Pluralist-Monism. Derived Category Theory as the Grammar of n-Awareness

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Summary

In this paper, we develop a mathematical model of awareness based on the idea of plurality. Instead of positing a singular principle, telos, or essence as *noumenon*, we model it as plurality accessible through multiple forms of awareness ("n-awareness"). In contrast to many other approaches, our model is committed to pluralist thinking. The noumenon is plural, and reality is neither reducible nor irreducible. Nothing dies out in meaning making. We begin by mathematizing the concept of awareness by appealing to the mathematical formalism of higher category theory. The beauty of higher category theory lies in its universality. Pluralism is categorical. In particular, we model awareness using the theories of derived categories and $(\infty, 1)$-topoi which will give rise to our meta-language. We then posit a "grammar" ("n-declension") which could express n-awareness, accompanied by a new temporal ontology ("n-time"). Our framework allows us to revisit old problems in the philosophy of time: how is change possible and what do we mean by simultaneity and coincidence? Another question which could be re-conceptualized in our model is one of soteriology related to this pluralism: what is a self in this context? A new model of "personal identity over time" is thus introduced.

Keywords: Awareness, mathematical model, derived categories, $(\infty, 1)$-topoi, higher category theory, homset, n-morphisms, weak equivalence, homotopy theory, n-declension, n-time, simultaneity, selves

1 Introduction

1.1 Pluralist-Monism and n-awareness

Among contemporary substance metaphysicians, the dominant view is *monism*, that is, the belief that there exists only one kind of thing in the universe (one kind of "stuff"). Monism comes in different guises. The most widespread doctrine is physicalism [1], which regards everything that exists as physical being. Notable alternatives are, for example, the "dual-aspect" monism of Baruch Spinoza [2] or the idealist monisms of Advaita Vedanta [3].

Monism could be differentiated from *pluralism*. The most well-known example being dualism, for example the belief in physical and non-physical ("mental") substances, often attributed to René Descartes [4]. Dualism lost much of its proponents in the course of the last decades. But also other, perhaps more exotic forms of pluralism exist. For example, the "three-world" view of Karl Popper [5] or the more radical pluralism of Bruno Latour [6].

However, there are good reasons to find monisms (and here in particular physicalism) as well as pluralism (and here in particular dualism) lacking: the former's seeming inability to account for awareness and the latter's seeming inability to account for the relation between different kinds of things, in particular the relation between awareness and the other substance(s). Terminologically consistent with traditional (critical) metaphysics, we refer to the being that is external to awareness as "noumenon" and adopt the quasi-Kantian view according to which a noumenon, while represented in awareness, exceeds any kind of "object-knowledge" acquirable via the senses but still needs to be posited as a "limiting concept" [7](A253/B310). But rather than playing the "metaphysical game" of proposing or defending a particular metaphysical doctrine that makes statements about the nature of this noumenoun, we instead propose a mathematical model to represent awareness and the relations between different forms of awareness. By this we hope to arrive at some interesting conclusions that possibly relate back to the noumenon. We will find that our model requires the noumenon to ground *pluralism* in awareness: While we stay silent on the ontology of this noumenon, it needs to be such that it affords an infinite variety of non-equivalent representations. Hence we refer to it as "pluralist-monism". This has consequences for an understanding of ourselves: who we are and what time is (for us).

More specifically, we take "1-awareness" to represent what one would simply call aware-

ness. 1-awareness is, for example, the ability to concentrate on an exam or a zoom call, while being implicitly directed to the environment and to future events. We need not stop at 1-awareness. If our mathematical approach is correct, it is natural to assume "higher forms" of awareness. We refer to "2-awareness" as the ability to simultaneously sustain any combination of 1-awarenesses, such as in the following example: Imagine you are having lunch in your home on a Saturday at noon in the year 2020 while simultaneously you are a Cambridge Apostle having a discussion in the Moral Sciences Club on a Saturday evening in 1888. Continuing, 3-awareness would be the ability to sustain any combination of 2-awarenesses. For example, simulatenously being aware of simulatenously being aware of having lunch in your home on a Saturday at noon localized in the year 2020 while being a Cambridge Apostle having a discussion in the Moral Sciences Club on a Saturday evening in 1884, and of giving a thesis defense on perfectoid spaces on a Friday at 10am in 2024, while being at your desk and finishing the first chapter of said thesis later this evening... And so on, up to "n-awareness".

How could such awareness be modeled mathematically, and given that it could, what language would be appropriate to express it? We use derived category theory and the construction of $(\infty, 1)$-topoi to represent such a pluralist-monism, which shows itself in a twofold way in our model: first, as the morphisms of an n-category; second, as a linguistic model of n-declension for n-time. We focus on relations between multiple awarenesses that give rise to the notion of "time". How to regard subjective experience against (the ontology of) time is one of the most puzzling open questions in philosophy. And any mathematical treatment of awareness should be able to shed some light on its relation to time. We propose a mathematical model of awareness, together with a grammar for a rudimentary language that helps to express temporal experiences. We then have established the background to revisit some open problems in the philosophy of time and sketch a way to resolve them within the proposed model.

An important consequence pertains to the problematic notion of personal identity over time. Rather than thinking about "selves" as enduring entities wholly present at every moment in time (i.e. like endurantists would), but also unlike the idea that we have (more or less well-defined) "temporal parts" (i.e. like perdurantists would), we present a structural "bundle-like" theory of the self of *weakly equivalent representations* (across times). This raises some questions regarding the notion of simultaneity and coincidence which shall also be tentatively answered.

1.2 Organization of the paper

The outline of the paper is as follows: In section 2, we model awareness by higher category theory (the "category of n-awareness", starting with $n = 1$). The objects of this category are representations of a noumenon; morphisms between such objects are maps between such representations.

We then distinguish "higher" categories, based on the category of 1-awareness. The set $Hom(X, Y)$ is the set of all morphisms between objects X and Y in a category. We would further want $Hom(X, Y)$ to be more than a set. We would like $Hom(X, Y)$ to be a topological space of morphisms from X to Y in order to give us some more structure to work with.

In line with pluralist ideas, we believe that different objects — through morphisms — affect each other ("they interact"). On a higher level ($k > 1$), these morphisms should be invertible, resulting in what could be called a "participatory universe" [16] on the level of k-awareness.

Invertible morphisms imply a "mirroring" of representations in all other representations it relates to – similar in spirit to the monadology of Leibniz [8]. This gives our model a notion of "self-reflexivity" on the level of the next higher category. Since $k - 1$-categories are contained as objects in a k-category, the property of self-reflexivity is "lifted" to the next higher level of awareness.

