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Aftermath Of The Nothing

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Abstract:

This article consists in two parts that are complementary and autonomous at the same time.

In the first one, we develop some surprising consequences of the introduction of a new constant called Lambda in order to represent the object “nothing” or “void” into a standard set theory. On a conceptual level, it allows to see sets in a new light and to give a legitimacy to the empty set. On a technical level, it leads to a relative resolution of the anomaly of the intersection of a family free of sets.

In the second part, we show the interest of introducing an operator of potentiality into a standard set theory. Among other results, this operator allows to prove the existence of a hierarchy of empty sets and to propose a solution to the puzzle of "ubiquity" of the empty set.

Both theories are presented with equi-consistency results (model and interpretation).

Here is a declaration of intent : in each case, the starting point is a conceptual questionning; the technical tools come in a second time

Keywords: nothing, void, empty set, null-class, zero-order logic with quantifiers, potential, effective, empty set, ubiquity, hierarchy, equality, equality by the bottom, identity, identification.

Part I

Lambda theory : Introduction Of A Constant For "Nothing" Into Set Theory: Most Noticeable Consequences And Perspectives.

1 Introduction

In this section, we present several immediate consequences of the introduction of a new constant called Lambda in order to represent the object "nothing" or "void" into a standard set theory. The use of Lambda will appear natural thanks to its role of condition of possibility of sets.
On a conceptual level, the use of Lambda leads to a legitimation of the empty set and to a redefinition of the notion of set. It lets also clearly appear the distinction between the empty set, the nothing and the ur-elements.
On a technical level, we introduce the notion of pre-element and we suggest a formal definition of the nothing distinct of that of the null-class. Among other results, we get a relative resolution of the anomaly of the intersection of a family free of sets and the possibility of building the empty set from "nothing". The theory is presented with equi-consistency results (model and interpretation).
On both conceptual and technical levels, the introduction of Lambda leads to a resolution of the Russell's puzzle of the null-class.
Finally, we suggest the possibility of the existence of a zero-order logic with quantifiers.

1.1 Why

Our aim is to clarify the real puzzle of Russell's conception of the null class as developed in the "Principles of Mathematics"[1]: 'But with the strictly extensional view of classes propounded above, a class which has no terms fails to be anything at all: what is merely and solely a collection of terms cannot subsist when all the terms are removed.'; Russell and Whitehead will formally express this inexistence in "The Principia Mathematica"[2]: '$\vdash\ .\neg\exists!\Lambda$'[1] .

[1] N.B. In Russell, Λ denotes the null-class, which is assimilated to *nothing*.

Russell could not accept the existence of the null class and assimilates it to "nothing", while recognising its technical utility, which is not conceptually satisfying for us. Notice that *in fine* Russell conceives the null class as the standard empty set (symbol: $\emptyset$): 'By symbolic logicians, who have experienced the utility of the null- class, this will be felt as a reactionary view. But I am not at present discussing what should be done in the logical calculus, where the established practice appears to me the best, but what is the philosophical truth concerning the null-class' [3].

Other logicians and mathematicians saw ontological difficulties with a class free of elements.
The first of them, Frege[4], strongly inspired Russell with his analytical philosophy approach in general, and his conception of the null-class in particular: 'When a class is composed of objects, when a set is the collective union of these, then it must disappear, when these objects disappear. If we burn down all the trees of a wood, we thereby burn down the wood'.
The fathers of the standard axiomatic set theory agreed with this view. So, in 1908, Zermelo [5] wrote: 'There exists a *fictitious* set, the null set, 0, that contains no element at all.'. In 1923, Fraenkel [6] added: 'For purely formal reasons, i.e. to be able to express some facts in a more simple and adequate manner, let us introduce here an improper set [*uneigentliche Menge*], the alleged set zero [*Nullmenge*] .../... It is defined by the fact that it does not contain any element; so it is not really a set, but it must be taken as such and be designed by 0'.
In his nominalist approach, Lesniewski[7] denies any kind of existence to classes in general and to the null-class in particular: 'I have always rejected, .../..., the existence of theoretical monsters such as the class of squared circles, being aware that nothing can be constituted of what does not exist'. Lesniewski only concedes the use of a nominal constant for denoting the *nothing*.
These quotations show that the doubts about the conceptual legitimacy of the null-class don't come only from detractors of set theory like Lesniewski, but mainly from several fathers themselves of the set theory!

We want to introduce here a clear distinction between the notion of empty set and the one of "nothing" (or "void"), that we will distinguish from $\emptyset$ via the symbol Λ.[2] The "nothing" must be conceived as the free space in any set (so also in the empty set): this is intuitively linked to the naive image of a set, as a "box" containing "objects" and where this is precisely possible because the box presents a "free space". This condition of possibility is also a condition of possibility in other fields, like the one of numbers and letters,

[2]In the use we make of it, Λ is not a class as in Russell; it is a new object that will be defined as a pre-element and a condition of possibility of sets, among others of the empty set.

see Pythagoras[8]: 'The void exists... It is the void which keeps the things distinct, being a kind of separation and division of things. This is true first and foremost of numbers; for the void keeps them distinct.' Here we see that the "nothing" clearly plays the role of *cut*.

The naive acceptance of the idea of "set" is then somehow validated in the case of the empty set: the empty set is a collection of "nothing".
Furthermore, this will allow the symbolic representation of the "empty space" that is intuitively present in any set, particularly in the traditional pictures of sets.

It would be natural to use the terminology of "inclusion" for the fact that the "empty space" Λ is "in any set". Nevertheless we show that the same symbol $\in$ can be used safely to express the fact of "belonging" to a set, for an object that is not Λ (and such an object is then called an "element" or a "set"), as well as the fact to be "the space Λ, present in a set" ("space" called "pre-element").

More precisely:

"$x \in y$" will express that x is an "element" of y only when $x \neq \Lambda$ (corresponding to the usual way of "belonging").
"$\Lambda \in y$" will express that Λ is "present in y"; and we use then the word "pre-element" instead of "element" to avoid any confusion.

Also, when more complex objects are constructed (via "terms", see section 2.2), the same kind of careful distinctions will be taken into account, as several interpretations are available. For example the usual singleton "$\{ a \}$" is simply "standard" in the universe "without Λ", while in the "completed universe" it will appear as something like "$\{ a , \Lambda \}$". This is further discussed in section 2.2).

However, even if the same $\in$-symbol is used in our theory, the roles of the elements/sets and of the unique pre-element are never confused; this immediately comes from the characteristic properties:

x is an "element" $\Longleftrightarrow \exists y \, y \in x$

x is a "pre-element" $\Longleftrightarrow [\, (\nexists y \, y \in x) \,\&\, (\forall z \neq x \; x \in z) \,]$

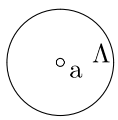
and these properties are guaranteed by the axioms (see section 1.2).

In addition, if the nothing-void is conceived as a potential, the Lambda theory is the first step forward to a theory where the notion of "potential membership" ("potentially belongs to") can be conceived. In that way, we can hope to handle the strange "ubiquity property" of the empty set (Theory in development).

Finally, we want to reduce significantly the ontological commitment of set theory. The classical axiom of existence becomes useless: there is no need to postulate the existence of a set any more (should it be the empty set) as Lambda (the "Void", the "Nothing") can be now seen as a generator of a hierarchy of standard sets.

Picture of a set and representation of Lambda

Lambda denotes the free zone around the element "a". The set pictured here is $\{a\}$ in the universe V of a standard set theory $\sum$. In the universe V_Λ of the Λ-theory $\sum_\Lambda$, the set pictured here is $\{a, \Lambda\}$.



1.2 How

Let's simply use the usual symbol $\in$ to express that Λ is "in $\emptyset$", in the same way as Λ is "in any set x". Starting from some set theory $\sum$ (in which the extensionality axiom holds and where $\emptyset$ exists), in the current first-order language $\mathcal{L} = (\in, =)$, we define a new theory $\sum_\Lambda$ in the expanded language $\mathcal{L}_\Lambda = (\in, =, \Lambda)$ (where Λ is a new constant symbol). This allows to give several distinct interpretations to the terms conceived in a classical way. Some of these new distinct interpretations produce interesting results, like: $\{\Lambda\} = \emptyset$, and relative "solutions" to the well-known "anomaly" of the usual phenomenon: "the intersection of an empty family is the universal class". If we call "sets" (in $\sum_\Lambda$) all the objects distinct from Λ, we expect that their behaviour is fundamentally the one described by $\sum$.

