Categorical Mental Imagery: Visualizing the 4th Spatial Dimension

Shanna Dobson
Zihang(Jim) Zhong

Abstract We present a colorful and novel discussion of mathematical techniques of visualizing a fourth spatial dimension. We first discuss notions of dimensionality including the homotopy dimension for objects in an $(\infty, 1)$-topos. We try to visualize the fourth spatial dimension using color, and illustrate this with four-dimensional ice-cream. We apply categorical negative thinking to what we have called ∞-visual epistemology. The aim is that visualizations of higher spatial dimensions can occur functorially. We illustrate with five images five conjectural methods for how to visualize the fourth spatial dimension.

Keywords n-functor; (-1)-category; higher category theory; mental imagery; phantasia

1 Introduction

The concept of *dimension* is an incredibly complex topological construction. Informally, the concept of dimension measures the expansiveness or size of an object. The notion of dimension contains "inside of it" several other concepts including covering, sidedness, refinement, and the space-filling ability of an object. The concept of dimension has many incarnations. Consequentially, many fields of mathematics have their own notion of dimension of particular spaces.

For example, a topological space has a topological dimension defined as follows:

Shanna Dobson
University of California, Riverside, 900 University Ave, Riverside, CA 92521
ArtCenter College of Design, 1700 Lida Street, Pasadena CA, 91103, e-mail: Shanna.Dobson@email.ucr.edu
Zihang(Jim) Zhong
ArtCenter College of Design, 1700 Lida Street, Pasadena CA, 91103, e-mail: zzhong1@inside.artcenter.edu

Definition 1. *([26]) A topological space X has Lebesgue covering dimension less than $n \in \mathbb{N}$ if every open cover U of X has a refinement V such that every element of X belongs to fewer than $n + 1$ elements of V. (Then X has dimension n if it has dimension less than $n + 1$ but not less than n.)*

Furthermore, the dimension of a topological space is invariant under homeomorphisms, making it a topological property. Given this definition of dimension in terms of refinements, a curious question naturally arises:

Question 1. *What is the dimension of empty space?*

This question appears to be a limit paradox [27], since it seems that only non-empty spaces would have a notion of a cover. A possible explanation is given by what is called negative thinking. Negative thinking is defined as follows:

Definition 2. *([22]) Negative thinking is a way of thinking about categorification by considering what the original concept is a categorification of...More generally, negative thinking can apply whenever you have a sequence of mathematical objects and ask yourself what came before the beginning?*

By negative thinking, we can *see* that the empty space has precisely dimension -1. Given such an answer, another curious question naturally arises:

Question 2. *How can one visualize a negative dimension?*

It seems that though an answer can be provided using a form of *negative thinking*, the answer provided is not easy to visualize.

Let us ask a similar question.

Question 3. *What is the dimension of a point?*

To answer this question, we must first define what a *point* is. A point has many incarnations as well, depending on the mathematical field from which we define it.

> As a topological space, the point space is the usual point from geometry: that which has no part. In more modern language, we might say that it has no structure - except that something exists. (So it is not empty!)...Up to homotopy equivalence, any contractible space qualifies as a point. [24]

Additionally, a point is

> a singleton set with a unique element •. It can be seen as a truth value that is true. It can even be understood as the (-2)-category. [24]

Finally, we offer a categorical interpretation of a point.

> As a category, we can interpret this [point] as a category with a single object • and a single morphism (the identity morphism on the object, which is not shown in the picture since it is automatic). This can be generalized recursively to higher categories: as an $(n + 1)$-category, the point consists of a single object • whose endomorphism n-category is the point (now understood as an n-category)...In the limit, the point can even be understood as an ∞-category, with a unique j-morphism for each $j \geq 0$ (each of which is an identity for $j > 0$). [24]

By applying negative thinking to our categorical definition of a point, we *see* that a point has dimension 0. Consequently, we can ask

Question 4. *How can we visualize dimension 0?*

As stated in our Introduction, the concept of dimension has many incarnations. Let us now consider the idea of dimension for objects in an $(\infty, 1)$-topos. The idea of dimension includes homotopy dimension, cohomological dimension, and the covering dimension defined above.

The homotopy dimension is defined as follows:

Definition 3. *([12] Corollary 7.2.1.12) If an $(\infty, 1)$-topos $\mathcal{X}$ is locally of homotopy dimension $\leq n$ for some $n \in \mathbb{N}$ then it is a hypercomplete $(\infty, 1)$-topos.*

The cohomological dimension is defined as:

Definition 4. *([16] Idea.) An object in an $(\infty, 1)$-topos is said to have cohomological dimension $\leq n$ if all cohomology groups of degree $k > n$ vanish on that object.*

Definition 5. *([12] Definition 7.2.2.18) For H an $(\infty, 1)$-topos and $n \in N$, an object $X \in H$ is said to have cohomological dimension $\leq n$ if for all Eilenberg-MacLane objects $B^k A$ for $k > n$ the cohomology of X with these coefficients vanishes:*

$$H^k(X, A) := \pi_0 H(X, B^k A) \simeq *.$$

We say the $(\infty, 1)$-topos H itself has cohomological dimension $\leq n$ if its terminal object does.

The following Corollary connects the homotopy dimension with the cohomological dimension.

Corollary 1. *([12] Corollary 7.2.2.30) If $\mathcal{X}$ has homotopy dimension $\leq n$ then it also has cohomological dimension $\leq n$.*

The converse holds if $\mathcal{X}$ has finite homotopy dimension and $n \geq 2$.

We have the following Theorem on the covering dimension for objects in an $(\infty, 1)$-topos.

Theorem 1. *([12] Theorem 7.2.3.6) If the paracompact topological space X has covering dimension $\leq n$, then the $(\infty, 1)$-category of $(\infty, 1)$-sheaves $Sh_{(\infty,1)}(X) := Sch_{(\infty,1)}(Op(X))$ is an $(\infty, 1)$-topos of homotopy dimension $\leq n$.*

Further examples of dimensions are the following, informally stated.

Example 1. *Vector Spaces: The dimension of a vector space is defined as the number of elements in its basis.*

Example 2. *Sheaf Theory: The dimension of a sheaf is the dimension of its vector space over a field and/or the dimension of its support.*

Example 3. *Fractality: a fractal dimension. M.L. Lapidus presents a new definition of fractality.*

Definition 6. *([13] Definition 12.1 A New Definition of Fractality) We propose to define "fractality" as the presence of at least one nonreal complex dimension with positive real part. [13]*

We now briefly address the concept of sidedness of an object, which is is a geometric topological concept that is very helpful in visualizing higher dimensions.

The sidedness of an object is encapsulated by its orientation. The idea is as follows: We say a typical two-sided surface has two local transverse orientations at every point. If there exists a path on the surface that changes the two local transverse orientations, then that surface globally has one side. If such a path does not exist, that is, no path interchanges the two local transverse orientations, then the surface is globally two-sided. The surface is additionally called orientable and admits an orientation. The surface is called non-orientable. Examples of non-orientable surfaces include the Möbius Strip, the Klein bottle, and the Roman Surface. Homological arguments make these definitions precise, but we will skip those for now.

Given this definition of sidedness, other question arises:

Question 5. *How do we define the inside and outside of a space?*

Question 6. *Can a space only have an inside and not an outside? and vice versa?*

Question 7. *How can we visualize an object with only one side? Would this be, metaphorically, like a mirror?*

The assignation of an inside of an object and an outside of an object is a topological notion as well. Recall that the interior of a set is the union of its open subsets. Informally, the inside of a shape can be thought of as consisting of all points that can be connected by a continuous path that does not intersect the boundary of the shape. Points are outside of the shape if they cannot be connected by a continuous path to points inside the shape without crossing the boundary. We further recall the boundary of a subset X of a topological space is thought of as $\partial X = \bar{X} \setminus int(X)$.

