see, for example, Fine 2012a: 56). And f_5 is not altogether irrelevant to f_3, as they do concern some of the same objects, viz., $\{s\}$. But f_5

[16] It is typically supposed that grounding is transitive. See, for example, Correia 2010: 262 and 2011: 3–4, Schnieder 2011: 451, and Raven 2012: 689 and 2013: 193. The transitivity of partial grounding is entailed by the systems of Rosen (2010: 115–16) and Fine (2012a: 55–56 and 2012b: 5–6). While this supposition has been challenged (see, for example, Schaffer 2012, Tahko 2013, and Rodriguez-Pereyra 2015), this debate is not settled. For replies to these arguments, see, for example, Litland 2013, Raven 2013: sec. 5, and Makin 2019.

concerns an object, viz., $\{s, \{s\}\}$, that f_3 does not. Granted, the sets f_3 and f_5 concern can be constructed by the set-building operation with the same raw materials. (More on this in the appendix.) But it just does not seem like the relationship between the cardinalities of $\{s, \{s\}\}$ and $\{s\}$ is relevant to, and so has any role to play in a metaphysical explanation of, why $\{s\} \in \{\{s\}, \{\{s\}\}\}$. So f_5 does not ground f_1.

Some might remain unconvinced that f_5 does not ground f_3. I don't see these individuals as *actively doubting* this claim. Rather, I see them as wanting to hear more in defense of it. In the appendix, I trace down the grounds of f_5 and identify a fact that more obviously does not ground f_3, whose irrelevance to why $\{s\} \in \{\{s\}, \{\{s\}\}\}$ is more obvious.

4. Odds and Ends

In this section I discuss a few odds and ends related to my results. First, I will consider and reply to an objection one might have to both of the cases I developed in the previous two sections. Second, I will explain how one can make sense of the notion of strict symmetry, and of the distinction between it and necessary symmetry, even when one eschews completions as elements of one's ontology. Third, I will take the first steps towards generalizing my main theses and arguments to n-ary relations for all $n \geq 2$.

Concerning the first odd or end, one might be tempted to point out, in reply to the second of the two examples I laid out in the previous two sections (which is the one that avoided the concern about the potential cheapness of logical truths), that it is relatively common in certain subliteratures, such as those on truthmakers and truthmaker semantics, to eschew disjunctive facts (see, for example, Russell 1919: 39, Wittgenstein 1922, Mulligan, Simons, and Smith 1984, Armstrong 1997, and Fine 2017: 562). One might, on this basis, argue that we should simply do away with disjunctively defined relations and/or completions of them. This would preclude R_2 from being a genuine relation, since it is disjunctively defined.

The first thing that should be noted is that the first example I discussed, R_1, shows that there are potentially necessarily symmetric but not strictly symmetric relations that are not disjunctively defined. Of course R_1 has its drawbacks as well. And at the moment, I have not found a non-disjunctive relation that avoids the drawbacks of R_1. I take it, however, that a theory of relations and their completions that is neutral with respect to whether disjunctively defined relations, and completions of them, exist has a *pro tanto* theoretical advantage over one that takes a stand on the matter either way. My argument, therefore, admittedly applies most forcefully only to those theories that are preferable in this respect. But such theories will presumably be contenders for an adequate theory of relations. They certainly shouldn't be *ignored*. As a result, their mere possible truth, along with the fact that they (or, at least, hyperintensional versions of them) entail that there are necessarily symmetric relations that are not strictly symmetric, is enough to show that these two notions of symmetry cannot be used interchangeably in discussions about relations, especially in today's hyperintensional post-modal environment.

Concerning the second odd or end, one can still make sense of the notion of strict symmetry, and of the distinction between it and necessary symmetry, even if one is unhappy admitting completions into one's ontology, by making use of Fabrice Correia's notion of *factual equivalence* (2010: 256 ff. and 2016). This notion is canonically expressed by the sentential operator '$\approx$'. In Correia's words, $p \approx q$ if p and q "say the same thing" (2010: 258) in the sense that they "describe the same facts or situations, understood as worldly items, i.e., as bits of reality rather than representations of reality" (2016: 103). Importantly, these "bits of reality" needn't be understood as facts or situations, which can be cleanly individuated. Factual equivalence claims can be taken as primitive. It could be that reality is just one big undifferentiated whole from an ontological point of view. Sense could nonetheless be made of certain hyperintensional nuances in the world with the help of the notion of factual equivalence. In the case of strict symmetry, one can accommodate the relevant nuances with the following definition of the notion.

