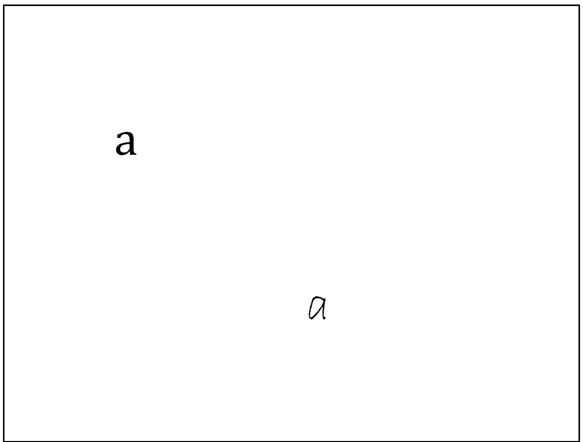
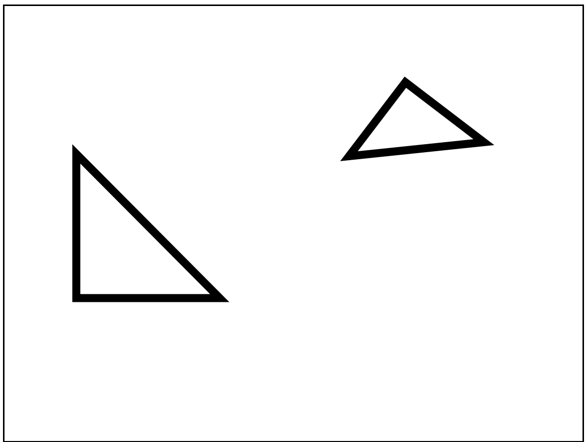
Øystein Linnebo. *Thin Objects* Oxford: Oxford University Press, 2018. Pp. x + 236. ISBN 978-0-19-964131-4

Here is a familiar pattern. We have a class of particular objects, which we call "tokens". This class is partitioned into sub-classes of tokens that are alike in some respect; to put it another way, there is an equivalence relation over the class of tokens. We also have a class of abstract types,[1] one for each equivalence class of tokens. Each abstract type stands for a commonality among the tokens in the corresponding equivalence class, which are said to be its instances.

Examples of the pattern are not hard to find. We might say of two token letters on the page that they are instances of the same letter type, or we might say of two token shapes that are similar (in the geometrical sense) that they are instances of the same shape type.



Two token letters of the same letter type.


Two token shapes of the same shape type.

Cardinal numbers, too, can be regarded as abstract types — here, arguably, the tokens are pluralities, and the equivalence relation is equinumerosity. Thus, for example, the plurality of cities in Wales and the plurality of species of flamingo are both instances of the cardinal number six.

Proceeding further into mathematics, we might say that each particular cyclic group of order six (e.g. $\mathbb{Z}/6\mathbb{Z}$ under addition, or the rotational symmetries of the regular hexagon under composition) is an instance of *the* cyclic group of order six.

[1] I call these things "abstract types" rather than simply "types" in an attempt to prevent any confusion between abstract types and the types of type theory.

I hope that my examples are sufficient to explain why some philosophers of mathematics (the "abstractionists", as Linnebo calls them) have thought that many mathematical entities are abstract types. Linnebo is clearly much influenced by earlier work by the "neofregeans" Bob Hale and Crispin wright.[2] One might also mention Kit Fine and Ed Zalta.[3] Linnebo's *Thin Objects* is the latest addition to the abstractionist literature — and it is a safe bet that the book will become *the* go-to work on the topic for researchers.

Thin Objects presents a theory of abstract types, and a theory of sets — for Linnebo, sets and abstract types are closely analogous. He discusses both metaphysical and epistemological issues. He also discusses the question of how determinate reference to abstract types and sets is possible. The book is both thorough and detailed — Linnebo is not one to leave loose ends. Somehow, it is also short (236 pages). Some parts of the book are intricate, but Linnebo does an excellent job at separating the difficult, more technical parts of the book from the easier, less technical parts — and the latter are written in exemplary clear prose. I wouldn't hesitate to include parts of the book in a seminar for upper-level undergraduates.

