Towards a Theory of Computation similar to some other scientific theories

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**ABSTRACT:** At first sight the Theory of Computation *i*) relies on a kind of mathematics based on the notion of potential infinity; *ii*) its theoretical organization is irreducible to an axiomatic one; rather it is organized in order to solve a problem: “What is a computation?”; *iii*) it makes essential use of doubly negated propositions of non-classical logic, in particular in the word expressions of the Church-Turing’s thesis; *iv*) its arguments include *ad absurdum* proofs. Under such aspects, it is like many other scientific theories, in particular the first theories of both mechanical machines and heat machines. A more accurate examination of Theory of Computation shows a difference from the above mentioned theories in its essential~~ly~~ including an odd notion, “thesis”, to which no theorem corresponds. On the other hand, arguments of each of the other theories conclude a doubly negative predicate which then, by applying the inverse translation of the ‘negative one’, is translated into the corresponding affirmative predicate. By also taking into account three criticisms to the current Theory of Computation a rational re-formulation of it is sketched out; to Turing-Church thesis of the usual theory corresponds a similar proposition, yet connecting physical total computation functions with constructive mathematical total computation functions.

**Keywords**: Theory of computation, Potential infinity, Problem-based organization of a theory, Doubly negated propositions, Non-classical logic, *Ad absurdum* proofs, Turing-Church thesis, Translation of the kind of logic, Constructive mathematics, Total computed function, Rational re-formulation of the theory.

1. The theoretical development of the Theory of Computation: two basic problems

Let us consider the Theory of Computation (TC) as a theory. Unfortunately, there is no common agreement whether TC is a scientific theory; also because ultimately there is no common agreement about the notion of a scientific theory (Stanford 2017, sect. 2.2).

However, TC is comparable with other theories, first of all, Mendeleev’s theory of classical chemistry. Mendeleev’s textbook on Chemistry starts by posing two basic problems; the first one is of a “quantitative nature”, which requires an operational study of all "the transformations of substances and the phenomena that accompany them" (Mendeleev 1906, p. 1). In TC we recognize the quantitative problem as the following one: “How many notions of computation can we define?” The second problem of Mendeleev’s theory is of a “qualitative nature”: “What is an element?” (Mendeleev 1906, pp. 20-21). In TC this kind of problem is easily recognized as the following one: “What is a computation?”.

Let us consider how Mendeleev solves these problems. After having examined the results of several chemical reactions and having attributed a valence number to each element (that represents the relational import of the theory), Mendeleev established based on the valences of the elements a periodic table of elements which represents the unitarian nature of all possible elements of matter. In such a way he answers the qualitative problem: an element is a component of this table.(Drago 2014) After having examined several formal notions of computability and proved that they are mutually equivalent (that represents the relational import of the theory), TC, through Church-Turing’s thesis (CTT), establishes the equality of whatsoever kind of formal computation with the intuitive notion of computability, *Ci*. That answers the qualitative problem of TC.

It was shown (Drago 2012) that some other scientific theories share the previous feature of Chemistry (albeit by reducing the two problems to only one). In the following list, a theory will be denoted by its author, its date of birth and its basic problem characterizing it: L. Carnot’s mechanics (the search of the invariants of the motion; 1783), Lagrange’ mechanics (the search of the easiest method of calculations in mechanics; 1878), Sadi Carnot’s thermodynamics (the efficiency of the heat/work transformations; 1824), Klein projective theory of geometries (what is a geometry, 1881), Einstein’s first paper on quanta (what is discrete in electromagnetism; 1905), Kolmogorov’s foundation of intuitionist logic (calculus of problems; 1932), Markov foundation of computable numbers (1962). A comparative analysis of the developments of the previous scientific theories, all based on a problem, suggests a common model of development. It is called a problem-based organization (PO) (Drago 2012).

The novelty of the present paper is essentially a comparison of the usual Theory of Computation with the new kind of theoretical organization to which we will refer in the following.

1. The theoretical development of the Theory of computation: Doubly negated propositions

 Let us summarise the main steps of the theoretical development of the above mentioned theories.