The behaviour of Λ will be governed (in $\sum_\Lambda$) by the two following axioms:

(1) Axiom of the *Pre-Element*: $\forall x\, (x \neq \Lambda \Rightarrow \Lambda \in x\,)$

(2) Axiom of the *Nothing-Void*: $\forall x \, (\neg \, (x \in \Lambda))$.

Notice that there can only be one "object" such as Λ, as axiom (1) is in contradiction with: $\exists y \neq \Lambda \, \forall x \, \neg \, (x \in y)$.

It is easy to construct (in a metatheory like Zermelo-Fraenkel) a model M_Λ for $\sum_\Lambda$, starting from a model M for $\sum$: we just artificially add a new element ("Λ") to the universe of M and extend adequately the $\in$-relation of M. The axiom of extensionality will still be applicable in M_Λ. It is easy, modulo some minimal conditions on $\sum$, to improve this result, namely to give an interpretation of $\sum_\Lambda$ in $\sum$ (instead of a *stricto sensu* "model" as just described), and to clarify the possibility of using Λ as parameter in several comprehension axioms: *inter alia* the example of separation, which is valid in M_Λ even for $\mathcal{L}_\Lambda$-formulas, once it is valid in M (for $\mathcal{L}$-formulas).

2 The theory

We start with a set theory $\sum$, expressed in $\mathcal{L} = (\in, =)$, and assume "$\top$", "$\perp$" (respectively "true", "false") as primitive symbols in our (classical) logic.

We expect $\sum$ to satisfy at least the 3 following conditions:

- $\sum \vdash$ EXT,
where EXT is the Extensionality axiom: $(\forall x \forall y \forall t (t \in x \iff t \in y)) \implies x = y$.

- $\sum \vdash \exists a \, \forall x \, (x \notin a)$; so "$\exists \emptyset$".

- $\sum \vdash \forall a \, \forall b \, \exists c \, \forall x \, (x \in c \iff (x = a \lor x = b))$;
(the classical "Pairing axiom").

Our theory $\sum_\Lambda$, in the language $\mathcal{L}_\Lambda = (\in, =, \Lambda)$ initially assumes the axioms described hereunder (2.1), but can surely be enriched based on the observation of the model M_Λ obtained by modification of M (see section 3). For convenient purposes, we introduce the following abbreviations:

- "$\forall^* x$" for "$\forall x \neq \Lambda$".

- "$\exists^* x$" for "$\exists x \neq \Lambda$".

Λ will be called "the Nothing" or "the Void"; and the "sets" are the objects x such that $x \neq \Lambda$.

For φ a formula in $\mathcal{L}_\Lambda$ (with "$\top, \bot$", allowed), φ^* will be obtained from φ by replacing in φ each $\forall$ by $\forall^*$ and each $\exists$ by $\exists^*$.

If Γ is a theory (list of axioms), Γ^* will denote the list of σ^*, with σ in Γ.

2.1 Axioms of $\sum_\Lambda$

(1) $\forall^* x(\Lambda \in x)$.

(2) $\forall x(x \notin \Lambda)$.

(3) σ^* for any axiom σ of $\sum$ (so $\sum_\Lambda$ "contains" $\sum$).

Remarks:

One can easily check that:

- $\sum_\Lambda \vdash$ EXT, i.e. EXT is applicable in the "full" universe (sets + Λ).

- $\sum_\Lambda \vdash \forall x(x \in \emptyset \Longleftrightarrow x = \Lambda)$, i.e. $\emptyset$ is the "singleton" of Λ (cf. hereunder our discussion about "terms").

2.2 Interpretations for terms

Usually, the term $\tau = \{x \mid \varphi\}$ is the name of the (unique via EXT) set b such that $\forall x(x \in b \Longleftrightarrow \varphi)$. In the theory $\sum_\Lambda$ however, we can now distinguish different interpretations for a term $\tau = \{x \mid \varphi\}$ based on a formula φ (in $\mathcal{L}_\Lambda$):

Definitions:

1) $\tau^* = \{x \mid \varphi\}^*$ is the unique set (if it exists) b (so b $\neq \Lambda$) such that : $\forall^* x(x \in b \Longleftrightarrow \varphi^*)$, or equivalently: $(\forall x(x \in b \Longleftrightarrow \varphi))^*$.

2) $\tau_\Lambda = \{x \mid \varphi\}_\Lambda$ is the unique set (if it exists) b (so b $\neq \Lambda$) such that : $\forall^* x(x \in b \Longleftrightarrow \varphi)$, or equivalently (in $\sum_\Lambda$): $(\forall x(x \in b \Longleftrightarrow (\varphi \vee x = \Lambda))$.

3) $\tau = \{x \mid \varphi\}$ is the unique object (if it exists) b (it could be Λ) such that : $\forall x(x \in b \Longleftrightarrow \varphi)$.

We will also use these indices "*" and "Λ" for the notations that abbreviate several classical terms, like:

$\{a\} := \{x \mid x = a\}$ (singleton)

$\{a, b\} := \{x \mid x = a \vee x = b\}$ (pair)

$\wp a := \{x \mid x \subseteq a\}$ (power set)

$\bigcup a := \{x \mid \exists y \in a, x \in y\}$ (general union)

$a \cup b := \{x \mid x \in a \vee x \in b\}$ (binary union)

$\bigcap a := \{x \mid \forall y \in a, x \in y\}$ (general intersection)

$a \cap b := \{x \mid x \in a \wedge x \in b\}$ (binary intersection)

With these clarifications, one can easily check that, in $\sum_\Lambda$:

- $\{\Lambda\} = \{\Lambda\}_\Lambda = \{\Lambda\}^* = \emptyset$: the empty set is the singleton of Λ.

- $\wp\Lambda = \wp_\Lambda\Lambda = \wp^*\Lambda = \emptyset$: the empty set is the Power set of Λ.

- $\bigcap_\Lambda \emptyset = \emptyset$: this constitutes a *relative* solution (Indeed, as we will see in section 4.4, it is the case that $\bigcap_\Lambda \Lambda = V$) to the well known classical "anomaly" of $\bigcap \emptyset = V$, that is in dissymetry with $\bigcup \emptyset = \emptyset$. In the Lambda theory, $\bigcap_\Lambda \emptyset = \bigcup_\Lambda \emptyset = \emptyset$.

- $\bigcap \emptyset = \Lambda$. In the same way, $\bigcup \emptyset = \Lambda$. So, once again we have a symetry between union and intersection of an empty family.

- Notice that $\bigcap^* \emptyset = V$, as in the "classical" situation.

3 Modelisation

3.1 The Idea

Working in Zermelo-Fraenkel as meta-theory, we can start with a model (in the *stricto sensu* sense, as in [9]) for $\sum$:

$M = (U_M, \in_M)$, where U_M is a set and $\in_M$ is a binary relation on M.

The desired model for $\sum_\Lambda$ is simply $M_\Lambda = (U_\Lambda, \in_\Lambda)$, where $U_\Lambda = U_M \cup \{\Lambda\}$ and $\in_\Lambda$ is the obvious extension of $\in_M$ such that: $\forall x \in U_M (\Lambda \in_\Lambda x)$

and $\forall x \in U_\Lambda \neg(x \in_\Lambda \Lambda)$, where Λ is some chosen element, not in U_M.

One can easily check that M_Λ models $\sum_\Lambda$.

The initial set theory $\sum$ should only satisfy the basic conditions described in section 2. When stronger theories $\sum$ are considered, new interesting properties appear in M_Λ, for example when $\sum$ satisfies the Power set Axiom, or other specific forms of comprehension. For further details, please refer to section 3.3. Examples: one can take (for $\sum$) ZF, or NF (Quine's New Foundations), or a "positive set theory" [10]. Furthermore, we can verify that for these "agreeable theories", there are corresponding comprehension axioms still applicable in M_Λ, even when the involved formula φ is in $\mathcal{L}_\Lambda$ (instead of in $\mathcal{L}$). As a consequence, Λ may appear as a parameter.
For example: the set $\{x \in a \mid \varphi\}^*$ exists in M_Λ when M is a model of ZF, even when φ is in $\mathcal{L}_\Lambda$; similarly, $\{x \mid \varphi\}^*$ exists in M_Λ when M is a model of NF (and φ is stratified): the reason is that by replacing in φ any atomic formula $x \in \Lambda, \Lambda \in x, x = \Lambda$, etc. by (the "ad hoc") $\perp$ or $\top$, one gets an equivalent formula in $\mathcal{L}$, stratified if φ was.

