II

Physical Geometry and Fundamental Metaphysics

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I.

Tim Maudlin's project sets a new standard for fruitful engagement between philosophy, mathematics and physics. I am glad to have a chance to be part of the conversation.

I think we can usefully distinguish two different strands within the project: one more conceptual, the other more straightforwardly metaphysical. The conceptual strand is an attempt to clarify certain concepts, such as continuity, connectedness, and boundary, which are standardly analysed in terms of the topological concept of an open set. The metaphysical strand is a defence of a hypothesis about the geometric aspects of the fundamental structure of reality—the facts in virtue of which the facts of physical geometry are what they are.

What I have to say mostly concerns the metaphysical side of things. But I will begin by saying just a little bit about the conceptual strand.

II.

In topology textbooks, expressions like 'continuous function', 'connected set of points', and 'boundary of a set of points' are generally given stipulative definitions, on a par with the definitions of made-up words like 'Hausdorff' (the adjective) and 'paracompact'. But as Maudlin emphasises, the former expressions can be used to express concepts of which we have enough of an independent grip to make it reasonable to wonder whether the topological definitions are even extensionally adequate. And he argues quite persuasively that we have no reason to believe that they are. The most important and general argument concerns the live epistemic possibility that physical space is discrete, for example by containing only finitely many points. According to Maudlin the target concepts, unlike their putative definitions in terms of 'open set', can have non-trivial application within physical space even if it is discrete. Maudlin offers an alternative system of definitions, based on the concept of a **line** (or of a **directed line**) which does not suffer from these problems.

The target concepts are somewhat specialised ones; while the ancient Greeks may have had them, they are at some distance from everyday life. For example, the concept of *connectedness* Maudlin is interested in is not obviously the same as the one I employ when I say that whereas my one-volume *Concise OED* is spatially connected, my two-volume *Shorter OED* is not: that judgment is surely consistent with the claim that even the one-volume book, when examined at sufficiently small scales, would reveal the same kind of geometric profile as an archipelago or a swarm of bees. To some extent, then, the project can be understood as one of 'conceptual synthesis' rather than conceptual analysis. On this way of thinking about it, the question is which natural and precise concepts there are in the vicinity of our rough-and-ready everyday geometric concepts; and Maudlin's answer is that in many cases, there are precise, natural, scientifically useful concepts, definable in terms of **linehood**, that are closer to the everyday concepts than anything definable in topological terms.

I am sympathetic to this claim, which is why I won't have much more to say about the conceptual strand of Maudlin's project. My main reservation involves a certain exclusive status Maudlin seems to be claiming on behalf of the concepts he has identified, which emerges in remarks like '[w]e conceive of geometrical spaces primarily by means of … lines' (p. 67). His view seems to be that for some collection of physical entities to count as a *space*, it must be possible, in a non-arbitrary way, to distinguish certain sets of them as **lines** satisfying the axioms for a Linear Structure. Without Linear Structure, there can be no such subject matter as *physical geometry* at all.

I think this is too demanding. Consider the following metaphysical hypothesis: there are finitely many 'points', whose structure determines a distinguished function that assigns a non-negative real number $d(x,y)$ to each pair of points x and y. These numbers obey the axioms for a metric space: $d(x,y) = 0$ iff $x=y$; $d(x,y) = d(y,x)$; $d(x,z) \leq d(x,y) + d(y,z)$. And that's it: there is nothing to determine any non-arbitrary criterion for counting certain pairs of points as "neighbours", and hence (since any finite Linear Structure can be fully characterised by saying which pairs of points in it are neighbours), there is nothing to determine any non-arbitrary assignment of a Linear Structure to the "points".

This hypothesis does have some unattractive features, which emerge when we ask *how it is* that the function *d* comes to be "distinguished" from all the other functions from pairs of points to real numbers. Answering this question in a satisfactory way may require adding some new entities to the fundamental ontology over and above the points.[1] Because of this, the hypothesis may prove less elegant and economical than some of its competitors. Nevertheless, the hypothesis does seem to be internally consistent, and consistent with the existence of creatures with evidence like ours. To the extent that developments in physics provide reason to question the standard assumption that spacetime is continuous, hypotheses like this belong on our menu of possible alternatives. If we accepted the conceptual claim Maudlin seems to be endorsing, we would have to say that if the hypothesis is true, there is no such thing as space or physical geometry. This seems odd. If the hypothesis is (conceptually) consistent with our own existence, it is surely also consistent with the existence of physical objects having shapes and sizes, standing at various distances from one another, and so forth. And if one concedes this much, it is hard to make sense of the thought that it is inconsistent with the existence of space. Even though there is one very interesting set of natural refinements of our everyday geometric concepts which can have no nontrivial application if the hypothesis is true, these are not the only potentially scientifically useful concepts in the neighbourhood. The discovery that truths involving our everyday geometric concepts are grounded in facts about a fundamental *d*-relation would be a surprising discovery in physical geometry: it would tell us something about the nature of space, not that space is an illusion.

[1] The demand for a nontrivial answer to this question can be motivated by the rejection of Heavy-Duty Platonism (see Field 1984 and §V below). There are many ways in which one could provide such an answer by invoking new physical ontology: the most flat-footed is simply to posit a collection of new physical entities carrying a structure isomorphic to that of the real line, related to pairs of points by a fundamental ternary 'd-relation'.

III.

On now to the metaphysical strand of Maudlin's project.

A central aim of metaphysics is finding out about the fundamental structure of the world—the actual, physical world. Of course physics is a vital ally in this inquiry, and given our present ignorance about key questions in physics, we should not expect to be able to make confident pronouncements about any but the most general aspects of the question. Nevertheless, we can make progress in exploring the space of coherent hypotheses about the fundamental structure of the world. And in doing so, it makes sense to devote special attention to hypotheses suggested by actually existing theories in physics, including both fully developed families of mathematically rigorous theories and more speculative suggestions thrown up by current research. We should be especially interested in developing general, flexible hypotheses about fundamental structure that can be filled in in different ways, so as to accommodate a wide range of possible developments in physics.