> **Strict Symmetry$^{\approx}$.** A relation R is *strictly symmetric* $=_{df}$ necessarily, for any x and y, $Rxy \approx Ryx$.

The difference between a necessarily symmetric relation and a strictly symmetric$^{\approx}$ one is that, for the former, Rxy and Ryx are merely necessarily equivalent, whereas for the latter,

Rxy and Ryx are *factually* equivalent. This characterization of strict symmetry allows one effectively to capture hyperintensional differences associated with what would be distinctions amongst certain completions of relations, without presupposing a background ontology of those completions.

Concerning the third odd or end, there presumably exist n-ary relations for $n > 2$. An example is *being between*, a ternary relation. Moreover, these relations can exhibit necessary and strict symmetry as well. And they can, like *being between*, be necessarily or strictly symmetric with respect to some, but not all, permutations of their relata.[17] Concerning this form of strict symmetry, Fine says,

> [T]he state of b's being between a and c is surely the same as the state of b's being between c and a; and so the ... relation of betweenness is strictly symmetric in its last two positions. (2000: 17)

To capture the ideas of permutation-relative necessary and strict symmetry for n-ary relations for all $n \geq 2$ (in a way that, as above, does not make reference to argument positions), let $[i_1\, i_2\, \ldots\, i_n]$ be the permutation that maps 1 to i_1, 2 to i_2, ..., and n to i_n, where $i_1, i_2, \ldots, i_n$ are pairwise distinct members of $\{1, \ldots, n\}$.

> **Permutation-Relative Necessary Symmetry.** An n-ary relation R is *necessarily symmetric* with respect to a permutation p of $\{1, \ldots, n\} =_{df}$ necessarily, for any $x_1, \ldots, x_n$, $Rx_1 \ldots x_n$ iff $Rx_{p(1)} \ldots x_{p(n)}$.[18]
>
> **Permutation-Relative Strict Symmetry.** An n-ary relation R is *strictly symmetric* with respect to a permutation p of $\{1, \ldots, n\} =_{df}$ necessarily, for any $x_1, \ldots, x_n$, if $\langle Rx_1 \ldots x_n \rangle$ exists, then $\langle Rx_1 \ldots x_n \rangle = \langle Rx_{p(1)} \ldots x_{p(n)} \rangle$ (or $Rx_1 \ldots x_n \approx Rx_{p(1)} \ldots x_{p(n)}$).

In these definitions, $p(x)$ is the result of applying the permutation p to x (so, for example, $[2\,1](1) = 2$. In the last one, $\langle \varphi \rangle$ is the appropriate completion of a single sort (fact, state

[17] Such relations are *merely partially* symmetric. Relations that are symmetric with respect to *every* premutation are *completely* symmetric. A completely *non*-symmetric relation is symmetric *only* with respect to the identity permutation (the permutation that leaves the arguments where they are). For further discussion of complete and partial (necessary) symmetry and of (necessary) non-symmetry, see Donnelly 2016: 84 ff.

[18] A conditional will not suffice for n-ary relations for $n > 2$, as it does in the case of binary relations. This is because there is an n-ary relation R for $n > 2$ (actually, there are multiple such relations) for which some permutation p of $\{1, \ldots, n\}$ is not such that $Rx_1 \ldots x_n$'s implying $Rx_{p(1)} \ldots x_{p(n)}$ implies that $Rx_{p(1)} \ldots x_{p(n)}$ implies that $Rx_1 \ldots x_n$. This is because a permutation's inverse for permutations of sets $\{1, 2, 3\}$, $\{1, 2, 3, 4\}$, ... is sometimes distinct from it. $[2\,3\,1]$'s inverse, for example, is $[3\,1\,2]$ and vice versa. (Thanks to Udayan Darji for a helpful discussion about this point.)