1. Sets

Linnebo develops (ch. 12) a version of the iterative conception of set. Suppose one speaks a language whose first-order quantifiers range over a domain $\mathcal{D}_0$ of *Urelemente*. According to Linnebo, one is then able to expand one's conceptual scheme, extending the domain of one's quantifiers to include one new set corresponding to each plurality of *Urelemente*. Call this new domain of quantification $\mathcal{D}_1$. One is then able to extend one's domain of quantification *again,* introducing new sets corresponding to the new pluralities of objects in $\mathcal{D}_1$. Call this new domain of quantification $\mathcal{D}_2$. One is then able to extent one's domain of quantification again, and again, and so on. Practical limitations aside, one could thus extend one's domain of quantification infinitely many times, take the union of the domains thus created, and then keep going …

In principle, proceeding in this way one could extend one's conceptual scheme so as to include sets of arbitrarily high rank. However, this still falls short of a of a truly *general* set theory — a set theory which covers sets of *all* ranks. So at this point, Linnebo develops a theory which describes *all possible further extensions of the domain.*

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2 See Wright's *Frege's Conception of Numbers as Objects,* or, for more recent work, Wright and Hale's *The Reason's Proper Study.*

3 See Fine, "Cantorian Abstraction: A Reconstruction and Defence," and Zalta's "Natural Numbers and Natural Cardinals as Abstract Objects."

To this end, he introduces a modal operator “$\Box$” whose meaning can be informally characterized thus (205):

$\Box p$ However the domain is further extended, *p*.

“$\Diamond$” is the dual. Linnebo shows that ZF is interpretable within his new modal version of set theory (216-22). Perhaps the most striking statement in Linnebo’s version of set theory is this (205):

$\Box\, \forall xx\, \Diamond\, \exists y\;\; Set(xx, y)$
However the domain is extended, given any plurality of objects, the domain can be further extended so as to include a set which contains precisely the members of the given plurality.

On this basis, Linnebo develops a plausible — and more importantly, precise — version of the of the view that the hierarchy of sets is “indefinitely extensible”.

2. Abstract Types

Let’s look at some abstract types.

Surely the most discussed case is that of the cardinal numbers. According to Wright and Hale, a cardinal number is an abstract type whose tokens are concepts. They further claim that every concept is an instance of some number, and that two concepts are instances of the same number whenever they are equinumerous. All of this is summarized by Hume’s Principle:

Hume’s Principle: $\forall F\, \forall G(\#F = \#G \leftrightarrow F \sim G)$

For somewhat technical reasons to do with his “dynamic abstractionism” — more on this presently — Linnebo prefers to work with pluralities rather than Fregean concepts (70-1).[4] So a Linnebovian will prefer a plural version of Hume’s Principle:

Hume’s Principle (Plural Version): $\forall xx\, \forall yy(\#xx = \#yy \leftrightarrow xx \approx yy)$

4 Linnebo does stress that his dynamic abstractionism *can* be made to accommodate types of concept, with a little elbow grease.

Assuming that we want to recognize zero and one as cardinal numbers, this requires us to include an empty plurality, and one-membered pluralities, in the domain of our plural variables (67).

Unlike Wright and Hale, Linnebo doesn't identify the natural numbers with the finite cardinals. Instead, he develops a view rather like term formalism, but with an abstractionist twist (ch. 10). We begin by observing that "5", "V", and "五" are alike in a certain respect — they play the same role in their respective numeral systems. We then identify the natural number five as the abstract type whose instances are these numerals — and likewise for the other natural numbers.

As I shall discuss in more detail later, Linnebo argues that each abstract type is ontologically dependent on its instances. He infers that each natural number exists only when tokens of it have been produced, from which it follows that only finitely many natural numbers exist. Linnebo adds that it is always *possible in principle* to create larger numerals — so the large natural numbers exist *in posse* though not *in esse* (187-8).

As I've said, Linnebo's book concerns sets and abstract types. One naturally wonders whether Linnebo is offering us a *complete* account of the ontology of mathematics: Does Linnebo make the ambitious claim that all mathematical objects are sets or abstract types?

Linnebo in fact draws back from making this ambitious claim — and *prima facie* there is good reason for doing so. Consider, for example, the complex numbers. There are strong Benacerrafian reasons for rejecting the identification of complex numbers with sets.[5] And complex numbers certainly don't seem to be abstract types. The question "What are the instances of $(1 + i)$?" is bizarre, because $(1 + i)$ is not of the right kind to have instances.[6]

On second thoughts, however, perhaps an abstractionist treatment of complex arithmetic is possible. As I have mentioned, we are used to the idea that algebraic structures can have instances — to repeat my example, *the* cyclic group of order six has as instances $\mathbb{Z}/6\mathbb{Z}$ under addition, and the symmetries of the regular hexagon under composition. Developing this thought, we may propose that the complex plane is a *type of field.* On this view, *individual* complex numbers are not types, but the complex plane as a whole is. And so perhaps an abstractionist account of the complex plane is credible after all.

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[5] In "What numbers could not be", Benacceraf argues against the identification of the natural numbers with sets. A similar argument can be given in the case of the complex numbers.