I take Maudlin's metaphysical proposals in this exploratory spirit. There are really two hypotheses, one more specific than the other. According to the less specific hypotheses, the fundamental entities include *points*, and the fundamental structure over these entities either includes, or very straightforwardly determines, a classification whereby some sets of points count as lines satisfying the axioms for a Linear Structure. (Or whereby some sets of ordered pairs of them count as directed lines satisfying the axioms of a Directed Linear Structure.) According to the more specific hypothesis, the relevant fundamental structure is a two-place relation among the points, which we can pronounce '*x* is earlier than *y*' (or 'x < y', or '*x* is in the past light cone of *y*', or '*y* is in the future light cone of *x*'); and the directed lines are defined in terms of this, as intervals of maximal totally ordered subsets of the extension of <.[2] Maudlin also suggests some additional fundamental ideology which

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[2] Maudlin omits 'intervals of'; but in view of axiom LS_2, which says that any interval of a line is itself a line, this must be intended.

could be added to either proposal: a fundamental property of *straightness* instantiated by some of the **lines**, and a fundamental binary relation of congruence (sameness of length) holding between some of the straight **lines**.

How do these bold metaphysical hypotheses bear on the conceptual side of the project? A strong reading of the conceptual claims would claim that concepts of continuity, connectedness, and boundary—perhaps even the concept of a space—only have nontrivial application if at least the weaker of the metaphysical hypotheses is true. But I doubt that this is what Maudlin intends. Surely he does not regard it as conceptually incoherent to suggest (as some have) that the description of the world in terms of spacetime points and their geometric structure is a "high-level" structure of some sort, as far from the fundamental as thermodymanics is from particle physics. It is better to think of the conceptual claims as having to do with the relations between the *concept* of a **line** and other geometric concepts, and as neutral about the question how all these concepts are anchored in fundamental metaphysics.

IV.

What does it mean to propound a hypothesis about the fundamental structure of the world? According to a standard approach, stating such a hypothesis involves (i) saying something about the *fundamental ontology*: the entities such that all facts ultimately boil down to facts about them; (ii) presenting a *fundamental ideology*: a catalogue of properties of, and relations among, the fundamental entities; and (iii) stating some *laws* which capture important general patterns in the instantiation of the fundamental properties and relations.

This mode of theorising raises a variety of further questions. Are we supposed to take the fundamental properties and relations seriously as entities, existing in the same fundamental sense as the objects that instantiate them?[3] Are we supposed to take the characterization of the laws as *laws* as adding something to the mere claim that they are true, and

[3] For the view that answers 'yes', see Armstrong 1978. In Dorr 2007 (§4) I describe two ways of thinking about these questions, 'physical nominalism' and 'structural nominalism', neither of which posits properties and relations as fundamental entities.

if so, what does it add? Are there cogent hypotheses about the fundamental structure of the world that cannot be stated in this form, because they posit fundamental facts which cannot legitimately be thought of as facts about the instantiation of properties and relations; and if so, how are these hypotheses to be formulated? These are all hard questions. Fortunately, we don't have to answer them to get started on the enterprise of formulating and comparing specific hypotheses inspired by theories in physics.

One worrying question that is less easily set aside is whether some apparent differences between such hypotheses are merely verbal. Suppose that, in spite of Maudlin's advice, we took seriously the hypothesis that topological *openness* belongs on the list of fundamental properties. Do we really have to choose between this and the hypothesis that takes topological *closedness* as fundamental, defining 'open set' as 'complement of a closed set' rather than vice-versa? It is hard to accept that there could be a genuine issue here.[4]

But this kind of worry is corrosive. Having got the idea, one will naturally start to wonder whether even superficially very different hypotheses about fundamental structure might not be mere notational variants. One will be tempted to look for some general principle according to which whenever there is a natural "translation" between two hypotheses about fundamental structure, or a natural "isomorphism" between the sets of possibilities they leave open, then there is no genuine difference between them. But short of verificationism, there is no known way of formulating such a criterion: it is just too obscure how understand this talk of translations and isomorphisms in such a way that it doesn't just beg the question whether the theories in question are genuine alternatives.[5]

[4] Sider (MS) defends a view according to which which there are genuine questions about which of two properties are fundamental even in cases like this. But Sider's way of thinking is hard to live with, especially when extended to semantic categories other than that of properties: for example, he is forced to think that there is a genuine question about whether 'and' or 'or' or both or neither is fundamental—a question whose answer we may not be in a position to know.

[5] The problem with the 'isomorphism' idea is clearest when the hypotheses we are dealing with happen to include very strong laws—strong enough to pin down the extensions of the fundamental properties and relations uniquely (up to isomorphism). For the set of structural possibilities left open by any such hypothesis has only one member, and there is trivially a unique 'natural isomorphism' between any two one-membered sets. But surely it is not true that any two such metaphysical hypotheses are notational variants of one another!

This doesn't mean that the concerns in question are never warranted. But it does suggest that we should bracket them if we want to get on with the inquiry: until we figure out some general principles for evaluating such claims, there is no point in opportunistically pronouncing that certain differences are merely notational whenever we find our patience wearing especially thin. No matter how gripped we are by these worries, once we have a particular hypothesis about the fundamental structure of the world on the table, we should search around for variants of it that seem simpler, more economical, or explanatorily more virtuous in some other respect. If the variants are genuinely and not merely verbally different, well and good; if not, then their existence still matters for the purposes of seeing how considerations like simplicity and economy bear on the really genuine questions about fundamental structure, whatever they are.

V.

The general question of fundamental metaphysics is 'What are the facts in virtue of which the world is the way it is?'. An important special case is 'What are the facts in virtue of which the facts of physical geometry are what they are?' One class of such facts that especially cries out for explanation are facts about geometrical relations whose relata include mathematical entities: for example, facts of the form *the ratio of the volume of region A to that of region B is real number x*. Of course, explanations have to stop somewhere. But there is something repugnant about the hypothesis—what Field (1984) calls 'Heavy Duty Platonism'—that such mixed mathematico-physical relations are fundamental.