of affairs, or proposition) for a given instance. *Being between*, for example, is necessarily symmetric with respect to the permutation [1 3 2] because corresponding instances of

- x_1 is between x_2 and x_3
- x_1 is between x_3 and x_2, equivalently, $x_{[1\,3\,2](1)}$ is between $x_{[1\,3\,2](2)}$ and $x_{[1\,3\,2](3)}$

are necessarily equivalent. It is strictly symmetric with respect to that permutation because they presumably correspond to a single fact, state of affairs, and proposition. But it is *not* necessarily symmetric with respect to, for example, the permutation [2 1 3], since corresponding instances of

- x_1 is between x_2 and x_3
- x_2 is between x_1 and x_3, equivalently, $x_{[2\,1\,3](1)}$ is between $x_{[2\,1\,3](2)}$ and $x_{[2\,1\,3](3)}$

are not necessarily equivalent. Nor is it strictly symmetric with respect to that permutation, since they presumably correspond to distinct facts, states of affairs, or propositions. To answer the question of whether strict symmetry is a more discerning notion than necessary symmetry in its full generality, then, one should establish the following two permutation-relative versions of (C1) and (C2).

(C1′) Every relation that is strictly symmetric with respect to a permutation p of $\{1, \ldots, n\}$ is necessarily symmetric with respect to p.

(C2′) There are relations that are necessarily symmetric with respect to a permutation p of $\{1, \ldots, n\}$ that are not strictly symmetric with respect to p.

It is a straightforward matter to generalize the proof of I gave of (C1) in section 1 and relativize it to an arbitrary permutation. And the examples I gave of necessarily symmetric relations that are not strictly symmetric in sections 2 and 3, R_1 and R_2, suffice to establish (C2′) as well, since there is a single permutation with respect to which each of R_1 and R_2 is necessarily symmetric but not strictly symmetric, viz., [2 1].

5. Concluding Remarks

The notions of necessary symmetry and strict symmetry come apart, at least in the context of hyperintensional accounts which individuate relations and their completions on the basis of relations' real definitions, rather than on the basis of whether they are necessarily coinstantiated/equivalent. I showed that, irrespective of how finely one individuates relations and their completions, every strictly symmetric relation is necessarily symmetric. But I argued that, on such hyperintensional accounts, there are necessarily symmetric definable relations that are not strictly symmetric. I did so by identifying specific examples of such relations and showing that completions of each by the same relata in opposite orders have different grounds, and therefore must be distinct. To the extent that my arguments have been convincing, they show that it is important to take heed of the distinction between necessary and strict symmetry when theorizing about relations, especially given that today's post-modal metaphysician is much more likely to countenance hyperintensionally individuated relations than yesterday's modal metaphysician was.[19]

Appendix

As I noted at the end of section 3, some might remain unconvinced that f_5 ([the cardinality of $\{s, \{s\}\} \geq$ the cardinality of $\{s\}$]) does not ground f_3 ([$\{s\} \in \{\{s\}, \{\{s\}\}\}$]). Again, I don't see these individuals as *actively doubting* this claim. Rather, I see them as wanting to hear more in defense of it. In what follows, I trace down the grounds of f_5 and identify a fact that more obviously does not ground f_3, whose irrelevance to why $\{s\} \in \{\{s\}, \{\{s\}\}\}$ is more obvious. This supports my claim that f_5 does not ground f_3. This irrelevance of the fact I identify to why $\{s\} \in \{\{s\}, \{\{s\}\}\}$ sheds light on why f_5 itself is irrelevant to why $\{s\} \in \{\{s\}, \{\{s\}\}\}$. In addition, however, it constitutes a further fact, distinct from f_5, which grounds f_2 but not f_1, thus providing independent support for my ultimate claim that f_1 and f_2 have different grounds, and are therefore distinct.

f_5 is a 'greater than or equal to' fact. That is, it is a fact about something's being greater than or equal to something. More specifically, it is about the cardinality of a set

[19]Thanks are due to Cody Gilmore who helped inspire this paper, and to Martin Glazier and Eileen Nutting, for providing helpful comments on early drafts. Thanks are also due to audience members at the 2021 APA Central Division Meeting, including Ben Caplan, Joop Leo, Jon Litland, Eileen Nutting, and Erica Shumener. I also wish to thank two anonymous referess for valuable comments which helped me improve the paper greatly.

being greater than or equal to the cardinality of a(nother) set. But what are the grounds of such facts? This question can be answered by looking at the definition of $\geq$. In this context, $\geq$ is straightforwardly defined disjunctively in terms of $=$ and $>$.