Using the technique of dimensional analogy, it turns out that in the fourth dimension, there is no concept of inside and outside.

> And, in the same way, three-dimensional beings (such as humans with a 2D retina) can see all the sides and the insides of a 2D shape simultaneously, a 4D being could see all faces and the inside of a 3D shape at once with their 3D retina.

Further illumination can come from the field of knot theory. Let S^1 denote the circle or a 1-sphere. Formally, a knot is defined as a particular embedding $S^1 \to \mathbb{R}^3$. It is known that, by changing crossings, any knot can be *unknotted* in the fourth dimension. That is, there exist no nontrivial knots in the fourth dimension.

This is formalized as follows: An embedding of the form $S^1 \to \mathbb{R}^4$ is called a trivial knot because it can be unknotted; that is, the trivial knots can be continuously deformed to S^1.

Question 8. *How can we visualize unknotting knots in $\mathbb{R}^4$?*

We will use color as the 4th-dimensional coordinate w to carry out the visualization. To work in color in the 4th dimension means we denote a point in $\mathbb{R}^4$ as a point of $\mathbb{R}^3$ that carries a color. Then we denote a point, say, pink, if it has w=0. We can then illustrate *moving into the 4th dimension* by changing the color of the point to, say, blue for the ana direction (positive real numbers), and cyan for the kata direction (negative real numbers). Working via color, we can see that we can change the crossings, and pull the underarc past the overarc of the crossing by first taking the underarc into the 4th dimension, where it changes to blue. Then we can come back to the 3rd dimension, so the underarc becomes pink. Thus, we can slide it over the overarc because the arcs are no longer living in the same point in $\mathbb{R}^4$.

We note that if Pink is a 2-category and the w-coordinate is color, to choose Pink as representing a point in the ana direction, we would have to rephrase all of this functorially (2-functorially). We discuss this further in Section 3.3: *The Fourth-Dimensional Ice-Cream Parlor*.

As our current study is focused on visualizing the fourth dimension, we will primarily work in Euclidean space and use the coordinate-centric definition of dimension in terms of the *minimum* number of coordinates required to describe a point in the Euclidean coordinate space. We will secondarily use the categorical definitions of dimensions defined above.

Canonical physics tells us that our universe/interface consists of four dimensions: three spatial dimensions and 1 dimension of time. In our current manuscript, we take a different perspective. We will study the fourth dimension as a spatial dimension, and attempt five consequential conjectures to visualize it.

Now, visualizing a fourth spatial dimension is incredibly challenging. A few questions naturally arise:

Question 9. *How can we attempt such a feat when we are immersed in 3D concepts, like inside and outside, which may not apply in a fourth dimension?*

Question 10. *Wouldn't we need a fourth dimensional eye-ball with a three dimensional retina in order "to see" the fourth spatial dimension?*

These are excellent questions! We might look towards the *Illumination Problem* for guidance, and attempt to visualize the fourth spatial dimension based on what could be seen by a light source in the fourth spatial dimension. In [6], we presented our notion of *sight as site*, visual sight as categorical site. Once our canonical visualization skills start to fail us, it might be incredibly useful to work categorically in order to "see."

It is customarily helpful to equate the fourth dimension with time. That is, the fourth dimension is temporal. Such an assignation can help us determine how we could move through the fourth spatial dimension, by considering how we move through the fourth temporal dimension. In our current universe, which abides by a particular class of physical laws, we seem to experience only slides of time designated as particular instances of "now," by the very fact that we are three-dimensional objects. By the properties of dimensionality, we are only able to perceive slices of the fourth (temporal) dimension. We have the remarkable property of being able to mentally transport ourselves to the past and to the future at will, a property that many non-human species do not have. Even more peculiar, we can *imagine* a possible future and a past that could have been. Thus, thinking of the fourth dimension temporally, we could say that each point in the fourth dimension contains your entire life. Thus, at each moment you could experience the totality of your life. Moving around in the fourth dimension would be like experiencing Everett's Many Worlds Theorem in action. Going further, it is uncertain if we would merely be moving along those timelines or if we would be using the *Everett phone*, which is a conjectural mechanism in which the Many Worlds can send information between each other; a remarkable feat!

In Charles Dickens' masterpiece *A Christmas Carol* [5], Ebeneezer Scrooge is able to walk amongst apparitions that were part of his past. He is even able to talk with one particular apparition, his business partner, Jacob Marley. The Ghost of Christmas Past visits Scrooge and takes him on a journey through his past. Scrooge sees Belle's husband say to Belle that he saw Scrooge "Quite alone in the world, I do believe." Scrooge is troubled by this memory of his past and asks the Ghost of Christmas Past to "remove [him] from this place!" Remarkably, the Ghost replies:

> I told you these were shadows of the things that have been. That they are what they are, do not blame me! [5]

To which, Scrooge passionately exclaims

> Remove me!...I cannot bear it! [5]

If we are able to experience the totality of our lives (many world lives) at each point in the fourth dimension, we might not bear it either[1]!

There exists another incredibly helpful way to think of how dimensions are constructed. This method is called "dragging" [30].

> To see how lower and higher dimensions relate to each other, take any geometric object (like a point, line, circle, plane, etc.), and "drag" it in an opposing direction (drag a point to trace out a line, a line to trace out a box, a circle to trace out a cylinder, a disk to a solid cylinder, etc.). The result is an object which is qualitatively "larger" than the previous object, "qualitative" in the sense that, regardless of how you drag the original object, you always trace out an object of the same "qualitative size." The point could be made into a straight line, a circle, a helix, or some other curve, but all of these objects are qualitatively of the same dimension. The notion of dimension was invented for the purpose of measuring this "qualitative" topological property.
>
> Finite collections of objects (e.g., points in space) are considered 0-dimensional. Objects that are "dragged" versions of zero-dimensional objects are then called one-dimensional. Similarly, objects which are dragged one-dimensional objects are two-dimensional, and so on. [30]

Thus, a fourth dimensional object is a "dragged" version of a three-dimensional object. By negative thinking, a zero-dimensional object is a "dragged" version of a -1-dimensional object. Now, dragging a -1-dimensional object sounds incredibly challenging! We can relate to Alice in Lewis Carroll's masterpiece *Through the Looking Glass* [3], who is having extreme difficulty in handling looking-glass cake:

> "What a time the Monster is, cutting up that cake!" [3]

[1] This could give a quite dark-humorous explanation of referring to the fourth dimension as *duration*; more like [en]duration.

As the Unicorn famously says,

> You don't know how to manage Looking-glass cakes," the Unicorn remarked. "Hand it round first, and cut it afterwards. [3]"

One can also use a boundary argument to visualize four dimensional objects:

- The boundary of a 1-dimensional object is two 0-dimensional objects.
- The boundary of a 2-dimensional object is four 1-dimensional objects.
- The boundary of a 3-dimensional object is six 2-dimensional objects.
- The boundary of a 4-dimensional object is eight 3-dimensional objects.

Thus, to visualize, say, a four-dimensional looking-glass cake, we could visualize the boundary as eight three-dimensional cubes. *This is quite unfriendly.*

Additionally, we can also employ color to perceive the fourth dimension. Let us formalize this as follows. We say a point in the fourth dimension (Euclidean space) can be represented by a 4-tuple of the form (x, y, z, w), where x, y, z are the standard Euclidean coordinate axes, but w is the fourth dimensional axis we call color; the color dimension. As such, we could say that two objects overlapped in the fourth dimension if their (x, y, z) coordinates were identical and they were the same color. For example, (x, y, z, w_{pink}) would not overlap with (x, y, z, w_{cyan}). Nor would objects with varying shades of Pink overlap. Thus to work via color in the fourth dimension means that a point in $\mathbb{R}^4$ is a point in $\mathbb{R}^3$ with color.