There are various philosophies of mathematics that would license a general demand to explain these mixed relations in terms of relations all of whose relata are concrete: nominalism, logicism, certain kinds of mathematical structuralism. Even the rather orthodox idea that all mathematical entities are to be identified with sets licenses the demand to some extent: for among the pure sets, there are various equally good candidates to be the set of real numbers, and it seems silly to suppose that the fundamental physical relations privilege one of these (e.g. Dedekind cuts of rationals considered as Wiener-Kuratowski pairs of von Neumann numbers). But whatever one's views about mathematical ontology might be, it is important to explore ways in which mixed relations might be defined in terms of fundamental relations whose relata are all physical objects. Whatever else it might be, mathematics is a useful representational tool; the fact that it is useful to theorise about the physical world by describing its relations to mathematical entities is not much of a reason to assume that these relations are metaphysically fundamental.

Relations between physical objects and real numbers are not the only mixed geometric relations that occur undefined in standard theories in physics. Such theories are often expressed in the language of differential geometry; and the most common approach this subject simply helps itself to the idea that some co-ordinate systems (functions from sets of points to n-tuples of real numbers) are 'legitimate' or 'admissible'. The intuitive picture is that the admissibility of a co-ordinate system consists in its being faithful to certain aspects of the intrinsic structure of the space, but the mathematics is silent about what this intrinsic structure consists in.

Maudlin is generally sympathetic to the project of analysing geometrical relations between the physical and mathematical realms in terms of relations intrinsic to the physical realm. For example, he likes the idea of analysing lengths in terms of some intrinsic notion of 'congruence'; and he is appropriately repulsed by the idea that notions like 'admissible co-ordinate system' might themselves be fundamental. Nevertheless, his opposition to Heavy Duty Platonism is not total. Maudlin's **lines** are *sets* of points: and while in the Relativistic part of the proposal the notion of a **line** gets analysed in terms of an 'earlier than' relation whose relata are just points, the fundamental property of straightness, and the fundamental relation of congruence, still seem to be instantiated by **lines**.

One way for opponents of Heavy Duty Platonism to build on Maudlin's work would be to look for further relations just among points in terms of which these affine and metrical properties of **lines** could be defined.[6] A very different approach, which seems to promise more generality, would take **lines** themselves to be concrete physical entities, every bit as real and fundamental as the points.[7]

The latter approach in turn comes in two versions. On the first version, we have something like classical mereology as part of our account of fundamental reality: **linehood** is a fundamental property instantiated by just some of the many mereological fusions of points.[8] On the second version, the fundamental entities are just the points and the **lines**. We have a fundamental relation of 'incidence' between points and lines: to be a **line** is just to be something upon which something is incident.

Some hold on a priori grounds that the true catalogue of fundamental relations will include a relation of *parthood* subject to the laws of classical mereology. These people will of course be drawn to the first version of the approach: if fusions of points are going to be in the ontology anyway, it seems more economical to identify the **lines** with some of these fusions, and identify incidence with parthood, rather than positing **lines** as an additional supply of mereological atoms. But I don't think that there are good a priori grounds to expect the fundamental structure of the world to include anything like mereological structure. Granted, if it doesn't, it may be hard to find entities in the fundamental ontology which we could plausibly identify with ordinary objects like chairs, tables, planets and people. But is there any reason to think that there are any such things, in the sense of

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[6] For example, we could, at least in a Euclidean context, define straightness in terms of a three-place "betweenness" relation, and the two-place congruence relation on **lines** in terms of a four-place relation among their endpoints, as in Tarski 1959 and Field 1980. Maudlin's main reason for resisting this kind of approach seems to be the claim that facts about long-range relations among points are *extrinsic* (and hence non-fundamental). As is clear from my discussion of the 'finite metric space' world in §II, I do not think that this is a good a priori constraint to put on our theorising.

[7] One could make do with just the points if one were willing to countenance fundamental 'plural properties' (or 'multigrade relations'): then linehood could be thought of as such a property, instantiated by many points at a time. Whether this is legitimate is a deep foundational question.

[8] If we wanted **directed lines**, we would need something more complicated—entities that stand to fusions of points as sets of ordered pairs of points stand to sets of points.

'there are' relevant to fundamental ontology? In my view, the sense of 'there are' in which it is obvious that there are chairs, tables, planets and people is something quite different.[9]

VI.

Once we have set aside our temptations to play the 'mere notational variant' card, we should be prepared to find that, even after we have settled on a general strategy like 'Take **lines** as fundamental!', there are many slightly different ways to implement the strategy in a hypothesis about fundamental structure. Once we have a particular hypothesis on the table, we can start tinkering with it to see if we can simplify its ontology or its ideology.

Maudlin's account of Relativistic spacetime embodies one such simplification. We might have thought that capturing this geometric structure would require a rich Linear Structure, with spacelike, timelike and lightlike **lines**, along with mixed **lines** made up of different kinds of segments. But as Maudlin shows, this is needlessly uneconomical: we can throw away all but the timelike-or-lightlike **lines** and still recover facts about the distances beween spacelike-separated points, by defining them in terms of facts about the lengths of timelike-or-lightlike **lines**. (And once we have done this, we can define **linehood** itself in terms of 'earlier than', provided we don't mind excluding spacetimes with closed timelike curves.)

Can we find other simplifications of a similar sort? Here is one idea: if we are eventually going to need a fundamental property of *straightness* that distinguishes a special substructure of **lines**, why not simplify the ideology and ontology by throwing away all the non-straight **lines**? In a relativistic spacetime, a specification of the straight timelike and lightlike **directed lines** and their congruence relation should be enough to pin down (up to a scale factor) the geometric structures required by standard presentations of the physics. Given the fundamental structure of straight **lines**, we could define derivative Directed Linear Structures containing non-straight lines—for example, we can build co-ordinate systems whose co-ordinate curves are straight **lines**, and define **lines** in a new broader sense as graphs of curves determined by quadruples of continuous co-ordinate functions.

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9 See Dorr 2005 and §1 of Dorr 2007.

