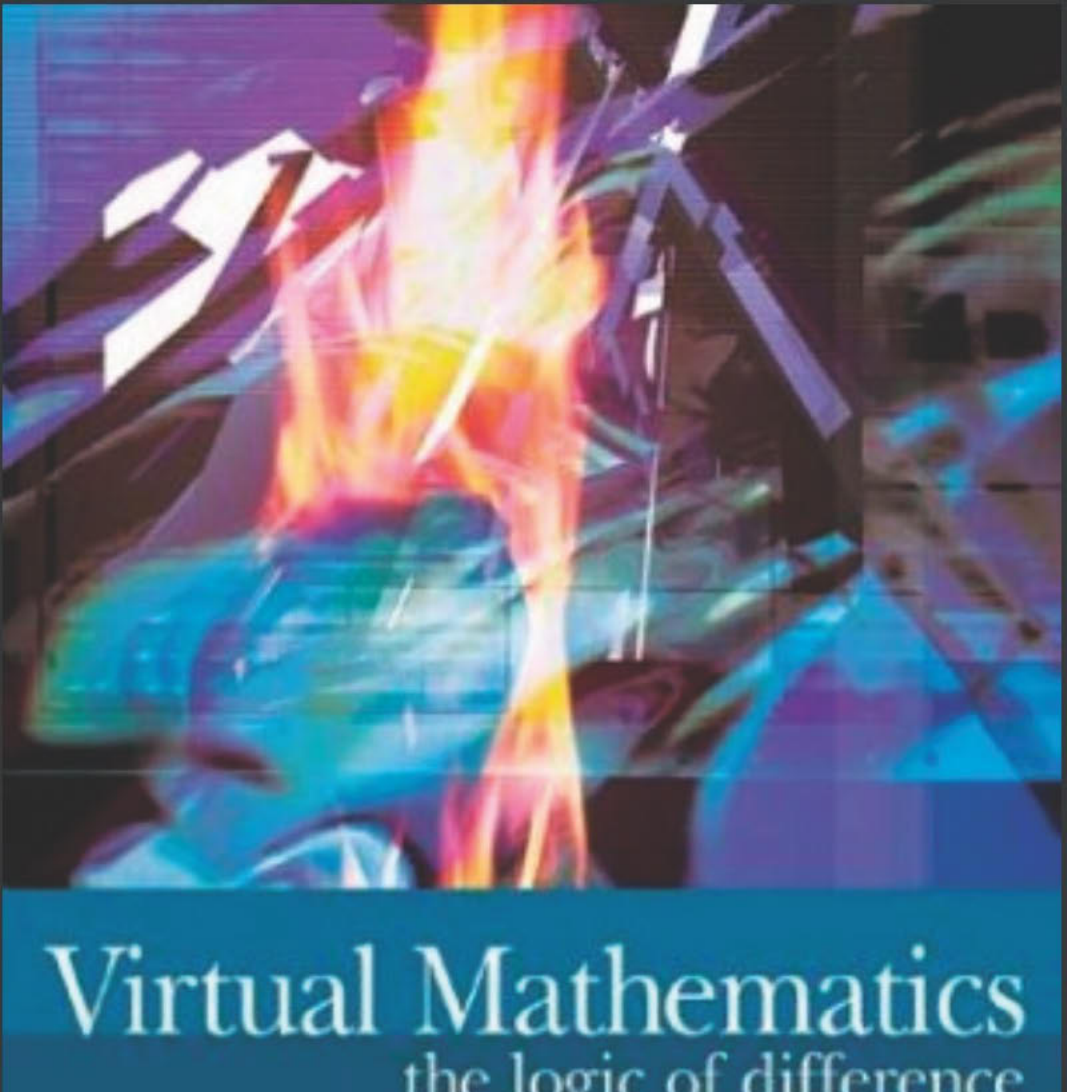

Virtual Mathematics

the logic of difference

EDITED BY SIMON DUFFY

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Contents

Notes on Contributors

Charles Alunni is director of the Laboratoire Disciplinaire 'Pensée des Sciences', and is responsible (with Alain Prochiantz) for the Gilles Châtelet Archives, at the Ecole Normale Supérieure (Paris, France). He is author of *Gaston Bachelard & la construction de la physique contemporaine* (Albin Michel, 2001); *Entretien avec Alain Connes* (Etudes, 2000); and of 'Diagrammes & Catégories comme prolégomènes à la question: Qu'est-ce que s'orienter diagrammatiquement dans la pensée?' *Théorie littérature enseignement* 22, *Penser par le diagramme: de Gilles Deleuze à Gilles Châtelet*, 2004.

Alain Badiou is Professor Emeritus in the Department of Philosophy at the Ecole Normale Supérieure (Paris, France), and the founder and president of the Centre International d'Etude de la Philosophie Française Contemporaine (Paris, France). He is author of *L'être et l'événement* (Seuil, 1988), *Being and Event*, translated by Oliver Feltham (Continuum, 2005); *Manifeste pour la philosophie* (Seuil, 1989), *Manifesto for Philosophy*, translated by Norman Madarasz (SUNY Press,1999); *Le nombre et les nombres* (Seuil, 1990); *Gilles Deleuze: Le clameur de l'être* (Hachette, 1997), *Deleuze: The Clamor of Being*, translated by Louise Burchill (University of Minnesota Press, 2000); and of 'Un, multiple, multiplicité(s),' *Multitudes* 1, mars 2000.

Gilles Châtelet (b. 1945-1999) taught at the University of Paris VIII (Saint Denis, France) and at the Institut des Hautes Etudes Scientifique (France). He is author of *Les enjeux du mobile: mathématique, physique, philosophie* (Seuil, 1993); *Vivre et penser comme des porcs: de l'incitation à l'envie et à l'ennui dans les démocraties-marchés* (Exils, 1998); and *Figuring Space: Philosophy, Mathematics and Physics*, translated by Robert Shore and Muriel Zagha (Kluwer, 1999). A second volume of *Les enjeux du mobile* is forthcoming from Éditions-Rue d'Ulm.

Manuel DeLanda is Adjunct Associate Professor of Architecture at Columbia University (New York, USA). He is the author of *War in the Age of Intelligent Machines* (Zone, 1991); *A Thousand Years of Nonlinear History* (Zone, 1997); *Intensive Science and Virtual Philosophy* (Continuum, 2002); and *Assemblage Theory and Social Complexity* (forthcoming), and a number of essays published in various journals and collections. He teaches two seminars at Columbia University, School of Architecture: 'Philosophy of History: Theories of Self-Organization and Urban Dynamics'; and 'Philosophy of Science: Thinking about Structures and Materials'

Simon Duffy is a Postdoctoral Research Fellow at the University of Queensland (Brisbane, Australia). He has a joint appointment in both Philosophy and the Centre for the History of European Discourses. His publications include 'The Logic of Expression in Deleuze's *Expressionism in Philosophy: Spinoza*. A strategy of engagement', in the *International Journal of Philosophical Studies* 12(1), 2004, and 'Schizo-Math. The logic of differen*t*/*c*iation and the philosophy of difference,' in *Angelaki: Journal of the Theoretical Humanities* 9(3), 2004. He has also translated a number of Deleuze's Seminars on Spinoza.

Robin Durie is Senior Research Fellow in the Health Complexity Group at Peninsula Medical School (Exeter, UK). He is the author of *Face to Face: Directions in Contemporary Women's Portraiture* (Scarlet Press, 1998); he is editor and co-translator of *Duration & Simultaneity* by Henri Bergson (Clinamen, 1999); he edited, and contributed the paper 'The Strange Nature of the Instant' to, *Time & the Instant: essays in the physics and philosophy of time* (Clinamen, 2001); and his published articles include 'The Mathematical Basis of Bergsonism,' *Journal of the British Society for Phenomenology* 35(1), 2004.

Aden Evens is Assistant Professor of Technical Communications at M.I.T (Massachusetts, USA). His publications include 'Math Anxiety' in *Angelaki: Journal of the Theoretical Humanities* 5(3), 2000; 'Sound Ideas,' in *A Shock to Thought: Expressions After Deleuze and Guattari*, edited by Brian Massumi (Routledge, 2002); and 'Concerning the Digital,' in *differences: a journal of feminist cultural studies* 14(2), 2003. His book on music and technology, *Sound Ideas: Music, Machines, and Experience*, will be published in Spring 2005 by University of Minnesota Press. When he's not thinking about math, technology, or philosophy, he makes music, some of which can be found on Montreal's constellation label, under the name *re:*.

Arkady Plotnitsky is Professor of English and Director of the Theory and Cultural Studies Program at Purdue University (Indiana, USA). He is the author of *The Knowable and the Unknowable: Modern Science, Nonclassical Thought, and 'The Two Cultures'* (Michigan, 2001); *Complementarity: Anti-Epistemology After Bohr and Derrida* (Duke, 1994); *In the Shadow of Hegel: Complementarity, History and the Unconscious* (University Press of Florida, 1993); and *Reconfigurations: Critical Theory and General Economy* (University Press of Florida, 1993); and editor (with Barbara Herrnstein Smith) of *Mathematics, Science and Postclassical Theory* (Duke, 1997).

Jean-Michel Salanskis is Professor of Philosophy at the University of Paris-X (Nanterre, France). He is the author of, amongst other books, *Le constructivisme non standard* (Presses Universitaire du Septentrion, 1999); *Modèles et pensées de l'action* (L'Harmattan, 2000); and *Sens et philosophie du sens* (Desclée de Brouwer, 2001); and recently of the two articles 'Les mathématiques chez x avec x=Alain Badiou', in *Penser le multiple* (L'Harmattan, 2002); and 'Phénoménologie lévinasienne (eth-analyse) du politique,' in *Levinas à Jérusalem. Interprétations philosophiques et perspectives religieuses* (Albin-Michel, 2005).

Daniel W. Smith is Assistant Professor of Philosophy at Purdue University (Indiana, USA). He has translated Deleuze's *Francis Bacon: The Logic of Sensation* (Continuum, 2003) and *Essays Critical and Clinical,* with Michael A. Greco (Minnesota, 1997) as well as Pierre Klossowski's *Nietzsche and the Vicious Circle* (Chicago, 1997) and Isabelle Stenger's *The Invention of Modern Science* (Minnesota, 2000). His published articles include 'Mathematics and the Theory of Multiplicities: Deleuze and Badiou Revisited', *Southern Journal of Philosophy* 41(3), 2003. He is currently working on a book-length study of Deleuze's philosophy.

David Webb is a Senior Lecturer in the Department of Philosophy at Staffordshire University (Staffordshire, UK). His published articles include 'Cavaillès, Husserl and the Historicity of Science', *Angelaki* 8(3), 2003; 'The Contingency of Freedom: Heidegger Reading Kant', *International Studies in Philosophy* 36(1), 2004; and 'The Complexity of the Instant: Bachelard, Levinas, Lucretius', *Time & the Instant: essays in the physics and philosophy of time*, edited by Robin Durie (Clinamen, 2001). He also edited and wrote the introduction to *The Birth of Physics* by Michel Serres, translated by Jack Hawkes (Clinamen, 2000); and is the author of *Heidegger: Ethics and the Practice of Ontology* (Clinamen, forthcoming).