$|x| \geq |y| =_{df} |x| = |y|$ or $|x| > |y|$.

And since the cardinality of $\{s, \{s\}\}$ $>$ the cardinality of $\{s\}$, ($\vee$I) implies that f_5 is fully grounded in

f_6: [the cardinality of $\{s, \{s\}\}$ $>$ the cardinality of $\{s\}$]

(It is not the case that the cardinality of $\{s, \{s\}\}$ $=$ the cardinality of $\{s\}$, and so this fact does not exist.) But what grounds f_6? What, in general, grounds the fact that a set has more members than a(nother) set?

One plausible answer to this question comes from looking at the grounds of equinumerosity facts between sets. Such equinumerosities can be defined in terms of *one-to-one correspondences*. In particular, $|x| = |y|$ can be defined as the claim that there is a one-to-one correspondence from set x to set y (and so one from y to x as well). This can be understood as a set-theoretic analog of Hume's Principle.

Hume's Principle. The number of Fs $=$ the number of Gs $=_{df}$ the Fs are equinumerous to the Gs.

The equinumerosity relation is understood to hold between the Fs and Gs iff there is a one-to-one correspondence that holds between the Fs and the Gs, where a one-to-one correspondence is a *total function* from the Fs to the Gs that is both *injective*, and *surjective*. The same would hold of two sets x and y.

One-to-One Correspondences. A total function g from x to y is *a one-to-one correspondence* ($g : x \xrightarrow[\text{onto}]{1-1} y$) $=_{df}$ both $g : x \xrightarrow{1-1} y$ and $g : x \xrightarrow{\text{onto}} y$,

where

Functions. A relation R is a *function* from a set x to a set $y =_{df}$ for any $z \in x$ and any $w \in y$ and $v \in y$, if Rzw and Rzv, then $w = v$ (i.e., R pairs no element of x with more than one element of y).

Total. A function g from x to y is *total* ($g : x \to y$) $=_{df}$ g maps every element of x to some element of y.

Injective Functions. A function g from x to y is *injective* ($g : x \xrightarrow{1-1} y$) $=_{df}$ g maps all distinct elements of x to distinct elements of y.

Surjective Functions. A function g from x to y is *surjective* ($g : x \xrightarrow{\text{onto}} y$) $=_{df}$ for every element z of y, g maps some element of x to z.

('$g : x \xrightarrow[\text{(onto)}]{(1-1)} y$' can be read, when in sentence position, as 'g is a (an injective/surjective) function from x to y'.)

With these definitions in hand, the above set-theoretic analog of Hume's Principle can be rewritten as follows.

$|x| = |y| =_{df}$ there exists a g such that $g : x \xrightarrow[\text{onto}]{1-1} y$.

The claim that $|x| > |y|$ can be similarly defined as the claim that no function from y to x is surjective, i.e.,

$|x| > |y| =_{df}$ for every g, it is not the case that $g : y \xrightarrow{\text{onto}} x$.

In simple terms, what this means is that any time one maps each member of y to a unique member of x, there will be at least one leftover member of x to which no member of y has been mapped.

I'll now trace down the grounds of f_6, with $>$ so understood, to confirm that it has a ground that is irrelevant to why $\{s\} \in \{\{s\}, \{\{s\}\}\}$, and so does not ground $[\{s\} \in \{\{s\}, \{\{s\}\}\}]$, or therefore f_1. Gideon Rosen (2010: 123) and Robert Schwartzkopff (2011: 362) propose that equinumerousity facts are grounded in appropriate facts about one-to-one correspondences. A set-theoretic version of their principle follows.