The opening pages of Maudlin's paper suggest that he would not be sympathetic to the idea that "rubber sheet geometry" is in this way derivative from affine geometry (straightness structure). He writes that

> The affine structure itself presupposes an even more basic organization of the points. The straight lines in a space are a subclass of the continuous curves, and the continuous curves are defined, mathematically, independently of the affine structure. So sitting at the bottom of this definitional hierarchy is a sub-metrical geometry, aspects of a space that do not depend on either the metric or the affinity. (Maudlin 2010, p. 63)

It seems to me that the mathematical sense in which topology is said to be 'more fundamental' than affine and metric geometry is quite different from the metaphysical sense of 'fundamental' we are concerned with. The mathematical 'fundamentality' of topology is a kind of generality: there are many kinds of mathematical structure within which there are natural definitions of properties obeying the topological axioms for 'open set'; this makes topology useful for capturing behaviour common to many mathematical structures. There is nothing in this to count against the hypothesis that the metaphysically fundamental facts about physical space are all facts about its affine or metric structure.

There are other kinds of potential simplifications we can consider once we start tinkering with the basic picture. For example, if we are going to have **lines** in the fundamental ontology in any case, we might consider simplifying the ontology by getting rid of points as fundamental entities, and doing everything with some fundamental relations among **lines**.

One strategy would take as fundamental the relation of two lines "overlapping"—intuitively, sharing at least one point in common. In terms of this, we can define what it is for two lines to *cross*, or "share exactly one point in common": λ_1 crosses λ_2 iff λ_1 overlaps λ_2, and there are lines λ_3 and λ_4 neither of which overlap λ_2 such that both λ_3 and λ_4 are parts of λ_1 and every line that is part of λ_1 overlaps either λ_3 or λ_4. (One line is *part* of another if every line that overlaps the former overlaps the latter.) If we are dealing with **directed lines**, we will want, instead, some fundamental relation like 'λ_1 overlaps λ_2 later than λ_3 does': we think of **directed lines** as determining an order (with ties) among the lines they cross, rather than an order among points. Given this fundamental structure, we should be able to code up points as equivalence classes of pairs of crossing lines.[10]

This reconstruction will break down in some simple Linear Structures. For example, there is the trivial Linear Structure consisting of a single, two-point line: obviously we cannot identify these points with two different equivalence classes of pairs of lines. More generally, in any Linear Structure in which the whole space is a **line** with endpoints, there are no pairs of lines that cross at an endpoint. There may be other more general codings which can accommodate these cases. And in any case, we should not dismiss a hypothesis about fundamental structure just because it admits a narrower range of possibilities than another account, especially if the possibilities in question are ones that we have no empirical reason to take seriously.[11]

Another approach would be to take parthood as primitive rather than defining it in terms of overlap. Indeed, we might not need anything else. I haven't got any neat proofs, but it seems likely that there is some large and interesting class of Linear Structures within which the facts about the subset relation among lines pins down the whole structure up to isomorphism.[12] Such a reduction of geometrical notions to mereological ones will be especially interesting to those (see §V above) who think they have a priori reason to include *parthood* on the list of fundamental relations in any case. However, as a matter of sociology, most of those who hold this view will also accept the mereological axiom of universal composition on *a priori* grounds, in which case their ontology will have to include fusions

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[10] To carry out this construction, we need to be able to say that *λ_1 and λ_2 cross at the same point where λ_3 and λ_4 cross*. There is always the option of taking this as a further fundamental relation. But at least in some well-behaved Linear Structures, it should be definable in terms of overlap. First, an obvious extension of the definition of *crossing* lets us define what it is for two lines to cross a third at the same point. And it seems intuitive that when λ_1 and λ_2 cross at the same point where λ_3 and λ_4 cross, we can find a line that crosses λ_1 and at least one of λ_3 or λ_4 at that point. (The only cases I can think of where this fails involve quite bizarre and degenerate Linear Structures.) If this holds, we can define 'λ_1 and λ_2 cross at the same point where λ_3 and λ_4 cross' as 'for some λ_5 that crosses λ_1 where λ_2 does, either λ_5 crosses λ_3 where λ_4 does and λ_1 crosses λ_5 where λ_3 does, or λ_5 crosses λ_4 where λ_3 does and λ_1 crosses λ_5 where λ_4 does.

[11] Cf. Maudlin on closed timelike curves.

[12] There are also physically interesting Linear Structures in which this is not so: for example, the Segment-Spliced Linear Structure of straight lines in Euclidean space.

of 𝔩𝔦𝔫𝔢𝔰 that are not themselves 𝔩𝔦𝔫𝔢𝔰, and their ideology will thus need a fundamental property to differentiate the 𝔩𝔦𝔫𝔢𝔰 from the non-𝔩𝔦𝔫𝔢𝔰.[13]

VII.

When we are trying to figure out how to divide our credence in a reasonable way between hypotheses about fundamental structure, considerations of simplicity will matter a lot. What we want is not just a short list of fundamental properties and relations, but a simple set of laws stated in terms of these properties and relations, in terms of which we can give satisfactory explanations of a wide range of evidence.

Making these discriminations will require a well-honed sense of what makes for a "satisfactory explanation". One important way in which laws can fail to make for satisfactory explanations is for them to take an 'as if' form. Someone who wanted to admit atoms but not subatomic particles into their fundamental ontology could write down a law of the form 'the motions of atoms are just as they would be if they were composed of subatomic particles obeying such-and-such laws'. Although such laws need not be especially complex in any obvious sense of 'complex', they do little to explain the phenomena that follow from them: the question '*Why* do the atoms move around like that?' cries out for an answer (see Dorr 2010, §4). Finding laws which avoid this kind of badness can be hard task. Suppose we are trying to write down a complete set of laws for some special-relativistic physics, as part of which we want to describe the structure of a Minkowski spacetime in terms of Maudlin's fundamental relations. It is not enough just to say that the extension of *earlier than* is a partial order, or that the intervals of its maximal totally ordered subsets form a Directed Linear Structure. This is far from sufficient to pin down the structure of Minkowski spacetime: if our laws said no more than this, they would be much too weak to do the necessary explanatory work. One thing we *could* say that would not be too weak is this: the set of spacetime points admits a co-ordinate system such that x is earlier than y

[13] Dorr (2004) argues on broadly a priori grounds that all fundamental relations must be symmetric. If this is right, some of the candidate fundamental relations we have considered can be dismissed without regard to considerations like simplicity: overlap is a candidate for fundamentality, while parthood is not. For the remainder of the present paper I will ignore this putative additional constraint on fundamental ideology.

iff the co-ordinates of x and y satisfy the standard co-ordinate definition of one point belonging to the past light cone of another. But this invocation of co-ordinate systems is worryingly reminiscent of the atom-lover's 'as if' law. In effect, we are saying that the facts about the *earlier than* relation are just *as if* spacetime had all this extra co-ordinate structure, related to the *earlier than* structure in a particular way. It would be much nicer if we could state some laws *directly* in terms of 'earlier than' which entail the existence of appropriate Lorentzian co-ordinate systems, in the same way that the "intrinsic" axiomatisations of Euclidean geometry developed by Hilbert (1899) and Tarski (1959) entail the existence of Cartesian co-ordinate systems.