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The chapter by Alain Badiou was presented at a conference in the spring of 2003. I gratefully acknowledge the permission of the author to publish the text of this paper in translation. In preparing the final draft of this paper, the version that recently appeared in Badiou 2004, translated by Ray Brassier and Alberto Toscano, was consulted.

1
Deleuze and mathematics

Simon Duffy

The collection *Virtual Mathematics: the logic of difference* brings together a range of new philosophical engagements with mathematics, using the work of French philosopher Gilles Deleuze as its focus. Deleuze's engagements with mathematics rely upon the construction of alternative lineages in the history of mathematics in order to reconfigure particular philosophical problems and to develop new concepts. These alternative conceptual histories also challenge some of the self-imposed limits of the discipline of mathematics, and suggest the possibility of forging new connections between philosophy and more recent developments in mathematics.

This component of Deleuze's work has, to date, been rather neglected by commentators working in the field of Deleuze studies. One of the aims of this collection is to address this critical deficit by providing a philosophical presentation of Deleuze's relation to mathematics; one that is adequate to his project of constructing a philosophy of difference and to the exploration of some of its applications.

This project developed as a result of encounters with an increasing number of researchers who have been working on the mathematical aspects of Deleuze's work and the key role that these play in his philosophy. By bringing this work together for the first time, this collection makes an important contribution to the emerging body of work that endeavours to explore the broad range of Deleuze's philosophy.

Deleuze is by no means the only contemporary philosopher to have engaged in work of this kind. For this reason the collection is not devoted solely to his work, but rather brings together a range of papers that address both the logic of these Deleuzian engagements between philosophy and mathematics, and the logic of other related efforts to mobilise mathematical ideas in relation to the history of philosophy in order to construct new philosophical concepts or to open up new lines of engagement between philosophy and mathematics. The collection includes papers by Alain Badiou, Gilles Châtelet and Jean-Michel Salanskis; and papers by David Webb on Michel Foucault and Jean Cavaillès, and by Charles Alunni on Albert Lautman and Gaston Bachelard. The bulk of the collection, however, is devoted to essays that directly address the work of Gilles Deleuze and the engagements that he undertakes between the discourse of philosophy and developments in the discipline of mathematics.

Deleuze's engagements with mathematics can be characterised in a general and schematic way as consisting of three different components, each of which is implicated by, and in turn implicates, the others. The explication of each of these components, and of its relations of implication, is required in order to develop an adequate understanding of these kinds of engagement.

1. The first component can be characterised as the history of mathematics relevant to each of the programmes or mathematical disciplines with which Deleuze engages, and the alternative lineages in the history of mathematics that are determinable in relation to them. An explication of how these 'histories' function as alternative lineages to the accepted (retrospectively constructed) history of mathematics that dominates the general understanding of the discipline, involves determining each of these alternative lineages as generated by a mathematical problem or problematic that challenges the self imposed limits of the discipline. The tensions that Deleuze characterises between the history of mathematics and the mathematical problematics that are extracted from it are particularly evident in the tension between what can be described as the axiomatised set-theoretical explications of mathematics and those developments or research programmes in mathematics that fall outside of the parameters of such an axiomatics; for example, algebraic topology, topos theory and differential geometry, to name but a few. Deleuze does not subscribe to what Corfield characterises as 'the logicists idea that mathematics contains nothing beyond an elaboration of the consequences of sets of axioms' (2003, 23). This tension between a set-theoretical axiomatics and the mathematical problematics that fall outside of its parameters can be understood as characteristic of the relation between what Deleuze and Guattari, in *A Thousand Plateaus* (1987), deem to be Royal or major science and nomadic or minor science. The nature of this distinction and its importance to the development of Deleuze's philosophy is explicated by Daniel W. Smith in chapter 8.

An understanding of each of the mathematical engagements which Deleuze undertakes throughout his work, therefore, requires a clear explication of the history of mathematics from which the specific mathematical problematic has been extracted, and of the alternative lineage in the history of mathematics that is generated in relation to it.

2. The second component of each of Deleuze's engagements with mathematics can be characterised as the explication of the manner by means of which these interventions in the history of mathematics are redeployed in relation to the history of philosophy. The mathematical problematics extracted from the history of mathematics are directly redeployed by Deleuze as philosophical problematics in relation to the history of philosophy. This is achieved by mapping the alternative lineages in the history of mathematics onto corresponding alternative lineages in the history of philosophy, that is, by isolating those points of convergence between the mathematical and philosophical problematics extracted from their respective histories. The redeployment of mathematical problematics as philosophical problematics is one of the strategies that Deleuze employs with respect to the history of philosophy. Deleuze actually extracts philosophical problematics from the history of philosophy, and then redeploys them either in relation to one another, or in relation to mathematical problematics, or in relation to problematics extracted from other discourses, to create new concepts, which, according to Deleuze and Guattari in *What is Philosophy?* (1991), is the task of philosophy.

3. The creation of new concepts by bringing together mathematical and philosophical problematics constitutes the third component of these Deleuzian engagements in mathematics. The implication of these three components in relation to one another constitutes the mechanism by means of which Deleuze's interventions in the history of mathematics serve in his project of constructing a philosophy of difference.

While Alain Badiou is clear about the relation that he figures philosophy bears to mathematics; that is, the foundational position of mathematics as ontology (2005), the role played by any mathematical ontology in the construction of Deleuze's philosophy of difference is certainly not stated as clearly from the outset, but is just as surely deployed as such by Deleuze. While the domain mobilised by Badiou in his philosophy of the event is set theory in its relation to category theory, that deployed by Deleuze is primarily algebraic topology, functional analysis and differential geometry. The very question of an ontology of mathematics requires that a mathematical problematic and a specifically ontological philosophical problematic be implicated in relation to one another; what Deleuze characterises in *Difference and Repetition* as 'reciprocal synthesis' (1994, 172). The Badiouian position, taking mathematics (or more specifically, set theory) as ontology already involves this kind of reciprocal implication between mathematics and philosophy.

According to this general schema, Deleuze can hardly be called a mathematician, nor is there a particularly Deleuzian mathematics; that is, despite engaging with particular mathematical problematics, Deleuze doesn't attach himself to a particular tradition or school of mathematics. When he and Guattari comment on 'the "intuitionist" school (Brouwer, Heyting, Griss, Bouligand, etc),' they insist that it 'is of great importance in mathematics, not because it asserted the irreducible rights of intuition, or even because it elaborated a very novel constructivism, but because it developed a conception of *problems*, and of a *calculus of problems* that intrinsically rivals axiomatics and proceeds by other rules (notably with regard to the excluded middle)'(Deleuze and Guattari 1987, 570 n. 61). Deleuze extracts this concept of the calculus of problems itself as a mathematical problematic from the episode in the history of mathematics when intuitionism opposed axiomatics. It is the logic of this *calculus of problems* that he then redeploys in relation to a range of episodes in the history of mathematics that in no way binds him to the principles of intuitionism. The relationship between Deleuze and the intuitionist school of mathematics is taken up by Aden Evens in chapter 11.

Deleuze is therefore very much interested in particular kinds of mathematical problematics that can be extracted from the history of mathematics, and in the relationship that these problematics have to the discourse of philosophy. He can therefore be understood to redeploy not only the actual mathematical problematics that are extracted from the history of mathematics in relation to the history of philosophy, but also the logic of the generation of mathematical problematics, that is, the calculus of problems, in relation to the history of philosophy. This is in order to generate the philosophical problematics which are then redeployed in his project of constructing a philosophy of difference. It is in relation to the history of philosophy that Deleuze then determines the logic of the generation of philosophical problematics as that characteristic of a philosophy of difference.

In order to present an adequate account of the engagements that Deleuze undertakes between developments in the discipline of mathematics and the discourse of philosophy, the mechanism of operation of this logic, as determined in relation to the discipline of mathematics and the mathematical problematics extracted from it, requires explication. Far from being a logic of the relation between Royal science and nomad science, between axiomatics and problematics, or of that between the history of mathematics and the mathematical problematics that are extracted from it; it is rather a logic of the generation of nomad science itself, or of each mathematical problematic itself. It is a logic that has proved incapable of being formalised by royal science or mathematics. However, and this is the irony of it, it is the logic that is characteristic of the advances or transformations in these disciplines. An explication of the mechanism of operation of this logic has been undertaken in chapter 7 in relation to the history of the differential calculus, and in chapter 11 by Aden Evens in relation to the concept of the surd in intuitionist mathematics. This logic, the logic of the calculus of problematics, is therefore not simply characteristic of the relative difference between Royal and nomadic science, or between the history of mathematics and its related mathematical problematics. It is rather characteristic of the very logic of the generation of each mathematical problematic itself. It is this logic that Deleuze redeploys in relation to the history of philosophy as a logic of difference in order to generate the philosophical problematics that he then uses to construct a philosophy of difference. Developing an understanding of the nature of this logic is the key to understanding Deleuze's engagement with the history of mathematics and his use of mathematical problematics throughout his work.

Another important characteristic that needs to be taken into consideration is the manner by means of which the kinds of engagement that Deleuze undertakes between the discourse of philosophy and developments in the discipline of mathematics can be repeated. Are the engagements that Deleuze undertakes with the discipline of mathematics exhaustive? Or is the logic of these engagements applicable elsewhere in relation to other developments in the discipline of mathematics? The purpose of doing so would be to characterise new mathematical problematics that can be directly redeployed as philosophical problematics in relation to the history of philosophy in order to construct new philosophical concepts.

The work of Manual DeLanda is significant in this respect insofar as it engages with the means by which the logic of the generation of problematics continues to affect the discipline of mathematics even after the reappropriation of particular problematics by an axiomatics, or indeed a formalism, that has been extended to accommodate them. DeLanda's work also involves the exploration of the subsequent departures from the discipline of mathematics and of the problematics that are thereby generated, opening up the potential for the creation of new philosophical concepts.

Work in this area thus entails an examination of more recent developments in mathematics in response to the mathematical problematics utilised by Deleuze; an examination of their continued status as problematics, or of the altered axiomatics of mathematics after reappropriation of those problematics. The aim would be to locate and characterise new mathematical problematics which can then be redeployed as philosophical problematics in relation to the history of philosophy, by isolating their points of convergence, in order to construct new philosophical concepts.

One of these developments is category theory, the programme that is 'challenging set theory to become the language of the dominant tradition,' in mathematics (Corfield 2003, 198). Category theory allows you to work on mathematical structures without the need first to reduce them to set-theoretical axiomatics. Although category theory does appear to be the historical continuation of set theory, the 'categorisation' of the concept of set is not a technical refinement but rather represents a profound conceptual change in mathematics (See Rodin 2004). Indeed category theory can be understood to have the potential to function as an alternative power of unification in mathematics to set theory (See Salanskis 2002, 102). The question of the formalisation of problematics, which category theory can perhaps be understood to pose, and of whether Deleuze's engagements with mathematics can be characterised in this way is taken up by Daniel W. Smith in chapter 8. Category theory 'began as a project to study continuous mappings within the programme of algebraic topology' (Corfield 2003, 198). Work on the latter was initiated by Henri Poincaré (b.1854 – 1912) as a project to help develop tools to study differential equations qualitatively. The importance of this aspect of Poincaré's work for Deleuze is taken up in chapter 7. The difference between algebraic topology and set-theoretic topology is that 'the latter is . ubiquitous in routine arguments and formulations, but the former is almost unreasonably effective in advancing mathematical understanding' (Macintyre 1989, 366). Deleuze's work should be understood to engage with the kinds of mathematical problematic associated with such transformations in the discipline of mathematics, whether or not they can be given a category-theoretic determination. It is these kinds of problematic that populate the innovative plane in mathematics that we are calling here 'virtual mathematics'

The essays assembled in this collection work together to characterise the way in which the three components of Deleuze's engagements with mathematics operate in his work, and to bring out the way in which Deleuze's philosophy exemplifies a particular kind of engagement between the discipline of mathematics and the discourse of philosophy. The collection seeks to promote this kind of engagement as a new understanding of this relation; one that recognises the difference between the two disciplines, and one that promotes further investigation of this difference.