 All the above mentioned scientific theories make use of propositions of a particular kind, i.e. doubly negated propositions[[1]](#footnote-1) that are not equivalent to the corresponding affirmative propositions since the contents of the latter ones lack evidence within the real world (DNPs). Eg, the same name of element, atom, literally means “not divided”. More precisely, Lavoisier defined this unknown entity by means of the following proposition: “If we link to the name of elements… the idea of last term arrived at by [chemical] analysis, all the substances which we were not able to decompose by any means are for us elements” (Lavoisier 1862-92, p. 7).

In the last century, the scholars of mathematical logic achieved a crucial result; i.e. the validity or not of the double negation law represents the best discriminating mark between classical logic and almost all non-classical kinds of logic, above all intuitionist logic (Prawitz and Melmnaas 1968; Grize 1970, pp. 206-210; Dummett 1977, pp. l7-26; Troelstra and van Dalen 1988, pp. 56ff.). Hence, in the case of a DNP, the failure of the double negation law qualifies it as belonging to non-classical logic, in particular, intuitionist logic.

TC includes a modal[[2]](#footnote-2) adjective, ‘partial’ in the same subject of study: “partial computable functions”. Notice that a modal word is equivalent to a DNP; e.g. possible = it is not the case that it is not”. That also holds ~~true also~~ in formal terms; modal logic may be translated by means of its S4 model into intuitionist logic. (Hughes and Cresswell 1996, pp. 224ff.) The meaning of this word “partial” is more clear when it is replaced by an equivalent DNP which deals with only what we objectively know; i.e. we do not know its undefined values, which in general cannot be tested, but only the determined final results of the computing process of a function; hence, the meaning of “partial” in objective terms is the following one: “not never [= only sometimes] defined”.

In addition, an important step of TC’s development includes a DNP. Although ignoring the logical features of a DNP, several authors have presented CTT through a DNP. For instance, Stephen Kleene: “… it cannot conflict..." (Kleene 1952, pp. 318-3l9). Alan Turing makes use of two modal words, “could” and ‘naturally’: “the [Turing machine’s] computable numbers *include* all numbers which could naturally be regarded as computable”;(Turing 1936, p. 249) Alonso Church avoids a DNP by means of a modal word: “it is thought to correspond satisfactorily...“ (Davis 1965, p. 90)[[3]](#footnote-3) Martin Davis does the same: "we have reason to believe that...",(Davis et al. 1995, pp. 68-69; without specifying the scientific meaning of his word “believe”; it actually means the DNP: "it cannot be false that it is…” Kurt Goedel: a "heuristic principle". (Davis 1965, p. 44) Equivalently, Emily Post: “A working hypothesis" (Davis 1965, p. 291).

As a consequence of the use of all the above DNPs, TC makes essential use of non-classical logic. This is the real reason for calling the subject of this theory employing a modal word, ‘computability’ (see the title of Davis et al. 1995); it means that the theory deals with what *may* be computed.

Notice that a DNP does not oppose ~~to~~ the corresponding affirmative proposition (to the negative one either); hence, it does not give a true/false mirror opposition that assures certainty to each step of a logical derivation. Hence, it does not properly pertain~~s~~ to a deductive theory, whose propositions are drawn from some axioms (AO). As a consequence, whereas AO theories are governed by classical logic, a theory based on DNPs is governed by intuitionist logic.

1. The theoretical development of the theory of computation: *ad absurdum* arguments and the final translation of the kind of logic

The author of a PO theory links a DNP to another DNP in order to constitute an *ad absurdum* argument (AAA) (revealed within the original text of the theory by words like e.g. “otherwise… not…”), whose conclusion is not other than a DNP. Moreover, this DNP may work as a premise for the next AAA, so that such arguments constitute a logical chain, as eg in Lobachevsky’s booklet on non-Euclidean geometry (Bazhanov and Drago 2010) and Sadi Carnot’s booklet on thermodynamics (Drago and Pisano 2000).

TC is one of the few mathematical theories whose textbooks at first glance present several *ad absurdum* arguments.

In the model of a PO theory, *the conclusion* of its chain of AAAs is *a doubly negated, universal predicate*. As a matter of fact, in each of the above mentioned theories such a predicate is equivalent to the main thesis AI of the intuitionist square of opposition, SOI (see the table of all its predicates and their implication relationships in Dummett 1977, p. 29). This conclusion represents a guess for solving the given basic problem and all correlated problems of the theory.