2 Artemis in the Fourth Dimension

We now practice working with the color dimension. Let us consider a snapshot of Artemis sleeping soundly [Figure 1].

Fig. 1 Artemis Sleeping [Image @ShannaDobson]

This is one of her favorite past times. Viewing this image from the perspective of our ordinary three-dimensional space, we can see Artemis is sleeping cozy and sweet.

Now, for an exciting challenge, we can also view this image in terms of our color dimension. We will view a point in $\mathbb{R}^4$ as a point in $\mathbb{R}^3$ that has a color. We denote bubblegum pink the ana direction in the fourth spatial dimension [the positive w-axis]. We denote royal blue the kata direction in the fourth spatial dimension [the negative w-axis]. We denote by the color snow white a w-coordinate that equals the value zero.

This image is now incredibly exciting to analyze! We could definitely say that Artemis is sleeping, as her snow white fur, fluffy tail, and whiskers have remained in the third dimension, since their fourth dimensional w-coordinate is zero.

But her bubblegum pink nose, paw pads, tummy, and ears have wandered into the Fourth Dimension! Sounds just like Artemis! Likewise, her royal blue eyes are also in the Fourth Dimension, but in the opposite direction! Perhaps this is what is called a *light sleep*.

Additionally, we could compose a series of three images of Artemis:

1. The first image displays Artemis asleep and she's entirely snow white.
2. The second image displays Artemis asleep and she's entirely bubblegum pink.
3. The third image displays Artemis asleep and she's entirely snow white again.

Fig. 2 Pink Artemis, Dreaming [Image @ShannaDobson]

We could imagine that the image of pink Artemis [Figure 3] corresponds to her dreaming. That is, whenever Artemis is asleep entirely pink, she is in a dream state. Thus, when Artemis dreams, she goes to the Fourth dimension. How fitting!

2.0.1 Cheshire Cat

Viewing Artemis sleeping while her pink nose and ears are in the fourth dimension conjures up an image resembling the Cheshire Cat, which is a brilliant character in Lewis Carroll's Alice's Adventures in Wonderland [2].

The Cheshire Cat has the remarkable ability to make his body slowly or suddenly disappear, leaving only his head, while in full conversation with someone, particularly Alice.

> ”Did you say pig or fig?” said the Cat.
>
> ”I said pig,” replied Alice; ”and I wish you wouldn't keep appearing and vanishing so suddenly; you make one quite giddy!.”
>
> ”All right,” said the Cat; and this time it vanished quite slowly, beginning with the end of the tail, and ending with the grin, which remained some time after the rest of it had gone.
>
> ”Well! I've often seen a cat without a grin,” thought Alice; ”but a grin without a cat! It's the most curious thing I eve saw in all my life!” [2]

Interestingly, we could say that the Cheshire Cat's vanishing denotes his moving into the fourth dimension. In particular, we could thus say that the Cheshire Cat is a marvelous creature that can simultaneously exist in three dimensions and in four dimensions at once! All the while remaining in full conversation with you.

Continuing thus, we could construct the entire set of characters in *Alice's Adventures in Wonderland and Through the Looking Glass*, including the Mad Tea Party, from the point of view of the color dimension.

Fig. 3 Cheshire Artemis [Image @ShannaDobson]

3 Four Dimensional Objects

3.1 Tesseract

One of the most well-studied four dimensional objects is the Tesseract, which is a four dimensional cube. As discussed in [11], Charles Hinton's popular method for constructing such a four dimensional cube is the following:

> The simplest form of Hinton's method is to draw two ordinary 3D cubes in 2D space, one encompassing the other, separated by an "unseen" distance, and then draw lines between their equivalent vertices...The eight lines connecting the vertices of the two cubes in this case represent a single direction in the "unseen" fourth dimension [11].

Let us repeat what is going on here. The eight connective lines represent *one direction* in the fourth dimension. It is canonical to assume that there are three directions associated to three-dimensional Euclidean space, corresponding to moving along the positive and negative coordinate axes. Thus, in four-dimensional Euclidean space, there would be eight directions, where the additional two directions, which Hinton coined *ana and kata* represent the two ways of moving along the novel fourth-dimensional w coordinate. Thus, the eight connective lines represent one of ana or kata.

3.2 Gedankenexperiment

We offer the following thought experiments as warm-up visualization practices before we present our novel illustrations.

3.2.1 Throwing a Dart in the 4th Dimension

Imagine you are in a room, and you are about to throw a dart across three-dimensional space to hit a dartboard. It would be possible for you to hit the dartboard since it is facing you without any obstacles, and there is no visual obstruction in the line of sight from you to the dartboard. You will probably say it would be impossible to hit the dartboard from the same spot if the dartboard is placed within an enclosed box. However, what if it was possible to hit the dartboard inside the box? That is,

Question 11. *What if, in the 4th dimension, there is one additional variable that allows you to manipulate the physical coordinate of "sidedness"?*

In that case, you will be able to traverse the dart physical coordinate to hit into the inside of the box. Recall, there are many interpretations of sidedness. The inside and outside of a surface are canonically dependent on perspective and a notion of origin. What could be considered the inside is usually from an *outside* perspective, and very well may be the reverse for a perspective originating from the *inside*.

3.2.2 Möbius Strip in Water

We can consider the following colorful illustration of sidedness in terms of w. Recall, a Möbius strip embedded in three-dimensional Euclidean space is a compact 2-manifold which only has one side. Let us hypothetically imagine putting a Möbius strip into water. The water is like negative space. The Möbius strip is like positive space. Now, as the Möbius strip is non-orientable, there exists a path on its surface that changes the two local transverse orientations. If there exists a transverse normal vector projecting outward from the Möbius strip, could the water project an inverse vector inward? By switching both of those directions, the strip will show its "insideness" in a topological sense. Whereas, the switching of local transverse coordinates is characteristic of non-orientability in $\mathbb{R}^3$, it is the very opposite in $\mathbb{R}^4$.

If the negative space is a body of water surrounding the Möbius strip, removing the strip instantaneously from inside the water will create a negative space. That negative space is the negative vector of the waterbody. Upon reversing the vector direction, only the negative space is visible. The same applies to the Möbius strip, where upon flipping the vector direction, only the exterior space is visible.

3.3 Fourth Dimensional Ice-Cream Parlor

"Topos tigers dream of looking-glass ice-cream" [6]

Imagine the following thought experiment: Let us imagine a fourth dimensional ice-cream parlor. *Where do we even begin?*
To start, we can work in the color dimension. That is, a point in $\mathbb{R}^4$ is a point in $\mathbb{R}^3$ with color.

Four things we know/assume immediately:

1. As a 4D creature has a 3D retina, they can see the outside faces and inside of the ice-cream parlor all at once.
2. The ice-cream would not be *inside* the ice-cream parlor.
3. Given that the concept of inside/outside is not well-defined for a 4D creature, it is entirely speculative that 4D creatures would consume anything, let alone brightly colored ice-creams.
4. In our 3D world, we need a temporal dimension to make ice-cream. We assume 4D creatures have a temporal dimension(s).

We will leave aside this consumption problem (item 3), and try and proceed mathematically.

Now, from our statement above, we know that two objects overlap in the fourth dimension if their (x, y, z) coordinates are identical and they are the same color. This thought experiment is tricky because in our 3D world, we differentiate ice-creams by their colors, which normally represent their flavor; i,e.pink-bubblegum flavored ice-cream is pink. However, we are also using color to represent the w 4th-Dimensional coordinate.