Alain Badiou provides a characterisation of the relation between philosophy and mathematics that sets the stage for further exploration of this relation using the work of Deleuze as a focus. Badiou mounts a defence of a particular kind of interrelation by distinguishing between two different styles of philosophy, namely the 'little style' and the 'grand style' The little style of philosophy figures mathematics in a secondary role to the specialisation 'philosophy of mathematics' which can be inscribed in the narrow genre of 'epistemology and history of science' the two operations of which are classification and historicisation. The grand style of philosophy, which Badiou champions, holds that mathematics directly clarifies philosophy, and not the inverse, and that it does so by forced (violent) intrusion in the intimate disposition of the questions of philosophy. He considers mathematics to be a necessary condition of philosophy itself, one that is 'both descriptively exterior and prescriptively immanent to philosophy.' Badiou suggests that the necessary task of philosophy is to find the new terms of the grand style.

Gilles Châtelet (b.1944 – 1999) is well known for his work on the relationship between the disciplines of mathematics, physics and philosophy, and for his interest in the work of Deleuze, who had a decisive influence on his philosophical development. The paper by Châtelet is previously unpublished, however it will appear in the second volume of his collected work, *Les enjeux du mobile*, that is forthcoming from Éditions-Rue d'Ulm. The notes for this paper were found on Châtelet's desk; it is what he was working on just before he died. These notes have been edited for publication by Charles Alunni, who provides the paper with an introduction. The paper betrays Châtelet's fundamental preoccupation with the diagrams of mathematical physics. Interest in 'diagrammatics' is one of the points of convergence between the work of Châtelet and that of Deleuze (See Deleuze and Guattari 1987, 141–8). According to Corfield, it is in higher dimensional algebra that 'diagrams are not just there to illustrate, [but] are used to calculate and prove results vigorously' (2003, 254). Just as algebraic objects once assisted in our understanding of topology, topological objects can now allow us to calculate in algebra. This is done by means of the manipulation of the projection of the topological object, or diagram, itself. Châtelet explores the implications of such a diagrammatics in relation to a particular example in knot theory.

Jean-Michel Salanskis seeks to renew the relation between mathematics and philosophy by liberating it from the alternative between a Heideggerian or phenomenological understanding, which considers mathematics to hinder any understanding of what stands beyond the objective, and an analytic treatment, which prefers pairing logic and philosophy rather than mathematics and philosophy. Salanskis considers Deleuze to have thought he could do this by proposing a general doctrine of what is as such, a metaphysics in the classical sense, which would be directly expressed in mathematical rather than logical terms. Salanskis puts forward the case that we can also promote the couple mathematics / philosophy in a more Kantian way, in reference to a renewed conception of the transcendental.

Deleuze's engagement with the work of the philosopher of mathematics Albert Lautman, who wrote in the 1930s and 1940s, and whose approach to a logic of mathematical problems and interpretations has inspired several contemporary French philosophers of mathematics, is introduced by Charles Alunni. Through the figures of Lautman and Bachelard, Alunni charts an alternative history to those developments that became the preoccupation of analytic philosophy. This includes an examination of the process of desubstantialisation in mathematical philosophy undertaken by Lautman through the development of the notion of duality, particularly in relation to the symmetry / dissymmetry distinction; and the developments associated with Bachelard's project of dialectial surrationalism, which identifies the foundational role of mathematics in the developments and transformations of scientific thought.

David Webb clarifies Foucault's conception of the historical *a priori* by returning to the work of Jean Cavaillès, the philosopher of mathematics repeatedly cited by Foucault as a significant figure in the French tradition of epistemology. Webb argues that the historical *a priori* emerged at least in part from a conception of the formal nature of thought developed (well before Foucault) specifically in opposition to phenomenology. Cavaillès' critique of Husserl's formal ontology promoted the idea of mathematics as a wholly distinct formal science that is neither grounded in the transcendental subject nor empirically derived. Moreover, it is characterised above all by its own historicity, which is similarly irreducible either to the historicity of the subject or to any form of cultural history. Webb argues that Foucault responds to Cavaillès' call for a break from the philosophy of the subject in favour of a philosophy of the concept (Cavaillès 1970). It is this move that various strands of French thought would go on to make in the latter half of the twentieth century.

Simon Duffy offers an historical account of one of the mathematical problematics that Deleuze deploys in *Difference and Repetition* (1994), and an introduction to the role that this problematic plays in the development of Deleuze's philosophy of difference. The episode in the history of mathematics from which this mathematical problematic is extracted is the history of the calculus and its various (alternative) lines (or lineages) of development, which were only put on a rigorous algebraic foundation towards the end of the nineteenth century. Arguments constructed on the basis of developments in Set Theory in the 1960s, specifically the controversial Abraham Robinson axioms that determine the distinction between Standard and Non-Standard analysis and which allow the pre-foundational proofs of the calculus to be verified, allow for the reintroduction of the relationship between mathematical and metaphysical developments of the calculus that were marginalised, to say the least, as a result of the determination of its rigorous algebraic foundation. Duffy argues that it is by means of the development of such an argument in *Difference and Repetition* that Deleuze determines a differential logic which is deployed, in the form of a logic of differen*t/c*iation, in the development of the logical schema of a theory of relations characteristic of a philosophy of difference. This logical schema provides one of the keys to understanding the relationship between Deleuze's philosophy of difference and the mathematical problematics with which it engages.

Daniel W. Smith examines the nature of the Deleuzian distinction between 'axiomatics' and 'problematics' as two different modes of formalization in mathematics. He argues that the fundamental difference between the two is that each has a different method of deduction: in axiomatics, a deduction moves from axioms to the theorems that are derived from it, whereas in problematics a deduction moves from the problem to the ideal accidents and events that condition the problem and form the cases that resolve it. As we have seen, this distinction is characteristic for Deleuze of the major or Royal science / minor or nomadic science distinction. Smith analyses the epistemological and ontological importance of these distinctions to the development of Deleuze's philosophy.

Aden Evens explores the use of symbols to designate mathematical concepts, which he argues is a practice that risks obscuring the problematic Idea from which the concepts are generated. He recounts a brief history of intuitionist mathematics that serves to demonstrate the nature of transformations in mathematics, that is, how mathematics advances, or moves ahead to encompass the new. Whereas formalist and realist schools of mathematics had effectively neutralized the 'surd' by treating it as one formalism among others, the intuitionists, led by L.E.J. Brouwer (b.1881 – 1966), challenged this interpretation. In the intuitionist calculus, the surd took on a kind of constructive or generative role. However, with time, the intuitionists created increasingly rigid formalisms to deal with the surd, such that by the middle of the century, its generative quality had again been neutralized. Evens claims that this logic of the surd, its constructive or generative role that in time is reappropriated by a certain formalisation, is characteristic of the kinds of transformations experienced by other disciplines as well as mathematics. The example that he offers is an analysis of the uncertainty principle of acoustics and the problematic nature of the accurate determination of a singular sound.

Arkady Plotnitsky examines the relation between Deleuze's philosophy and the developments in mathematics made by the work of Bernhard Riemann (b.1826 – 1866). Riemann's mathematics is considered to be a *conceptual* mathematics, which distinguishes it from the set-theoretical mathematics that dominated the field at the time. Plotnitsky also argues that category theory is much closer to the mathematical programme of Riemann than to set theory. Further, that in fact the fields of algebraic topology and differential geometry, from which category theory emerged, developed in the wake of Riemann's work and were greatly influenced by it. It is within this context that Plotnitsky explores the significance of Riemann's mathematical concepts to the conceptual architecture of Deleuze's philosophy.

Manuel DeLanda situates the work of Deleuze as a precursor to the work that is currently being done by analytic philosophers of science on the concept of 'phase space', such as Bas Van Fraasen. The notion of 'state space' or 'phase space' is associated with the visual or geometric approach to the study of differential equations pioneered by Poincaré and greatly developed in the last two decades thanks to the availability of computers as visualization tools. DeLanda discusses the ontological analysis that Deleuze derives from the work of Albert Lautmann, who was greatly influenced by Poincaré; he considers its advantages over the analyses done by Van Fraasen; and demonstrates its significance to an understanding of Deleuze's neo-materialist philosophy.

Robin Durie gives an account of the role of the problem in determining the relation between mathematics and ontology, or metaphysics, in Deleuze's philosophy. He outlines a series of developments in mathematics that determine the field from within which a number of concepts that play a decisive role in Deleuze's philosophy emerge. These include the development of the differential calculus, from which differential geometry and the calculus of variations emerged; the innovative work of Riemann on n-dimensional space and multiplicities; the contribution of Poincaré's qualitative theory of non-linear differential equations to the development of topology; and René Thom's catastrophe theory, which is one recent such development. Durie argues that the principle that underpins each of these mathematical developments is their fundamentally relational approach to mathematics. He identifies the problem that Deleuze's engagement with mathematics poses to philosophy as the problem of how to become relational. Durie suggests that it will be by forging relations with the emerging sciences, such as complexity theory, that new problems will emerge for philosophy.

2
Mathematics and philosophy

Alain Badiou

In order to address to the relation between philosophy and mathematics it is first necessary to distinguish the grand style and the little style.

The little style painstakingly constructs mathematics as the *object* for philosophical scrutiny. It is called the little style for a precise reason, because it assigns mathematics to the subservient role of that which supports the definition and perpetuation of a philosophical *specialisation*. This specialisation is called the 'philosophy of mathematics' where the 'of' is objective. The philosophy of mathematics can in turn be inscribed under the area of specialisation that supports the name 'epistemology and history of science', an area to which corresponds a specialised bureaucracy in the academic authorities and committees whose role it is to manage the personnel of researchers and teachers.

But in philosophy, specialisation is invariably the means by which the little style insinuates itself. In Lacanian terms, this occurs through the collapse of the discourse of the Master, which is rooted in the signifier of the same name, the S1 that gives rise to a signifying chain, onto the discourse of the University, that perpetual commentary which adequately represents the second moment of all speech, that is, the S2 which only exists by making the Master disappear under the commentary which exhausts it.

The little style of the philosophy of mathematics, and of its epistemology, strives for such a disappearance of the ontological sovereignty of mathematics, its instituting aristocratism, its unrivalled mastery, by confining its dramatic and almost incomprehensible existence to a generally dusty compartment of academic specialisation.