The Schwartzkopff-Rosen Principle. If $|x| = |y|$, then $[|x| = |y|]$ is fully grounded in [there exists a g such that $g : x \xrightarrow[\text{onto}]{1-1} y$].[20]

Using the same basic approach, 'greater than' facts would be grounded by appropriate true universal generalizations about functions, viz.,

The Greater-Than Principle. If $|x| > |y|$, then $[|x| > |y|]$ is fully grounded in [for every g, it is not the case that $g : y \xrightarrow{\text{onto}} x$].[21]

[20]See Donaldson 2017 for an in-depth discussion of this principle.

[21]My conclusions in this section do not depend on $[|x| > |y|]$ being grounded in [for every g, it is not the case that $g : y \xrightarrow{\text{onto}} x$] rather than these facts being identified. These assumptions yield the same grounds of f_5, and equally strong cases that f_5 has a ground that f_3 lacks. See fn. 8.

f_6, then, would be grounded by

f_7: [for every g, it is not the case that $g : \{s\} \xrightarrow{\text{onto}} \{s, \{s\}\}$].

f_7 is a universally quantified fact, and such facts are typically taken to be grounded in their instances.

($\forall$I) If $\varphi(a)$, then $[\forall x\, \varphi(x)]$ is (partially) grounded in $[\varphi(a)]$.[22]

f_7 will be grounded in as many facts as there are functions on sets in the domain of R_2 that meet the condition it specifies. There are two (total) functions from $\{s\}$ to $\{s, \{s\}\}$ that fail to be surjective.

$g_1 : \{s\} \rightarrow \{s, \{s\}\}$ where $g_1(s) = s$
$g_2 : \{s\} \rightarrow \{s, \{s\}\}$ where $g_2(s) = \{s\}$

(In words, g_1 is the function that maps the sole member of $\{s\}$ to the member s of $\{s, \{s\}\}$, while g_2 is the function that maps the sole member of $\{s\}$ to the member $\{s\}$ of $\{s, \{s\}\}$.) f_7, therefore, would be grounded by the following facts.

f_8: [it is not the case that $g_1 : \{s\} \xrightarrow{\text{onto}} \{s, \{s\}\}$]
f_9: [it is not the case that $g_2 : \{s\} \xrightarrow{\text{onto}} \{s, \{s\}\}$]

What are the grounds of f_8 and f_9? It is tempting to explain each by appealing to the fact that, in plain English, $\{s, \{s\}\}$ has more elements than $\{s\}$. But remember that this is the fact whose grounds we are currently engaged in delineating, viz., [the cardinality of $\{s, \{s\}\} >$ the cardinality of $\{s\}$], i.e., f_6. But there are facts which effectively specify the exact number of elements of each of the sets which f_8 and f_9 concern which involve only the ideological resources of first-order logic with identity, and they plausibly (at least partially) ground f_8 and f_9. We need only look at these grounds of either f_8 and f_9 to eventually find a ground of f_6 that is not a ground of f_3. In the case of f_8, these facts are:

f_{10}: $[\exists x\, (x \in \{s\} \,\&\, \forall y\, (y \in \{s\} \rightarrow y = x))]$

[22] This principle is from Schnieder 2011: 460–61. See Fine 2012b: 59–62 for difficulties with, and a solution to, the formulation of a principle giving the full grounds of universal quantifications. Fortunately, I only need to rely on the above principle, which gives only some of the partial grounds of such facts, and which does not face this issue.

f_{11}: $[\exists x\, \exists y\, (x \in \{s, \{s\}\} \,\&\, y \in \{s, \{s\}\} \,\&\, x \neq y \,\&$
$\forall z\, (z \in \{s, \{s\}\} \to (z = x \vee z = y)))]$

f_{10} and f_{11} are existentially quantified facts, and such facts are typically regarded as being fully grounded in their instances.