A possible way out is to categorify colors in two-ways: colors as 1-categories and colors as 2-categories. Such a classification could account for the color of ice-creams and the color of w. We have recently conjectured that Pink is a 2-category [6]. Thus,

Conjecture 1. *w-coordinate colors are 1-categories, and ice-cream colors are 2-categories.*

Thus, ice-cream colors are defined in terms of enrichment. One way to account for the fact that the inside = the outside of the ice-cream, perhaps ice-cream colors are reflections. Then, perhaps, categorically speaking,

Conjecture 2. *Fourth-dimensional ice-cream is a functor that associates to every w-color-coordinate an ice-cream color.*

In particular,

Conjecture 3. *Fourth-dimensional ice-cream is a functor, a reflector functor.*

Another way is the following:

We consider the category Set where the objects are sets of 4-tuples (x, y, z, w) and the 1-morphisms are total functions between the sets. Then, four-dimensional ice-cream is a functor that associates to every 4-tuple a color.

Conjecture 4. *Ice-cream is a set-valued functor of points [10] on **Set** where the Hom-sets are similarity transformations.*

Thus, we have reduced and refined the incredibly challenging problem of visualizing a fourth-dimensional ice-cream parlor, to visualizing ice-cream colors as higher-categorical diagrams. Working functorially in the w-color-coordinate, we can thus conjecture universal properties of dimensionality, which is bright and promising.

Continuing thus, we could construct the entire Mad Tea Party categorically.

We note that many techniques for working in higher dimensions will bypass entirely any visualization of those higher dimensions, and instead focus on the axiomatization/equations alone towards a consistent theory independent of visualization. In this sense, visualization is seen as an impediment to the axiomatization of the mathematics of a fourth-dimensional object.

We now take a slightly different approach and focus on the visualization of the fourth spatial dimension, until our skills fail us. Then we will try categorical methods to ”see” the fourth spatial dimension in other ways.

4 A Topology of Mental Imagery

We open with a question we cannot answer:

Question 12. *Why is it so difficult for us to visualize a fourth spatial dimension, and, more precisely, spatial dimensions higher than 3?*

We often think that we have an incredible and limitless ability *to imagine*. It is as if we can construct a *space* of mental imagery as boundless as Fantastica.

Fig. 4 Functorial Ice-Cream [Image @ShannaDobson]

> ...In Fantastica, there are no measurable distances so that 'near' and 'far' don't at all mean what they do in the real world. They vary with the traveler's wishes and state of mind. Since Fantastica has no boundaries, its center can be anywhere - or to put it another way, it is equally near to, or far from, anywhere. It all depends on who is trying to reach the center. [9]

But visualization of higher mathematical concepts such as a fourth spatial dimension, might make us think otherwise.

One can ask if there exists a mechanism wherein the *space* of mental imagery affords certain computational complexities, so that when that limit is reached, something like synesthesia results. It is as if too many dimensions of visualization are compressed into mental imagery, knottedly; like a Grothendieck topology on a category of mental imagery, though mental images have no extensive spatium. The same could be said for the neo-cortex's role in imagination.

More precisely stated,

Question 13. *What exactly is the topology of the space of mental imagery?*

Question 14. *What is the role of computational complexity in imagination and mental imagery?*

Question 15. *What is the mechanism for storing and recalling mental images of higher information, when our neurons, our basis for computation, are three dimensional objects?*

Question 16. *What possible insight can we gleam from the Phantasia spectrum, particularly Hyperphantasia, on the colorful process of the imagination, storage, and recall of higher dimensional images?*

William Burroughs is famous for producing passages of text which cannot conjure up an ”image in the mind.”

> ...they have place their thesis beyond the realm of fact since the words used refer to nothing that can be tested. The words used refer to nothing. The words used have no referent. [1]

We have a slightly different situation: The words [mathematical symbols] used have a precise referent, but we cannot visualize it; it exceeds visualization. The words have no visual referent.

Many questions follow:

Question 17. *Can we only mentally visualize what we already know?*

Question 18. *Do we not have enough imagination to visualize fourth-dimensional complexity?*

Applying negative thinking, one can ask what came before mental imagery and imagination.

4.0.1 An Argument from Active Inference

There exists a plausible explanation using the active inference formalism, centering on generative models and what is called *Zones of Bounded Suprisal* [14] [15] (Private correspondence Dr. Hector Manrique, Dr. Michael Walker).

> Sentient creatures create models of the sensed world, these generative models are mutable and attuned to their specific environments by adaptation via natural selection because of their active exploration and the ensuing capitalization on the sensory feedback that acting upon the world produces in brains that are neurobiological inference machines.
>
> Our species, Homo *sapiens*, inhabits a perceptual 3-dimensional world and hence the generative models deployed to make sense of visual reality is 3-dimensional. Making sense of, or even mentally representing, a 4-D visual reality requires generative models that have been selected in response to a sensory 4-D world. In Manrique & Walker [14] [15] we introduced the Zones of Bounded Surprisal (ZBS) concept that roughly refers to the disparity from predictions the surprisal of which is bounded for any given species and correspondingly apprehended by it. A species like ours that has a wide ZBS can register and encode observations that differ considerably from predictions, whereas those species with a narrow ZBS are impervious to observations that even slightly differ from what they had predicted. That is to say, they maintain their predictions in toto, even in the face of contradictory sensory information. Plausibly, the difficulty of even imagining objects in 4-D space may well derive from the outlandishness of matters that fall outside our ZBS. We hypothesize that any species like ours that experienced 4-D matters often could potentially develop generative models adapted to this sensorium and thus register, envisage, and imagine such matters because they would fall inside its ZBS. However, previous exposure to 4-D visual stimuli would be required in order to generate the appropriate world-models.
>
> Perhaps the remarkable capacities of the human brain's short-term and long-term working memory, long-term episodic, semantic and procedural memory, and prospective memory

("of the future"), facilitate our awareness ("in our mind's eye") not only of the forward passing of time and our memory of time passed (cf., Ebenezer Scrooge), but also of the frustration, on waking from sleep, that we remember having experienced during dreams when we "saw" ourselves repeating behaviour over and over again, trapped out of time "in a loop" as it were (cf., Alice and Red Queen's race), instead of our behaviour developing in the temporal way that it "ought" to have done from a normative temporal perspective taken from stored memories of lived experiences and appropriate world models; it is as if our dreaming neurobiological inference machine (brain) had gone into a default mode and failed to visualize or envisage forward behavioural developments in future time."

To *picture in our mind* a certain idea/object usually means to construct a visual representation of that idea/object where that representation is not spatially extensive; it does not exist in a spatium. The question is, well where does this mental image exist? Oftentimes mental imagery can be fleeting. Well, could we then say that transient existence is still existence *somewhere*? Are some thoughts too quick to imagine?

To put our questions more bluntly,

Question 19. *If thoughts are electro-chemical impulses produced by firing neurons creating propagation of electrical signals, then what does it mean to see one's thoughts? on any scale of duration of these thoughts?*

These are terribly strange questions and another follows.

Question 20. *Where is the mental image of the fourth spatial dimension?*

When we are asked to visualize an ice-cream parlor in our normal world [consisting of three-dimensions of space and 1 dimension of time], being mindful of the Phantasia spectrum, we can usually *picture in our mind*:

- The decor of the parlor;
- The sparkling colors of the ice-cream;
- The rich flavors of the ice-cream;
- The sight of colorful condiments like sprinkles, whipping cream, and cherries;
- Our visualizations may even contain smells.
- We can smell the images of the chocolate syrups and the freshly baked waffle cones.