In fact, this operation with which the little style alone is associated is recognisable by the hold that it exerts over its object, because it produced what could be called a castrated mathematics; this hold therefore is only made through history and classification. To classify and historicise are indeed the two operations of all little styles, when the goal is to eliminate a frightening master-signifier.

I would immediately like to give a genuinely worthy example of the little style. Let us say a great example of the little style. I refer to the 'philosophical remarks' that conclude a truly remarkable book, entitled *Foundations of Set Theory*, a book whose second edition, from which I quote, dates from 1973. I say that it is great, among other things, in that it is written by three first-rate logicians, and mathematicians: Abraham Fraenkel, Yehoshua Bar-Hillel and Azriel Levy. The concluding philosophical paragraph of the book directly states that:

> Our first problem regards the ontological status of sets Since sets, as ordinarily understood, are what philosophers call universals, our present problem is part of the well-known and amply discussed classical problem of the ontological status of the universals. (Fraenkel et al. 1973, 331–2)

Let us note at once three things in this short paragraph, to which any follower of the little style would at once, without malice, give their blessing.

Firstly, one wonders not at all about what mathematics could mean for ontology, but rather about the specific ontology of mathematics. In other words: mathematics is a particular case of a ready-made philosophical question, and not a challenge to or the undermining of this question, even less its paradoxical or dramatic solution.

Secondly, what is this ready-made philosophical question? It is actually a question that concerns logic, or the capacity of language; specifically the question of universals. It is only through a preliminary reduction to logical and linguistic problems that mathematics is forcibly incorporated into a specialised objective area of philosophical interrogation. This is a fundamental characteristic of the little style.

And thirdly, the philosophical problem is by no means instituted or provoked by the mathematical: it has an independent history; it was, as the authors remind us, a prominent feature of 'medieval discussions' It is a classical problem, to which mathematics represents the opportunity for a modernized regional adjustment.

This is what will be seen in the labour to classify the responses:

> The three main traditional answers to the general problem of universals are known as *realism, nominalism* and *conceptualism*. We shall not deal here with these lines of thought in their traditional version, but only in their modern counterparts known as *platonism*, *neo-nominalism* and

> *neo-conceptualism* In addition, we shall deal with a fourth attitude which regards the whole problem of the ontological status of universals in general and of sets in particular as a metaphysical pseudo-problem. (Fraenkel et al. 1973, 332)

Clearly, the philosophical incorporation of mathematics in the little style is a *neo-classical* operation. It supposes that mathematics can be treated as the object of specialised philosophical consideration; that this treatment necessarily proceeds through the consideration of logic and language; that it is compatible with ready-made philosophical categories; and that it leads to doctrinal classifications sealed by proper names.

In philosophy, the deployment of such a neo-classicist approach has an old technical term: scholasticism.

As far as mathematics is concerned, the little style is a regional scholasticism.

The perfect example is given in a lecture by Pascal Engel, professor at the Sorbonne, which is reproduced in the book, *L'objectivité mathématique* (Engel 1995, 133–46). Engel, in the course of a grammatical digression concerning the status of statements, manages to use, in regard to the philosophy of mathematics, no less than twenty five classificatory syntagms. These are, in their order of appearance in this little jewel of scholasticism: Platonism, ontological realism, nominalism, phenomenalism, reductionism, fictionalism, instrumentalism, ontological antirealism, semantic realism, semantic antirealism, intuitionism, idealism, verificationism, formalism, constructivism, agnosticism, ontological reductionism, ontological inflationism, semantic atomism, holism, logicism, ontological neutralism, conceptualism, empirical realism, and conceptual Platonism. This astonishing work of labelling does not seem to have exhausted the categorical permutations, far from it. It is undoubtedly infinite, which assures scholasticism of an abundant future, even if, as is required by its ethic of intellectual 'seriousness', one always works in teams.

Nevertheless, it is possible to make a quick survey of modern scholasticism in the company of our three initial authors. First, they propose definitions of each fundamental orientation. Then they cautiously indicate that there are, as we have seen with Pascal Engel, all kinds of intermediate positions. Finally, they nominate the purest champions of the four camps.

Let us take a closer look at them.

To begin with, the definitions. In the following passage, the word 'set' is to be understood as designating any mathematical configuration that can be defined in rigorous language.

> A Platonist is convinced that to each well-defined (monadic) condition there exists, in general, a set, or a class, which comprises all and only those entities that fulfil this condition. [The Platonist moreover is convinced that this set, or this class, is] an entity of its own right of an ontological status similar to that of its members.

> A neo-nominalist declares himself unable to understand what other people mean when they are talking about sets unless he is able to interpret their talk as a 'façon de parler' The only language [that the neo-nominalist] professes to understand is a calculus of individuals, constructed as first-order theory. (Fraenkel et al. 1973, 332)

> There are authors who are attracted neither by the luscious jungle flora of Platonism nor by the ascetic desert landscape of neo-nominalism. They prefer to live in the well-designed and perspicuous orchards of neo-conceptualism. They claim to understand what sets are, though the metaphor they prefer is that of construction (or inventing) rather than of singling out (or discovering) which is the one cherished by the Platonists They are not ready to accept axioms or theorems that would force them to admit the existence of sets which are not constructively characterizable. (Fraenkel et al. 1973, 334–5)

In short, the Platonist admits the existence of entities that are transcendent to human constructive capacity and that are indifferent to the limits of language. The nominalist only admits the existence of verifiable individuals that fulfil a transparent syntactic form. The conceptualist requires that all existence is subordinated to an effective construction, itself dependent on the existence of previously evident, or constructed entities.

Church or Gödel can be cited as uncompromising Platonists; Hilbert or Brouwer as unequivocal conceptualists, and Goodman as a fanatical nominalist.

The most radically agnostic hypothesis remains, that which always comes in fourth position. Following the theses: 'Sets have a real existence as ideal entities independent of the mind' (thesis 1), 'Sets exist only as individual entities validating linguistic expressions' (thesis 2) and 'Sets exist as constructions of the mind' (thesis 3), always comes, in fourth position, the supernumerary thesis: 'the question of how sets exists has no independent meaning outside a given theoretical context'.[1]

> The prevalent opinions [that is, Platonism, nominalism and conceptualism] are caused by a confusion between two different questions: the one whether certain existential sentences can be proved, or disproved, or shown to be undecidable, within a given theory, the other whether this theory as a whole should be accepted. (Fraenkel et al. 1973, 337)

Carnap, theorist of this clarificatory orientation, suggests that the first problem, depending on the resources of the theory in question, is purely technical; and that the second comes down to a practical question, based upon such variable criteria as:

> likelihood of being consistent, case of maneuverability, effectiveness in deriving classical analysis, teachability, perhaps possession of standard models, etc. (Fraenkel et al. 1973, 337)

It is by confusing these two questions that one is led to posing meaningless metaphysical problems, such as for example: 'Do non-denumerable infinite sets exist?' – a question which, by introducing existence absolutely and not relatively to a theory, leads to entirely sterile controversies.

Clearly then, the little style is played in all four corners, according to whether the existence of mathematical entities is adopted as a realist, linguistic, constructive or purely relative maxim.

But this is because it was initially supposed that philosophy refers to mathematics by the critical examination of its objects; that these objects must be investigated in regard to their existence; and that finally there are four ways of considering existence: as intrinsic; as only the correlate of a name; as a construction of the mind; or as a variable pragmatic correlate.

Everything else is the grand style. Which, in a word, stipulates that mathematics directly clarifies philosophy, rather than the inverse, and that it does this by forced, even violent, intervention into the intimate operation of questions.

Allow me to begin with five majestic examples of the grand style: Descartes, Spinoza, Kant, Hegel and Lautréamont.

Descartes, *Regulae ad directionem ingenii,* rules for the direction of the mind, rule 2.

> These considerations make it obvious why arithmetic and geometry prove to be much more certain than other disciplines: they alone are concerned with an object so pure and simple that they make no assumptions that experience might render uncertain; they consist entirely in deducing conclusions by means of rational arguments. They are therefore the easiest and clearest of all the sciences and have just the sort of object we are looking for. Where these sciences are concerned it scarcely seems humanly possible to err, except through inadvertence.
>
> Now the conclusion we should draw from these considerations is not that arithmetic and geometry are the only sciences worth studying, but rather that in seeking the right path of truth we ought to concern ourselves only

> with objects which admit of as much certainty as the demonstrations of arithmetic and geometry. (Descartes 1985, 366)

For Descartes, mathematics is clearly the paradigm of philosophy, a paradigm of certainty. Let us add that it is by no means a logical paradigm because it is not the proof which gives the paradigmatic value of mathematics to the philosopher, it is the absolute simplicity and clarity of the mathematical object.

Spinoza. Appendix of the first part of the *Ethics*, the text dear to Louis Althusser.

> So they maintained it as certain that the judgments of the Gods far surpass man's grasp. This alone, of course, would have caused the truth to be hidden from the human race to eternity, if Mathematics, which is concerned not with ends, but only with the essences and properties of figures, had not shown men another standard of truth (*aliam veritatis normam*).
>
> [W]e have such sayings as 'So many heads, so many attitudes' 'everyone finds his own judgment more than enough', and 'there are as many differences of brains as of palates.' These proverbs show sufficiently that men judge things according to the disposition of their brain, and imagine, rather than understand them. For if men had understood them, the things would at least convince them all, even if they did not attract them all, as the example of mathematics shows. (Spinoza 1985, 442, 5)

For Spinoza, it is no exaggeration to say that mathematics governs the historical destiny of knowledge, and thus the economy of freedom, or beatitude. Without mathematics, humanity is in the night of superstition, itself summarised by the maxim: there is something we cannot think. To which it is necessary to add that mathematics instructs us on the most important point: that what is thought truly is immediately shared. Mathematics indicates that the comprehension of anything whatsoever is radically undivided. To know is to be absolutely and universally convinced.

Kant, *Critique of Pure Reason.* Preface to the second edition.

> Mathematics has, from the earliest times to which the history of human reason reaches, in that admirable people the Greeks, travelled the secure path of a science. Yet it must not be thought that it was as easy for it as for logic – in which reason has to do only with itself – to find that royal path, or rather itself to open it up; rather, I believe that mathematics was left groping about for a long time (chiefly among the Egyptians), and that its transformation is to be ascribed to a revolution, brought about by the happy inspiration of a single man in an attempt from which the road to be

> taken onward could no longer be missed, and the secure course of a science was entered on and prescribed for all time and to an infinite extent.
>
> A new light broke upon the first person who demonstrated the isosceles triangle (whether he was called 'Thales' or had some other name). For he found that what he had to do was not to trace what he saw in this figure, or even trace its mere concept, and read off, as it were, from the properties of the figure; but rather that he had to produce the latter from what he himself thought into the object and presented (through construction) according to a priori concepts. (Kant 1998)

Thus Kant thinks, firstly, that mathematics has secured for itself from its origins the sure path of a science. Secondly, that the creation of mathematics is an absolute historical singularity, a 'revolution', so much so that its emergence should be singularised: it is due to the felicitous thought of a single man. Nothing, we will see, is more opposed to an historicist or culturalist explanation. Thirdly, once opened, the path is infinite, in time as well as in space. This universalism is a concrete universalism because it is a trajectory of thought, which can always be retraced, irrespective of the time or the place. And fourthly, Kant sees in mathematics – the return of its paradigmatic function – the first conception of a knowledge which is neither empirical (it is not what is seen in the figure), nor formal (they are not the pure, static and identifiable properties of the concept). Mathematics thus opens the way for the critical representation of thought: that is to say knowledge as non-empirical production or construction. A sensible construction, but adequate to the constituting *a priori*. That is to say that Thales is the supposed name of a revolution for the whole of philosophy. Kant represents nothing other than the examination of the conditions of possibility of that which Thales constructed.