Admittedly, the analogy between the obviously bad 'as if' law and the law that says that there is an admissible co-ordinate system is far from perfect. To properly assess the stringency of the demand for explanatorily satisfactory laws, we will need a more thoroughly worked out account of what the relevant kind of badness consists in. (Dorr 2010 contains some suggestions.) But even at this stage, when we are considering competing lists of fundamental relations, it is clearly worth our while to see which lists allow us to state satisfactory "intrinsic" laws, and which require us to resort to suspicious devices like existential quantification over co-ordinate systems.

I don't know how well Maudlin's favoured fundamental relations do by this criterion. I don't know how to write down "intrinsic" laws about these relations strong enough to pin down, say, the distinctive geometric structure of Minkowski spacetime, or of a vacuum solution to Einstein's field equation for general relativity; but that isn't to say it can't be done. Still less do I know how to add additional fundamental relations to describe some sort of physical content in spacetime—say, the electromagnetic field—in such a way that I could state a satisfactory intrinsic system of laws encompassing both the geometry and the physics. Figuring out whether these things can be done is a big task, which may require considerable technical ingenuity.

VIII.

Theories of fundamental structure based on Maudlin's ideas are attractive. But the right slogan for this stage of our enquiries is 'Let a thousand flowers bloom'. As part of this horticultural endeavour, we should pay special attention to hypotheses which take metaphysical inspiration from the mathematical tools used in existing mathematical physics. For we can hope, by doing this, to find systems of fundamental relations for which the task of extracting explanatorily satisfactory laws from existing theories in mathematical physics will be especially easy.

The standard mathematical apparatus used for stating physical theories about spacetime is that of differential geometry. Mostly, everything is done on the assumption that spacetime forms a *smooth manifold*. Smooth manifolds are mathematical objects somewhat richer than mere topological spaces: we can think of them as capturing the structure of a slightly less amorphous kind of rubber sheet, which can be deformed only by gentle stretches and squeezes which never introduce anything like a "kink" or "corner". On the most common approach, this structure is given by specifying a set of admissible coordinate systems, or 'atlas' for the space. But let me sketch a somewhat less well-known approach, which I think may be better adapted to metaphysical purposes.[14] On this way of proceeding, what we are given is a privileged class of *smooth* functions (also known as 'C^∞ functions'), subject to certain axioms, within the set of all functions from the points of the space to real numbers, or 'scalar fields'.[15] The scalar fields form a mathematical "ring", in the sense that we can define well-behaved notions of addition ($(f+g)(p) =_{df} f(p)+g(p)$) and multiplication ($(fg(p) =_{df} f(p)g(p)$). The smooth functions are required to be a

[14] My main source here is chapter 4 of Penrose and Rindler 1984. Thanks to Frank Arntzenius for pointing me towards this.

[15] For the record, here are the axioms for an n-dimensional smooth manifold, as given by Penrose and Rindler (1984, §4.1):
(1) Whenever F is a C^∞ function from $\mathbb{R}^m$ to $\mathbb{R}$, and $f_1 \ldots f_m$ are smooth, the function $F(f_1, \ldots, f_m)$ defined by $F(f_1, \ldots, f_m)(p) = F(f_1(p), \ldots, f_m(p))$ is smooth.
(2) If for every point p there are smooth functions h and f such that $h(p) > 0$ and $hf = hg$, then g is smooth.
(3) For each point p there are smooth functions $h, x_1, \ldots, x_n$ such that (i) $h(p) > 0$; (ii) whenever $h(p_1) > 0$ and $h(p_2) > 0$ and $x_1(p_1) = x_1(p_2)$ and … and $x_n(p_1) = x_n(p_2)$, $p_1=p_2$; and (iii) for every smooth g, there is a $C\infty$ function F from $\mathbb{R}^n$ to $\mathbb{R}$ such that $hg = hF(x_1, \ldots, x_n)$.

subring of this ring: that is, adding and multiplying smooth functions always yields another smooth function. The *constant* functions are in turn a subring of the ring of smooth functions.[16]

Armed with the primitive distinction between smooth and non-smooth scalar fields, one can define further classes of mathematical objects associated with the manifold. A *smooth vector field* on the manifold is a function V from smooth scalar fields to smooth scalar fields such that (i) $V(f) = 0$ when f is a constant function; (ii) $V(f + g) = V(f) + V(g)$, and (iii) $V(fg) = fV(g) + gV(f)$ (the Leibniz product rule). We can define addition on smooth vector fields by $(V_1 + V_2)(f) =_{\mathrm{df}} V_1(f) + V_2(f)$, and multiplication of smooth vector fields by smooth scalar fields by $(fV)(g) =_{\mathrm{df}} fV(g)$. We define a *smooth covector field* on the manifold as a function ω from smooth vector fields to smooth scalar fields that is "C^∞-linear", in the sense that $\omega(V_1 + V_2) = \omega(V_1) + \omega(V_2)$ and $\omega(fV) = f\omega(V)$. Finally, *a smooth tensor field of rank m,n* is a function that takes m smooth covector fields and n smooth vector fields and yields a smooth scalar field, and that is C^∞-linear in each argument. In physics, we pick out some of these mathematical entities as (somehow) physically distinguished. For example, in a (false) theory of a fundamental continuous fluid, a smooth vector field might be physically distinguished as 'the fluid velocity field'. In electromagnetism, a smooth tensor field of rank 0,2 is physically distinguished as the *electromagnetic field tensor*. One of the physically distinguished smooth tensor fields (also of rank 0,2) that makes an appearance in almost every physical theory stated in the language of differential geometry is the *metric*. The metric tensor field plays a special relation in the analysis of geometric facts: for example, notions like the length of a path or the volume of a region are standardly defined in terms of it. But from the point of view of the laws, the metric is just another physically distinguished smooth tensor field. To state the physical laws, we define up various further fields (e.g. the Ricci tensor field, the stress-energy tensor field) in terms of our original list of physically distinguished fields, and state equations (e.g. Einstein's field equation) involving the results.