(∃I) If $\varphi(a)$, then $[\exists x\, \varphi(x)]$ is fully grounded in $[\varphi(a)]$.[23]

So f_{10} and f_{11} are fully grounded in the relevant instances, i.e.,

f_{12}: $[s \in \{s\} \,\&\, \forall y\, (y \in \{s\} \to y = s)]$
f_{13}: $[s \in \{s, \{s\}\} \,\&\, \{s\} \in \{s, \{s\}\} \,\&\, s \neq \{s\} \,\&$
$\forall z\, (z \in \{s, \{s\}\} \to (z = s \vee z = \{s\}))]$

Subsequently applying (&I) to f_{12} and f_{13} yields the following more ultimate grounds of these facts.

f_{14}: $[s \in \{s\}]$
f_{15}: $[\forall y\, (y \in \{s\} \to y = s)]$
f_{16}: $[s \in \{s, \{s\}\}]$
f_{17}: $[\{s\} \in \{s, \{s\}\}]$
f_{18}: $[s \neq \{s\}]$
f_{19}: $[\forall z\, (z \in \{s, \{s\}\} \to (z = s \vee z = \{s\}))]$

Things have become rather complicated, so, before proceeding further, I'll provide an updated diagram of the grounds of f_1 and f_2. See figure 3 below. For simplicity, I depict only f_{14}, f_{16}, and f_{17} from the list above. The universally quantified facts f_{15} and f_{19} will be grounded in their non-vacuous instances.[24] But we can leave them aside, since some of the facts in the list above concerning only particulars do not appear to ground $[\{s\} \in \{\{s\}, \{\{s\}\}\}]$, despite the fact that they concern Socrates or sets constructed from him. Consider f_{16}. It would be strange if part of a metaphysical explanation for why

[23] This principle can be found as stated in Correia (2011: 5) and Schnieder (2011: 460). Fine (2012b: 59–60) provides a reason to add in the condition that a exists as well, where this existence predicate is primitive and not defined as it usually is in terms of existential quantification and identity. But this does not affect the intended result of my application of this condition, and so I ignore it for the sake of simplicity.

[24] Entities that vacuously satisfy the conditions specified in the universally quantified facts f_{15} and f_{19} will include members of the domain of R_2 that aren't members of $\{s\}$ and $\{s, \{s\}\}$, respectively, and for that reason surely will themselves be or have grounds that are irrelevant to the fact that $\{s\} \in \{\{s\}, \{\{s\}\}\}$. For simplicity, and because there is precedent to include in at least some universally quantified facts' grounds only their non-vacuous instances (see, e.g., Skiles 2015: 731), I would consider only the conditions' non-vacuous instances.