3.2 Interpretation of $\sum_\Lambda$ in $\sum$

The interpretation of $\sum_\Lambda$ in $\sum$ here developed guarantees the equi-consistency of $\sum$ and $\sum_\Lambda$; the converse interpretation (of $\sum$ in $\sum_\Lambda$) is obviously given by the initial universe of $\sum$. The construction described in 3.1 is the classical model-theoretic one. However if equi-consistency only is considered, this construction can be improved and we can give a direct interpretation of $\sum_\Lambda$ in $\sum$.

Just take, in the universe U of $\sum$, a copy U' of that universe, such that $U' \neq U$; this allows to choose an object in $U \setminus U'$, and we call this object "Λ". The usual technical trick to get such a U' and Λ (consider f.ex. $U' := U \times \{\emptyset\}$, and $\Lambda := (\emptyset , \{\emptyset\})$ is perfectly available here (modulo our conditions on $\sum$; cf. section2).

Of course we transfer isomorphically the $\in$-relation on the universe U to the universe U', so that $(U', \in')$ satisfies $\sum$. As universe for our interpretation of $\sum_\Lambda$ in $\sum$, we take then the class $U_\Lambda := U' \cup \{\Lambda\}$, and apply on it the obvious class-relation $\in_\Lambda$ defined by:
$x \in_\Lambda y$ iff $[(x \in U' \wedge y \in U' \wedge x \in' y) \vee (x = \Lambda \wedge y \in U')]$.
The conclusion is now similar to the one of 3.1: $(U_\Lambda , \in_\Lambda)$ interprets $\sum_\Lambda$ (in $\sum$).

3.3 Enriched Theories

We already mentioned in 3.1 that M_Λ (model in 3.1 or interpretation in 3.2) presents new interesting properties, when stronger theories $\sum$ are considered.
This is particularly the case for comprehension schemes, and can be explained very simply modulo the following technical remarks.

From the construction, it is obvious that, in M_Λ, any atomic formula containing the symbol Λ ($x \in \Lambda$, $\Lambda \in \Lambda$, $\Lambda \in x$, $x = \Lambda$, etc) is equivalent to $\perp$ or $\top$, if the variables are supposed to represent objects distinct from Λ.

As a consequence, for any sentence σ (sentence: formula without free variable) in the language $\mathcal{L}_\Lambda$, the sentence σ^* is equivalent, in M_Λ, to a sentence $(\tilde{\sigma})^*$, where $\tilde{\sigma}$ is obtained from σ by replacing each atomic formula containing the symbol "Λ" (adequately) by "$\perp$" or "$\top$", the choice being determined by the axioms (concerning Λ) of $\sum$ (see 2.1).

Examples:

One will replace "$x \in \Lambda$", "$\Lambda \in \Lambda$", "$x = \Lambda$" by "$\perp$"; and "$\Lambda \in x$" by "$\top$".

This elementary fact proves the following technical lemma:

If $\sum \vdash (\tilde{\sigma})^*$, then $M_\Lambda \models \sigma^*$.

This has interesting consequences on several so-called "comprehension schemes"; three examples are described below:

1) stratified comprehension, i.e. the scheme of axioms:

σ (the universal closure of): $\exists a \forall x (x \in a \iff \psi)$, for each stratified formula ψ.
Let's consider here even a stratified ψ in $\mathcal{L}_\Lambda$ (ψ "stratified" for $\mathcal{L}_\Lambda$ is obtained from a stratified formula in $\mathcal{L}$, where one or more free variables have been replaced by "Λ"). Then it is clear that $\tilde{\varphi}$ is again stratified (in $\mathcal{L}$ this time).

So, if $\sum$ is the system NF (cf.[9]), then $M_\Lambda \models \sigma^*$, even when σ is a stratified comprehension axiom with ψ in $\mathcal{L}_\Lambda$.

2) Separation and Replacement (as in ZF):

for σ an instance of one of these classical schemes, with Λ now admitted as parameter in the involved formulas ψ, we obviously have that $\tilde{\sigma}$ is again an instance of the same scheme (in $\mathcal{L}$ this time).

Let us briefly detail this for separation (the case of replacement is analogous):

Let's consider σ (the universal closure of): $\forall b \exists a \forall x (x \in a \iff (x \in b \wedge \varphi))$, with φ in $\mathcal{L}_\Lambda$ (our formula ψ here is: $x \in b \wedge \varphi$). Then $\tilde{\sigma}$ is (the universal closure of): $\forall b \exists a \forall x (x \in a \iff (x \in b \wedge \tilde{\varphi}))$, which is again an axiom of separation (in $\mathcal{L}$).

Conclusion: if $\sum$ is ZF, then M_Λ satisfies the versions of Separation* and replacement* that admit Λ as parameter. (i.e. involved formulas ψ in $\mathcal{L}_\wedge$)

3) Positive Comprehension:

several such systems have been proposed and studied; a description and references can be found in [9].
The basic idea is to consider comprehension for "positive" formulas, i.e. formulas not allowing negation (nor, of course, implication); notice that "$\perp$" and "$\top$" are considered as positive formulas. The corresponding scheme is then made of sentences σ (universal closure of): $\exists a \forall x (x \in a \iff \varphi)$, for any positive φ.

Now, let's allow also positive formulas φ in $\mathcal{L}_\Lambda$. It is obvious that $\tilde{\varphi}$ is again positive (in $\mathcal{L}$), so that $\tilde{\sigma}$ is still in the same scheme.

So, if $\sum$ is a positive set theory (one of the existing variants), then M_Λ satisfies the version of $\sum^*$ that allows Λ as parameter in the comprehension scheme.

Synthetic conclusion:
Our model/interpretation construction (cf. 3.1, 3.2) gives equiconsistency results for several "enriched" theories; more precisely:

for $\sum$ satisfying specific "comprehension schemes" (as described above), we have the equiconsistency between $\sum$ and $\sum_\Lambda^+$, where: $\sum_\Lambda^+$ is $\sum_\Lambda$ enriched with Γ^*, Γ being one of the types of schemes 1), 2), 3), that admits here Λ as parameter in the involved formulas φ.

4 Interest, Nature & Properties of Λ

4.1 Terminology

From an ontological point of view, we insist on the fact that here (in $\sum_\Lambda$), we clearly distinguish two types of objects:

- the "sets", *elements* x characterized (equivalently) by: $x \neq \Lambda$; $\Lambda \in x$.

- the "void" or "nothing" or *pre-element* Λ characterized by our axioms (1), (2) (section 1).

4.2 Internal and External Condition of Possibility

Lambda is a condition of possibility of elements (in the state of affairs, sets) in two ways:
- as an internal condition of possibility, Λ enables a set to contain elements. Lambda is the fundamental constituent of any set. This is expressed by the axiom of the pre-element. Indeed, in order for a set to contain other sets, an available space is necessary. Without the internal condition of possibility Λ, a set would be an atom, an ur-element, because there would be no way to make a distinction between the elements of a set.
- as an external condition of possibility, Λ also allows to have different sets. Indeed, the "Nothing" plays the role of cut, the *physical* separation between sets.

Thanks to this statute of condition of possibility, we think that the use of Lambda in set theory and the construction of $\sum_\Lambda$ are not artificial.

4.3 Lambda and the "contradictory property"

The contradictory property is traditionally sufficient to define $\emptyset$: $\{x : x \neq x\} = \emptyset$. Here in $\sum_\Lambda$, that property offers some more possibilities:

- $\{x : x \neq x\} = \Lambda$.

- $\{x : x \neq x\}_\Lambda = \{x : x \neq x\}^* = \emptyset$.

4.4 Lambda versus the Null-Class (or Empty Set $\emptyset$)

It is fundamentally clear that Λ is not $\emptyset$, as the first is the (unique) pre-element, while the second is an element (or set) (cf. 4.1). This has, of course, many consequences on their respective behaviours; we give here some interesting examples, involving cases where they behave in an analogous manner, as well as cases where they don't. For the notations (terms): cf. 2.2.