[16] Note that there is no need to add topology as an additional primitive structure over and above smoothness: we can define the open sets to be (unions of) the sets $\{p: f(p) > 0\}$, where f is smooth. See Penrose and Rindler 1984, p. 181.

If we are in the business of exploring alternatives to Heavy Duty Platonism, we will need to find some nontrivial way of answering the question *what it is* for a given function from spacetime points to real numbers to be smooth, or for a given function from pairs of vector fields to scalar fields to be the metric. One could attempt to analyse these these mixed properties and relations in terms of relations all of whose relata are *points*. But this looks very challenging. A general moral we can draw from Maudlin's theory of lines is that it helps a lot to have a fundamental ontology that contains some entities besides the points. A conservative approach would enrich the fundamental ontology by adding some new entities which behave like sets, or mereological fusions, of points. But as I have already said, I don't think mereology has any special status when we are doing fundamental ontology. If we want to posit a fundamental relation subject to laws which would make it sensible to pronounce it 'part of', we must do so on the same a posteriori grounds for which we would posit any other piece of fundamental structure. We should be careful not to overlook alternatives to, and generalisations of, mereological structure just because of their unfamiliarity.

What I want to suggest is that instead of positing fundamental entities which behave like *sets* of points, we should consider positing fundamental entities which behave like *functions from points to real numbers*. I will call these putative entities 'Scalars'. But it is important that they are not supposed to be *identical* to scalar fields in the ordinary sense. The latter are mathematical functions: according to orthodoxy, sets of ordered pairs of points and real numbers. The Scalars, by contrast, are concrete physical entities whose fundamental relations to points and to one another somehow determine a natural correspondence between them and scalar fields.

Fully filling in this theory will require specifying the fundamental relations which confer this structure on the Scalars. There are various ways to do this. For the sake of definiteness, let's suppose we use (a) a ternary 'sum' relation among Scalars; (b) a ternary 'product' relation among Scalars, and (c) a ternary relation 's_1 and s_2 coincide at p'. In terms of these relations we can state laws of plenitude which 'say' that there is a Scalar corresponding to each function from points to real numbers, in the same sense in which the laws of classical mereology 'say' that there is a region corresponding to each set of points.[17]

Taking this ontology and ideology as a starting point, it is a relatively straightforward matter to craft a detailed hypothesis about fundamental structure based on some existing physical theory couched in the language of differential geometry. First we will need to capture the "smooth manifold" structure of spacetime by introducing a new fundamental property *smoothness* which distinguishes a special class of Scalars, subject to some distinctive laws corresponding to the axioms of footnote 16.[18] And then we will need some further fundamental relations corresponding to the physically distinguished fields. For example, if the mathematical physics we are trying to recover talks about a distinguished vector field V (such as the fluid velocity field), we can add a corresponding fundamental binary relation over Scalars: *V maps s_1 to s_2*.

Things get trickier when the physics involves distinguished tensor fields which cannot be defined in terms of distinguished vector and scalar fields. A flat-footed approach would add two new systems of fundamental entities corresponding to the space of all vector fields and the space of all covector fields, with fundamental relations among these entities corresponding to physically distinguished tensor fields. But this seems ontologically extravagant, and fortunately, may not be necessary. First of all, we can do without covector fields as fundamental entities. In any differential manifold there is a special function d

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[17] Let me give a sketch of a way this might be done. First some definitions: (i) A Scalar is *nowhere-negative* iff it is the product of some Scalar with itself. (ii) $s_1 \leq s_2$ iff $s_2 = s_1 + s_3$ for some nowhere-negative s_3.. (iii) s is *rational* iff it is contained in every nonempty set of Scalars that is closed under the operations of addition, multiplication, taking the additive inverse and taking the multiplicative inverse (when one exists). (iv) s is *constant* iff whenever s' is rational, either $s \leq s'$ or $s' \leq s$. Given these definitions, we can then write down axioms which say that the constant Scalars have the structure of the real line, and a second-order axiom (or first-order axiom schema) which says that for every function F from points to constant Scalars, there is a unique Scalar that coincides at every point p with $F(p)$.

Some nominalists will find the use of set theory (or second-order logic) in the definition of 'rational' unacceptable: they will need some additional fundamental ideology, such as a fundamental property of *constancy*. However, Dorr (2010) defends a view on which such uses of mathematics in stating physical laws can be acceptable even if there are (fundamentally speaking) no mathematical entities.

[18] Arntzenius (MS) shows how to state nominalistic versions of axioms (1) and (3), which eliminate the need to quantify over functions on $\mathbb{R}^m$.

that maps each smooth scalar field f to a smooth covector field df, defined by $df(V) = V(f)$. A covector field is said to be *exact* iff it is the result of applying d to some smooth scalar field. Although not every smooth covector field is exact, each smooth tensor field of rank m,n is fully determined once we know what scalar field it yields as output when given any m *exact* covector fields and n smooth vector fields as input. So if, for example, we are trying to reconstruct a physically distinguished tensor field T of rank 2,0, we can do so using a fundamental three-place relation among Scalars, corresponding to the mathematical relation $T(df_1, df_2) = f_3$.