Yet, when asked to visualize a fourth dimensional ice-cream parlor [consisting of four spatial dimensions, and, say, two time dimensions [2]], the mental image may not appear. We feel it is incredibly difficult "to see" it. Where is the image? In [7] we posited a notion of ∞-visual epistemology to consider this very situation, where there are epistemological obstructions to visualizing an image. We might proceed by a

[2] We are assuming, perhaps completely incorrectly, that higher dimensional worlds contain associated temporal dimensions as well. In the case of our fourth-dimensional ice-cream parlor and the assignation of two time dimensions, we are assuming that two time dimensions would capture what 1 dimension of time captures in our world consisting of three dimensions; *movement through*. It sounds terrifying to be able to move through a space the points of which correspond, temporally, to the equivalent of experiencing every moment of your life(s) simultaneously; yet another image that is difficult to image!

visual analogue of dimensional analogy. But trying to drag 3-dimensional ice-cream cones to produce a fourth-dimensional ice-cream cone seems incredibly daunting.

4.1 Negative Thinking for Visualization

Perhaps a categorical perspective on visualization might aid us visualizing seeming limit paradoxes of a -1-dimension and a fourth spatial dimension. One can immediately ask:

Question 21. *How can we interpret visualization categorically?*

In [6] we have investigated visual sight as categorical site; what we call *sight as site*. We discussed *sight as site* using representable functors in a *pro-object*-formalism and posited the color *Pink* as a 2-category, among other conjectures involving color as representability. We illustrated functorial sight through our novel constructions of Categorical Ozma and Cinderella, the Site of Oz, and a pro-*Through the Looking-Glass*.

As an extension of our work on *sight in terms of representability* (representable functors), we should be able to visualize a fourth spatial dimension functorially. Thus, we can perhaps visualize a -1-dimension and a fourth spatial dimension by treating these visualizations as n-functors for $n = -1$ and $n = 4$.

We recall the definition of an n-functor, a definition which increases in difficulty as n gets larger:

Definition 7. *([21] Idea) An n-functor is simply a functor between n-categories. Similarly, an ∞-functor is a functor between ∞-categories.*

A few examples of n-functors are the following [21]:

Example 4. *Implication: a (-1)-functor*

Example 5. *Function: a 0-functor*

Example 6. *Functor: a 1-functor*

Example 7. *2-Functor, includes pseudo functor*

Example 8. $(\infty, 1)$*-functor*

Example 9. (∞, n)*-functor*

We expand on the (-1)-functor [implication] and the 2-functor in hopes that it might aid our visualization attempts below.

> An implication may be either an entailment or a conditional statement; these are closely related but not quite the same thing.
>
> 1. Entailment is a preorder on propositions within a given context in a given logic...
> 2. A conditional statement is the result of a binary operation on propositions within a given context in a given logic...

> You can think of entailment as being an external hom (taking values in the poset of truth values) and the conditional as being an internal hom (taking values in the poset of propositions)... [20]

> As a (-1)-category is simply a truth value, so a (-1)-functor is simply implication. [20]

A 2-functor is informally defined as follows:

Definition 8. *([19] 1. Idea)*

> *A 2-functor is the categorification of the notion of a functor to the setting of 2-categories. At the 2-categorical level there are several possible versions of this notion one might want depending on the given setting, some of which collapse to the standard definition of a functor between categories when considered on 2-categories with discrete hom-categories (viewed as 1-categories). The least restrictive of these is a lax functor, and the strictest is (appropriately) called a strict 2-functor.*

We now recall the definition of an (n, r)-category in higher category theory.

Definition 9. *([17]) An (n, r)-category is a higher category such that, essentially:*

- *all k-morphisms for $k > n$ are trivial.*
- *all k-morphisms for $k > r$ are reversible.*

Using this definition along with negative thinking, we can imagine what a $(0, 1)$-category looks like.

Definition 10. *([18] 1. Idea) In the context of higher category theory / (n, r) categories, a poset [partially ordered set] is equivalently regarded as a $(0, 1)$-category.*

$(0, 1)$-categories play a major role in logic, where their objects are interpreted as propositions, their morphisms as implications and limits/products and colimits/coproducts as logical conjunctions and and or, respectively.

Question 22. *How far can we take negative thinking? Can we use it to consider what, categorically, came before color? Color as a dragged object?*

Question 23. *Can we use it to consider what, categorically, came before dimension? Fourth dimension as a dragged object?*

5 Five Illustrated Techniques for Visualizing the Fourth Spatial Dimension

We now present our ideas of visualizing the fourth spatial dimension. Each of the five ideas has a different conception of what the w-coordinate *denotes*. Thus, each of the five visualization techniques represents a different way of accessing the fourth spatial dimension. As such, we call these *concept points*.

5.1 Concept Point 1

Our first concept point is the following:

Conjecture 5. *Concept Point 1: Inside/Outside is a perspective of simultaneity: infinity perspective.*

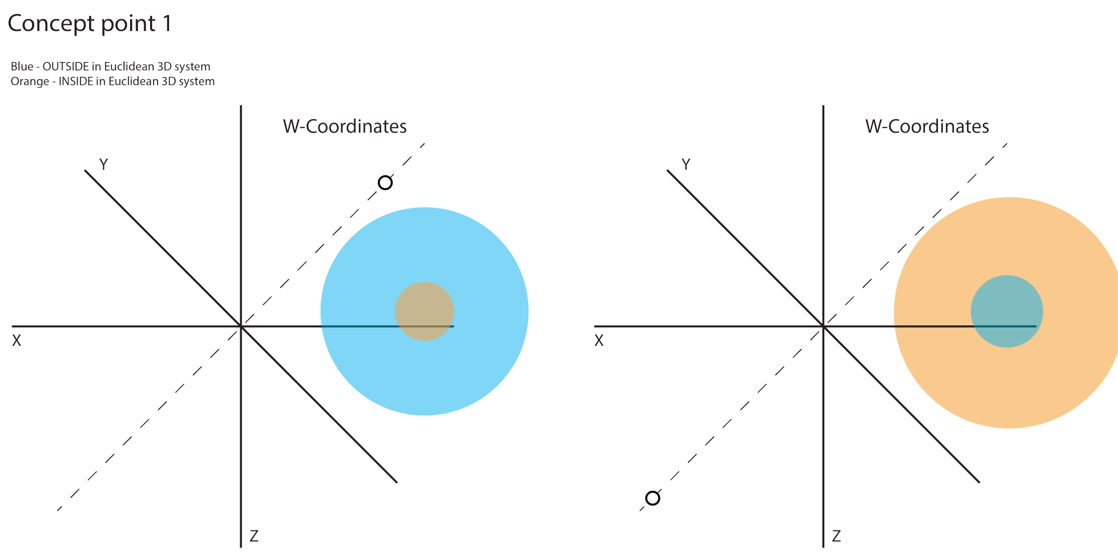

Fig. 5 First characterization of the fourth dimension via a perspective of inside/outside simultaneity: ∞-perspective. [Image @Zihang(Jim) Zhong]

Recall, the three-dimensional representation of the four dimensional Klein Bottle has one-side and no volume. If we move along what *should be* the inside of the bottle, we would eventually reach the outside and then arrive again at the inside, while the bottle still only has one opening in three-dimensional Euclidean space. Thus, the inside of the bottle seamlessly connects with the outside of the bottle. Importantly, this illustrates that there is no concept of inside/outside of this bottle. Perhaps the Klein bottle does not have a complex geometric configuration in three-dimensional space. Rather, four-dimensional space simply reads it as having a seamless connection from its outside to its inside.