However, having invertible morphisms implies strict structural equivalence. But this we feel cannot be quite right as this would destroy any plurality. The philosophical problem underlying this dilemma is the following: If the universe evolves toward a system of equivalent representations, how to prevent it eventually becoming "totalitarian" (at least from a structural perspective), neglecting individuality?

In order to guarantee both self-reflexivity and pluralism, we work in an enriched category called a "derived category" [9], which contains a model structure that allows for a new notion of equivalence, called a "weak equivalence". We say that two objects are weakly equivalent if one can be "continuously deformed" into the other; this being a notion of homotopy theory, a branch of mathematics concerned with various deformation equivalences. Having merely *weak* equivalences means that having invertible morphisms between objects makes them the same only "up to" [31] homotopy equivalence (axioma-

tized in derived category theory[1]). Having invertible morphisms as weak equivalences gives a sense of pluralist-monism: Objects interact with each other and are weakly equivalent representations, but still possess an irreducible type of individuality. Simply stated, *weak equivalence guarantees plurality.* We conclude by generalizing our setup to $(\infty, 1)$-topoi to look at objects with higher homotopy.

In the next section 3, we then focus on some of the potential consequences of our mathematical model. We first sketch the formulation of a new grammar of n-declension which follows from our mathematical model of n-awareness and its concomitant extension to n-time. The grammar does not merely illustrate (or describe) the structure of awareness. By having a linguistic system to actually express experiences, this tells us much about the way how we concretely engage with the world (as individuals) – analogous to the idea of a language game [12], but updated to a "language scheme", inspired by the mathematical model proposed.

Further implications are discussed with respect to traditional problems of temporality. *Time* in our framework is modeled as nested hierarchies of "k-times" ($k \leq n$): 1-awareness implies a notion of 1-time, 2-awareness implies a notion of 2-time, ... , and n-awareness implies a notion of n-time. On our reading, temporality is just a consequence of a structure of representations. It is at most an ordering property. This is not an altogether new idea [10], but our mathematical model bears some natural consequences that haven't, to our knowledge, been developed so far in the literature: because of our structural definition of n-time, there is no problem of temporal coincidences. A 2-awareness contains its 1-awarenesses, which are "simultaneously present" in 2-awareness. Events can be simultaneous to each other indeed because simultaneity is a homotopy equivalence.

Finally, we turn to a question deeply related to all soteriologies across all religions: what is a "self"? Without a clear notion of self, it becomes meaningless to ask any "mortal questions" [11]. *Prima facie*, our structure sustains such a multiplicity without assuming personal identity over time. We replace the idea of personal identity over time with an updated form of the "bundle theory of self"[2] [13], based on a pluralist-monism and using

[1]Technically, we represent the noumenon as an abelian group [14] and form a complex [15] of these groups, which is informally a sequence of compositions. Then we "chain" the complexes together using a chain map. These complexes are the objects in the derived category. Thus we have upgraded the representation of the noumenon from an object in Cat to an object in the derived category.

[2]The "bundle" refers to a collection of (weakly equivalent) awareness-objects which are related via

the language of homotopy theory.

The notion of "self" is replaced with $Hom(X, Y)$. But since this is assumed to be an *enriched* $Hom(X, Y)$ – the topologically structured collection of all morphisms from X to Y (that are only *weakly equivalent*) – it is by definition a plurality. There is no conception of "enduring over time", as if time were somehow exterior to experience; as if time were constantly present allowing us to move through it (however *that* happens); or as if time were a well-ordering or counter. By contrast, we claim that time is a homotopic concept, that there is simultaneity only up to homotopy; and that selves exists only up to homotopy. The $(\infty, 1)$-topoi is the space where can do homotopy theory. It is therefore our meta-language and the space of all possible awareness.

2 Model

2.1 Basics of higher category theory

A mathematical model should express this in a minimal and conceptually coherent way. For this, we appeal to category theory. (Higher) Category theory, while seemingly abstract, offers a flexible framework in which to define "relations", arguably the basic structural property of awareness [17, 18, 19, 20]. This is why we wish to model n-awareness, not with respect to a substantial (e.g. temporal) ontology, but with respect to structure – as if the structure alone provided the ontology. We will use a specific approach, called derived category theory and the theory of $(\infty, 1)$-topoi, to relate this or our notion of a pluralistic monism.

In mathematics, category theory formalizes the notion of "structure" in a very general way. A category is defined by its *objects*, A, B, and *morphisms* (structure-preserving maps) between those objects, $A \to B$, satisfying certain requirements (composing associatively and the existence of an identity[3]). We propose to change the "category number" of mathematical models to study awareness. Awareness is sometimes treated as a "0-category", e.g. functions between domains (or more generally: sets). One prominent approach along these lines is machine-state functionalism which identifies "mental states" with functional

morphisms.

[3]Two examples are the category of sets where this amounts to the fact that morphisms are functions, or the symmetric monoidal category where this condition means to satisfy hexagonal identity.

relations between sensory inputs, behavioral outputs and other "mental states" [21]. Importantly, functionalism also assumes that mental states are individuated with respect to the relations they have to other mental states.

Carrying this idea further, we propose "n-categories" to study awareness. A "1-category" contains objects and "1-morphisms" which are morphisms between these objects. The objects of the 1-categories are possible 1-awarenesses – we call those objects "representations of the noumenon". But unlike functionalism, we do not wish to enter the debate surrounding the question *what* these things are (e.g. brain states [22] or fundamental constituents of the universe [23, 24]), but only ask *how they relate*. Our hope is that by establishing a general model to represent how awareness relates to itself, we can conjecture certain interesting properties about awareness and ontology. This method is different from the way that the metaphysics of awareness is usually done. We do not start with a "working definition" of what awareness is, but a give a model of what it *does*.

The 1-category of awareness not only consists of experience-involving objects but also of the relations between them. This means that, for example, me being aware of the zoom call right now is contained (as representation) in a category made up of myriads of (potential) other experiences – sensing the joy in your voice, or being afraid of your dismissive reactions to what I want to say – but also my (implict) perceptions of environmental going-ons, memories, and other (possibly explicit) background experience; and similar for the examples of attending a meeting of the Cambridge apostles in 1888 (this experience is related to my potential experience of listening to Wittgenstein a few years later), or for the experience of a thesis defense in 2024 (this experience is related to my actual experiences of writing the thesis now), etc. See Fig. 1 (left).