 At this point, the problem is solved for what can be solved through some logical arguments. Afterwards, the author of the scientific theory wants to come back to make use of both the usual direct proofs and mathematical formulas, including an exact equality symbol, and hence belonging to classical logic. He achieves the goal by *translating the doubly negated, conclusive predicate into the corresponding affirmative predicate*, which can represent a mathematical formula with equality well.

The above translation of the universal doubly negated predicate may be justified in the following way: since at this step of the theoretical development the conclusive predicate is supported by ~~a~~ formal reasoning constituted by some AAAs, the author feels justified in translating it into its corresponding affirmative predicate. By translating his final predicate, Markov suggested that this step undergoes two requirements: this final predicate has to come out an AAA and be decidable.(Markov 1962, p. 5)

Notice that this translation results to be an (implicit) application of Leibniz’ principle of sufficient reason (PSR) – whose antecedent is itself a universal doubly negated predicate (“Nothing is without reason”), whereas the consequent is an affirmative one (“Everything has a reason”) which belongs to classical logic.(Drago 2017) It is easy to recognize through the same above mentioned Dummett’s table that this logical translation of the intuitionist version of thesis AI into the classical thesis A implies the translation of the entire kind of predicate logic from intuitionist logic to classical logic. Hence, the application of the PSR constitutes the inverse translation of the well-known ‘negative translation’ from classical logic into the intuitionist one, obtained by adding *grosso modo* two negations to each predicate (Troesltra van Dalen 1988, pp. 56ff.). That also implies a change of theory’s organization, from the previous PO to AO, which is the suitable organization for linking together mathematical derivations based on exact equalities from few hypotheses.

Is there a conclusive doubly negated predicate within the theoretical development of TC? After having proved that all formal definitions of a computation define the same set of functions (Odifreddi 1986, pp. 87-101), TC’s textbooks compare these definitions of computation, *Cf*, with the intuitive notion, *Ci*. Textbooks present their relationship as an “equivalence”; this word means the following universal predicate: “It is not true that *Ci* is not the formal notion of a computation, *Cf*”, *¬*(*Ci* ≠ *Cf*); i.e. a DNP as it occurs in a PO theory. Hence, also the theoretical development of TC includes a conclusive doubly negated predicate, exactly CTT.

Moreover, textbooks~~’~~ aim is to state the corresponding affirmative predicate, ie the equality, *Cf* = *Ci*; which is the starting point of the subsequent deductive development of TC in classical logic. As a fact, they state this equality by appealing to a so-called “thesis”, the word inside the CTT[[4]](#footnote-4). In both mathematics and theoretical physics the word “thesis” is commonly intended as the declaration of a subsequent theorem. Yet, one cannot *formally* equate an intuitive notion, *Ci*, with *Cf*. We have to conclude that, although seemingly plausible, the theorem proving this “thesis” cannot exist (Kleene 1952, p. 318; for a more accurate debate of this question see Shapiro 2006; Folina 1998; Folina 2006) [[5]](#footnote-5).

Some scholars remarked that TC makes use of a notion – ie “thesis”– which, by lacking the following theorem, represents a strange element of a theory, which instead as a whole is a highly formalized theory. The following words of Galton lucidly characterize the situation:

We thus seem to have the curious circumstance of a corpus of exact mathematical results being held up as evidence for a thesis which is impossible to state clearly while maintaining its distinctness from those results.(Galton 1996, p. 141)

In conclusion, some formal steps of the theoretical development of a PO theory are recognized inside the theoretical development of TC; but the last step of a PO development (an application of PSR on the final predicate) is lacking; this difference is so relevant to make uncertain previous at glance characterization of TC as a PO theory. On the other hand, an AO of TC is not possible, because from the intuitive notion, *Ci*, included by CTT no exact derivations can be drawn. As a consequence, it seems that TC cannot be organized in one of the only two possible kinds of organization, AO and PO. As a matter of fact, along many decades, scholars unsuccessfully tried to obtain a specific organization of TC.[[6]](#footnote-6)

 This fact raises the question: Is TC an informal theory (in agreement with the merely instrumental character that traditional mathematicians attribute to it)? In the past, in order to avoid this humiliating appraisal, a long-time debate on the nature of TC born. Its main subject was its crucial step, CTT.