Lambda (Λ)	Null Class ($\emptyset = \{\Lambda\}$)
$\bigcap \Lambda = V$	$\bigcap \emptyset = \Lambda$.
$\bigcap_\Lambda \Lambda = V$	$\bigcap_\Lambda \emptyset = \emptyset$.
$\bigcap^* \Lambda = V$	$\bigcap^* \emptyset = V$.
$\Lambda \cap \emptyset = \Lambda$	$\emptyset \cap \emptyset = \emptyset$.
$\Lambda \cap \Lambda = \Lambda$	$\emptyset \cap \emptyset = \emptyset$.
$\{\Lambda\} = \emptyset$	$\{\emptyset\} \nexists$.

As we have announced in section 2.2 as well, the classical anomaly of the intersection reappears at a deeper level, at Lambda level. Nevertheless we find interesting to see that it does not appear on the level of the sets any more.

4.5 Lambda versus Ur-elements

The use of "nothing" also enables the distinction of the empty set from ur-elements (or "atoms"), which are generally considered as kinds of empty sets:
with u for an ur-element and x for a set, the expression $u \in x$ (which can be true or false) is syntactically admitted, while the expression $x \in u$ is not syntactically allowed.
Of course, "no thing" belongs to an ur-element, even not Lambda. This is precisely why an ur-element is a kind of atom. But we can make the distinction between Lambda and an ur-element too. Lambda belongs to the empty set, and more, Lambda belongs to every set. As we have defined it, as condition of possibility of elements, Lambda is a *pre-element*.
Finally, Lambda is different from the empty set by definition, and by its behaviour as we have seen in point 4.4.
Notice that Λ also enables the distinction $\emptyset$ from any ur-element.

4.6 Lambda as Generator of Sets

It is "ontologically interesting" to notice that, while we presented here $\sum_\Lambda$ as constructed on the basis of $\sum$, where "$\emptyset$" is already present, we can easily give an autonomous direct presentation of $\sum_\Lambda$ where $\emptyset$ would be "generated" (for example as $\{\Lambda\}$ or $\wp\Lambda$, as we have seen) if $\sum_\Lambda$ assumes the Pairing axiom or the Power set axiom.

So it is possible to make redundant the classical axiom of existence; there is no need any more to postulate the existence of a set. Moreover, we can build the hierarchy V_ω of sets starting from Lambda (assuming the Pairing

Axiom), or even the Von Neumann Hierarchy (of the well-founded sets), if $\sum$ extends ZF:

- Let V_0 be Λ.
- For any ordinal number β, let $V_{\beta+1}$ be the Power set of V_β. So, V_1 is $\wp(\Lambda) = \emptyset$.
- For any limit ordinal λ, let V_λ be the union of all the V-stages so far: $V_\lambda := \bigcup_{\beta\leq\lambda} V_\beta$.

The class V is defined as the union of all the V-stages: $V := \bigcup_\alpha V_\alpha$.

For those who consider the need to postulate some primitive entities to be problematic, we hope that Lambda will appear as a more attractive entity than the *empty set* or any other *primitive set*. Indeed, in this way, the theory is completed by the bottom, in a "minimal way" (Lambda being "nothing"), and is more in adequacy with a possible "mathematical reality" or at least with "formal possibilities".

4.7 Lambda and Simplification of the Axiom of Infinity

Another nice consequence of the use of Λ is the possibility - modulo a slight modification - of simplifying the classical axiom of infinity, as used in ZF.

That axiom starts with an initial set b (often, but not necessarily, $\emptyset$) and postulates the existence of an infinite *set* ("x"):

$$\exists x(b \in x \wedge \forall y\,(\,y \in x \Rightarrow y \cup \{y\}\ \in x)).$$

In $\sum_\Lambda$, the part "$b\ \in\ x$" can be removed, as Λ is "omnipresent" as pre-element in any set. So that the axiom of infinity can be reformulated as:

$$\exists^* x\,\forall y(y\ \in\ x\ \Rightarrow\ y \cup \{y\}\ \in\ x).$$

4.8 Lambda as solution to the Puzzle of the Null-Class

We have seen that the Puzzle of the Null-Class as found in Russell consists in the dichotomy between the technical legitimacy of the use of the Null-Class and its ontological illegitimacy.

In the "Principia", Russell justifies this ontological illegitimacy in two ways:
- the null-class does not exist because it does not contain anything.
- the null-class does not exist because it cannot belong to any class.[3]

[3] A.N. Whitehead, B. Russell. *Principia Mathematica To *56.* Ćambridge University Press, (1997) p.227.

In the “Principles”, Russell did not really succeed in giving conceptual legitimacy to the null-class; in the “Principia”, he does not even try to do it. However he seems to be satisfied in some way with the conceptual use of the “nothing” since he reduces the null-class to it. Indeed, if the “Nothing” had no legitimacy at all, the reduction of the null-class to the “Nothing” would make no sense.

The trick of Lambda theory here consists in starting from and exploiting this *conceptual* legitimacy of the “Nothing” in order to give it a *technical* legitimacy as well.

The way to give technical legitimacy to the “Nothing” is the introduction of the axiom of the “pre-element”.
While Russell justifies the inexistence of the null-class by denying it the privilege of belonging to another class, the axiom of the “pre-element” says that Lambda denotes the “Nothing” because it belongs to *any* set. In the spirit of the definition of the inclusion of the empty set in any set, we could say that, since Lambda denotes the “Nothing”, there is no set to which Lambda cannot belong. So Lambda belongs to the empty set too.
The empty set becomes the set that contains only Lambda.

The fact that our intuitive wishes about an adequate behaviour of Lambda can be formally axiomatized and lead to equiconsistency results gives a technical legitimacy to the notion of “void”.

In this way, not only the null-class seems to acquire complete (conceptual *and* technical) legitimacy in set theory, but the “Nothing” does too.

5 Zero-Order Logic with quantifiers

The Lambda theory is expressed in the standard first-order logic. It means that the quantified variables x, y, z... must be instantiated by first-order objects.
Second-order logic, in addition to individuals, quantifies variables that range over relations (properties) and functions too.
Second-order logic is extended by higher-order logics and type theory.
In the other direction, we have the zero-order logic. It is often assimilated to propositional calculus because quantification is not possible on variables of propositions. But zero-order logic is sometimes also presented as a first-order logic without quantifiers. A finitely axiomatizable zero-order logic is isomorphic to the propositional logic. With axiom schema, it is a more expressive system than propositional logic (cfr. the system of primitive recursive arith-

metic).
As a consequence of the introduction of Lambda in the language of a standard first-order logic, we think that we can consider the possibility of the existence of a zero-order logic with quantifiers that range over *pre-elements* only. As pre-element, Λ is a zero-order entity. Indeed, Lambda appears to be the smallest constituent that can be added to a set theory.
In this case, propositional calculus becomes the fundamental structure of any logic.

6 Conclusion

"*Nothing*" is added to set theory. This means: the introduction of the constant Lambda denoting the "Nothing" (the Void) in the language of set theory does not imply that some new thing (in the sense of: "new set") is added to the theory. The Nothing is subjacent to the standard set theory as a pre-element. It is a condition of possibility for the elements of the theory.
The Nothing has two functions:
- the first one is the function of *internal* condition of possibility. This enables a set to contain elements. Lambda is the fundamental constituent of any set, and this is expressed by the axiom of the pre-element;
- the second one is the function of *external* condition of possibility. Lambda plays the role of the *physical* space, of cut, between sets and allows to have distinct sets; this can be more specifically studied by formal ontology.

We believe that, as condition of possibility of sets, Lambda is more fundamental than sets and should not be reduced to a technical artifice. The Lambda theory is a natural approach of set theory and probably introduces the smallest, the minimal constituent that can be added to a set theory.

The use of Lambda leads to several interesting and/or surprising conceptual and/or technical results.

1) Introducing the Nothing in the language of set theory allows to distinguish the Nothing from the empty set, solving what we called the puzzle of Lambda in Russell's approach (see section 1.1).

2) The notion of "set" acquires here a real ontological dimension. In the *naive* conception of a set, a set is a collection of elements, and therefore it is not easy to give conceptual legitimacy to the empty set. The Lambda theory legitimates and gives an emblematic status to the empty set, now defined in a positive way: it is the only set that contains only Nothing.