But one need not finish here. One could "increase the category number" to study possible relations between relations. A "2-category" contains objects, 1-morphisms, and 2-morphisms, which are morphisms between the 1-morphisms. In Fig. 1 (middle) we not only have morphisms between objects but also morphisms between the morphisms. The collections of all morphisms from A to B form a set called the "homset", $Hom(A, B)$. In a 2-category, each homset itself carries the structure of a category – a collection of objects and morphism satisfying certain requirements – and thus morphisms between such homsets can be regarded as morphisms between 1-categories. This higher dimensional structure allows the 2-category to represent two moments of awareness "at once", represented structurally. See Fig. 1, (middle). Continuing further, a "3-category" contains objects, 1-morphisms,

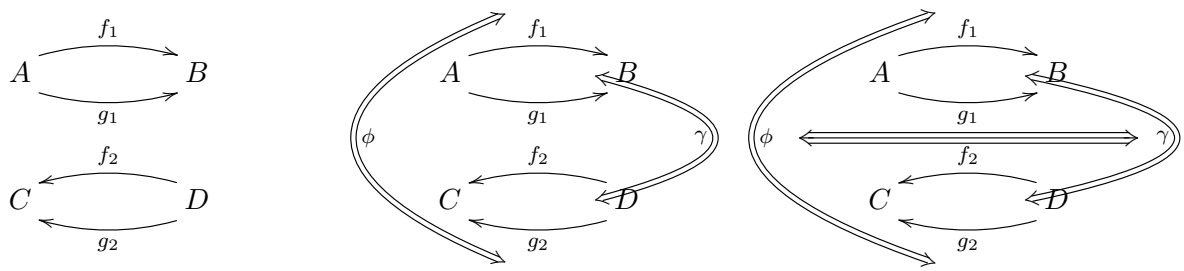

Figure 1: A graphical overview for n-awareness, for $n = 1, 2, 3$. Left: 1-category consisting of objects A, B, C, D (representations of the noumenon) and possible 1-morphisms, f_i, g_i between A and B (respectively between D and C). Middle: 2-category which also includes 2-morphisms, $\phi : f_1 \to f_2, \gamma : g_1 \to g_2$ between 1-morphisms (only two are shown). Right: 3-category which also includes 3-morphisms between 2-morphisms (only one is shown).

2-morphisms, and 3-morphisms between the 2-morphisms, where to 3-morphisms can be seen as relations between 2-categories. (Fig. 1, right).

It follows that an n-category contains objects, 1-morphisms, 2-morphisms, ..., up to n-morphisms between the $(n-1)$-morphisms. Just as 2-awareness "contains" 1-awareness, this structure is nested. With hindsight to later chapters, let us note that the grammar of our normal language is not apt to represent the complexities implied by higher category theory. We shall thus introduce the idea of n-declension later on in the paper.

2.2 Further structure on the categories

Now that the basic idea is explained, we need a little more structure on our categories in order to better capture the notion of awareness:

- **self-reflexivity**
  At a higher level, we would like our morphisms to be *invertible*. That is, not only do we want a map from $Hom(A, B)$ to $Hom(C, D)$, we would like the map to be reversible too — *actio est reactio*. An (∞, k)-category is an infinity category in which all morphisms higher than k are invertible. For our model, we chose to work in $(\infty, 1)$ categories, where all k-morphisms (for $k > 1$) are invertible; this means

that the respective objects have the "same" structure (i.e. up to isomorphism). This also means that any object of k-awareness "reflects" all the other objects it is related to, and looking at the whole category (or at the respective object in the next higher category), it can be said that awareness is "self-reflexive". We thus represent n-awareness as self-reflexive multiplicity, starting from a presentation of the original noumenon (i.e. the object of 1-awareness).

- **plurality, not mere multiplicity**
  But such a total invertibility would (structurally) imply a strict *equivalence* between instances of awareness (up to isomorphism). Phenomenologically, this would mean that my awareness now is (in some sense) equivalent to my awareness yesterday, and even worse: *your* awareness tomorrow is (in some sense) equivalent to *my* awareness two days ago. This conflicts with our intuition that awareness is a unique experience of a unique self. One could refine this strong notion of equivalence and develop the idea of a "weak equivalence" between objects.

Instead of having just a *set* of morphisms from one object to another (i.e. the homset), we want our categories to have the structure of a *topological space* of morphisms from one object to another.[4] This is the reason we later work in the *topoi setting*, so that we can have a *space* of maps between the objects we wish to study. We thus need to upgrade our category to an enriched category which allows us to use *homotopy theory* to make these ideas more precise. We now state the most important concepts, the interested reader can find a more in-depth discussion of the mathematical steps in the appendix:

1. *Homotopy theory* gives us a (relative) notion of equivalence that allows us to understand n-awareness as nested "simultaneous presence" of m moments of 1-awareness, thereby avoiding certain problems that are related to time. Homotopy theory, which studies deformation equivalences called homotopies, is defined as follows: Two continuous functions from one topological space to another are called homotopic if one can be continuously deformed into the other. *Homotopy groups* extend this notion to equivalences between topological spaces, recording information about the holes in

[4] Others have previously hypothesized, for slightly differrent reasons, that a mathematical model of consciousness should have at least the structure of a topological space [25, 26].

each space.[5] There is a remarkable freedom in reducing strong equivalences, such as the claim that object A equals object B, to deformation equivalences, such as the claim object A is "deformation equivalent" to object B because one can create the homotopy map which makes these objects homotopy equivalent. We extrapolate that to personal identity over time: You are not strictly "self-same", with some substantialist notion of self. But, using our model of homotopy types, you could be homotopy equivalent to a "structural equivalent" of yourself. It is from this homotopy setting that we derive the notion of a weak equivalence.

2. *Triangulated and derived categories.* Let A be a Grothendieck abelian category (e.g., the category of abelian groups). We define $K(A)$ to be the homotopy category of A whose objects are complexes [15] of objects of A and whose homomorphisms are chain maps modulo homotopy equivalence. The weak equivalences are quasi-isomorphisms defined as follows: A chain map $f : X \to Y$ is a quasi-isomorphism if the induced homomorphism on homology is an isomorphism for all integers n. $K(A)$ is endowed with the structure of a *triangulated category.* A triangulated category has a translation functor and a class of exact triangles which generalize fiber sequences and short exact sequences.