Since CTT cannot be surely considered as *the result of proof* and *an axiom* either, it was intended either as *a definition*, at the cost of emptying the subsequent theory of any innovative content; or rather as *an empirical hypothesis*, at the cost of emptying its proposition of any theoretical content; or rather as *a proposal for the sake of argument*, ie. a “thesis”, at the cost of dismissing the usual meaning of this word in the common mathematical and logical arguing.

As a result, the theoretical systematization of the subjects to which current TC refers~~,~~ appears still blocked at a first stage, i.e. the stage of defining its basic notions.[[7]](#footnote-7)

However, this debate ignored the non-classical logic nature of all word versions of CTT. In the above, we have remarked that CTT is a DNP because all its word versions are DNPs and it essentially states an equivalence *¬*(*Ci* ≠ *Cf*). As such, it can play the role of the final doubly negated predicate of a PO theory to be discovered. Yet, the current meaning of it is the corresponding affirmative version (*Ci* = *Cf*); this fact reminds us that within a PO theory the final doubly negated predicate is then translated by an application of the PSR into the corresponding affirmative predicate. Therefore, current CTT implicitly and ambiguously plays two roles; the role of the final predicate of a PO theory and the role of the affirmative version which is obtained after its translation through an application of PSR.

These facts suggest that a PO of TC is possible, provided that one distinguishes in it two main contents, that concerning the role of the final predicate and that concerning the role of (a similar proposition of current meaning of CTT) explicitly affirming equality. In the following, a suggestion for such a PO of CT will be presented. Yet, a new organization of CT has to take into account other unsatisfactory aspects of present TC; next Sect. will illustrate them. To avoid them, re-formulation of TC as a PO theory has introduced some basic modifications of the current theory.

**4. Three more criticisms to current CT**

Let us consider the history of recursive functions. As it is well known, elementary recursive theory, relying on the notion of potential infinity (PI), originated through the celebrated Goedel’s paper of the year 1931. Then Ackerman invented a recursive function that is different from all elementary recursive ones. The novelty was obtained by essentially including the actual infinity (AI). After this result, TC’s theorists developed a general recursion theory relying on AI through some idealistic notions (ie unbounded minimalization).

The application of CTT states the equivalence between the set of the partial computable functions with the set of general recursive functions, which rely on AI. ; hence, also the former functions rely on AI. As a consequence, current TC, concerning the set of partial computable functions, is too wide in scope for the set of the functions relying on the notion of only PI (despite a student, by making attention to the Turing Machine naively supposes that TC is bounded to make use of notions relying on PI only).

As a second criticism, let us consider the word ‘partial’ inside the denomination ‘partial computable function’. It means that we do not know for which numbers such a kind of a function is defined. [[8]](#footnote-8) Sundholm has stressed that this notion is inappropriate to TC:

However, strictly speaking, *partial function* is an oxymoron. The adjective `partial' acts as a [structural] *modification* [of its subject, ie the noun function] that takes us out of [the usual definition of] functions, rather than as a qualifying property among functions. A ‘partial function’ is [actually] no function, since it is not defined for every element of the domain. The syntactic form “partial function' is misleading. Instead, in recursion theory, one could better speak about recursively enumerable functional relations, whether total or not. The reading 'function that is partial recursive', on the other hand, would appear to indicate a (total) function that for some reason is not fully recursive, but only partially so. (Sundholm 2014, p. 3)[[9]](#footnote-9)

After half a century since their introduction, should we avoid partial computable functions?[[10]](#footnote-10) A theorist contested TC’s scholars’ decision of basing the programming language upon partial computable functions:

There is a dichotomy in language design, because of the halting problem. For our programming discipline, we are forced to choose between

* 1. Security - a language in which all programs are known to terminate.
	2. Universality - a language in which we can write
		1. all terminating programs
		2. silly programs which fail to terminate

and, given an arbitrary program we cannot in general say if it is (i) or (ii).

Five decades ago, at the beginning of electronic computing, we chose (B). If it is the case, as seems likely, that we can have languages of type (A) which accommodate all the programs we need to write, bar a few special situations, it may be time to reconsider this decision. (Turner 2004, p. 767)

 Moreover, he has proved that this decision is not unavoidable. By choosing only total functions he suggested an innovative programming language, ie “Total functional programming”15.