3) More generally, the Lambda theory also allows to redefine the concept

of set: a set can contain sets or objects thanks to the free space denoted by Λ. The *naive* acceptance of set is in some way validated in the case of the empty set: the empty set is a collection of "Nothing".

4) The Lambda theory also allows to build the empty set by means of the axiom of pairing and also by means of the axiom of the power set applied to Lambda: $\emptyset = \{\Lambda\} = \wp(\Lambda)$. In classical ZF set theory, the existence of a set is to be postulated; in Lambda theory, the first set is built from "Nothing".

5) In the Lambda theory, the axiom of the existence of the empty set or the construction of the empty set by means of a contradictory property becomes useless. "Something", or rather a *pre-thing* belongs to the empty set: the "Nothing", and remember that the empty set is the only set to which only Lambda belongs.

6) It also allows to distinguish the empty set from ur-elements, which are generally considered as kinds of empty sets. No thing belongs to an ur-element, not even Lambda. This is why an ur-element is a kind of atom. Now, no thing belongs to Lambda either. But we can make the distinction between Lambda and an ur-element: Lambda belongs to the empty set, moreover, Lambda belongs to every set.

7) Finally, the Lambda theory is the first step forward to a theory where the notion of "potential membership" ("potentially belongs to") can be considered and formalized.

So, the use of Lambda in set theory appears natural and could even be seen as necessary.

For the theory, the balance in terms of investment and profits is clearly positive: the investment is *quasi-null*, the gains are numerous.

Part II
Theory of the Potential: Some Consequences Of The Introduction Of An Operator Of Potentiality Into Set Theory.

7 Introduction

This section shows the possibility of the existence of a hierarchy of empty sets and proposes a solution to the puzzle of "ubiquity" of the empty set via the introduction of an operator of potentiality into set theory [12]. The introduction of this operator of potentiality indeed allows to mathematically distinguish the concepts of potentiality and effectiveness, and consequently to see sets in a new light. It also leads naturally to interesting questionning on equality and identification, with some unexpected results.

The starting point, genealogically, was a tentative to resolve the following conceptual question:

- Does a hierarchy of empty sets exist to a somehow similar extent as the hierarchy of infinite sets demonstrated by Cantor.[13] Assuming the answer is yes, then how can we demonstrate the existence of a hierarchy of empty sets?

The idea of making a distinction between two possible aspects of a set: its effective content and its *potential* content, appeared to be a promising solution to demonstrate the existence of this hierarchy of empty sets. From there, the challenge consisted in giving technical legitimacy to this new notion of *potential* introduced in the frame of a classical first-order set theory. This is what is developed in the first paragraphs of this section.

In addition to the possibility of highlighting the existence of a hierarchy of empty sets when considered under their potential aspect, another consequence of the introduction of the operator of potentiality into classical set theory is the possibility to clarify the real puzzle of the *ubiquity* of the empty set in particular, and of any set in general. What do we mean by *ubiquity*? If we remove the elements of a set, we get an empty set. And different non empty sets from which we remove all elements, will give the *same* empty set. This is also expressed by the fact that the empty set is included in any set, according to the interpretation of the formal definition of the inclusion: since the empty

set does not contain anything, it is true that $\forall x \in \emptyset, x \in y$.
The puzzle consists in the fact that the same object is so to say "localized" at different places at the same time.
Indeed, according to Leibniz's law of the identity of the indiscernibles,[13] two empty sets can only be one and the same empty set.

The introdution of the operator of potentiality will make the indiscernibles discernible. The main interest of this paper is precisely the discovery of a way or a "possibility" to refine the notions of equality and identity.

8 How Is Introduced The Operator Of Potentiality?

In the usual first-order set theories (like ZF, NF and other more recent alternative set theories,[14]), the distinction between the "element" and the "collection" aspect of a set is generally not taken into consideration. Other theories, however (like VBG, Kelley-Morse, etc.[15]), clearly distinguish the notions of "set" and "class", and use thus a more "sophisticated" language (with two types of variables); here we also use several types of variables.

In our theory, several types of variables will also be used. The clear distinction of the 2 aspects element versus collection both on the effective level (classical nature and status of sets/objects) and on a potential level (the newly evidenced nature and status of sets/objects) allows to demonstrate the existence of a hierarchy of empty sets, brings a solution to the puzzle of ubiquity, by re-defining the notions of equality and identification.

8.1 The Idea

In theories where the "individuals" (called "sets") do have the "double nature" of "element" and "collection" (in the expression "$a \in b$" , the individual a "shows" its "element" aspect, while b shows its "collection" aspect), we add to the usual language $\mathcal{L} : (\in, =)$, an operator of potentiality denoted by the symbol $\Diamond$. It allows the universe to be extended via the introduction of new objects, of a different nature, and for which we will use the symbol $\Diamond$ and parameters p, q, r, ... as (upper/lower) index.

In summary:

- x will denote a standard object (a "set" in the initial set theory); it will be called a "schizo-object" or "schizo-set"(simultaneously element and collection).

- $\mathscr{C}$ will be a fixed class (in the initial set theory), called Class of Contexts (the "parameters p, q, r, ..." here above) that must satisfy the following minimal conditions: it must contain the empty set ($\emptyset \in \mathscr{C}$), and an infinite strict $\subset$ chain (notations: $\subseteq$ is the usual large inclusion: $a \subseteq b$ iff $\forall x \in a, x \in b$; $\subset$ is the strict inclusion: $a \subset b$ iff $a \subseteq b$ and $a \neq b$). For convenience, we introduce the restricted quantifier "$\forall_{\mathscr{C}}$" with the obvious meaning: $\forall_{\mathscr{C}} p \equiv \forall p \in \mathscr{C}$.
- $x^p_\Diamond$ will denote a *potential* (non-standard) element of some standard set y, according to some context p (see section 1.2.2) and some axiomatic rules (see section 3.1); it will be called *ghost-element*. It means that x does not belong *effectively* (in a standard way), to some set y, but that it belongs *potentially* to y.
- $y^\Diamond_p$ will denote the *potential* (non-standard) content of the set y relatively to the "context" p (p is supposed to be an "effective" set); $y^\Diamond_p$ is the "collection" of what can be "put" in y in relation to p. Such a collection $y^\Diamond_p$ will be called *hole-collection*.
When necessary, for clarity purposes, the index p, q, r, ... will be used with effective sets too.

We thus have three universes U, $U^\Diamond$ and $U_\Diamond$ interconnected by the relation "$\in$" modulo some rules described hereafter.

There is no relativization of quantification: same quantifiers apply to usual sets and to hole-collections so that, sometimes, sets and hole-collections can share the same content (thanks to axiom 4 and to the use of the Scott's equality by the bottom ($\overline{\overline{\barwedge}}$)) even though they belong to disjoint universes.

We extend the language $\mathcal{L} = (\in, =)$ of the initial set theory to a new "typed" language $\mathcal{L}^\Diamond_\Diamond$ that allows two new types of variables: "$x^\Diamond_p$" and "$x^p_\Diamond$" (where "x" is a variable of the language $\mathcal{L}$).

In a first time, before axiomatisation and modelization, we show already some rules that we expect these objects to obey:

- If y is a standard set/object: $x^p_\Diamond \in y$ is legal, $y \in x^p_\Diamond$ is not legal ($x^p_\Diamond$ cannot be placed at the right of the relation $\in$). This is why $x^p_\Diamond$ is *element* only.

- If y is a standard set/object: $y \in x^\Diamond_p$ is legal, $x^\Diamond_p \in y$ is not legal ($x^\Diamond_p$ cannot be placed at the left of the relation $\in$). This is why $x^\Diamond_p$ is *collection* only.
The *effective/standard* content (the y) of $x^\Diamond_p$ is the potential content (the $y^p_\Diamond$) of x.

And we suppose of course that the universes U, $U^{\Diamond}$ and $U_{\Diamond}$ are disjoint. A fortiori, $x \neq x_p^{\Diamond}$, $x \neq x_{\Diamond}^p$ and $x_p^{\Diamond} \neq x_{\Diamond}^p$.

The relations $\in$ and $=$ will be extended to the enlarged universe "$U \cup U^{\Diamond} \cup U_{\Diamond}$", and it is worth to note that the relations $\subseteq$, $\supseteq$ will make sense also between "sets" (x) and "hole-collections" ($y_p^{\Diamond}$), as between "hole-collections". So that, when both "inclusions" hold ($\subseteq$ and $\supseteq$), "equality of content" is realized; and in that case we will use Scott's "equality by the bottom" symbol: $\overset{=}{\wedge}$.