   Localization by quasi-isomorphisms preserves this triangulated structure. Bousfield localization [29], particular to triangluated categories, allows us to make more morphisms count as weak equivalences and this is formally how we get from 1-awareness to n-awareness. One of the axioms of a triangulated category states that given the diagram in Fig. 2, where A, B, C and A', B', C' form exact triangles, and the morphisms f and g are given such that the square $ABB'A'$ commutes, then there exists a map $C \to C'$ such that all the squares commute. This triangulated category is a categorization of a set-theoretic *ordinal.* A set is an ordinal number if it is transitive and well-ordered by membership, where a set T is transitive if every element of T is a subset of T. Ordinals locate within a set as opposed to cardinality which references merely size.

   The category of A represents all possible awarenesses (related by composition). There

[5] We say that two topological spaces, X and Y, are of the same homotopy type or are homotopy equivalent if we can find continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to the identity map idX and $f \circ g$ is homotopic to idY.

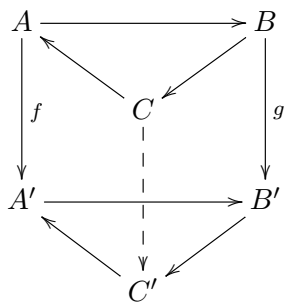

Figure 2: A triangulated category composed of two exact triangles A, B, C and A', B', C' and three commuting squares $ABB'A'$, $AC'C'A'$, and $BB'C'C$, with a non-unique fill-in [44].

are an infinite number of 1-awarenesses in this category. Differentials define a 1-awareness relating with another 1-awareness. The composition of any two A's is the zero map. This is the group law, the return to identity. Let B be the abelian group of 2-awarenesses. The differentials define relating with another 2-awareness. The diagram commuting means the 2-awareness is related with the 1-awareness. In essence, this diagram represents a higher-dimensional notion of commutativity through the map $C \to C'$ and the three squares commuting. By higher-dimensional, we mean the three commuting squares and the two exact triangles together form a cone construction. There is a relation here between the higher commutativity and the set-theoretic ordinal.

3. *Infinity-topoi.* A topos (Greek for "place") is a category which behaves like the category of sets but also contains a notion of localization. Topoi are modeled after Grothendieck's notion of a sheaf on a site [45].[6] Formally, a topos is a category equivalent to the category of sheaves of sets on a site. A prototypical example of a topos is the category of sets, since it is the category of sheaves of sets on the one point space.[7]

[6] A sheaf is a tool used to pack together local data on a topological space

[7] Informally, topoi are "nice" categories for doing geometry that act like models of intuitionist type theory. They are abstract contexts "in which one can do mathematics independently of their interpretation as categories of spaces." [9, 27]

An $(\infty, 1)$-topos is a generalization of topos to higher category theory, where higher categories allow for k-morphisms between $(k-1)$-morphisms.[8] Specifically an $(\infty, 1)$-topos is an $(\infty, 1)$-category C which satisfies three conditions: C is presentable (with kappa filtered colimits), locally cartesian closed, and satisfies a descent condition (where an object in C is sent to the slice category C/u). The objects in $(\infty, 1)$-topoi are generalized spaces with higher homotopies that carry more structure. A prototypical example would be $(\infty, 1)$-topoi of $(\infty, 1)$-sheaves [33]. The $(\infty, 1)$-topos is, informally, the meta-language of our derived category since our derived category is a homotopy category and the $(\infty, 1)$-topos is, informally, a place where one can do homotopy theory. For our topoi model, we generalize 1-awareness objects to 1-awareness objects with higher homotopies carrying more structure. Specifically, instead of modeling 1-awarenesses as objects in a Grothendieck abelian category, we model 1-awarenesses as sheaves over a site.

3 Consequences of the model

3.1 Language

We now turn to some important problems that are prompted by this treatment. Presently, we do not have a way to talk (or write) about n-awareness – our language is antiphrastical. But rather than declaring n-awareness to be merely an artifact of misguided language use (e.g. an illusionary result of playing a "language game" à la Wittgenstein [12]), we believe that it is important to actually find a (quasi-)linguistic system that *could* express it. This is due to the fact that having such a language is more than a mere "gloss" over the basic (abstract) structure of the world but an important precondition of (concretely) engaging with it. The linguistic system we envision could be called a *"language scheme"*. Similar as metaphysics could be regarded a result of language (mis-)use, questions of ontology could be rephrased in terms of the mathematical structure used in our model. We are "upgrading" Wittgenstein's language games to language schemes which we regard as new meta-language. In mathematics, "schemes" are generalizations of algebraic varieties [34]. We use our model of n-declension and take as grammatical primitives commutative ring spectra. So, instead of (linguistic) meaning being derived from whatever game is at play,

[8] Higher category theory investigates the generalizations of ∞-groupoids to directed spaces [41].

we say meaning is derived from whatever scheme is at play. We now outline the basic grammatical structure of this scheme.

First, we propose a novel "grammar of multiplicity". Modern English language contains three grammatical cases (subjective, objective, and possessive) with different declensions for each case. We extend this grammatical structure by adding a number to the respective pronoun, indicating the simultaneous presence of different "me's" within an experience of multiple awareness (Tab. 1). Formally, the resulting "n-declension" allows for all combinations of "n, $n-1$, ..., 1-declensions" to be present in a sentence. So for example, a statement compatible with double-awareness could contain a "1-I" as well as a superposed "2-I" as subject. Analogously, "n-conjugation" could be defined. This is one way the grammar of sentences could sustain the presence of n-awareness.

Second, we envision a novel temporal ontology. Spatio-temporally multiplied awareness is obviously not reconcilable with theories which posit awareness to be bound to one particular location in (*physical*) space and time. However, *non-physicalist* theories (outside of space-time) seem to be able to accommodate the concept of n-awareness. Many mystics reported a similar kind of experience, in particular in the religious traditions from the East and West. If awareness is *not* bound to a single region in space and time, this suggests that awareness cannot be understood as an emergent property of localized physical systems.

We want an ontology that reconciles both the physicalist as well as the non-physicalist understanding. This calls for a novel ontology which could accommodate multiple forms of awareness throughout different points in time. Our model conceives of a temporal multiplicity with a categorified model of n-time, evidenced in the proposed language scheme by way of n-inflection for n-conjugation. Modern English is spoken in local, linear time, yet it allows the inflectional change of verbs by way of conjugation. We extend the idea of language spoken in linear time, conjugated over three tenses, to one of n-conjugation as follows: Instead of using only past, present, future, and their perfect correspondences, present-perfect, past-perfect, future-perfect, we allow for 1-past, 2-past, ..., 1-present, 2-present, ..., 2-present-perfect, 2-future-perfect, n-future-perfect etc, which is what we call n-time. This generalizes the discussion on temporal experience, which has traditionally been expressed in terms of a "tensed" experienced time vis-a-vis an "untensed" physical parameter time.