That takes care of the tensors of rank m, 0 (those whose arguments are all covector fields), but still leaves us with no way to deal with other tensor fields short of enriching the fundamental ontology with new entities corresponding to the space of all vector fields.[19] But there is a trick that we can use to avoid this. As I said above, in physics the list of physically distinguished fields normally includes a special smooth tensor field g of rank 0,2, the "metric". Like any smooth tensor field of rank 0,2, g determines a function Φ_g from smooth vector fields to smooth covector fields, defined by $\Phi_g(V_1)(V_2) =_{df} g(V_1,V_2)$. And in almost all physical theories, the metric is required to be "non-degenerate", which means that Φ_g must be a *bijection* between the smooth vector fields and the smooth covector fields. The upshot is that we can use the metric to go back and forth as we please between covector fields the vector fields to which they are mapped by Φ_g, and thus between tensor fields of rank m, n and tensors fields of rank $m+n$, 0. In this friendly context at least, we can reconstruct tensor fields of all sorts using fundamental relations all of whose relata are Scalars. To characterise a physically distinguished tensor field T of rank m,n, we will posit a fundamental $m+n+1$-place relation over Scalars, corresponding to the mathematical relation $T(df_1, \ldots, df_m, \Phi_g^{-1}(df_{m+1}), \ldots, \Phi_g^{-1}(df_{m+n})) = f_{m+n+1}$.[20]

(I have been ignoring complications induced by the arbitrariness of units. For example, if we don't want to have a metaphysically privileged unit of distance, we shouldn't

[19] See Arntzenius MS for one way to do this.

[20] This applies equally to the metric itself, which will be captured by a fundamental ternary relation among Scalars, corresponding to the mathematical relation $g(\Phi_g^{-1}(df_1),\Phi_g^{-1}(df_2)) = f_3$, or equivalently, $\Phi_g^{-1}(df_1)(f_2) = f_3$.

really want our fundamental relations to pin down the metric tensor *uniquely*—rather, there should be a one-dimensional family of "equally good candidates" to be the metric tensor, each corresponding to a choice of unit. The obvious way to achieve this neutrality involves adding more argument places to the fundamental relations, by analogy with the move from a 'length' relation between lines and numbers to a ternary 'length-ratio' relation between pairs of lines and numbers. I won't go into the details.)

If I were advertising the ontology of Scalars as a way of vindicating nominalism, you would have a right to be suspicious. In respect of the fundamental relations they instantiate and the characteristic laws which govern those relations, Scalars do not look much like paradigmatically concrete objects. On the other hand, spacetime points have by now come to be generally accepted as 'concrete' in spite of the fact that (according to many theories) they are governed by laws which make them behave just like certain mathematical entities. For my part, I don't care at all about the labels 'concrete' and 'abstract'. What I care about is finding an economical fundamental ontology and ideology in terms of which I can state strong and simple laws. I don't mind borrowing ideas from mathematics about what this structure might look like, and I am not at all worried that in doing so I will somehow have started down a slippery slope at the end of which is the fully-fledged Platonism which incorporates all of mathematics into the fundamental ontology.

It is worth noting, though, that Scalars are not really so dissimilar to other putative entities that have generally been accepted as concrete. Spacetime *regions* are usually taken to be no less nominalistically kosher than spacetime points. But one way to think of the Scalars corresponding to functions whose values lie between 0 and 1 is as "fuzzy regions", to which points can belong to different degrees. That doesn't seem so strange, does it? Admittedly, it is harder to get any such intuitive purchase on the rest of the Scalars. But if we were really worried about this, we could make do with the more restricted set of Scalars—by using some smooth bijection between [0,1] and the real line, we could treat them as proxies for the full set of scalar fields, refitting the fundamental relations underpinning physically distinguished fields in such a way as to take this representation into account.[21]

IX.

Here, then, is a possible fundamental structure reality might have: an ontology of points and Scalars, with fundamental 'sum' and 'product' relations among Scalars giving them the structure of a ring; a fundamental 'coincidence at a point' relation; a three-place relation on Scalars corresponding to the metric tensor field; and perhaps some more relations on Scalars corresponding to other physically distinguished vector and tensor fields. This is certainly a less intuitive picture than one based on Maudlin's **lines**. But I doubt this kind of intuitiveness is very important in fundamental metaphysics, especially if the sacrifice of intuitiveness makes it possible to state intrinsic, explanatorily satisfactory versions of physical laws.

Now that we have a hypothesis on the table, we can consider simplifying its ontology and ideology in various ways.[22] I will mention four possible simplifications.

(i) We could do without the idea of a privileged unit Scalar. This would mean thinking of various operations on Scalars, including multiplication, as making sense only relative to an arbitrary choice of a given constant Scalar to serve as unit: some of our fundamental relations will then require an extra argument place to capture this unit. In fact, this move lets us avoid the need for a fundamental relation corresponding to multiplication of Scalars altogether: all we need is a fundamental property of *constancy*. For the addition facts,

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[21] The theory I am sketching has a variant in which Scalars really are just regions, not of standard four-dimensional spacetime, but of a five-dimensional space whose points behave like ordered pairs of old-style spacetime points and real numbers. Arntzenius (MS) makes a strong case that in the context of gauge-symmetric field theories, one should take the points of some such higher-dimensional space (a fibre bundle space) to be fundamental, if one includes points in the fundamental ontology at all.

[22] I am still ignoring the demand from Dorr 2004 that fundamental relations be symmetric. It is certainly possible to define the asymmetric relations I have been talking about in terms of symmetric ones, but so far I have not looked for especially simple ways of doing this.

together with the facts about which Scalars are constant, suffice to fix the extension of the four-place relation '$s_1 s_2 = s_3$ relative to the choice of constant scalar s_4 as unit'.[23]

(ii) Instead of a generous ontology with Scalars corresponding to arbitrary functions from points to real numbers and a fundamental property of *smoothness* picking out some of them as special, we could try to get by with a more economical ontology in which we only have Scalars corresponding to smooth scalar fields in the first place. The cost of doing this is that it is not obvious how to state, in an explanatorily satisfactory way, a 'plenitude' axiom guaranteeing that there as many Scalars as we want there to be. Doing this should not be too difficult if we don't mind laws which appeal to mathematical ontology or higher-order logic; but evaluating such laws raises some difficult issues I gestured towards in §VII.[24]

(iii) If the topology of the whole spacetime is non-compact (e.g. because it is infinite in extent), we might squeeze out a little more ontological economy by restricting ourselves not only to smooth Scalars, but to smooth Scalars which are nonzero only within some compact region.[25]

(iv) If we are going to have Scalars in the ontology in any case, it is natural to ask whether we can make things a bit more unified by getting rid of the separate category of *points*. This is easily done if we have the rich ontology of Scalars corresponding to arbitrary functions. We can reconstruct points as special sets of Scalars: intuitively, those that are zero everywhere except for the given point. The ring structure of the Scalars is enough to tell us which sets of Scalars correspond to points in this way: the relevant sets are those that (a) are closed under addition; (b) are closed under multiplication by any Scalar; (c) contain at least one nonzero Scalar, and (d) have no proper subsets meeting conditions (a-c). (These are called the *minimal ideals* of the ring.) We will no longer need a separate fun-

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[23] Cf. the axiom system for the real numbers devised by Tarski (1936, §61), in which '1' is primitive and multiplication must be defined (using second-order methods).