8.2 Extensionality, Equality, Equality by the Bottom, Identity, Identification

In standard set theories, the Extensionality axiom is often presented as following [see 1]:

EXT: $(\forall t(t \in x \Longleftrightarrow t \in y)) \Longleftrightarrow x = y$.

i.e. as a combination of:

1) $(\forall t(t \in x \Longleftrightarrow t \in y)) \Longrightarrow x = y$

the extensional identity, which is an expression of Leibniz Law of Identity of Indiscernibles; and:

2) $x = y \Longrightarrow (\forall t(t \in x \Longleftrightarrow t \in y))$.

which is the predicate calculus equality, or Law of Indiscernability of Identical (Law of Substitution).

The Scott's equality by the bottom ($x \overset{=}{\wedge} y$) or equality of content (extensional identity) is defined as being the double inclusion: $x \subseteq y \wedge x \supseteq y \Longrightarrow x = y$.

The axiom EXT precisely says that Scott's equality and ordinary equality (that of identification) coincide.

Logics (replacement) explains that ordinary equality implies Scott's equality ($= \Longrightarrow \overset{=}{\wedge}$).
Ext axiom adds the converse implication ($\overset{=}{\wedge} \Longrightarrow =$).

Identification is associated with ordinary equality. The ordinary equality is indeed a congruence, an equivalence with the substitution (or replacement) property. The Scott's equality by the bottom is just a relation of equivalence. In this case, it will only be question of identity of the objects concerned.

8.3 Ur-elements and True Empty Set

The "objects" of type $x^p_\Diamond$ will actually have the behaviour of "ur-elements" (as $y \in x^p_\Diamond$ is not legal). In contrast to this, those $a^\Diamond_p$ that have an empty content will be identified by the theory to a "unique true empty set", and so will not be ur-elements, for which the question of content does not make sense; see for example [16].

8.4 Context

In theories such as ZF, mathematicians work in some delimited context, such as V_α (the set of all well-founded sets of rank $< \alpha$), for some large enough ordinal α. The principle of the theory of the potential consists in the delimitation of contexts of work. We will call $\mathscr{C}$ the class of the allowed contexts.
If we don't work so when the starting set theory of our model is ZF for example, if the potential elements of a set x are simply the elements of the complement of x in the universe of the theory, the Russell's paradox arises in the theory of the Potential.
The way used to indicate a context is the introduction of a parameter in the notation x so as to have: $x^p_\Diamond$ or $x^\Diamond_p$ (for x a set).

As we have said above, we will use the letters p , q , r for the parameters.

8.5 Process

Thanks to our operator of potentiality, we get a mean of localization of "empty sets" on a potential level and so solve the anomaly of the "ubiquity" of the empty set (and the solution is extended to non-empty sets as well); and we are able to make the distinction between what we call "effective" (standard) elements (z) of a set x and what we call "potential" (in a non modal way, as we will show) elements ($z^p_\Diamond$) of the same set x. In this way, it will surprisingly be possible to show that two sets equal on an effective level ($x = y$) can be different on a potential level ($x^\Diamond_p \neq y^\Diamond_q$), and conversely, that sets different on an effective level ($x \neq y$) can be equal on a potential level, ($x^\Diamond_p = y^\Diamond_q$).

Another astonishing consequence of this approach is the possibility to highlight the existence of a "hierarchy" of empty sets on a potential level, thanks to the following property: $p \subset q \Longrightarrow \emptyset^\Diamond_p \subset \emptyset^\Diamond_q$.

9 Conditions on $\sum$

We start with a set theory $\sum$, expressed in $\mathcal{L} = (\in, =)$, and assume "$\top$", "$\perp$" (respectively "true", "false") as primitive symbols in our (classical) logic.

We expect $\sum$ to satisfy at least the 4 following conditions:

- $\sum \vdash$ EXT,
where EXT is the Extensionality axiom: $(\forall t(t \in x \Longleftrightarrow t \in y)) \Longrightarrow x = y$.

- $\sum \vdash \exists a \forall x\,(\,x \notin a\,)$; existence of the empty set $\emptyset$.

- $\sum \vdash \forall a \forall b \exists c \forall x\,(\,x \in c \Longleftrightarrow (\,x = a \vee x = b\,)\,)$; the classical "Pairing axiom".

- $\sum \vdash \forall a \forall b \exists c \forall x\,(\,x \in c \Longleftrightarrow (\,x \in a \vee x \in b\,)\,)$;
(Finite union).

These minimal conditions will allow us to keep extensionality in the extended universe and to satisfy our wishes. In addition, we will get a mutual interpretability of $\sum$ and $\sum_{\Diamond}^{\Diamond}$.

10 Formalized theory $\sum_{\Diamond}^{\Diamond}$ of Potential

10.1 List of Axioms of Potential and comments

- Axiom 1: $b \subseteq p \Longrightarrow (\,x_{\Diamond}^{p} \in b \Longleftrightarrow x \in b_{p}^{\Diamond}\,)$. This is the axiom of "Switch" element/collection (interconnection between the nature of element and the nature of collection). If a set b has a potential to have additional members $x, y...$, then that shows that those additional members $x, y...$ have the potential to belong to b.

- Axiom 2: $x \subseteq p \subset q \Longrightarrow x_{p}^{\Diamond} \subset x_{q}^{\Diamond}$. This is the axiom of strict Hierarchy. The leitmotiv behind this is: "the greater the context, the greater the potential". It will reveal an easy way to build a hierarchy of "empty sets" on a potential level.

- Axiom 3: $a \subseteq b \Longrightarrow b_{p}^{\Diamond} \subseteq a_{p}^{\Diamond}$. This is the axiom of Reversing. It corresponds to the natural expectation that "the bigger the set, the less it has potential elements".

- Axiom 4: $a_{p}^{\Diamond} \subseteq p$. This is the axiom of Localization of potential content. It ensures that $\in$ and $\subseteq$ can hold between the elements of U and $U^{\Diamond}$ despite of the fact that these two universes are disjoint. So it is clear that $a_p^\Diamond$ contains ordinary, *effective* elements. In light of axiom 1, these *effective* elements of $a_p^\Diamond$ appear to be the *potential* elements of a.

- Axiom 5: $\exists z \,(\, z \in x_p^\Diamond \,) \Longrightarrow x \subseteq p$. This is the axiom of Contextualization. It ensures that if x is not in the good context, it will have no potential content.

- Axiom 6: $x_p^\Diamond \subseteq y_q^\Diamond \wedge x_p^\Diamond \supseteq y_q^\Diamond \Longrightarrow x_p^\Diamond = y_q^\Diamond$. This is the axiom of Extensionality for hole-sets. Notation: $\mathrm{EXT}^\Diamond$.

- Axiom 7: $x_\Diamond^p = y_\Diamond^q$ iff $p = q$ and $x = y$. This is the axiom of Equality on the universe of the $x_\Diamond$.

- Axiom 8: $\emptyset_p^\Diamond \doublebarwedge p$. This is the axiom of maximum potential content, complement of axiom 4. Our expectation being that, when considered in a determined context, $\emptyset$ contains potentially all the elements of this context. Indeed, since the empty set $\emptyset$ does not contain any effective element, $\emptyset_p^\Diamond$ is the set $a_p^\Diamond \subseteq p$ that contains all the elements of p. In other words, $\emptyset_p$ $(\emptyset \subseteq p)$ contains potentially all the elements of p.

- Axiom 9: $x \in a_p^\Diamond \Longrightarrow x \notin a$. This is the axiom of *strict* potentiality. "Potential" elements of a set are elements that do not belong effectively to the set (according to the context considered). The *effective* elements x of $a_p^\Diamond$ are potentially in a $(x_\Diamond^p \in a)$. So we see that $\Diamond$ is not modal. Indeed, in a modal approach, one would expect $x \in a \Longrightarrow x \in a_p^\Diamond$.

10.2 Theorems

Theorem 1 : $p \subset q \Longrightarrow \emptyset_p^\Diamond \subset \emptyset_q^\Diamond$, by axiom 2. It gives a strict hierarchy of empty sets and an infinite chain $\emptyset_1^\Diamond \subset \emptyset_2^\Diamond \subset ...$ thanks to the conditions on $\mathscr{C}$.