Table 1: Grammar of a language which could express n-awareness, extended from ordinary English case structure.

			$n = 1$	$n = 2$	$n = 3$	$\cdots$	$n = k$
subjective	I (we)	$\rightarrow$	1-I (1-we)	1-I (1-we)	1-I (1-we)	$\cdots$	1-I (1-we)
				2-I (2-we)	2-I (2-we)		2-I (2-we)
					3-I (3-we)		$\vdots$
							k-I (k-we)
objective	me (us)	$\rightarrow$	1-me (1-us)	1-me (1-us)	1-me (1-us)	$\cdots$	1-me (1-us)
				2-me (2-us)	2-me (2-us)		2-me (2-us)
					3-me (3-us)		$\vdots$
							k-me (k-us)
dep. possessive	my (our)	$\rightarrow$	1-my (1-our)	1-my (1-our)	1-my (1-our)	$\cdots$	1-my (1-our)
				2-my (2-our)	2-my (2-our)		2-my (2-our)
					3-my (3-our)		$\vdots$
							k-my (k-our)
indep. possessive	mine (ours)	$\rightarrow$	1-mine (1-ours)	1-mine (1-ours)	1-mine (1-ours)	$\cdots$	1-mine (1-ours)
				2-mine (2-ours)	2-mine (2-ours)		2-mine (2-ours)
					3-mine (3-ours)		$\vdots$
							k-mine (k-ours)

3.2 Revisiting problems of temporality

Different temporal ontologies have been proposed throughout the ages, without coming to definite conclusion. Rather than proposing yet another metaphysical framework, we wish to concentrate on particular problems that feature prominently in recent and not so recent discussions. We have posited a structural model of awareness using derived category theory and hope that approaching these problems structurally will lead to a remedy.
The problems we wish to focus are the following:

1. **How is change possible?** Change is a manifest everyday experience. The idea that "change" does not really exist goes back to the works of Parmenides and the Eleatic school of philosophy. While this philosophy has been influential up to this day, for example, in the metaphysical thought of Martin Heidegger [46], the our common day experiences seems much better captured in the "everything flows" of Heraclitus [47]. More specifically, in the philosophy of time the idea of a "tenseless" vs. a "tensed" time is a prominent distinction introduced by John McTaggert [48].

   The American philosopher David Lewis revived the problem of change for the philosophy of time in the problem of temporary intrinsics [49], 198f.: *For instance shape: when I sit, I have a bent shape; when I stand, I have a straightened shape. Both shapes are temporary intrinsic properties; I have them only some of the time. How is such change possible? I know of only three solutions.*
   According to David Lewis the problem of temporary intrinsics could be solved in three ways. Either one acknowledges that there *are no* intrinsic properties, just "disguised relations"; or one believes that only those properties that exist at the *present* moment are real, whereas properties that an object seem to have had are, in some sense, fictional (this position is known as "presentism"); or one accepts that objects have genuine temporal parts (e.g. the me-yesterday, the me-now, and the me-tomorrow). The latter solution to the problem of temporary intrinsics has been deemed the only viable solution to the problem of temporary intrinsics which is not "incredible" [49] and started the appreciation of "perdurance" theories in the modern philosophy of time at the expense of so-called "endurance" theories that conceive of persisting wholes without temporal parts [50]. In addition to Lewis' metaphysical rejunevation, much support for a "perdurance-like" theory seems to come from science, in particular Einstein's theory of relativity. Perdurance theory, so it is often

but not uni-vocally believed, squares well with the believe that space-time forms a four-dimensional continuum as described by the special theory of relativity [50].

2. **What is simultaneity?** The second problem worth mentioning in this respect is the problem of simultaneity (coincidence). It seems that, when discussing n-awareness we believe in the simultaneous presences of two experiences. This seems to violate the basic intuition that no two objects could occupy the same place in time unless they are the same (or unless they share temporal parts: the statue and the clay have temporal parts that overlap[9]). But it also seems to be in conflict with basic principles of physics understanding according to which there can be no "absolute" notion of simultaneity.

   However, note that the problem of simultaneity is mainly a (conceptual) "design issue" which stems from a linear notion of time where simultaneity is conceived in terms of ("temporal") coincidence, or, alternatively, from the treatment of time in the framework of Minkowski space-time, the so called "fourth dimension". We offer a structural solution and define simultaneity by commutativity of diagrams of chain comlexes in a homotopy category, and respectively by the "up to" notion.

   Commutativity classifies the equivalence of all possible ways to get to a destination. Take, for example, the chain map between complexes (see Fig. 4 in the appendix). The composition means that there are two ways to get to B_{n-1}, and there is no structural difference in choosing one way over the other. There is no indicated starting point or canonical progression. Rather, we see all possible paths, and even infinite paths are alluded to. The "up to" notion grants a universal equivalence which structurally corresponds to commutativity. For instance, making the statement that all morphisms are equivalent "up to" homotopy means that they are equivalent with respect to homotopy. There is no substantial way to distinguish one morphism over any other one. In a sense, commutativity in our diagrams is algebraically sustaining the "up to" notion. Equivalence (and hence simultaneity) is never truly absolute.

3. **Temporal coincidence and synchronous reference?** While perdurantism claims to solve problems of temporal coincidence (which are only problems if designed to be so), it has its own problems when trying to account for the acts of synchronous

[9] Another argument in favor of perdurantism.

and asynchronous reference. For instance, if I touch my hand on a hot stove and get burned, it takes "time" for my system to register "pain". My feeling pain "now" is a result of my action in the immediate past. One could claim that all of "present" feelings are results of actions in the past, be it immediate or not. As such, there is no such thing as synchornous reference, only asynchronous. This means it is important to question how received ontologies (such as perdurantism or endurantism) handle the question of asynchronous reference. At least prima facie, it seems they can't: "There is no logically necessary connection between events at different times; therefore nothing that is happening now or will happen in the future can disprove the hypothesis that the world began five minutes ago." [51] By contrast, our model provides a structural framework for synchronous and asynchronous reference through the representation of the homset: a model based on pluralism might the right answer for the question of how events are "temporally" related to each other.