In three-dimensional space, object A can visually obstruct object B if object A is ”in front of” object B along the line of sight. Also, an object has a clearly defined *inside* and a clearly defined *outside*. In the fourth dimension, both of these characteristics are inhibited. That is, there exists no physical blocking of any perspective from switching between ”in front of” and inside/outside; there is no self-intersection of the neck of the bottle.

We can ask how we could switch between perspectives that are physically blocking one perspective in Euclidean 3-D space? One conjecture of how this can happen is if both the inside/outside can be observed simultaneously from the same perspective. Thus, there would be no object collision; no object blocking. To this end, we let the w-coordinate dictate how much of each of the perspectives unfold or fold between the inside and the outside. As the w-coordinate changes it would correspond to traveling on the inside of the object, and eventually reaching the outside. Thus, in Figure 2, the w-coordinate dictates the *unfolding* between the inside and the outside of any given object.

In the illustration (Figure 5), we used two colors (blue and light orange) to show the inside and the outside of the same object. As the w-coordinate changes, one "side" would enlarge and the other "side" would shrink in proportion. In this two-dimensional illustration, the two sides might appear on top of each other. But, in four-dimensional space, they are not. One can try to imagine viewing both perspectives simultaneously, as clear as day in the four-dimensional world.

Another possibility is the following: In three-dimensional Euclidean space, the the 3-tuple (x, y, z) controls the positional factors of the points in the space. Could the w-coordinate control the rotational factors as well? By changing the w-coordinate, it will rotate each individual coordinate on the entire grid inside the area of effect. By rotating the w-coordinate you will eventually return all the rotational factors of each individual coordinate to its original w-coordinate, thus completing the inside-to-outside-to-inside transformation. This characterization is like the rotational equivalent of unknotting, earlier alluded to. In terms of negative thinking, we can think of rotation as an (-1)-functor.

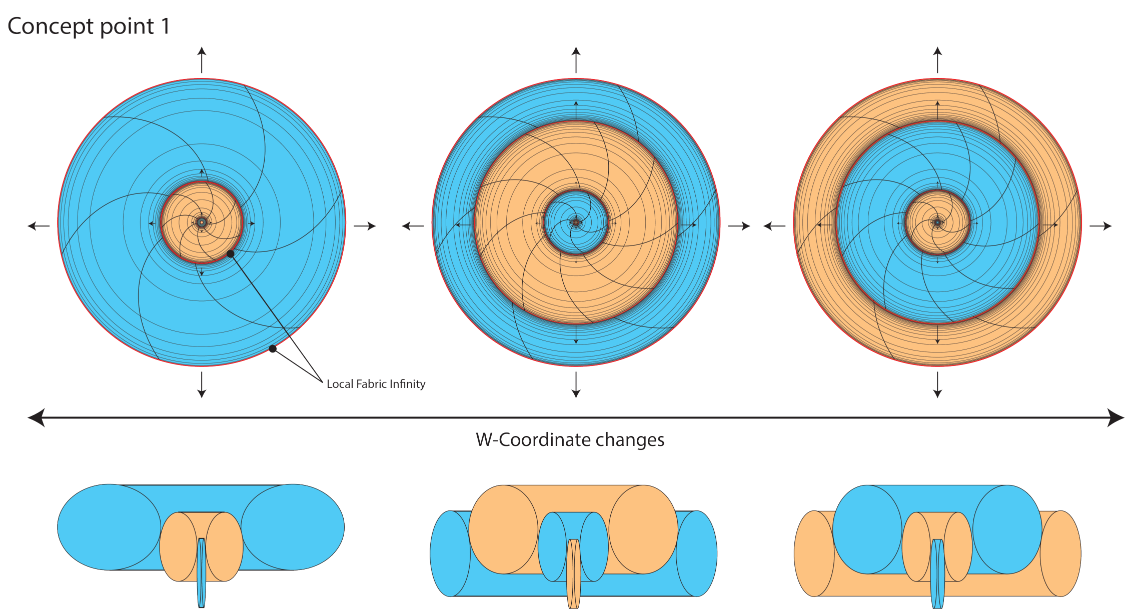

Fig. 6 Second characterization of the fourth dimension via a perspective of inside/outside simultaneity: ∞-perspective. [Image @Zihang(Jim) Zhong]

In the second illustration (Figure 6), the top row of images depicts an observer looking at a four-dimensional object from a one-point perspective. The blue region denotes the ”outside” of the given object in a Euclidean three-dimensional space. The light orange represents the “inside”. As the w-coordinate changes, the object will shift in between the inside and outside perspectives.

We can attempt to imagine this shape as a distortion in the surface (fabric) of the four-dimensional object, rather than as a donut-like shape. Any object would be so distorted in the 4th dimension. For example, consider an amazon box that has been opened and flattened. It has 6 distinct faces. Considered as a three-dimensional object in three-dimensional Euclidean space, not all of the edges connect with each other. However, in the fourth dimension all the edges should be connected with each other.

One of the ways to visualize this is to distort our vision in a circular manner. In order to transition from outside to inside, the perspective will be compressed to almost infinity before becoming less distorted as the inside perspective appears and becomes larger. The local fabric near the infinitely compressed surface is marked with red circular lines, and shows the seamless connection between inside and outside perspectives. The circular black contour lines show how both perspectives get more compressed the closer they are to the the local fabric near the infinitely compressed surface. The entire cycle of *perspective uncompressed and re-compressed* continues infinitely as the w-coordinate moves. But only at any given w-coordinate does a certain perspective become the largest. Any shrinking perspective will either shrink to the side(when w is moving in a direction), or shrink towards the middle (when w is moving in the inverse direction).

The second row of image shows side views of this movement, where the inside perspective gets larger towards the middle and then becomes compressed to the side. Theoretically this means you can see every single infinite inside/outside perspective! This shows how there is no blocking of perspective in four-dimensional space.

This is just one way of attempting to show how a fourth-dimensional perspective would unfold in a two-dimensional illustration. This infinity behavior can also be described as a sort of tunnel vision. As the w-coordinate changes, different parts of the tunnel become closer or further away. However, every other piece of the tunnel will be the inside/outside perspective, with one able to see the entire tunnel when they travel in one direction, and vice versa.

5.2 Concept Point 2

Our second concept point is the following:

Conjecture 6. *Concept Point 2 is the dual-axis perspective: mirror movement.*

Revisiting the question of perspective, one can ask what is a dual perspective of looking at the inside/outside? The above conjectures explored how perspectives and

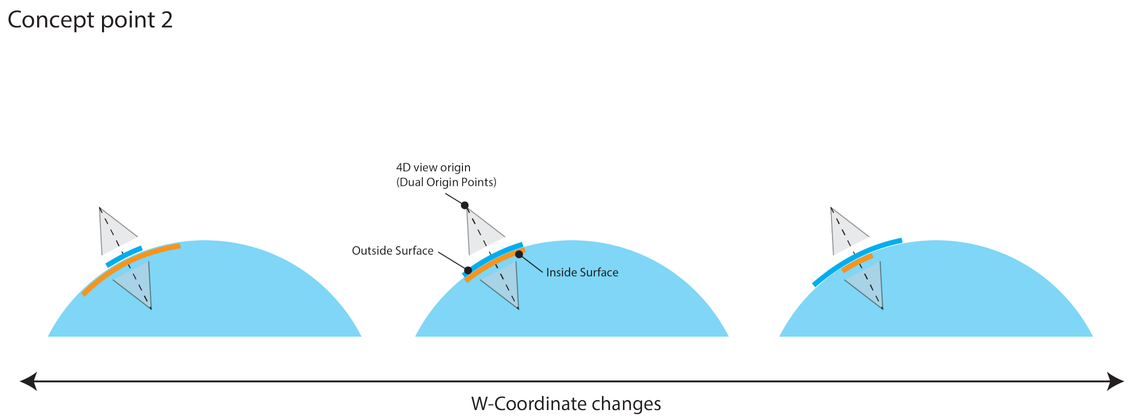

Fig. 7 Characterization of the fourth dimension via a dual-axis perspective of mirror movement. [Image @Zihang(Jim) Zhong]

w-coordinates could be positioned to allow unobstructed view of both the inside and outside simultaneously. But what exactly is the inside and outside in this sense?