 A third criticism. Why an intuitive notion, *Ci*, plays a decisive role inside a formal mathematical theory~~?~~. For the sake of clarity, let us answer the following question: What means to build a scientific theory in order to define an intuitive notion?. A plain answer is the following one: if at a certain step of its development a theory formally obtains the wanted result, from this time on the

notion at issue is no longer an intuitive one, because it was already translated through formal means into a formal element of a scientific context;[[11]](#footnote-11) alternatively, if the theory cannot define it as a formal notion, the trivial conclusion is that the theory is merely unsuccessful. Since manifestly cannot prove CTT and hence give a unique, formal definition *Cf* of *Ci* , apparently this appraisal applies to TC.

 However, let us examine closer this question. If the intuitive notion of *Ci* inside CTT has to be preserved by the theory, it represents a stumbling block to a scholar inspecting TC’s foundations. Indeed, a theory, including as a basic notion an intuitive, hence inaccurately defined notion, surely cannot be organized as an AO, since no accurate logical inference of an axiomatic theory is possible by starting from an intuitive notion. Alternatively, this intuitive notion may be considered as the definition of a program of research for solving the following problem: is *Cf* = *Ci* true? This suggestion addresses considering TC as a PO theory aimed at solving this problem. According to the model of a PO theory, this theory obtains an affirmative predicate - from which all the consequences may be deduced according to classical logic - not before having applied the PSR for translating the conclusive DNP into the corresponding affirmative predicate. Thus, TC has to translate the equivalence(= not difference) *¬*(*Cf* ≠ *C*i) between the intuitive notion of computability and the formalized one, into an *equality Cf* = *Ci* through an application of the PSR. Yet, according to Markov the application of PSR is correct when the predicate is decidable and results from the conclusive AAA.(Drago 2012) But an AAA arguing on an intuitive notion *Ci*, can proceed in a merely *informal* way (e.g. of it is the AAA: “Otherwise, our mind errs”). In sum, the previous predicate is not decidable. Hence, no correct translation of this predicate of equivalence into an equality predicate is possible through an application of the PSR.

In conclusion, no intuitive notion can be included by a scientific theory, by CTT too.

5. Towards a rational re-construction of TC according to both the model of a PO theory and constructive mathematics

As a consequence of the above analysis let me suggest a proposal for a new formulation of TC according to the PO and avoiding the above three criticisms.

To reiterate the kind of organization suitable to answer the basic problem of its historical origin, the new TC has to choose the PO. In addition, TC is first of all as several scholars have stated (see eg Svozil 2006, p. 492) - a theory of computing physical machines and at last a physical theory. As a consequence, in order to avoid all idealistic notions - that a priori appear inappropriate to a theory of operative computations -, it has to choose PI.[[12]](#footnote-12)

Furthermore, a new formulation of TC does not need to include the modal words “computable” and “computability” – expressing a subjective viewpoint rather than an objective viewpoint -, but only the word “computed”, which indicates what a usual experimental scientific theory refers to, i.e. provable and reiterable facts. In particular, the new TC has to discard partial computable functions, for rather considering only objectively defined notions of computation, i.e. the total functions.

Then let us start this new theory. The first step of the theoretical development of a PO theory is the declaration of its basic problems. By following Mendeleev’s suggestion the first problem is the qualitative one: “What is a computation?”, or better, “What is a *physical* computation?” .

Then, let us put the quantitative question: “How many notions of a computation, based on PI only exist there?” (or better, the henological question: “How to unify the multitude of formal definitions of physical computation?”). Here, fortunately, Turing was capable to answer the question for physical computations through his celebrated theorem suggesting the Universal Turing Machine[[13]](#footnote-13). Hence, the notion of a physical computation *CP* is well-defined on PI only.