Hegel. The important Remark, in the *Science of Logic*, that follows the development of the infinity of quantum.

> But in a philosophical respect the mathematical infinite is important because underlying it, in fact, is the notion of the genuine infinite and it is far superior to the ordinary so-called metaphysical infinite on which are based the objections to the mathematical infinite. (Hegel 1969, §540)
>
> It is worthwhile considering more closely the mathematical concept of the infinite together with the most noteworthy of the attempts aimed at justifying its use and eliminating the difficulty with which the method feels itself burdened. The consideration of these justifications and characteristics of the mathematical infinite which I shall undertake at some length in

> this Remark will at the same time throw the best light on the nature of the true Notion itself and show how this latter was vaguely present as a basis for those procedures. (Hegel 1969, §543)

The decisive point this time is that, for Hegel, mathematics and philosophical speculation share a fundamental concept, which is the concept of the infinite. In particular, the destitution of the metaphysical concept of the infinite, which is to say the destitution of classical theology, is initially undertaken by the determination of the mathematical concept. Hegel obviously has in mind here the creation of the differential and integral calculus in the seventeenth and eighteenth centuries. He wants to show that the true concept of the infinite, the dialectical concept, makes its entry on the historical scene under the auspices of mathematics. The method is remarkable: it is necessary to examine the contradictory labour of the concept within the mathematical text itself. Indeed, the concept is active and deployed, it ruins the transcendent theological concept, but it is not knowledge realised by its own action. Unlike the theological infinite, the mathematical infinite is the same as the good infinite of the dialectic. However, it is only the same according to the difference which is to not yet know itself as the same. In this instance, as in Plato or in my own work, philosophy comes down to informing mathematics of its own speculative grandeur. In Hegel this takes the form of a detailed examination of what he calls the 'justifications and determinations' of the mathematical concept of the infinite, an examination which, for him, returns to a detailed reading of the conceptions of Euler and Lagrange. In this reading, one sees how the mathematical concept of the infinite, which for Hegel is still impeded, still in the grip of the 'difficulty with which the method feels itself burdened', carries with itself the affirmative resource of a genuinely absolute conception of quantity.

Let us finish this survey of the grand style as it should be, on the border between philosophy and the poem. Isidore Ducasse, known as the Count of Lautréamont. Like Rimbaud and Nietzsche, Lautréamont, under the post-romantic name of Maldoror, wants to bring about a denaturing of man, a transmigration of his essence, a positive becoming-monster. In other words, an ontological deregulation of all the categories of humanism. In this task, mathematics plays a crucial auxiliary role. *Songs of Maldoror,* Second song:

> 0 austere mathematics! I have not forgotten you since your learned teachings, sweeter than honey, distilled themselves through my heart like refreshing waves. Instinctively, since the day of my birth, I have aspired to drink from your spring more ancient than the sun and I still continue to

> frequent the courtyard of your solemn temple: I, the most faithful of your initiates. There used to be a vacuum in my soul, a something, I know not what, dense as smoke; but wisely and religiously I mounted the steps that lead to your altar, and you dispelled that gloomy shroud as the wind blows a butterfly. In its place you set an extreme coldness, a consummate prudence and an implacable logic. Arithmetic! Algebra! Geometry! Imposing trinity! Luminous triangle! He who has never known you is without sense! He merits the ordeal of the most cruel tortures for in his ignorant carelessness there is a blind contempt. But you, 0 concise mathematics, by the rigorous fetters of your tenacious propositions and the constancy of your iron-bound laws you dazzle the eyes with a powerful reflection of that supreme truth whose imprint is manifest in the order of the universe. Your modest pyramids will endure longer than the pyramids of Egypt, those ant-hills erected by stupidity and slavery. The end of all centuries will yet see, standing upon the ruins of time, your cabalistic ciphers, your terse equations, and your sculptural lines, enthroned at the vengeful right hand of the Omnipotent, while in despair like jets of water the stars will sink into the eternity of a horrible and universal night; and while man, grimacing, thinks of settling his accounts with the last judgment. I thank you for the numberless services you have rendered me. I thank you for the unfamiliar qualities with which you have enriched my intelligence. Without you I might perhaps have been overcome in my struggle against man. (Lautréamont 1965, 86–90)

This text is very striking. It develops around mathematics a kind of icy sanctification, which tends to recall the dialectical significance of the great Mallarméan symbols: the star, 'cold from forgetfulness and desuetude',[2] the mirror 'frozen in [its] frame' the tomb, the 'solid sepulchre where all things harmful lie',[4] the 'hard lake haunted beneath the ice / By the transparent glacier of flights never flown' [5] One may speak readily of a wintry anti-humanism. With Lautréamont however, the 'extreme coldness' of mathematics is coupled with a monumental aspect, a kind of Masonic symbolism of eternity, the 'luminous triangle', the 'constancy of your iron-bound laws' the pyramid. Just as Nietzsche, in order to overcome Christ in favour of Dionysus, made Zarathustra speak in the language of the Gospels (in truth, I say it to you, etc), so too does Lautréamont, in order to impose the montrous becoming of exhausted and defiled man, speak the language of the Old Testament and of Masonic esotericism. Within this framework, mathematics, organised into algebra, arithmetic and geometry – that is into 'laconic equations', 'cabalistic ciphers' and in 'sculpted lines' – renders an indispensable service: it imposes a kind of implacable eternity against the humanist conception of man. Mathematics is, in effect, 'more ancient than the sun', and it will remain intact 'upon the ruins of time' Mathematics is the discipline and the severity, the immutability and the reflection 'of the supreme truth'. From this to saying that it is the inscription of being itself, there is only one step; a step that, as you know, I have taken. But for Lautréamont, mathematics is something even better: it is what provides the intellect with 'alien qualities' This point is essential: between the human intellect and mathematics, there is no intrinsic harmony. The exercise of mathematics, its 'sweeter than honey' lessons, is the exercise of an alteration, of an estrangement of the intelligence. And it is above all by this resource of strangeness that mathematical eternity subverts ordinary thought. We grasp here the profound reason for which, without mathematics, without the infection of conventional thinking by mathematics, Maldoror could not have prevailed in his fundamental struggle, the struggle against humanist man, the struggle to bring forth beyond man the free monster which he is capable of becoming.

On all these points, between wintry anti-humanism and the transhuman advent of truths, I believe myself to be the only authentic disciple of Isidore Ducasse. Why then declare myself a Platonist, as I like to do, rather than a Ducassian, or a son of Maldoror?

It is because Plato does not say anything different.

Just as Isidore Ducasse, Plato affirms that mathematics is that by which *doxa* is undone, that by which the sophist is defeated, that without which there could never arise, beyond existing humanity, those philosopher-kings who, in the conceptual City constructed by Plato, have the allegorical name of overman. This is clarification that in order to have any chance of seeing these philosopher-kings appear, it is necessary to teach young people arithmetic, plane geometry, stereometry and astronomy for at least ten years. Mathematics has for Plato this admirable quality, which is, of course, its essential value: it sets its sights on pure essences, on the idea as such. But also its utility can be reduced to the only pragmatics of any value for a man who has risen beyond man, namely war. See, for example, *The Republic* (1987), book 7, 525c (which I have taken the liberty to retranslate):

Socrates: Our overman is both philosopher and captain?
Glaucon: Correct.
Socrates: Then a law must be passed. And immediately.
Glaucon: A law? Why a law, by God? Which law?
Socrates: A law which stipulates the teaching of higher arithmetic, my

simpleton Glaucon. But it will be necessary to devote onese firmly.

Glaucon: To devote oneself? To what end?

Socrates: Take a young fellow who wants to become admiral of th fleet, or minister, or president, or something of that ilk. young hotshot who studied at LSE or Yale. Do you imagin he'll be rushing to enrol at the institute of higher arithmetic It will be necessary to seriously talk it up, let me tell you.

Glaucon: I can't imagine what we're going to tell him.

Socrates: The truth. Something harsh. For example: 'My friend, if yo want to become a minister or admiral, it is first necessary t stop being such an agreeable young man, an affected yuppi of sorts. Take numbers, for example, do you know wh numbers are? Not for running your miserable finances. No for counting the beef and eggs that your parents peddle at th port! But number such as you contemplate it in its etern nature, only by the force of your affected intellect disinfecte by me! Number as it is in war, in the terrible reckoning o weapons and corpses. But above all, number such as it brin about a complete upheaval of thought, erasing becoming an approximation in order to face being just as it is, and in i truth.'

Glaucon: When you set on him with your speech, in my opinion, th pretentious lad will take off, green with fear.

This is what I mean by the grand style! Arithmetic as stellar and warlik inhumanity.

It should come as no surprise that today a systematic attack i being waged against mathematics from all sides. Just as against politics in the name of State management and economics; or against art, in th name of cultural relativity; or against love, in the name of sexual prag matics. The epistemological specialisation of the little style is only a involuntary component in this attack. We have no choice: to defen ourselves, 'we' who speak on behalf of philosophy itself, and th supplementary step that it can and must take, we must find the new term of the grand style.

Let us first summarise the teaching of our admirable predecessors.

As we saw with them all, the confrontation with mathematics is a absolutely necessary condition of philosophy itself, a condition that is a once descriptively external and prescriptively immanent for philosophy And this holds even where there are enormous divergences as to wha constitutes the fundamental project of philosophy. To create a new conception of politics for Plato. To extend the scope of absolute certainty to the essential questions of life for Descartes. To attain the intellectual love of God for Spinoza. To know exactly where the border between faith and knowledge lies for Kant. To expose the becoming-subject of the Absolute for Hegel. To disfigure and overcome humanist man for Lautréamont. In each case it is a question of giving 'thanks' to rigorous mathematics. Philosophy can be: a rationalism tied to transcendence, from Descartes to Lacan; a vitalist immanentism, from Spinoza to Deleuze; a pious criticism, from Kant to Ricœur; a dialectic of the Absolute, from Hegel to Mao Tse-tung; or an aesthetic creationism, from Lautréamont to Nietzsche. For the founders of each of these lineages, it remains that the cold radicality of mathematics is the necessary exercise through which is forged a thinking subject adequate for the transformations that it will have to undergo.

It is no different to this for me. I have assigned philosophy the task of constructing the reception in thought of its own time, of refracting incipient truths through the unique prism of concepts. Philosophy must intensify and gather together, under the sign of systematic thinking, not just what its time imagines itself to be, but what its time is – without knowing it – capable of. In order to do this, I too had to laboriously register my own lengthy 'thank you' to rigorous mathematics.

Let us formulate the maxim abruptly – no relation between the grand style of philosophy and mathematics is to say: no philosophical grand style at all.