This conjecture places the dual perspective in a three-dimensional sense to try to visualize how a four-dimensional perspective would work. To be *inside of an object* in the literal sense refers to an inner space of an object, which is usually not visible directly from the exterior. If we use *this* as the basis of inside/outside then the dual perspective would simultaneously be positioned opposite of each other, with one viewing another directly with the surface in between. The *w*-coordinates in this sense would shrink or enlarge one of the sides and the other depends oppositely on the *w* direction in which it moves.

We can try to imagine this as if we are shrinking/enlarging the focal length on a camera. In three-dimensional space, nothing physically of that object is changing. But, in four spatial dimensions, one perspective of that certain view is enlarging, and in another perspective the surface is shrinking. Going further, we can now think that one surface is physically shrinking, and another identical surface, but from the other perspective view, is physically enlarging in four-dimensional space.

The illustration above (Figure 7) shows how the dual perspective would be positioned in viewing a surface of the sphere. The illustration shows the inside and outside surface of the same section of sphere, which the *w*-coordinates controls. As the *w*-coordinate changes from left to right, the light orange region representing the inside of that surface shrinks, and the outside of that surface enlarges. But, on the three-dimensional side (the blue sphere) the surface is not stretched. The dual perspective points are facing directly toward one another, and they are in a state like an entanglement, where an entanglement takes the form of a categorical adjunction. Else, they could be in a state resembling superposition; where one moves the other follows.

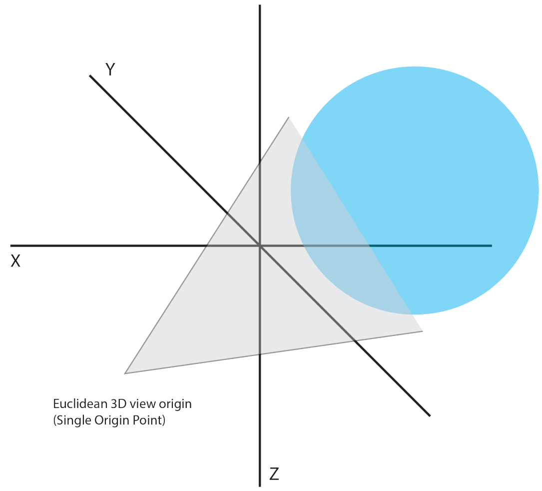

Fig. 8 One [Ice-cream] Cone of Vision in Three Dimensions. [Image @Zihang(Jim) Zhong]

We provide an additional perspective drawing (Figure 8), which illustrates one cone of vision in three dimensions, contrasted with two cones of vision in four dimensions.

5.3 Concept Point 3

Our third concept point is the following:

Conjecture 7. *Concept Point 3 is a spherical perspective.*

This conjecture explores what we call the *single-origin volume-perspective* in the fourth-dimension. Now that we have explored our conjectures of multiple-origin perspective, what if the perspective origin in the fourth-dimension is still one point? What could the w-coordinate denote in order to show the transition between, say, the inside and the outside of an object? Moreover, what would the inside/outside represent in this perspective?

In this conjecture, the fourth-dimensional perspective will no longer be based on a single point of origin that is projecting a singular two-dimensional-depth view. Rather, the depth will be three-dimensional. The perspective will be a single origin projecting a spherical perspective. The x, y, z axes *wrap around* the perspective.

One may ask what would a 3D perspective in 4D space look like. To start with, the perspective could look like a sphere, with the x, y, and z axes becoming a loop that connects with itself. Any movement in those coordinates essentially controls the sphere's size and location (how big it expands/contracts). If the inside/outside still has a similar meaning to the conjectures previously, then the w-coordinate could denote the rotation of each three-dimensional Euclidean axis. Since we can revolve in

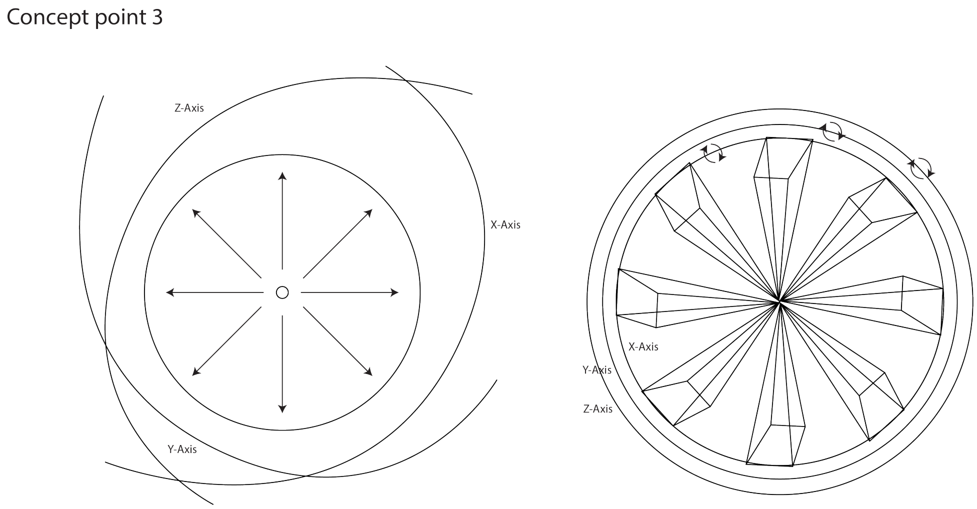

Fig. 9 Characterization of the fourth dimension via a spherical perspective. [Image @Zihang(Jim) Zhong]

between each individual coordinate to rotate to the inside and to the outside anyway possible, it makes the w-coordinate become three individual variables: w_1 controls the x-axis rotation; w_2 controls y-axis rotation; and w_3 controls the z-axis rotation. At any given space, each of the w_i can rotate to any angle, while the remaining w_i rotate to the opposite angle. Since each axis is like an infinite loop, what this does is twists the object into showing the inside/outside of the object without the forms ever breaking off with one another. We can imagine this *twisting* behavior categorically as a reflector functor.

The illustration above (Figure 9) shows how the x, y, and z axes' position is *wrapped around* the center perspective. Changing any of the axes essentially enlarges or shrinks their sphere of influence. The illustration on the right shows how w_1, w_2, and w_3 controls each individual axis. We note that the three variables are not directly related. For example, rotating a certain portion of the x-axis 180 degrees to the inside only distorts its connection with the y-axis, whose rotational value still shows the outside perspective. The ”axes” behave more spherical(planar)-like then like linear-axes.