 As a fact, many notions of a mathematical computation have been suggested.[[14]](#footnote-14) One may try to prove, through constructive mathematical tools based on PI, the possible equivalence between any pair of formal notions of mathematical computation among the great variety of them, e.g elementary recursion theory, constructive theory of Diophantine equations, constructive version of Lambda Calculus, etc. so to try to define a common mathematical notion of constructive computation, *CM*. I do not know whether such equivalence proofs already exist, but at a first glance they seem possible. (However, no surprise if it will be proved that such an “absolute” mathematical notion *CM* does not exist; the final situation may be similar to the result of a comparison of two operatively defined sets of the (constructive) real numbers, the set of rational numbers and the set of algebraic numbers: their intersection is a proper subset of each one). Hence, the second problem of the mathematical theory of computation, i.e. the quantitative problem, is an open problem. I proceed by assuming that there exists the solution of this problem.[[15]](#footnote-15)

Then the wanted re-formulation continues by tackling the question of whether all mathematical notions of computation *CM* and the physical notion of a Turing machine computation, *CP*, are equivalent. In a constructive setting, this problem corresponds to the “hard half” problem of TCC. A priori, the *CM*, whose nature is that of a construction by human mind according to essentially the principle of non-contradiction, may be essentially different from *CP*, since the latter one includes essentially the relationship with reality. In other terms, here we are led to consider the question of whether a Turing machine through computations can generate a contradiction.

One easily proves the above equivalence through an AAA: surely no contradiction in Turing machine is possible, otherwise, the same physical reality would represent a contradiction, contrarily to the millennial human experience of calculations and physical operations[[16]](#footnote-16). The conclusion is a DNP: “It is impossible that Turing computation gives different results from the mathematical computations”; or, in other terms: “The two kinds of computation are equivalent”.

At this point the two requirements of PSR’s applicability are satisfied: the predicate is decidable; moreover, it comes from an AAA. Then the application of PSR to the above DNP obtains the corresponding affirmative proposition: “Both physical and mathematical notions of total computation define the same set of functions”. At last, a similar proposition to the usual Turing-Church thesis connects physical total computations with constructively mathematical, total computations.

Afterwards, on change the kind of logic: from the intuitionist one of previous development of the theory into the classical logic of the subsequent part of the theory, which is developed in a deductive way. In such a way also the qualitative problem is solved.

A last remark. According to this suggestion of TC as a PO one cannot call the equality of the two notions a ‘thesis’ in the usual meaning, i.e. as the declaration of a next theorem proving it. Indeed the above AAA only proves their equivalence; and however, the last step of the development of a PO theory is not this AAA but the application of PSR, which is rather an inductive affirmation. It is an implicit reference to this inductive step that led theorists of TC to attribute to CTT an unusual logical power with respect to the traditional theses of ~~the~~ mathematical proofs.