To see how we would circumvent these problems note that, just as perdurantism uses the structure of the fourth dimension (i.e. the theory of Minkowski) for its ontology, n-awareness uses that of $(\infty, 1)$-topoi. We replace the idea that objects are bearers of (intrinsic or extrinsic) properties by the idea that "properties" are identical to the morphisms (relations) between objects within a category. "Change" in this context is tantamount to the addition (removal) of morphisms between objects in a category.
The traditional distinction between perdurantism and endurantism could be illustrated by the notion of a so-called "coequalizer":

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"] \arrow[dr, "q'"'] & Q \arrow[d, dotted, "u"] \\ & & Q' \end{tikzcd}$$

In category theory, a coequailizer refers to a single object (a "colimit") associated to the different morphisms f and g between objects X and Y, such that $q \circ f = q \circ g$. Furthermore, the objects Q is universal, meaning it is unique "up to" an isomorphism u[10]. It follows that *properties* (i.e. morphisms within a category) can be associated to a single and unique (up

[10]Isomorphisms are permutations of morphisms; the "up to" phrase exemplifies how distinct objects in the same class can be considered equivalent under a particular condition [31]

to isomorphism) *object*. Whereas perdurantism, translated into the language of category theory, is about change between such properties (i.e. the addition of new morphisms), endurantists refer to the unchanging (persisting) object defined by them.

This also gives us a handle on the problem of simultaneity: Our view of simultaneous awareness is not aptly perdurantist, although superficially it seems a perdurantist representation with the n-declension of 1-her, 2-hers, ... n-her, but these are relational properties and not necessarily temporal parts. It can be asked how different Mary-tomorrow is from Mary-today given that Mary-tomorrow has more morphisms? If Mary refers to the objects of a category and the properties are its morphisms, then saying that Mary has "changed" is merely to say that Mary has added connections/morphisms. We say that Mary-today is the same as Mary tomorrow "up to" isomorphism.

Analogous to how perdurantists resolve the problem of coincidence by noting that temporal parts can surely "overlap" without implying that the two objects that overlap are identical, we note that categories (given they have at least some basic structural features, e.g. are topological) could too be said to "overlap". But this does not commit us to perdurantism as ontological position. The worry that our notion of "simultaneously being aware" commits us to a strong notion of simultaneity which is in conflict with physics. We instead choose to model time in terms of an equivalence relation using homotopy theory – "time" is not an absolute (ontological) notion, but instead refers to a relative (epistemic) ordering scheme of experiences. Thus, to every level of awareness, there corresponds a level of time. 1-awareness corresponds to 1-time. 2-awareness corresponds to 2-time etc. There is only a composition law not uniquely defined up to a homotopy of time. So simultaneity is a homotopy equivalence and homotopy equivalences are neither perdurantist nor endurantist.

3.3 Soteriology

In order to talk meaningfully about soteriology, we need a meaningful concept of self. We do have such a concept: the self as homset. To be self-reflexive would be to have invertible morphisms; offering a geometrical version of relationlism, rather than a set-theoretic one.

Our model can be summarized as follows: We take 1-awareness as the presentation of a ("pluralist-monist") noumenon. To every 1-awareness we can associate (one or more) 1-morphisms which constitutes its "1-time". 1-time is but the relation in which awarenesses — perceived as present, past, future, possible or actual — stand. Every 2-awareness has

its 2-time AND 1-time, since it contains 2-morphisms as well as 1-morphisms. Continuing, we have n-awareness containing n-time. This is new because current models of temporal ontology still frame time as a static, 1-dimensional phenomena, static enough so that we claim to have a "personal identity over time." By contrast, we posit that this notion of "over time" is better conceptualized in terms of the relations that figure in n-categories. We extrapolate to personal identity over time. You are not self-reflexive with some ill defined notion of self. But, using our model of homotopy types, you could be homotopy equivalent to your other versions of yourself. It is from this homotopy setting that we derive the notion of a weak equivalence.

The triangulated category (Fig. 2) is our structural representation of soteriology. It is a higher-dimensional notion of commutativity. Simply speaking, this means there is more than one canonical way to reach divinity. Having such a higher notion of commutativity in a homotopy category means that there are many paths that are homotopy equivalent to the canonical path to divinity. As such, our soteriology is pluralistic. Soteriology is not a cardinal issue, it is an ordinal issue, having more to do with how close or far one is to (one's own) divinity.

This invites the following thought: Instead of taking n to infinity and reflecting, similar as in the Vedic tradition, the size (i.e. cardinality) of this infinity, one could look for the highest *ordinal* of infinity, reflecting that awareness is not about size but about order — to be aware of the divine means being able to localize oneself within the (levels of the) infinite. Penitence is about navigating this distance. "Cardinality" is about size (e.g. the size of a set); But "closeness" is an ordinal concept. How to get there from many ways — that's homotopy.

4 Discussion

Our work posits an ontology of plurality in three ways: structurally we begin with the noumenon represented by the objects of 1-awareness and invertible n-morphisms which represent a multiplicity. Second, our notion of n-time represents a temporal multiplicity and gives rise to a n-declension. Third and finally, our soteriology is pluralistic; our concept of ("the one") self is a homset.

Pluralism is categorical. We are taking the monism of structure and making it plurastic by working in a derived setting appealing to the infinity topoi (with chain complexes with

localization giving more weak equivalences, cf. appendix). Similar ideas (in a less technical setting) can be found in the systems of Leibniz [8] or, perhaps, the Yogacara school of Buddhism [52]. It could be objected that our model really expresses a (self-reflexive) *monism*, since we ground the model in a seemingly universal form of structural representation, thus making any soteriology which keeps with individuals difficult to sustain. We disagree. Our model implies — due to its notion of "weak equivalence" – a form of pluralism.

$(\infty, 1)$-topoi play a central role in our model. In general, topoi are models of internal logic, which means that any[11] logical statement could be internalized. $(\infty, 1)$-topoi are types with internalized descent datum [38]. All conceivable relations are built inside the framework. The $(\infty, 1)$-topoi is our meta-language and the space of all possible awareness. To categorize soteriology means to internalize soteriology.

[11] except for those dependent on the law of the excluded middle and the axiom of choice [36].