[6] Linnebo discusses the complex numbers on pp. 24-5; conspicuously, he doesn't claim that they are sets or abstract types.

4. Dynamic Abstractionism

Here is a well-known challenge for abstractionists.

The abstractionist will surely want to identify ordinal numbers with abstract types. In this case, the tokens are well-ordered sequences, and the equivalence relation is isomorphism. The following general principle is analogous to Hume's Principle:

$$\forall R_1\, \forall R_2\, (R_1 \text{ is a well-ordering} \wedge R_2 \text{ is a well-ordering} \rightarrow (\mathrm{Ord}(R_1) = \mathrm{Ord}(R_2) \leftrightarrow R_1 \sim R_2))$$

The problem, of course, is that this principle is inconsistent — this is a version of the Burali-Forti Paradox. It is no straightforward matter to develop a consistent theory of ordinals-as-abstract-types. Worse still, the inconsistency of the above principle threatens to discredit Hume's Principle by association (54).

Linnebo develops (ch. 3) a response to this challenge that parallels his treatment of set theory. Let us suppose that we start with a domain $\mathcal{D}_0$. We can then extend the domain, by adding in an ordinal number corresponding to each well-ordering of elements of $\mathcal{D}_0$. Call the new domain of quantification $\mathcal{D}_1$. We can then introduce still more ordinals, so as to ensure that there is an ordinal type for each well-ordering of elements of $\mathcal{D}_1$. This procedure can be repeated indefinitely $\mathcal{D}_2$, $\mathcal{D}_3$, $\mathcal{D}_4$, …, $\mathcal{D}_\omega$, … . As in the case of the sets, one can then use modal logic to state a *general* theory of the ordinal numbers as abstract types — a theory which does not give rise to the Burali-Forti paradox.

Linnebo calls this "dynamic abstractionism" — we might say that Linnebo is giving an "iterative conception of abstract type" which parallels the more familiar "iterative conception of set". This is, I think, the single most distinctive aspect of Linnebo's theory of types — setting it apart from the preceding work by Wright and Hale.

5. Dynamic Abstractionism and Metaphysical Ground

In a recent article,[7] I raised an objection to the view that abstract types are dependent on their tokens. Setting aside some of the technicalities, the objection was as follows. Suppose it is claimed

7 Donaldson, "The (Metaphysical) Foundations of Arithmetic?"

(a) that an abstract type is ontologically dependent on its tokens, and (b) that a cardinal number is a type of plurality. These two claims together imply that each cardinal number *n* is metaphysically dependent on each *n*-membered plurality – each *n*-some, so to speak. Thus, for example, the cardinal number three is metaphysically dependent on each threesome — including for example the threesome consisting of one, two, and three. Assuming (c) that a plurality is ontologically dependent on its members, we are led to the conclusion that three is ontologically dependent on itself – a conclusion repugnant to most metaphysicians. Whatever one makes of the claim that God is *causa sui,* surely numbers should be denied that distinction.[8]

Linnebo's book promises a solution to the problem. Let us suppose that the cardinal numbers are generated by an iterative process of abstraction, as Linnebo claims. And let us suppose, for the sake of illustration, that the initial domain contains just two objects, *a* and *b*. Then the first few domains in the sequence are as follows:

$\mathcal{D}_0 = \{a, b\}$
$\mathcal{D}_1 = \{a, b, 0, 1, 2\}$
$\mathcal{D}_2 = \{a, b, 0, 1, 2, 3, 4, 5\}$
$\mathcal{D}_3 = \{a, b, 0, 1, 2, 3, 4, 5, 6, 7\}$
…
…

We will then deny that each cardinal number *n* is metaphysically dependent on *any n-some whatever;* rather, he will say that a "new" cardinal number generated at stage *k* is grounded by the pluralities in $\mathcal{D}_{(k-1)}$ from which it is generated. Thus, in our example, three is grounded by each threesome drawn from $\mathcal{D}_1$, but it is not grounded by threesomes containing itself. Thus, the problem evaporates.

This is a very elegant solution to the problem, in particular because Linnebo's dynamic abstractionism is independently motivated. It seems to me that it is an important and impressive feature of Linnebo's theory of abstract types.

But perhaps I am now going further into the weeds than a reviewer should. So I will wrap things up here by repeating that *Thin Objects* will surely become the "go to" book on abstract types, and is essential reading for researchers in the philosophy of mathematics and metaphysics.

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8 I should add that the claim that ontological dependence is irreflexive is somewhat controversial. See Elizabeth Barnes' "Symmetric Dependence" for discussion.

References

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