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1. In the following I will underline the negative words belonging to a DNP for making easier an inspection by the reader. Notice that the current usage of English language exorcises DNPs since they are considered to be specific of primitive languages. Moreover, some linguists maintain that those who speak by means of DNPs want to be, for instance, unclear. (Horn 2002, pp. 79ff.; Horn 2010, pp. 111-112) Instead many other languages (Latin since two millennia) make use of DNPs in a correct way, although allowing some exceptions: a rhetorical double negations or psychologically emphatic doubly negations, which are equivalent to which may be equivalent to an affirmative or a negative proposition. [↑](#footnote-ref-1)
2. A modal word will be underlined with dots. [↑](#footnote-ref-2)
3. Most textbooks make use of the word "equivalence", without independently define it. Notice that the same word is used also by most Thermodynamics’ textbooks illustrating the principle concerning all kinds of energy. Actually, this word equivalence means: “It is not true that it is not the same...", i.e. it is a DNP. [↑](#footnote-ref-3)
4. As a historical fact all formal notions of computation resulted to be extensionally equivalent. This result plus CTT led Kurt Goedel to define ‘computability’ as an “absolute notion” from all notions of computation (Goedel 1990, p. 150). In theoretical physics the same adjective “absolute” is attributed to the temperature measured by the ideal Kelvin thermometer. But the latter notion is derived from a universal theorem (Sadi Carnot’s), whereas in TC a similar theorem (stating *Cf* = *Ci*) is no more than supposed (although Goedel believed in it; Shapiro 2006, p. 122-123). [↑](#footnote-ref-4)
5. Kripke’s attempt (Kripke 2013) attributes to the notion of deducibility in logic an intermediate role between the notion of mathematical computability and the notion of physical computability. Hence, its argument relies on two intermediate steps: one is a theorem (Goedel completeness theorem for the first-order predicate calculus of predicates with identity); the other one is again a “thesis”, Hilbert’s one (about the relation of mathematical calculations and logical deductions); which in its turn lacks of a proof. [↑](#footnote-ref-5)
6. Present TC seems an application of a Suppes’ philosophical attitude. Patrick Suppes accepts that a rigorous axiomatic of a physical theory cannot exist, because an axiomatic should include also axioms connecting theoretical propositions and experimental data (called ‘connection axioms’). But, given the informal nature of the experimental reality, thus kind of axioms cannot exist. “A Suppes’ axiomatic” is then constituted by no more than a mathematical predicate applied to a *mathematical model* of reality. Under this light present Theory of Computation seems a formal corpus of exact, mathematical results which is applied to a mathematical model of the physical reality of computing machines, i.e. Turing Machine. But this presentation of TC is unsatisfactory, because (by privileging the mathematical aspect of TC with respect to the physical aspect) characterizes TC as a merely abstract theory. [↑](#footnote-ref-6)
7. This present theoretical situation of TC reminds that of thermodynamics after its birth as a completed theory (1850). Thermodynamics was the first theory lacking of the basic notion of the AO organization of theoretical physics of this time, i.e. the force-cause. However, in order to organize the new theory according to the only known organization of a theory, AO, it was organized by means of “principles” (implicitly of an AO theory) at a heavy logical cost: Sadi Carnot’s result was qualified as the second principle, whereas as first principle was stated the conservation of energy that had been discovered 25 years later the former one. “By what magic has our stream risen higher than its source?” (Bridgman 1941, p. 10). As a consequence of this clumsy organization, a debate on the definition and the theoretical role of the intuitive notion of energy - as unifying together heat and all kinds of works – born and persisted along half a century. Henri Poincaré (1901, ch. VIII) concluded the debate by stating that (the intuitive notion of ) energy “is a convention made to suit our convenience, but with no further significance”. Percy Bridgman (1952, pp. 26-27) recalls Poincaré argument: if we were ever confronted by a situation in which the conservation of energy apparently failed, we would at once save the situation by inventing a new form of energy. In other terms, the former equality is manipulable, although in exceptional circumstances. Thermodynamics as a theory has to operate through the mathematical formula of the conservation of energy as it is known at its time. [↑](#footnote-ref-7)
8. Moreover, once one base TC on the problem “What is a computation?”, the partial functions have to be discarded as an inappropriate answer. [↑](#footnote-ref-8)
9. TC’s generalization of the usual notion of a completely defined function into a partial one apparently parallels

the generalization that a theoretical physicist performs on each result he receives from an experimental physicist. While

this experimental result is a truncated number - hence a rational number with a finite number of digits, hence it is at all

known -, the former physicist, in order to make easier through calculus his calculations on it, extrapolates it into a real

number; which, being composed by an infinite number of non-periodical digits, as say π, is not completely known (if

not in an idealistic, Platonist way); in such a way the dense set of rational numbers is changed into the compact set of

real numbers. Yet, whereas a theoretical physicist verifies his calculations performed through real numbers, by

comparing them with the rational numbers of measurement data (within an acceptable range of approximation), TC’s

theorist does not know whether his generalization into partial functions is confirmed or not by hard facts (surely not by

the mutual equality of all notions of a computable function - the intuitive one too -, an equality which actually is