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Appendix

To later introduce the derived categorical setting, we will first explain the spirit of *quasi-isomorphisms* using the notion of quasi-categories formulated by Andre Joyal [39]. A quasi-isomorphism is a morphism $A \to B$ of chain complexes[12] such that the induced morphisms $H_n(A, \cdot) \to H_n(B, \cdot), H^n(A, \cdot) \to H^n(B, .)$ of homology groups [13] are isomorphisms for all n.

An important invariant of a mathematical space is encoded by its *homology* group - the number of holes in that space. As such, they provide a means to compare spaces. For X a topological space, a set of topological invariants $H_0(X)$, $H_1(X)$,..., called the homology groups of X, represent the homology of X. The number of k-dimensional holes in X is encoded by the kth Homology group $H_k(X)$. For instance, $H_0(X)$ encodes the "path connected" components of X, where a (0-dimensional) hole encodes if the space is disconnected.

As an example, let us examine the homology groups of S^1, the 1-dimensional sphere (which is really just a circle). Take X to be S^1. X is connected and has one 1-dimensional hole and no other holes for $k > 1$. The homology groups of X take the form:

$$H_k(S^1) = \begin{cases} \mathbb{Z} & k = 0, 1 \\ \{0\} & \text{otherwise} \end{cases}$$

Take X to be S^2, the 2-dimensional sphere (which is just the surface of a ball). S^2 is connected and has just one 2-dimensional hole. The homology groups of X are represented as:

$$H_k(S^2) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ \{0\} & \text{otherwise} \end{cases}$$

Quasi categories are homotopoi [39] which possess rich general structures and do not necessarily have a uniquely defined composition of morphisms. Quasi-categories are like ordinary categories in that they are certain simplicial sets which contain objects (the 0-simplices of the simplicial set) and morphisms between these objects (1-simplices). Unlike categories, however, morphisms can be composed, but the composition is well-defined only up to still higher order invertible morphisms. This means that all possible morphisms which

[12] respectively of cochain complexes.

[13] respectively of cohomology groups.

serve as the composition of two 1-morphisms are related to each other by 2-morphisms called 2-simplices, which resemble homotopies. It turns out that every Kan Complex [28] is a quasi-category. So from this Kan Complex we build the notion of the chain complex which we will use in the next section.

It is important that we ask the following: If the composition law between two 1-morphisms need not be uniquely defined, what does that mean phenomenologically? That is, what does 1-morphism being unique up to 2-morphism mean in terms of identity over time? It could be argued that a unique composition of morphisms gives an inadequate notion of a personal identity over time because there is only one morphism that serves as that particular composition. By contrast, leaving the composition not uniquely defined allows for n-awareness, where the n-awarenesses represent all possible morphisms related to each other by higher invertible morphisms.

Homotopy theory gives us a (relative) notion of equivalence that allows us to understand n-awareness as "simulatenous presence" of n moments of 1-awareness, thereby avoiding certain problems that a related to time. Homotopy theory, which studies deformation equivalences called homotopies, is defined as follows: Two continuous functions from one topological space to another are called homotopic if one can be continuously deformed into the other. *Homotopy groups* extend this notion to equivalences between topological spaces, recording information about the holes in each space.[14] There is a remarkable freedom in reducing strong equivalences, such as the claim that object A equals object B, to deformation equivalences, such as the claim object A is "deformation equivalent" to object B because one can create the homotopy map which makes these objects homotopy equivalent. We extrapolate that to personal identity over time: You are not self-reflexive, with some substantialist notion of self. But, using our model of homotopy types, you could be homotopy equivalent to a "structural equivalent" of yourself. It is from this homotopy setting that we derive the notion of a weak equivalence.

We mention two reasons that we work in the *derived categorical* setting. One reason is that knowing the homology of a space does not give complete information about its homotopy type. This is seen by that fact that there exist topological spaces X and Y such that $H_i(X)$ is isomorphic to $H_i(Y)$ for every i, but X is not homotopy equivalent to

[14] We say that two topological spaces, X and Y, are of the same homotopy type or are homotopy equivalent if we can find continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to the identity map idX and $f \circ g$ is homotopic to idY.

$$\begin{tikzcd} X \arrow[r, "\tilde{f}_0"] \arrow[d, "X\times\{0\}"'] & E \arrow[d, "\pi"] \\ X \times I \arrow[r, "f"] \arrow[ur, dotted, "\tilde{f}"] & B \end{tikzcd} \qquad \begin{tikzcd} Y & X \arrow[l, "\tilde{f}_0"'] \\ Y^I \arrow[u, two heads, "p_0"] & A \arrow[l, "f_0"] \arrow[u, "i"'] \arrow[ul, dotted, "\tilde{f}"] \end{tikzcd}$$

Figure 3: Homotopy lifting and extension property for topological spaces E and B. In the leftmost figure, the homotopy lifting property allows homotopies in the space B to be uplifted to the space E for any homotopy $f : X \times [0,1] \to B$ and for any map $\tilde{f}_0 : X \to E$ such that $f_0 = \pi \circ \tilde{f}_0$. A lifting $\tilde{f}$ corresponds to a dotted arrow giving a commutative diagram. In the rightmost figure, the homotopy extension property extends certain homotopies defined on a subspace to a larger space. The homotopy extension property is dual to the homotopy lifting property [42].

Y. The derived category remembers the entire complex, which is crucial to our model of n-awareness, and the consequent model structure gives us nice classes of morphisms which axiomatize homotopy theory. Another reason is that the derived category setting allows us to *localize* in the category setting. Localization is a formal process of adding inverses to a space. A category can be localized by formally inverting certain morphisms, such as the weak equivalences in the homotopy category of a model category. We use a special case of localization called Bousfield localization [29], which assigns a new model category structure with more weak equivalences to a given model category structure. So Bousfield localization allows us to make more morphisms count as weak equivalences and this is formally how we get from 1-awareness to n-awareness.

To axiomatize homotopy theory, we use the construction of a Quillen model structure [40]. A model structure on a category consists of three classes of morphisms: weak equivalences, fibrations, and cofibrations. Weak equivalences are quasi-isomorphisms, maps which induce isomorphisms in homology. Cofibrations are maps that are monomorphisms that satisfy the homotopy extension property. Fibrations are maps that are epimorphisms that satisfy homotopy lifting property (Fig 3). In the derived setting, quasi isomorphisms are used as the class of weak equivalences, fibrations mimic surjections, and the cofibrations mimic inclusions. From this model structure, we will define the notion of simultaneity.