In 1973, Lacan, using a 'we' that, for all its imperiousness, included psychoanalysts no less than psychoanalysis, declared: 'mathematical formalization is our goal, our ideal' (Lacan 1998, 119). Following the same rhetoric, where the 'we' now comes to designate philosophers and philosophy, I will say: 'mathematics is our obligation, our alteration.'

None of the advocates of the grand style ever imagined that the philosophical identification of mathematics had to proceed by way of a logicising or linguistic reduction. Suffice it to say that for Descartes, it is the intuitive clarity of ideas that founds the mathematical paradigm, and by no means the automatic character of the deductive process, which is only the scholastic and indifferent part of things. Similarly, for Kant, the historical destiny of mathematics as the construction of the concept in intuition constitutes a revolution that is independent of the achieved destiny of logic, which, since its founder Aristotle, has done nothing more than be repeated. Hegel scrutinises the foundation of a concept, that of infinity, disregarding the apparel of proof. Lautréamont certainly appreciates the iron necessity of the deductive process and the coherence of figures, but what ultimately counts for him is the power of eternal survival and the icy discipline of mathematics. As for Spinoza, it is from the ontology underlying mathematics that he awaits salvation. That is, these advocates are those who think being outside of any consideration of a meaning or an end, and who only become attached to the cohesiveness of consequences.

There is not a single word about language in all this.

Let us not hesitate to say in passing that on this point Wittgenstein, as cunning as he is in his loquacious hysteria, and despite the beautiful proportions of the *Tractatus,* undeniably one of the masterpieces of anti-philosophy, should no less be held to be one of the creators of the little style, whose principles he sets out with his customary brutality. Thus in *Tractatus* 6.21: 'A proposition of mathematics does not express a thought' (1992, 65). Or worse still, in *Remarks on the Foundation of Mathematics* (1978), we find a kind of trite pragmatism, which is very fashionable nowadays:

> I should like to ask something like: 'Is it usefulness you are out for in your calculus? – In that case you do not get any contradiction. And if you aren't out for usefulness – then it doesn't matter if you do get one.' (sec. II, par. 80: 104e)

We forgive Wittgenstein, but not those who take shelter behind his aesthetic ruse – of which the core is moral, even religious – to adopt once and for all the little style, and to vainly attempt to throw to the lions of indifference, our modern lions, those who intend to remain faithful to the grand style.

In any case our maxim is: *philosophy must enter into logic via mathematics, and not into mathematics via logic.*

Which in my work became: mathematics is the science of being *qua* being. Logic pertains to the coherence of appearance. And if the study of appearance also mobilises certain areas of mathematics, it is quite simply because, according to an intuition formalised by Hegel but which actually goes back to Plato, it is of the essence of being to appear. This still retains the form of all appearance in a mathematisable transcendental order. Here again however, it is transcendental logic, which is a part of mathematics related to contemporary sheaf theory, that subordinates formal or linguistic logic, which is ultimately only a superficial translation of the former.

Taking up again the 'we' I used earlier, I will say: 'mathematics teaches us about what must be said concerning what is; and not about what is permissible to say concerning what we think there is.'

Mathematics is a weapon for philosophy, a formidable machine of thought, a catapult trained on the bastions of ignorance, superstition and mental servitude. It is not at all a docile grammatical region. For Plato, mathematics is what allows us to break free from the sophistical dictatorship of linguistic immediacy. For Lautréamont, it is what liberates us from the moribund figure of the human. For Spinoza, it is what breaks with superstition. But you have read their texts. Today some would lead us to believe that mathematics is itself relative, prejudiced and inconsistent, unnecessarily aristocratic, or alternatively subservient to technology. You should be aware that this is propaganda against what has always been the most implacable enemy of spiritualist approximations and rowdy proclamations of scepticism, these insipid allies of flamboyant nihilism. Mathematics indeed is unaware of what it means to say: 'I cannot know'. Spiritualist categories such as those of the unthinkable and of the unthought, of what exceeds the meagre resources of human reason, or the sceptical categories according to which we cannot really resolve any problem, nor respond to any serious question, are categories whose existence mathematics does not acknowledge in its own realm.

Science in general is not reliable on this point. Quentin Meillassoux has convincingly argued that physics offers no barricade against spiritualist, even obscurantist, speculation and biology, this wild empiricism disguised as science, even less so. It is only in mathematics that one can say, unequivocally, that in so far as thought formulates a problem, it is definitive that thought can solve it, that thought will solve it, however long it takes. For it is also in mathematics that the maxim 'Keep going!' a maxim which satisfies the needs of ethics, has the most consistency. How else are you to explain why a problem posed by Fermat more than three centuries ago can be solved today? And why we still engage in proving or disproving conjectures first proposed by the Greeks more than two thousand years ago? Yes, mathematics conceived in the grand style is warlike, polemical, fearsome. And it is by donning the contemporary matheme like a coat of armour that I have undertaken, alone at first, to undo the devastating effects of philosophy's 'linguistic turn'; to trace a line of demarcation with phenomenological religiosity; to re-found the metaphysical triad of being, the event and the subject; to take a stand against poetic prophes ing; to identify generic multiplicities as the ontological form of the tru to assign a place to Lacanian formalism; and, more recently, to deplo the logic of appearing.

Let us say that, as far as we are concerned, mathematics is alwa more or less equivalent to the bulldozer with which we remove the rubbl that prevents us from constructing new edifices in the open air.

The principal technical difficulty is undoubtedly the assumptio that mathematical competence requires years of initiation. Whence th temptation, for the philosophical demagogue, not to whisper a word mathematics, or to act as though the basic rudiments were enough make its subject-matter comprehensible. In this regard, Kant did not s a very good example, by letting generations believe that they could gras the essence of mathematical judgement through a single example lik $7 + 5 = 12$. This is like somebody saying that one can grasp the relatio between philosophy and poetry by reciting:

> Humpty Dumpty sat on the wall,
> Humpty Dumpty had a great fall.
> All the king's horses and all the kings men
> Couldn't put Humpty together again! [6]

After all, they are just lines of verse, just as $7 + 5 = 12$ are numbers.

It is striking that in philosophical texts of general ambition, or o the grand style, the quotation of poems is regarded as self-explanator but not at all the quotation of a piece of mathematical reasoning. Nobod seems to think it acceptable to dispense with Hölderlin or Rimbaud, o Pessoa, in favour of Humpty Dumpty, no more, to tell the truth, than t replace Wagner with Julio Iglesias. But as soon as there is mathematics either the reader loses interest, or he believes that it is of the little styl of epistemology, of the history of science, of specialisation.

This was not the point of view of Plato, nor that more generally o any of the great philosophers. Plato frequently quotes poets, but he also quotes theorems, perhaps easy by today's standards, but certainly difficult for his contemporaries. As for example, in *Meno*, the construction o the square whose surface is double that of a given square.

I claim the right to mathematical citation, provided that it is appropriate to the philosophical theses in which it is inscribed, and that all the elements required for its comprehension are readily available. Give us an example, I hear you say. Well, I will not give you an example of an example. For the real examples, integrated into the movement of thought; I have already provided hundreds of them. I will mention two of these movements instead, for your exercise: In chapter 4 of *Le nombre et les nombres* (1990), the presentation of Dedekind's doctrine of number. Or meditation 7 of *L'être et l'événement* (1988), meditation on the point of excess. Consult them, read them, using naturally the reminders, the cross-references and the glossary that I have provided. If someone does not understand, they can write to me exactly *what* they do not understand – [otherwise we're simply dealing with the excuses for the reader's laziness]. We understand the sentence of Anaximander, an elegy of Rilke, a seminar of Lacan on the real, but not the proof, produced two thousand five hundred years ago, that there is an infinity of prime numbers. This state of affairs is radically antiphilosophic, and serves only those partisans of the little style.

I have spoken of bulldozers and rubble. Of which contemporary ruins do they happen to be? I think Hegel saw it before anyone else: in its essence, mathematics proposes a new concept of the infinite. And on the basis of this concept, it authorises an immanentisation of the infinite, its separation from the One of theology. Hegel also saw that the algebraists and analysts of his time, Euler or Lagrange, were not completely clear on this point – it is indeed with Baron Cauchy that a little order may be established in the thorny issue of the limit of a series, and not until Cantor before a little more light is thrown on the ancient question of the actual infinite. Hegel thought that this quandary was due to the fact that the 'true' concept of the infinite belonged to speculation, and that mathematics was, all things considered, only the blind bearer, or the unconscious act of its birth. The truth is that the mathematical revolution, rendering explicit what had always been implicit in mathematics since the time of the Greeks – that is, the thorough rationalisation of the infinite – was yet to come, and, in a certain sense, will always be yet to come. Since we still do not know how to reasonably 'force' the type of infinite proper to the continuum. Nevertheless, we do know why mathematics radically subverts empiricist moderation and elegant scepticism. It is because mathematics produces the possibility of thinking that the infinite is the native element of rational thought, and not its unfathomable exterior. Mathematics is that from which there is no reason to confine thought within the ambit of finitude. With mathematics, as Hegel would have said, we know that the infinite is nearby.

Then, can one object: if we already know the result, why not be satisfied with it? Why still engage in the dry labour of familiarising ourselves with new axioms, with unprecedented proofs, with difficult concepts, with inconceivably abstract theories? It is because the infinite such as mathematics exposes it to philosophical will, is by no means a stable and irreversible given. The historicity of mathematics is nothing other than the labour of the infinite, than its ongoing and unpredictable re-exposition. Just as a revolution, French or Bolshevik, cannot exhaust the formal concept of emancipation, even though it presents its real, no do the mathematical avatars of the thought of the infinite exhaust the speculative concept of infinite thought. The confrontation with mathematics must constantly be reconstituted, because the idea of the infinite only manifests itself through the moving surface of its reconfigurations. And it is all the more essential that our ideas of the finite, and thus the philosophical virtualities of finitude, are retroactively displaced and reinvigorated through those crises, revolutions and the repentances that affect the mathematical schema of the infinite. There is here a moving front, a struggle as silent as it is relentless, and nothing announces, there no more than elsewhere, the advent of perpetual peace.

What is there in common, in regard to the most subtle consequences for thinking, between the infinity of prime numbers as conceived by the Greeks, the fact that a function tends towards the infinite, the limit of a series whose indices tend towards the infinite, the infinitely small of non-standard analysis, the regular or singular infinite cardinals, the existence of a number-object in a topos, the capture and projection onto a family of sets by a functor of an untotalisable collection of algebraic structures, and hundreds of other formulations, concepts, schemes and theoretical determinations? Probably something that touches upon the fact that the infinite is the intimate law of thought, its natural medium because anti-natural. But in another sense, they have nothing at all in common. Nothing that would allow one merely to repeat and to maintain only allusive and simplified relations with mathematics. This is because to use the words of our late friend Gilles Châtelet, the mathematical elaboration of thought is not of the order of an unfolding or of pure consequence. It comprises decisive but previously unknown gestures. It is necessary to begin again, because mathematics is always beginning again, transforming its abstract body of concepts. It is necessary to begin studying, writing, understanding again that which is most complicated in the world, that whose abstraction is the most insolent, because the philosophical struggle against that which unites finitude and obscurantism thrives on this recommencement.