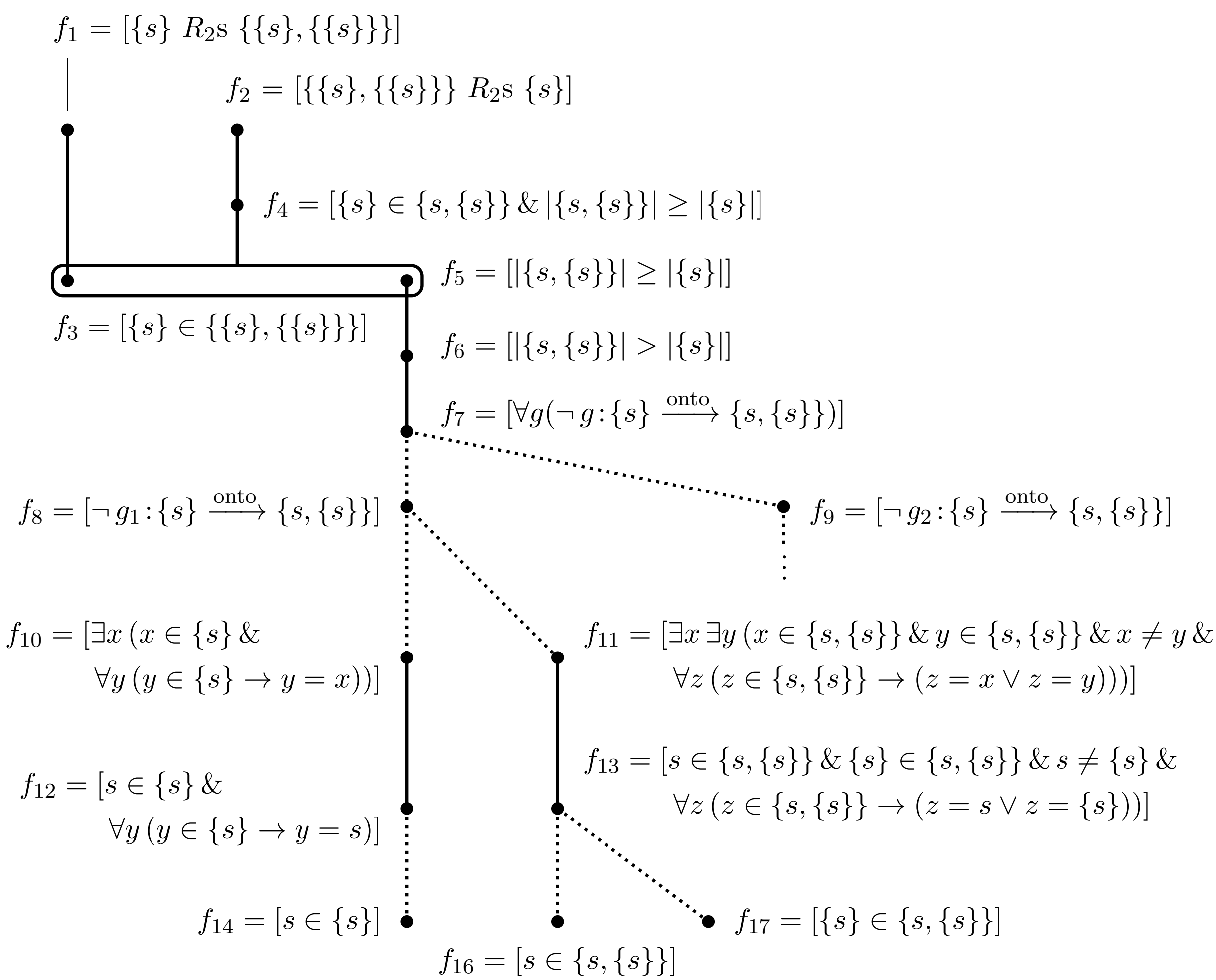

Figure 3: Grounds of f_5 (for R_2). A dotted line running in a downward direction from a node x to another node y, which may run through one or more other nodes, indicates that x is partially but not fully grounded in y.

$\{s\} \in \{\{s\},\{\{s\}\}\}$ (f_3) is that $s \in \{s,\{s\}\}$. The claims, after all, concern altogether different objects. Granted, they are or are constructed from the same object, viz. Socrates, via the set-forming operation. But neither fact seems to be a suitable ground of the other.

It is important not to confuse facts like f_3 and f_{16} with the objects they concern. The sets they concern are *constructible*, in the sense that they can be constructed out of more basic elements of the ontology (as in Fine 1991: sec. 2). Constructible objects can plausibly be taken to be grounded in the things from which they are constructed.[25] If this is so, then each

[25]Fine (1991) does not say as much, though he says that an explanatory relationship of the same directionality holds of our *reasons for admitting* constructible entities into our ontology: "Some of the objects of [a constructional] ontology are accepted (i.e., included within the ontology) on the grounds that they are

of the sets facts f_3 and f_{16} concern (viz., $\{s\}$ and $\{\{s\},\{\{s\}\}\}$ in the case of f_3 and $\{s,\{s\}\}$ in the case of f_{16}) will be ultimately grounded by Socrates, and perhaps also the set-forming operation. These sets may even lack any other grounds. But this does not undermine the idea that f_3 and f_{16} have different grounds. The sets f_3 and f_{16} concern are not f_3 and f_{16} *themselves*. f_3 and f_{16} are rather facts about membership claims holding amongst these objects.