supported by a claimed “thesis” which lack of a formal proof). [↑](#footnote-ref-9)
10. The philosophy of TC’s theorists to consider the greatest number as possible of functions, even the partial ones, seems to repeat the philosophy of 19th Century mathematicians of calculus, wanting to embrace all possible real functions - also the pathological ones - under an intuitive notion of a “law”, or a “correspondence”, between two intuitively conceived “sets”. Their original aim was to find out through these new functions the most general definitions of the two basic operations of calculus, ie derivative and integral. After two centuries of an enormous inventive effort, someone correctly remarked that the variety of pathological real functions (even if bounded to be monotone functions! Zamfirescu 1981) is so great that they make impossible any universal definition of both derivative and integral operations; so that a scholar (Feferman 2000) suggested that the pathological functions should be called “monsters”, to be removed from the corpus of the mathematics of a working mathematician. Moreover, the philosophy of TC’s theorists also seems to ignore the wise attitude of thermodynamic theorists, who started their theory by selecting inside the general set of the irreversible transformations which hardly can be mathematized, the irreversible transformations only. [↑](#footnote-ref-10)
11. This is the case of Information Theory. There, a merely mathematical definition of “information” is cursorily justified and then from it all consequences are drawn, by disregarding any question of how the mathematical notion of “information” adequately corresponds to the intuitive notion, since the latter notion, by apparently including many semantic meanings, has surely much more wide import than that of the former one. The same holds true for the intuitive notion of energy, which in any thermodynamic discourse has to be referred to the formal notion of the formula of conservation of energy and eventually to the variation of the internal energy. [↑](#footnote-ref-11)
12. I motivate the choice for PI also by means of a historical note on the developments of both physical computations and mathematical computations. Pithagoras’ discovery of the calculations leading to “forbidden” irrational numbers led to dismiss arithmetical computations as the foundational ones for mathematics. Rather, geometry was considered a well-founded theory based on the geometrical constructions, considered at that times as physical operations (in geometry the results of measurements were left merely indicated and the only mathematical function was the proportion, whose ratios do not produce irrational numbers). In the 17th Century born the calculus of infinitesimals (which were metaphysics-laden notions); their marvellous successes led to think that its symbolic mathematical calculation is privileged with respect to physical computations, first of all the geometrical computations (Lagrange wanted to reduce the entire geometrical theory to calculus; Lagrange 1773). In the 19th Century it was necessary a (Lazare Carnot-Cauchy and Weierstrass) “reform” in order to consider calculus as not an idealistic thinking, but primarily an operative process. However mathematicians persisted in attributing to it a primary role within the entire mathematics and within theoretical physics too. Within the philosophy of mathematics this privilege constituted a cultural bias (also leading to devaluating TC). By instead avoiding this cultural bias, one easily recognizes that mathematical computations, although can be formalized through appropriate objective symbols (written by paper and pencil or on a blackboard) are first of all mental, hence subjective acts, possibly including notions relying on AI; hence they cannot be put before physical computations, otherwise science would be a theory of possibly idealistic abstractions. . [↑](#footnote-ref-12)
13. Notice that this result was possible just because all machines are considered as operatively determined and hence appealing to PI only, i.e. without any idealistic notion. Otherwise, the answer would had been an idealistic model of machine whose computation operations could be approached by the computation operation of any real computing machine. [↑](#footnote-ref-13)
14. Notice that such a problem essentially concerns functions. Instead, what is relevant for each physical machine is its computation process, i.e. the computation of a result from a given input (like a compass with a given amplitude gives a circle of a fixed radius *r*); whereas by widening its amplitude we obtain a new circle with a greater radius *R*). In more general terms, one may consider a computer as performing a function connecting through its kind of computation a whatsoever input with a corresponding output. In more general terms, it is possible to consider a computation function which is specific for such a computer; but this general notion is not essential for a physical CT. Instead a mathematical computation cannot be theorized on a single number. This one has to be included within an infinite set. In the mid of 19th Century the notion of a function was obtained by Lobachevsky and Dirichelet by generalizing the geometrical line; subsequently this notion was generalized to an idealistic notion allowing all kinds of “monsters” (Feferman 1999). [↑](#footnote-ref-14)
15. Andrew Horsten (Horsten 2006) suggests an excellent analysis of the traditional TCC through both intuitionist logic and epistemic logic. It is remarkable that his Theorem 1 of p. 257 (suggested to him by A. Troelstra) is at all foreseeable from the viewpoint of TC as a PO theory: “For every sentence ** of the language of classical arithmetic, if **(**) is provable in *HA* + *ICT*, then is provable in *PA*. logic , where the CTT is essentially the PSR”. (where ** is the ‘negative translation’ and *ICT* is a formulation of CTT in intuitionist logic). Indeed the addition of the PSR to *HA* means to change the logic from the intuitionist one to the classical one. [↑](#footnote-ref-15)
16. This statement is similar to the statement which played an essential role for building mechanics and thermodynamics, in particular their respective theories of machines: “It is impossible a perpetual motion”. Notice that it was established after the millennial unsuccessful attempts to build a machine producing motion without receiving work. [↑](#footnote-ref-16)