This is why Mallarmé was mistaken on at least one point. Like all great poets, Mallarmé engaged in a tacit rivalry with mathematics. He tried to show that the poetic line saturated with images, if contained within the naked rhythm of thought, can hold as much or more truth than the extra-linguistic inscription of the matheme. This is what made him write, in a draft of *Igitur*:

> The infinite emerges from chance, which you have denied. You mathematicians expired – I am projected absolute. I was to finish an Infinite. (1982)

The idea is clear: Mallarmé acuses mathematicians of having denied chance, and thus of fixing the infinite in the hereditary rigidity of calculation. This rigidity, in *Igitur*, is symbolised by the family. Whence the poetic, anti-mathematical operation which, Mallarmé believes, binds the infinite to chance, and is symbolised by the dice-throw. Once the dice are thrown, and whatever the result, 'the infinite . escapes the family' The result of which being that mathematicians expire, and along with them goes the abstract concept of the infinite, in favour of the impersonal absolute which becomes the hero.

But what Mallarmé did not see, is that the operations by which mathematics reconfigures the infinite in thought are constantly affirming chance through the contingency of its recommencement. Philosophy is thus called upon to gather together, to conjoin, the poetic affirmation of the infinite which is drawn metaphorically from chance, and the mathematical construction of the infinite, which is drawn formally from an axiomatic intuition. The injunction to mathematical beauty intersects with the injunction to poetic truth. And equally the inverse.

There is a very short poem by Alvaro de Campos, a heteronym of Fernando Pessoa. De Campos is a scientist, an engineer, who draws the moral from what I have just been saying. Here is this poem, which you can immediately learn by heart:

> Newton's binomial is as beautiful as the Venus de Milo.
> The trouble is few people are aware of it. (1971, 115)

Style, the grand style, is simply to be aware of it.
Translated by Simon Duffy

Notes

1 [*Trans*. I have followed Brassier and Toscano in presenting each of these theses in terms of 'sets'. See Badiou 2004, 6.]

2 [*Trans*. from the poem 'One toss of the dice will never abolish chance' by Mallarmé (1994, 144).]

3 [*Trans*. from 'Herodiade' by Mallarmé (1994, 30).]

4 [*Trans*. from 'A funeral toast' by Mallarmé (1994, 45).]

5 [*Trans*. from 'Several Sonnets II' by Mallarmé (1994, 67).]

6 [*Trans*. – Badiou's manuscript actually cites a variation on the chorus famous French patriotic song from 1914 called 'Quand Madelon', wor Louis Bousquet, music by Camille Robert, performed by the singer Ba

Quand Madelon vient nous servir à boire	When Madelon has just served us a
Sous la tonnelle, on frôle son jupon	Under the arbour, we brush past her coat
Et chacun lui raconte une histoire	And each tells her a story
Une histoire à sa façon	A story in his own way.
La Madelon.	

I have followed Brassier and Toscano in replacing the French with a known rhyme that serves the same purpose for the essay. See Ba 2004, 17.]

3
Interlacing the singularity, the diagram and the metaphor

Gilles Châtelet *(edited by Charles Alunni*)*

Introduction by Charles Alunni

> you on whom the future counted so much, you didn't fear to put fire to your life
> We will wander for a long time around your example.
> It is necessary to return . All will have to be started again.
> René Char, *Dans l'atelier du poète*.[1]

The paper presented here for the first time is derived from a lecture that Gilles gave at the École Normale Supérieure in Paris, in a seminar series whose general title was *Possible Worlds*.[2] Not personally having been able to be present, and Gilles having prepared his talk on a series of scattered notes, I asked him, during April, to write his notes up as a paper for publication. The reverberations that I had heard from this lecture, in particular through the report of one of our common students, encouraged me to be pressing on this point. As always, Gilles had captivated a public which, at the beginning, and for reasons of differences in philosophical position, was far from accepting of him a priori. His speculative power, doubled by his 'heroic fury', had shaken the listeners of the rue d'Ulm. Several months passed, without us ever finding ourselves in a situation to resume the point of this project, and without me ever knowing whether Gilles had the least intention to carry it out. Less than two weeks before his suicide, I prepared one of the lectures of my own seminar at the table of a cafe where we were in the habit, for approximately five years, of periodically meeting.[3] Alexis de Saint-Ours, our common student, then came to find me, accompanied by Gilles whom he told me he had 'convinced to come as far as here'. Rather a rare thing, Gilles carried a briefcase. After having settled opposite me, he opened it and produced a small bundle of documents, which he showed me without allowing me to consult them, and all the while saying: 'You see, I listened to you: I am about to finish my paper'

A few days later, in the early morning of June 15, 1999, th announcement of the appalling news was communicated to me by our ol mutual friend Bernard Besnier, 'Director of Studies' (*caïman*) at the EN in Fontenay-Saint Cloud. When, a few days later again, we found ou selves with his sister and her closest friends in his apartment on boule vard Rochechouart, we found his work table entirely occupied b bundles of documents and opened books, which showed that Gilles wa working there at the time when he killed himself; these were the element of the paper in question. In agreement with our friends, I decided to col lect them to try to reconstitute the final text of the paper that was t become his last fundamental contribution. The task proved to be long an difficult. He who admired William Burroughs so much (and whom h had met during his stay in the United States) proceeded to compose hi manuscripts by an operation of textual *cut- up*.[4] Refusing any use of th computer, he developed a handwritten manuscript on which was the glued other pieces of printed text (quotations photocopied then cut ou and stuck, mixed with other pieces of his own texts that he was in th habit of typing). The difficulty was made even worse both by a system atic absence of any numbering of the documents, and by the use of 'secret' code, marked on the top of a page and on separate paper frag ments (of the type Φ1, Φ2α, etc. .). Lastly, as with the photocopies o whole books which he made use of and bound, he excluded the title pag from the reproduction (making it sometimes difficult to identify th author and the work), in the same way, in his manuscripts Gilles practi cally never indicated the source of his quotations. In the text which inter ests us, this was particularly the case for his references to the 'kno theorist, Louis H. Kauffman. Although working on the same sources a Gilles, it has taken approximately two months of work to rebuild thi kind of textual and theoretical *puzzle* in the form of a paper.

The manuscript, as with all Gilles' manuscripts, is deposited wit the *Gilles Châtelet Archives* at the École Normale Supérieure in Paris, which is, by convention, under the responsibility of myself and Alai Prochiantz. These *Archives* were able to be repatriated to rue d'Ul thanks to the *École Polytechnique* and to the mediation of the forme director of his department of mathematics, François Laudenbach. Thi was made possible following the donation which was made to the Écol Normale Supérieure by Doctor Edwige Bourstyn-Châtelet, sister o Gilles, and to whom the oeuvre was bequeathed. For which I'd like t take the opportunity here to express my sincere gratitude.

Ulm, Paris, March 200

Notes

1 [*Trans.* toi sur qui l'avenir comptait tant, tu n'as pas craint de mettre le feu à ta vie ./ Nous errerons longtemps autour de ton exemple./ Il faut revenir .Tout est à recommencer'.]
2 Meeting of April 15, 1999. [*Trans.* – at the ENS in rue d'Ulm, Paris.]
3 For the meetings of the 'Pensée des sciences' seminar, held twice a month at the École Normale Supérieure (Wednesday evenings from 8 to 10.30pm), we had founded a kind of ritual which consisted in (and consists in still today) continuing the debates over late victuals. The debates were so animated that the owner called us 'the folk group' (*le groupe folklorique*). This bistro, located opposite rue d'Ulm, used to be known as *Le Normal Bar*. In the sixties, it was already a meeting place: that of Jacques Lacan and his group.
4 Remember that it was between 1958 and 1960, at the time of his Parisian stay at the famous Beat Hotel of rue Gît-le-Cœur, that Burroughs became impassioned with the results of this technique of the cut-up developed by the painter and poet Brion Gysin.
5 Call number Ined.01.

Gilles Châtelet

If the allusive stratagems can claim to define a new type of systematicity, it is because they give access to a *space where the singularity of the diagram and the metaphor may interlace*, to penetrate further into the physico-mathematic intuition and the discipline of the gestures which precede and accompany 'formalisation' This interlacing is an operation where each component backs up the others: without the diagram, the metaphor would only be a short-lived fulguration because it would be unable to operate: without the metaphor, the diagram would only be a frozen icon, unable to jump over its bold features which represent the images of an already acquired knowledge; without the subversion of the functional by the singular, nothing would come to oppose the force of habit.

We would thus like to undertake research which would give priority to certain key axes, which would analyse the increasingly crucial role played by the allusive stratagems in the articulation of the intuitive practices of two different domains or disciplines: Physics and Mathematics, Geometry and Algebra, an articulation *that does not embody a relation of instrumentality of one practice over the other.*

A) *Analysis of the relationship between philosophical metaphors and scientific metaphors.*

Aristotle already noted that the metaphor could be understood as a 'syllogism to complete': it is precisely this *invitation to complete* that permits that which is not actually presented there to be shown. The metaphor allows one to think between the lines and thus is not only a linguistic impertinence necessarily devoted to precariousness and quickly absorbed by convention.

In philosophy, metaphors are not content to play a subsidiary role, which one could, if absolutely necessary, do without, but often appear as centrepieces of what Jean Ladrière calls the 'support of the line of argument': by creating the effect of *veracity*, this support establishes itself as complementary to studied deduction by means of logic in the narrow sense that ensures the *transfer of the supposed truth.*

These metaphors of a particular type reign over a whole context and globally command a whole system of more traditional metaphors devoted to the local illustration of propositions. Without what might be referred to as '*orchestrating metaphors*', the propositions would appear isolated, even if they respected the habitual protocols of sequences.

It is precisely this veracity and this allusive capacity that nourishes the argumentative support found at the heart of the intuitive practices of the most formalised sciences.

Whether they are scientific or philosophical, metaphors organise the key points of reactivation and acceleration, the fulcra, the 'Archimedes levers' able to retain a whole context and propel a whole set of concepts to a higher speed, allowing for example in physics or mathematics the almost instantaneous transit of deductive chains of considerable length.

This great proximity between the 'scientific' metaphor and the 'philosophical' metaphor gives rise to the thought that each of these fields expresses two different, yet capable of being articulated, modes of intervention of allusive stratagems. Thus, one can observe that the philosophical metaphor is in a relation of *rivalry-complicity* with conceptual grasp – with the reciprocal *threat of overflowing*: the incontinent proliferation of the metaphor, the domestication of metaphorical impertinence by conceptual grasp.

This relation of rivalry-complicity brings the philosophical metaphor closer to the scientific metaphor, if one recalls that *the latter is a part of the implicit text which accompanies any demonstrative development.* This implicit text – whose importance we emphasised in

Figuring Space (Châtelet 2000)[6] – allows an overall view of this development, entering into a relationship of alliance and rivalry with the official text presenting the *procès verbal* of the demonstrations. If it seems difficult to speak about the *procès verbal* of the demonstrative in philosophy, one can nevertheless point out that the philosophical metaphor is organised in *regulated sequences* ensuring an effect of convergence, of allusions and large unsteady oscillations (*mises en bascule*), an effect intended to force conviction, just as the implicit text allows anticipation of the already acquired stages of a proof.

The crucial strategic character of the metaphor – whether in science or philosophy – lets us suspect that it would help in better determining the proximities and differences of these two fields of rationality; this is why we propose to analyse it in several precise cases.