It is plausible that constructible sets have the members they have because they are constructed in the way they are. On this view, $[s \in \{s,\{s\}\}]$ would be grounded in the fact that the set-forming operation applies to Socrates in the final step of the construction of $\{s,\{s\}\}$, and $[\{s\} \in \{\{s\},\{\{s\}\}\}]$ would be grounded in the fact that the operation applies to $\{s\}$ in the final step of the construction of $\{\{s\},\{\{s\}\}\}$. The following diagram depicts the constructions of $\{s,\{s\}\}$ and $\{\{s\},\{\{s\}\}\}$. See figure 4 below. Note that neither $\{s,\{s\}\}$ nor

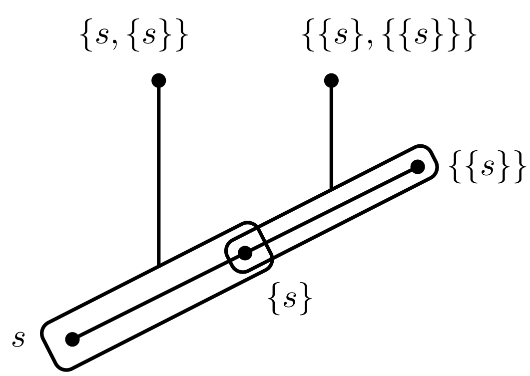

Figure 4: The construction of $\{s,\{s\}\}$ and $\{\{s\},\{\{s\}\}\}$. A solid line running in a downward direction from a node x to another node y, indicates that x is constructed by a single application of the set-forming operation to y. A solid line connecting a node x to a solid box enclosing nodes $y_1, y_2, \ldots$ indicates that x is constructed by a single application of the set-forming operation to $y_1, y_2, \ldots$

$\{\{s\},\{\{s\}\}\}$ is involved in the construction of the other — not in the final step or in any other step. As a result, these considerations do not support the idea that one of $[s \in \{s,\{s\}\}]$ and $[\{s\} \in \{\{s\},\{\{s\}\}\}]$ grounds the other, and, indeed, speak against it. It seems reasonable to

constructed from other objects within the ontology". I don't take Fine's claim to be the same as or support my claim about the grounds of such objects. But the adoption of both claims *would* yield a satisfyingly harmonious picture.

expect the membership of a constructible set x to ground the membership of a constructible set y *only if* the set-forming operation is applied to x to yield y, either immediately or mediately. And this is not the case for $\{s, \{s\}\}$ and $\{\{s\}, \{\{s\}\}\}$.[26] In addition, no other sort of reason to think that either of $[s \in \{s, \{s\}\}]$ and $[\{s\} \in \{\{s\}, \{\{s\}\}\}]$ grounds the other is forthcoming. I conclude that f_6, and therefore f_5, has a ground, viz., $[s \in \{s, \{s\}\}]$, that f_3 lacks. This may explain why f_5 is irrelevant to why $\{s\} \in \{\{s\}, \{\{s\}\}\}$, and therefore why f_5 does not ground f_3. But it also constitutes independent support for the claims that f_1 and f_2 have different grounds, and are therefore distinct.

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[26] f_{16} ($= [s \in \{s, \{s\}\}]$) might be foundational (i.e., it might have no grounds). But even if it is not, it is presumably ultimately grounded in facts which play no role in providing a metaphysical explanation of $[\{s\} \in \{\{s\}, \{\{s\}\}\}]$ (f_3). This would mean that f_3, and so f_1 on the one hand and f_{16}, and so f_4, f_3, and f_2 on the other hand are, at least partly, ultimately grounded in different portions of reality. This is helpful to note because it provides an explanation for why some of the non-fundamental facts higher up in the grounding chain ascending from f_{16}, such as f_5, fail to ground f_3. It is this different portion of reality which grounds f_{16} that is responsible, for example, for why f_5 does not fully ground f_1. (By saying that f_1 and f_2 are ultimately grounded in different facts, I mean to remain neutral about whether metaphysical foundationalism, understood as the claim that every non-fundamental fact is fully grounded in some fundamental facts or others (see Dixon 2016: 446 and Rabin and Rabern 2016: 363–64), is true. I mean that either (i) any collections of fundamental facts Γ and Δ, the former of which fully ground f_1 and the latter of which fully ground f_2, are distinct, or, alternatively, (ii) if one or both of f_1 and f_2 fail to be fully grounded in any fundamental facts, that there is some infinite descending chain of grounds, each member of which (at least partially) grounds one of f_1 and f_2 but not the other.)

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