Theorem 2 : $p \subseteq q \Longrightarrow a_p^\Diamond \subseteq a_q^\Diamond$.
Suppose $p \subseteq q$, and distinguish the following cases:
- $a \subseteq p$:
* if $p = q$, obviously, $a_p^\Diamond = a_q^\Diamond$;
* if $p \subset q$, then by axiom 2: $a_p^\Diamond \subset a_q^\Diamond$;
- $a \nsubseteq p$, so by axiom 5: $a_p^\Diamond \doublebarwedge \emptyset$, and so $a_p^\Diamond \subseteq a_q^\Diamond$.

Theorem 3 : $\emptyset_p^\Diamond \subseteq p$, by axiom 4. This is the localization of the empty set.

Theorem 4 : $x_p^\Diamond \subseteq \emptyset \Longrightarrow x_p^\Diamond = \emptyset_\emptyset^\Diamond$; proof:

Suppose $x_p^{\diamondsuit} \subseteq \emptyset$.
By axiom 4: $\emptyset_\emptyset^{\diamondsuit} \subseteq \emptyset$.
So $x_p^{\diamondsuit} \overline{\overline{\barwedge}} \emptyset_\emptyset^{\diamondsuit}$ and by axiom 6, $x_p^{\diamondsuit} = \emptyset_\emptyset^{\diamondsuit}$.
This is the theorem of the empty *hole*. Via this theorem, all the empty hole-sets are identified to $\emptyset_\emptyset^{\diamondsuit}$.

Theorem 5 : among all the potential forms of $\emptyset$: $\emptyset_p^{\diamondsuit}$, $\emptyset_\emptyset^{\diamondsuit}$ is the only "true" empty one, i.e. $(\forall x\,,\, x \notin \emptyset_p^{\diamondsuit}) \iff p = \emptyset$.
Proof:
- if $p = \emptyset$, by axiom 4: $\emptyset_\emptyset^{\diamondsuit} \subseteq \emptyset$, so $\forall x\,(\, x \notin \emptyset_\emptyset^{\diamondsuit}\,)$.
- if $p \neq \emptyset$, then $\emptyset \subset p$, so by axiom 2: $\emptyset_\emptyset^{\diamondsuit} \subset \emptyset_p^{\diamondsuit}$, which contradicts $\forall x\,(\, x \notin \emptyset_p^{\diamondsuit}\,)$.

Theorem 6 : $\emptyset_p^{\diamondsuit}$ is the maximum of the $a_p^{\diamondsuit}$, $(a_p^{\diamondsuit} \subseteq \emptyset_p^{\diamondsuit})$. By axioms 4 and 8.

Theorem 7: $\forall_{\mathscr{C}}\, p\,(\, p_p^{\diamondsuit} = \emptyset_\emptyset^{\diamondsuit}\,)$.
Proof:
By axiom 4, $p_p^{\diamondsuit} \subseteq p$. This implies, by axiom 9 (for $a = p$): $p_p^{\diamondsuit} \subseteq \emptyset$. Then, by theorem 4: $p_p^{\diamondsuit} = \emptyset_\emptyset^{\diamondsuit}$.
We will call this the theorem of the self-context.

10.3 Examples and Observations

In ZF, let's take $\mathscr{C} = \{\, V_\alpha \mid \alpha\, ordinal\,\}$: this is a strict chain that induces the strict chain of hole-sets $(\emptyset_{V_\alpha}^{\diamondsuit})_{\alpha\, ord.}$.

In NF, let's take $\mathscr{C} = V$: this is a lattice, producing another lattice of hole-sets: $(\emptyset_p^{\diamondsuit})_{p \in V}$.

With some extra-properties (obvious by theorem 2):

- $\emptyset_{p\cap q}^{\diamondsuit} \subseteq \emptyset_p^{\diamondsuit} \cap \emptyset_q^{\diamondsuit}$.

- $\emptyset_p^{\diamondsuit} \cup \emptyset_q^{\diamondsuit} \subseteq \emptyset_{p\cup q}^{\diamondsuit}$.

Remark about $\emptyset_{\diamondsuit}^{\emptyset}$.
$\emptyset_{\diamondsuit}^{\emptyset}$ is a particular element as it cannot belong to any set: $\emptyset_{\diamondsuit}^{\emptyset} \in x$ is impossible; indeed: by axiom 4, we have $\emptyset_{\diamondsuit}^{\emptyset} \subseteq \emptyset$, and by axiom 1: $\emptyset_{\diamondsuit}^{\emptyset} \in x \iff x \in \emptyset_\emptyset^{\diamondsuit}$.
Comparing the theory of the Potential with the Lambda theory [6](where Lambda, the *nothing*, belongs to any set, including the empty set), it rather appears that $\emptyset_{\diamondsuit}^{\emptyset}$ is a kind of *anti-Lambda*.

11 Modelisation

Let M be a model for our initial set theory $\sum$ (in $\mathcal{L} = (\in, =)$). $M^\Diamond$ will be the set of the "formal objects" $x_p^\Diamond$, with x in M and p in $\mathscr{C}$ (a class in M satisfying the minimal conditions explained in 1.2.1), and $M_\Diamond$ will be the set of the "formal objects" $x_\Diamond^p$, with x in M and p in $\mathscr{C}$. Technically, one can simply take for $x_p^\Diamond$ the triple : $(\ x\ ,\ p\ ,\ 1)$ and for $x_\Diamond^p$, the triple : $(\ x\ ,\ p\ ,\ 2)$. And we can suppose wlog that the universes of M, $M^\Diamond$ and $M_\Diamond$ are mutually disjoint. We have in the universes of $M^\Diamond$ and $M_\Diamond$ as many sub-copies of M as we have contexts p, q, r...

The universe of M^* (the model for $\sum_\Diamond^\Diamond$) will be given by the union of the universes of M, $M^\Diamond$ and $M_\Diamond$.

The structure M^* will be given by:

(1) the minimum relation $\in^*$, extending $\in_M$ such that:

- $x \in^* y \iff x \in_M y$

- if $\forall z \in_M y\, ,\, z \in_M p$, then $x \in^* y_p^\Diamond \iff (\, x \in_M p \wedge x \notin_M y\,) \iff x_\Diamond^p \in^* y$

(2) the minimum equivalence relation $=^*$, extending $=_M$ such that:

- $x =^* y \iff x =_M y$

- $\forall z\, (\, z \in^* x_p^\Diamond \iff z \in^* y_q^\Diamond\,) \iff x_p^\Diamond =^* y_q^\Diamond$

- $x_\Diamond^p =^* y_\Diamond^q \iff (\, x =_M y \wedge p =_M q\,)$

A straightforward verification allows to see that M^* indeed satisfies our axioms of Potential!

In particular is $=^*$ indeed a congruence, the "replacement" aspect being guaranteed by our conditions on the use of $\in$ between objects of different types (see 1.2.1.).

This model fundamentally uses the difference "$p \setminus x$" to play the role of $x_p^\Diamond$.
The extension of $\in_M$ to $\in^*$ simply specifies that "$z \in^* x_p^\Diamond$" whenever "the contextualization is satisfying", i.e. "$x \subseteq p$" and "$z \in p \setminus x$". Further does $z_\Diamond^p \in^* x$ then happen exactly when "$z \in^* x_p^\Diamond$".

Remark: for the sake of simplicity, our relative consistency proof (here above) was presented in the model-theoretical style. But it is interesting to notice that actually the theories $\sum$ and $\sum_{\Diamond}^{\Diamond}$ are mutually interpretable, as our conditions on $\sum$ guarantee that we can construct the adequate copies of the universe in the theory itself.

12 Discussion on Identifications

Our approach leads us to interesting questionnings about equality and identity, including some cases of Leibniz Law of identity of indiscernibles infraction.

12.1 "Equalities" and Identifications in $\sum_{\Diamond}^{\Diamond}$

In the first-order language of a standard theory, $\in$ and $=$ are fundamental. Now in the theory of Potential, we have introduced the symbol $\overline{\overline{\curlywedge}}$, the Scott "equality by the bottom".

It allows us to compare the content of objects of different nature such as standard sets (sets with effective content) and hole-sets (sets with potential content).

So here, we meet several kinds of identities and identifications:

1) in $\sum$: EXT: $x \overline{\overline{\curlywedge}} y \Longrightarrow x = y$.