5.4 Concept Point 4

Our fourth concept point is the following:

Conjecture 8. *Concept Point 4 is a three-dimensional retinal perspective path.*

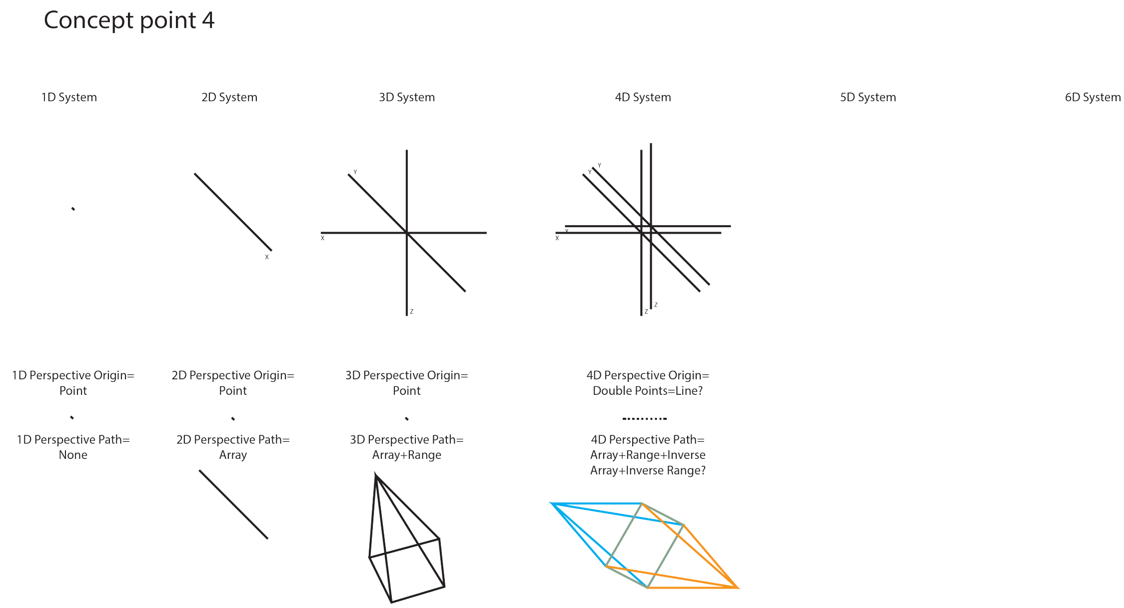

Fig. 10 Characterization of the fourth dimension via a three-dimensional retinal perspective path. [Image @Zihang(Jim) Zhong]

In a 1-dimensional Euclidean space, the perspective is one-point and it has no depth. In a 2-dimensional Euclidean space containing 1-dimensional lines, the perspective still originates from one point. Thus, this is still one-point perspective, but that perspective path is a line, which now has depth. In a three-dimensional Euclidean space containing planes, the perspective originates from a point, but the perspective path is now two-dimensional with depth (Figure 10).

Continuing, in the 4D physical system, with the conjectures mentioned above, there exists either two origins of perspective, or there can exist a single point of origin, but with a three-dimensional perspective. Each of these has a three-dimensional perspective path, which is like a three-dimensional retina. Thus, these perspectives both have depth, but also something additional. We call it distortion for now. The distortion that is associated with w-coordinate dictates which shape is less distorted and which is more, just like depth dictates the distance of an object from the designated perspective.

We extend our construction to the fifth dimension for future conceptions. In a fifth-dimensional Euclidean space, the perspective path is four-dimensional. Thus, there could be something that starts to dictate multiple symmetries like the class of similitudes, where multiple Euclidean systems could start to appear. In a six-dimensional Euclidean space, the perspective path is five-dimensional. This is so complicated to imagine, but perhaps there could exist entanglements as complex perspectives like a *5-depth*. Categorically, this could resemble a 5-functor.

5.5 Conjecture Point 5

Our fifth and final concept point is the following:

Conjecture 9. *Concept Point 5 is topological positive/negative space.*

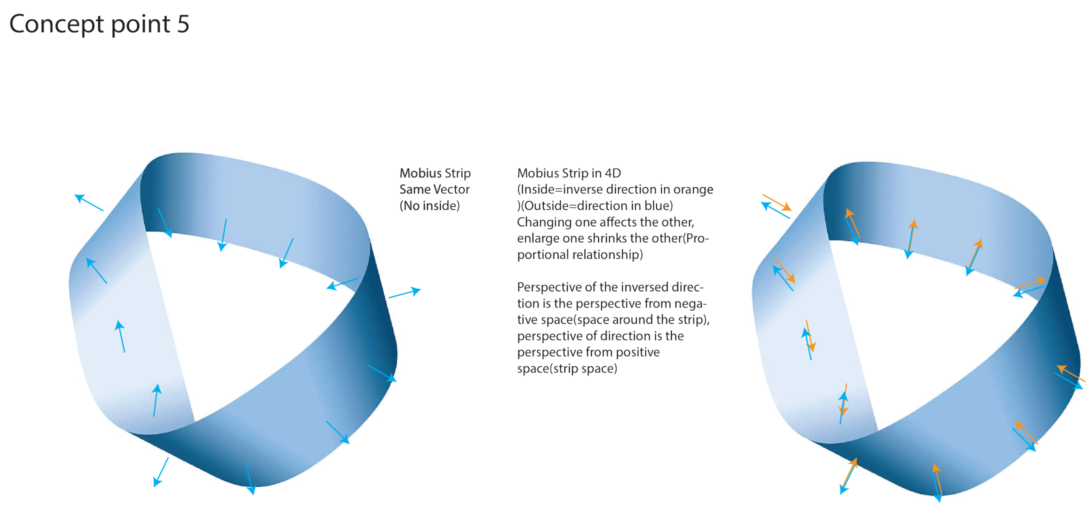

Fig. 11 Characterization of the fourth dimension via topologized positive/negative space. [Image @Zihang(Jim) Zhong]

Recall, a Möbius strip embedded in three-dimensional Euclidean space is a compact 2-manifold which only has one side. Thus, there is no inside (inverted direction). We can ask if the fourth dimension could introduce a new variable that can allow us to topologize the notion of positive/negative space. Up to this point the definition of inside/outside still relies on the concept of a three-dimensional space reference. This conjecture combines the previous concept of dual perspective, but now the perspective is tied to the direction of the normal vector. Up to this point the definition of inside/outside still relies on the concept of a three-dimensional space reference. This conjecture combines the previous concept of dual perspective, but now the perspective is tied to the direction of the normal vector. As there exists a natural embedding of the Möbius strip into $\mathbb{R}^4$, the Möbius strip will never *unknot* its half-twist. However, if the normal vector of the Möbius strip is projecting *outward*, could the negative space that surrounds the strip be projecting an inverse normal *inward*. If so, by switching both of those directions simultaneously, the strip will show its "insideness" in a topological sense.

The illustration above (Figure 11) attempts to illustrate this conjecture. The extra variable in this case is the normal vector direction, which is represented with orange. Any enlargement of the vector in a certain direction will cause the shrinkage of the inverse vector in the opposite direction. As the w-coordinate moves, these vectors interchange, and at a certain stage, the entire Mobius strip will be inverted.

Utilizing categorical negative thinking, we can imagine the negative space as a body of water surrounding the Möbius strip. Removing the strip instantaneously from inside the water will create a negative space. That negative space is the negative [normal] vector of the waterbody, and upon flipping/interchanging the vector's direction only the negative space is visible. The same applies to the Möbius strip, where when flipping/interchanging the vector's direction only the exterior space is visible. There exists a play of which normal vector is visible, resembling sight as site [6]. This is another way of showing that inside = outside in the fourth dimension; that is, the concept of inside/outside does not exist.

6 Conclusion

We have presented our consequential conjectures, which are colorful illustrations of possible interpretations of fourth spatial-dimensionality. It appears that visualizing a fourth spatial dimension is like a limit paradox: have we discovered a limit to the concept of mental imagery? If so, imaginative visualizations could be restricted by our generative models; that, indeed, we cannot imagine the fourth dimension because we are not equipped with the requisite sensory apparatus to receive information from a fourth spatial dimension. It would be very exciting to explore various exercises in *picturing the fourth dimension*, to increase the complexity of mental-imagery-making in our generative models.

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