[24] See Dorr 2010 for further discussion.

[25] Since this will mean getting rid of constant Scalars, there is no longer any use for the fundamental property of *constancy* contemplated under simplification (i). Its work could be done instead by a fundamental binary relation, s_1 *is constant wherever* s_2 *is nonzero*.

damental relation of 'coincidence at a point': we can say that s_1 and s_2 'coincide at the point represented by minimal ideal S' iff $ss_1 = ss_2$ for each $s \in S$. This won't work if we have adopted simplification (ii), getting rid of all but the smooth Scalars: since no smooth Scalar can be zero at all points but one, the ring of smooth Scalars doesn't have any minimal ideals. But a closely related construction still will: we can represent points as *maximal* ideals—sets of scalars that meet conditions (a-c) and are not proper subsets of any other set meeting conditions (a-c), except for the set of all Scalars. Under this representation, each point is represented by the set of all Scalars that are *zero* at that point. This will be reflected in our new definition of 'coincidence at a point': for s_1 and s_2 to 'coincide at the point represented by maximal ideal S' is for $s_1 - s_2$ to be a member of S. Whichever way we do it, the upshot is that points are superfluous. The addition and multiplication structure of the Scalars—or even more minimally, the addition structure together with the facts about which Scalars are *constant*—is enough to pin down a unique smooth manifold (up to diffeomorphism).[26]

A fundamental ontology comprising nothing but Scalars is somewhat alien to our ordinary ways of thinking. We are used to associating fundamentality with smallness of size; whereas to the extent that it makes sense think of Scalars as having sizes at all, most if not all of them are enormous. The vision is as different as can be from that of Humean Supervenience (Lewis 1986). But this is not such a novelty: theories of 'gunky spacetime' (e.g. Arntzenius 2008) propose an ontology in which big and small objects are on a par, and in which points don't exist at all except as constructions out of regions.

Could there be people like us, having evidence like ours, in a world where the only fundamental things were Scalars? I think it will be hard to deny that there could be, at least if one is comfortable with the idea that people are going to turn out to be non-fundamental entities in any case, and with a broadly functionalist picture of properties like personhood. The idea that there could be people in such a world challenges the as-

[26] Geroch (1972) suggests using the ring of Scalars (an 'Einstein Algebra') as an alternative to an ontology containing points, although it is not clear whether he regards this as anything more than a useful notational variant. Earman (1989) also promotes the approach (under the name 'Leibniz algebras'), although I don't think it has the particular advantages over the standard ontology of points which he claims on its behalf.

sumption that non-fundamental entities are "built" out of fundamental entities as walls are built out of bricks. But that is an assumption that needs to be challenged in any case.[27]

There are many other variants of the Scalar-based ontology which we could consider, and which we might be led to take seriously by looking at the details of particular physical theories. For example, it isn't really crucial that Scalars behave like functions from points to *real numbers*. There are other kinds of value-spaces that would do equally well for the purposes of defining vector and tensor fields, and which might natural candidates to use if we were going to need them anyway for the purposes of physics.[28]

Let me end with a moral that may have wider application. Separating the investigation of the metaphysical foundations of physical geometry from the investigation of the metaphysical foundations of physics as a whole might work well enough as a simplifying device. But ultimately, we just care about the fundamental structure of the world. And given how intimately geometry is bound up with the rest of physics, it would be foolish to assume there will be any useful way to separate off the geometric aspects of the fundamental structure from the rest. When we are investigating the metaphysical foundations of geometry, we will do well to keep an eye on the question how the structure we are describing could be enriched so as to capture a fully-fledged physics. And when we are investigating the metaphysical foundations of the parts of physics that go beyond mere ge-

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[27] I don't want to give the impression that you need to buy in to a framework of 'fundamental' and 'non-fundamental' entities (or kinds of quantification) in order to take the ontology of Scalars seriously. We could claim that chairs, people, and so on just *are* certain Scalars—e.g. Scalars that are zero at points "occupied" by the object in question, or Scalars that are nonzero at such points. I don't see that such an identification is any more problematic than the identification of people with regions of spacetime (see Sider 2002, §4.8). There is the issue that there is a vast multiplicity of Scalars which seem equally well qualified to be identified with any given ordinary object; but this is just another instance of the Problem of the Many, no different in principle from the difficulty in deciding on the exact borders of the region identical to a given ordinary object.

[28]Also, I don't think it is really crucial that we be able to make sense in an absolute way of comparisons between the values of Scalars at different points: I am hopeful that it would be enough for the Scalars to have the structure of the space of smooth sections of a fibre bundle carrying a *connection* that lets one make sense of a *local* notion of 'constancy'. (See Maudlin 2007, chapter 3 and Arntzenius MS for explanations of these notions.) This would be natural given the role of such fibre bundles in modern physics. But at present I don't have a good sense of how to state explanatorily satisfactory laws about entities with this less-rich structure.

ometry, we should avoid the all-too-common mistake of treating space and time as if they were a metaphysically unproblematic backdrop to which we can freely appeal in explaining properties like mass and charge. Most of the decisions that one must make in formulating a fully-worked out hypothesis about the fundamental structure of reality will already have been made by the time one has figured out how to account for space and time; if one has carried out this part of the task properly, filling in the rest of the picture should be plain sailing.[29]

[29] Thanks to Harvey Brown for spurring me to take this on, to Tim Maudlin and Scott Sturgeon for helpful comments, and especially to Frank Arntzenius, for many hours of discussion without which I wouldn't have the least clue what to think about these matters.

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