In the antecedent, we have the equality by the bottom, defined by means of $\in$: $\forall t\,(t \in x \Longleftrightarrow t \in y)$; in the consequent, we have the ordinary equality (identification).

2) in $\sum_{\Diamond}^{\Diamond}$:
- EXT (for standard, effective, objects).
- EXT$^{\Diamond}$: $\forall t\,(t \in x_p^{\Diamond} \Longleftrightarrow t \in y_q^{\Diamond}) \Longrightarrow x_p^{\Diamond} = y_q^{\Diamond}$. This is expressed by axiom 6: $x_p^{\Diamond} \subseteq y_q^{\Diamond} \wedge x_p^{\Diamond} \supseteq y_q^{\Diamond} \Longrightarrow x_p^{\Diamond} = y_q^{\Diamond}$, the axiom of Extensionality for hole-sets.

3) The relation $\overline{\overline{\curlywedge}}$ can be seen as a "*multi-objects*" or "*mixed*" equality that applies to "sets" of the same type or not: it is indeed allowed between x and y, or between $x_p^{\Diamond}$ and $y_q^{\Diamond}$ with identification then (respectively by EXT and by EXT$^{\Diamond}$), and between x and $y_q^{\Diamond}$ but in that case with no identification via $=$.

4) a particular case of "identity of content" is given by axiom 8: $\emptyset_p^\diamondsuit \,\bar{\bar{\wedge}}\, p$. (they both contain the same elements) despite of the fact that they are different ($\emptyset_p^\diamondsuit \neq p$).

12.2 Divergent Identifications & Leibniz Law of Identity of Indiscernibles Infraction

We discuss now some rather unexpected phenomena where different ways of "identification" possibilities do not coincide.

Case 1: different non empty sets that are potentially identical.

- Proposition 1: If $\mathscr{C}$ satisfies (in $\sum$) the following hypothesis, that there exist sets y , z and distinct sets a , b , p , q all in $\mathscr{C}$, realizing:

$b = a \cup \{ y \}$
$p = a \cup \{ z \}$
$q = a \cup \{ y , z \}$

with $y \notin a$ and $z \notin a$,

then $a_p^\diamondsuit = b_q^\diamondsuit$.

Proof:
By Axiom 9, $a_p^\diamondsuit$ and "a" are disjoint . But by Axiom 4, $a_p^\diamondsuit \subseteq p$. So, by the definition of "p", $a_p^\diamondsuit \subseteq \{ z \}$. But $a_p^\diamondsuit$ cannot be empty because $a \subset p$, so (by Axiom 2) $a_a^\diamondsuit \subset a_p^\diamondsuit$. So $a_p^\diamondsuit \,\bar{\bar{\wedge}}\, \{ z \}$ and by EXT$^\diamondsuit$ (ordinary equality") $a_p^\diamondsuit = \{ z \}$.
An analogous argument shows that $b_p^\diamondsuit \,\bar{\bar{\wedge}}\, \{ z \}$, so that $a_p^\diamondsuit \,\bar{\bar{\wedge}}\, b_q^\diamondsuit$ and by EXT$^\diamondsuit$: $a_p^\diamondsuit = b_q^\diamondsuit$ (while $a \neq b$).

Comment: here the potential levels p , q are distinct: $p \neq q$; to compare with proposition 2.

- Proposition 2: Take for $\sum$ the classical theory ZF (with axiom of foundation here), and for $\mathscr{C}$ the class of the sets V_α of the Von Neumann hierarchy. Consider the variant of the model for $\sum_\diamondsuit^\diamondsuit$ (as constructed in section 4) where the role of "$p \setminus x$" used to interpret $x_p^\diamondsuit$ would now be held by $V_\alpha \setminus V_{\rho x}$, where ρx is the classical "rank of x", i.e. the smallest ordinal β such that $x \subseteq V_\beta$. Then there exist distinct sets x , y indiscernible at the same potential level: $x_\alpha^\diamondsuit = y_\alpha^\diamondsuit$.

Proof: just take $x \neq y$ with $\rho x = \rho y$, and $\alpha > \rho x$; then $x_{\alpha}^{\Diamond} = y_{\alpha}^{\Diamond}$.

Comment: compared to proposition 1, we work here at the same potential level.

Case 2: identical sets that are potentially different: $a = b$ but $a_p^{\Diamond} \neq b_q^{\Diamond}$ with $p \neq q$.

A contrario to the case 1, sets equal when considered on their effective aspect (content) can be differentiated when considered on their potential aspect. Example: $\emptyset = \emptyset$ but $\emptyset_p^{\Diamond} \neq \emptyset_q^{\Diamond}$ when $p \subset q$, by axiom 2. The most remarkable illustration of this is the hierarchy of empty sets. This is of course a theorem of $\sum_{\Diamond}^{\Diamond}$.

Let us note that we cannot have $a = b$ and $a_p^{\Diamond} \neq b_p^{\Diamond}$ because the equality relation is a congruence and thus: $a = b \Longrightarrow a_p^{\Diamond} = b_p^{\Diamond}$.

13 Solution to the Set's Ubiquity Puzzle

The trick of the theory of the Potential consists in relying a and $a_p^{\Diamond}$ in such a way that the effective content of $a_p^{\Diamond}$ be the potential content of a.

In other words, $a_p^{\Diamond}$ is a considered under the angle of its potential content according to a context p. It is guaranteed by axiom 1.

There are for a set a as many $a_p^{\Diamond}$, $a_q^{\Diamond}$, $a_r^{\Diamond}$..., as there are different contexts p, q, r in which a is included.

The *problem* of the ubiquity of any set is then solved. It is possible to make the distinction between the different copies of a same set a according to the set/context in which it is included.

14 Hierarchy of Empty Sets

The initial question, which could be viewed as quite ambitious, was unexpectedly found to lead to additional interesting research like the possibility to introduce and legitimate the use of an operator of potentiality in the language of a first-order theory.

But the demonstration of the existence of this hierarchy of empty sets remains an interesting result. According to axiom 8, a hole-empty set $\emptyset_p^{\Diamond}$ is always identical by the bottom to a level p in the hierarchy of levels, and in general to a non-empty set which is the corresponding level. So we have a hierarchy of hole-empty sets. These non-empty hole-empty sets contain standard elements x. But according to axiom 1 (switch effective/potential elements), the potential elements $x^p_\Diamond$ belong to the standard empty sets $\emptyset_p$. So we have a hierarchy of empty sets according to their potential content.

15 Conclusion

Our initial questionning on the possible existence of a hierarchy of empty sets led us to introduce the operator of potentiality $\Diamond$ into a set theory satisfying minimal standard conditions. We saw that the key of the theory is the notion of *context* p. It allowed us to build new objects, *ghost-elements* ($x^p_\Diamond$) and *hole-collections* ($x^\Diamond_p$). Thanks to them, in addition to the possibility to build a hierarchy of empty sets, we can solve the puzzle of the ubiquity of sets and show that sets with same effective content can have different potential content. Conversely, we can have different non empty sets with the same potential content.

Our approach lead us to an interesting questionning about equality and identification. As it applies to two different kinds of objects, the relation of equality by the bottom $\overline{\overline{\wedge}}$ of the theory of Potential can be seen as a kind of *mixed* equality.

We saw that the *true* empty set is $\emptyset^\Diamond_\emptyset$, i.e. the only set that contains neither effective nor potential elements.

It is clear that the operator of potentiality is not modal.

Back to our initial questionnings, let's stress a fundamental difference in our approach compared to that of Cantor. In Cantor, the tool (equipotence) precedes the concept (hierarchy of infinite); in our approach, the concept (hierarchy of empty sets) precedes the tool (operator of potentiality).

Finally a word on two works in progress.

(1) We think that an alternative presentation should be possible. We think that it should be possible to use a unique relation $\in^\Diamond_p$, on the universe of effective sets (and $x \in^\Diamond_p y$ would so to say correspond, in an adapted way, to "$x \in y^\Diamond_p$" and to "$x^p_\Diamond \in y$"), to introduce reasonable concepts for "pseudo-modal" formulas "$\Diamond_p \varphi$" and terms "$\{\, x \mid \varphi \,\}^\Diamond_p$".

(2) We compared (end of section 4) our theory of Potential with the Λ-theory [17], and wonder whether some common theoretical extensions would

be possible.

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