We let A be a Grothendieck abelian category, such as the category of abelian groups.[15] The Grothendieck abelian category is an AB_5 category with a generator. AB_5 categories are AB_3 categories (abelian categories possessing arbitrary coproducts) in which filtered colimits of exact sequences are exact [32]. The category of abelian groups is a prototypical example of a Grothendieck category, with generator the abelian group $\mathbb{Z}$ of integers. The category of abelian groups has as objects abelian groups and as morphisms group homomorphisms. We use Grothendieck categories because we need a category universally enriched over abelian groups to model n-awareness. In particular, we take this group to model awareness, with an:

- identity element, which serves as a basic notion of "self"
- inverse which illustrates the "back-reaction" (reflectivity) for each element
- associativity which defines an "order" of perception
- a closure property which defines the "privacy" of awareness

Groups encode symmetries. But what is 1-awareness an symmetry of? We hypothesize that it encodes a complexity class of Turing degree 0 [43].

We then construct a new "derived category", $D(A)$, whose objects are complexes of objects of A and whose morphisms are chain maps. $D(A)$ contains a model structure that will be our model of n-awareness.[16]

Firstly, we define a chain complex. A chain complex $(A_\bullet, d_\bullet)$ is a sequence of abelian groups $..., A_0, A_1, A_2, A_3, A_4, ...$ connected by homomorphisms (called boundary operators or differentials) $d_n : A_n \to A_{n-1}$, such that the composition of any two consecutive maps is the zero map. Explicitly, the differentials satisfy $d_n \circ d_{n+1} = 0$, or with indices suppressed, $d^2 = 0$. A chain complex has the form:

$$\cdots \xleftarrow{d_0} A_0 \xleftarrow{d_1} A_1 \xleftarrow{d_2} A_2 \xleftarrow{d_3} A_3 \xleftarrow{d_4} A_4 \xleftarrow{d_5} \cdots$$

Secondarily, a chain map f between two chain complexes $(A_\bullet, d_\bullet), (B_\bullet, d_\bullet)$ is a sequence f_n of homomorphisms $f_n : A_n \to B_n$ for each n that commutes with the differentials on the

[15] Grothendieck worked on unifying various constructions in mathematics. For instance, the Grothendieck group construction is the most universal way of constructing an abelian group from a commutative monoid. [30]

[16] For a more detailed exposition see the work of A. Caldararu [44].

$$
\begin{tikzcd}
\dots & A_{n-1} \arrow[l, "d_{A,n-1}"'] \arrow[d, "f_{n-1}"'] & A_n \arrow[l, "d_{A,n}"'] \arrow[d, "f_n"'] & A_{n+1} \arrow[l, "d_{A,n+1}"'] \arrow[d, "f_{n+1}"'] & \dots \arrow[l, "d_{A,n+2}"'] \\
\dots & B_{n-1} \arrow[l, "d_{B,n-1}"'] & B_n \arrow[l, "d_{B,n}"'] & B_{n+1} \arrow[l, "d_{B,n+1}"'] & \dots \arrow[l, "d_{B,n+2}"']
\end{tikzcd}
$$

Figure 4: A commutative diagram of a chain map between two chain complexes $(A_\bullet, d_\bullet), (B_\bullet, d_\bullet)$, with a sequnce of morphisms $f_n : A_n \to B_n$ and differentials $d_B, n \circ f_n = d_A, n \circ f_{n-1}$.

two chain complexes, where $d_B, n \circ f_n = d_A, n \circ f_{n-1}$. A chain map takes the form of the commutative diagram in Fig. 4. $(f_\bullet)_* : H_\bullet(A_\bullet, d_{A,\bullet}) \to H_\bullet(B_\bullet, d_{B,\bullet})$ on preserves cycles and boundaries, so f induces a map on homology.

Let A be a Grothendieck abelian category (e.g., the category of abelian groups). We define $K(A)$ to be the homotopy category of A whose objects are complexes of objects of A and whose homomorphisms are chain maps modulo homotopy equivalence. The category $K(A)$ has a model structure in which the cofibrations are the monomorphisms (categorical generalizations of injective functions) and the weak equivalences are the quasi-isomorphisms defined as follows: A chain map $f : X \to Y$ is a quasi-isomorphism if the induced homomorphism on homology is an isomorphism for all integers n. $K(A)$ is endowed with the structure of a *triangulated category*. A triangulated category has a translation functor and a class of exact triangles which generalize fiber sequences and short exact sequences. [17] Localization by quasi-isomorphisms preserves this triangulated structure. One of the axioms of a triangulated category states that given a diagram:

$$
\begin{tikzcd}
A \arrow[rr] \arrow[dd, "f"] & & B \arrow[dl] \arrow[dd, "g"] \\
& C \arrow[ul] \arrow[dd, dashed] & \\
A' \arrow[rr] & & B' \arrow[dl] \\
& C' \arrow[ul] &
\end{tikzcd}
$$

[17] Diagram of two exact triangles A, B, C and A', B', C' and three commuting squares $ABB'A'$, $AC'C'A'$, and $BB'C'C$, with a non-unique fill-in [44].

where A, B, C and A', B', C' form exact triangles, and the morphisms f and g are given such that the square $ABB'A'$ commutes, then there exists a map $C \to C'$ such that all the squares commute, where this fill-in is not unique [44]. This Triangulated category is a categorization of a set-theoretic ordinal. A set is an ordinal number if it is transitive and well-ordered by membership, where a set T is transitive if every element of T is a subset of T.

Through a localization process, called "Bousfield localization" [29], the derived category $D(A)$ of the initial abelian category A is obtained by "pretending" that quasi-isomorphisms in $K(A)$ are isomorphisms. Specifically, the localization is constructed as follows: morphisms in $D(A)$ between $A\cdot$ and $B\cdot$ will be 'roofs' [44], with f, g morphisms in $K(A)$ and f a quasi-isomorphism. This roof represents $g \circ f^{-1}$.

The category of A represents all possible awarenesses (related by composition). There are an infinite number of 1-awarenesses in this category. Differentials define a 1-awareness relating with another 1-awareness. The composition of any two A's is the zero map. This is the group law, the return to identity. Let B be the abelian group of 2-awarenesses. The differentials define relating with another 2-awareness. The diagram commuting means the 2-awareness is related with the 1-awareness. In essence, this diagram represents a higher-dimensional notion of commutativity through the map $C \to C'$ and the three squares commuting. By higher-dimensional, we mean the three commuting squares and the two exact triangles together form a cone construction. There is a relation here between the higher commutativity and the set-theoretic ordinal.