B) *Analysis of the role of allusive stratagems in contemporary physico-mathematics: a new conception of notation.*

Recent spectacular developments in *Knot Theory* (*Théorie des Nœufs* (sic!)), the work of Vaughan Jones marking the turning point, renders manifest a profound articulation between Geometry, Algebra, Topology and Physics.[7]

These developments should not only be appreciated as distinguishing themselves by 'varied applications' – as one terms the extensive character of a transfer of technology from one discipline to another – but as falling under a tradition of implementation of a *graphic reason* in the exact sciences.

We have already noted that the 'orchestrating metaphors' were able to exceed [to dissolve] the duality of the deductive and the argumentative by establishing a new relation between illustrating and the illustrated. This is also the case for some contemporary research which completely renews *the very notion of indexation.* No explicit intuition accompanies the 'classical' behaviour of calculations: in formulae of the type $\sum_i =$, the set of indices is neutral and the indexation remains completely external to the development of these calculations, behaving like a 'notation' which is completely indifferent to that which it notes. These formulae remain captive to a linear successiveness, x_1 then x_2 then x_3 etc., an artificial sequence a little analogous to the chain of verbs *veni, vidi, vici* where the temporal order of the processes of enunciation (*énonciation*) replicate exactly the order of the processes of the statement (*énoncé*).

The contemporary point of view makes the notation *concrete* by identifying it with a diagram already used in an *a priori* foreign domain (knot theory [*théorie des nœufs* (sic)[8]]). This domain thus 'evokes' gestures which are classical for it, but completely new in the domain where it is imported as 'notation' Thus certain *a priori* not very suggestive complicated formulae of tensor calculus, can be condensed in a fulgurating way and launch new calculations.[9]

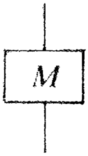
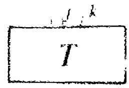
This upsets the very notion of indexation which becomes *bi-dimensional* in freeing itself from the successive: it is very much a victory of the hand that comments on itself, the indexation no longer being delivered by an external 'set', but by a process of deformation and modification of diagrams.[10] This confronts us with a remarkable situation: theorems of mathematics make it possible to support the notation for this same mathematics (See Kauffman 1991, 15).

We propose to analyse in detail this *revenge* of the hand which is no longer content to drone out x_1 then x_2 then x_3 etc., as prescribed by linear successivity, but can play on all the routes permitted by the (interlacing) tracery. The notation *contaminates* to some extent the calculations, in order to create a new context like literary metaphor.

Let us recall again that *Figuring Space* (Châtelet 2000) concludes by stressing that the knots (*nœufs* [sic]) and the (interlacing) tracery are reduced neither to an ornament, nor to a particular chapter of the topology of ordinary Space, but introduce a new mode of intervention of the geometrical figure, just as they had introduced a new manner of making the image penetrate the text in order to avoid too linear a reading of it, and thus expressing *an ability to rupture* which is a reminder of the type of intervention of metaphors already described; this seems to be associated with an aptitude of the 'tied' to interweave an 'over-under' with a cursory reading which *passes* simply from right to left or from left to right. The 'tied' puts in question the traditional opposition between habitual, totally undifferentiated, geometric space and. . strongly differentiated (high and low, right and left. .) 'psychological' space (Ernst Mach) which induces *evaluations and orientations.*

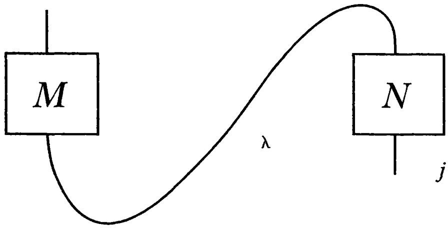
Matrix algebra already used the high-low opposition, and Einstein's convention of tensor products $\sum T_i^{\lambda\mu} T_{\lambda\mu}^j$ clearly showed the subsidiary role of the silent index: it diverted attention to the *intrication* of this *opposition* and of the *successivity* of the summation.

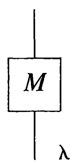
By proposing for a matrix

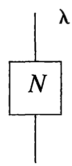
$M = (m^i_j) =$  $\delta =$  ou $T = T^{ijk} =$ 

one accentuates the operation of disappearance of the silent indices in favour of *incidental and emergent features,* and of a *compact reading of products.*

Thus, a product of matrices becomes:

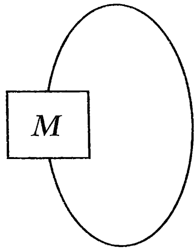
$\sum M^i_\lambda N^\lambda_j \longleftrightarrow$ 

To sum is no longer to drone out but to connect 

with 

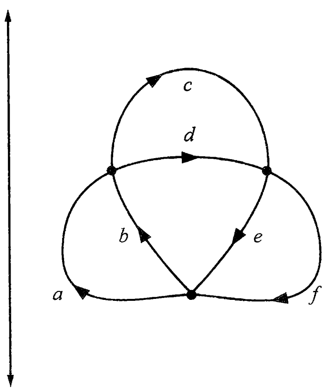
the product becoming 

The formula becomes very spectacular for the *trace*:

$\sum M_i^i \longleftrightarrow$ 

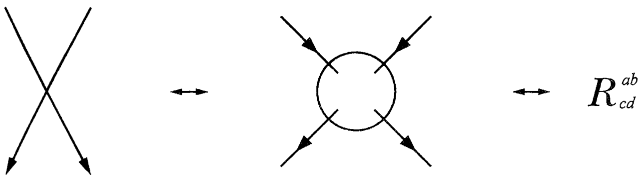
Current mathematical physics succeeded in uniting operations already very powerful by themselves: all the 'imagery' of Feynman diagrams and the diagrams of homological algebra and algebraic topology which give priority to the point of view of arrows and above all *blocks of arrows* (exact series, sequences. . cf. C) at the expense of the classical point of view of 'alpha and omega sets' (*ensembles de départ et d'arrivée*). They induce a new grasp of the relation between the image and the calculation: to think from the start at the level of blocks, is to capture the operativity to a greater degree – which is not without recalling the 'global effects' of the orchestrating metaphor described in paragraph A.

One is thus led to associate each knot (*nœuf* [sic]) – more precisely, its projection on a plane – with a tensor expression [thus benefiting from all of the autospatiality]:

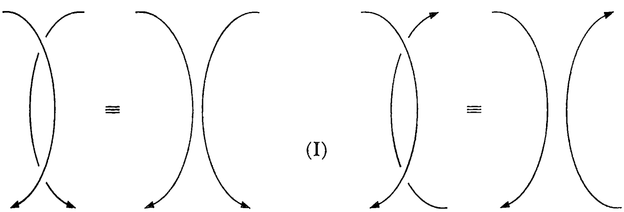


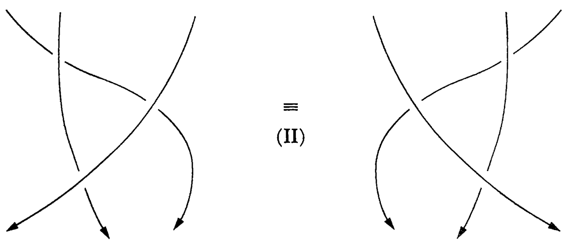
$$\longleftrightarrow \quad T(k) = \sum_{a,b,c,d,e,f} R^{ba}_{dc}\, R^{dc}_{ef}\, R^{ef}_{ba}$$

But there is more [to this link between geometry and algebra]: a third component comes [to be connected and] to complete the new notation – the intersections of the projection of the knot (*nœuf* [sic]) can also be seen as collision diagrams of particles:

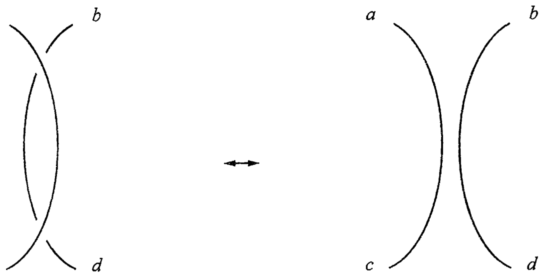


One can show that deformations of the graph do not affect the type of knot (*nœuf*); they can be classified as follows:





These deformations resulting directly from the concrete intuition of the sliding of knots (*nœuf*) materialised by bits of string, induce classical tensor relations concerning the constraints of Quantum Field Theory (and reciprocally). It is thus necessary to take all of this terminology of categories of 'braids', of 'ribbons' which irrigate algebra with geometrical allusions seriously. All these effects are multiplied ten-fold by association with Quantum Field Theory; one would be tempted to say that the two operations (algebraic and quantum) reinforce one another, mathematical arrows becoming physical, and *reciprocally.* Thus, the deformation:



is *equivalent* to the unitarity of matrices associated by convention (I). (II) is *equivalent* to a classical relation of statistical Mechanics.

We see to what point we are far from the classical figuration and illustration which always lead to a kind of dissymmetrical predation of the concrete by the abstract. We already knew that the knot (*nœuf*) is not captured by an intuition of volume or of a given container: it does not occupy a 'place' in our space – there is no *outside* and *inside* of a knot (*nœuf*). The knot (*nœuf*) is not a figure: it is, if you like, an experiment in autospatiality. This is why it so upsets the indexation of physical formulae which, to it, seemed *a priori* foreign. Indexation is no longer reduced to the external evaluation of a collection, but becomes the protagonist of an experiment which secretes its own overflow.

It reveals the grasp of Feynman diagrams as convenient conventions which associate integral calculus with a reproductive imagery of real particle collisions as definitively null and void, leaving the capacity of these diagrams for auto-procreation in suspense.

To index diagrams by knots (*nœuf*) is not to associate two imageries operating by resemblance, but to grasp in a single act the two dynamics of allusion (the collisions not 'resembling' knots (*nœuf*)). The conditions of the *knot* (*nœuf*) *diagram* become here *physical conditions* by the transfer of operations (and reciprocally).

The success of this synthesis of indexation by knots (*nœuf*) would certainly have been appreciated by C.S. Peirce, who liked to say that 'algebra is nothing other than a kind of diagram', stressing that it had the privilege of articulating three functions: that of an *icon* (similarity in reality between its signifier and its signified – *resemblance*); that of an *index* (contiguity in reality between signifier and signified – *auto-overflowing*) and that of a symbol (*instituted*, learned, contiguity between signifier and signified – *convention*), the most perfect being the one where these functions 'are in as equal proportion as possible' [Cf. Châtelet 2000, 188 n. 40].

This indexation by knots (*nœuf*) incarnates this ideal of equilibrium between image, allusion and calculation.

One can now appreciate the great subversive proximity of the unsteadily oscillating relation, illustrating-illustrated, of the orchestrating metaphor, and this driven by allusive stratagems, particularly in the case of the indexation-knot (*nœuf*).

One could detect here an invaluable pivot point: that which would refuse the quartering denounced by Heidegger; that of an informative language aiming at the most massive and most rapid production of messages, and sanctioned by 'yes-no' decisions, aiming to force nature to appear in a calculable objectivity, to debit compact and irreducible units of signification, and of a language of plastic tradition, able to stammer, and which lets things appear.

C) *The revolution of Grothendieck as the articulation between concrete geometrical and concrete algebraic.*

Let us recall that, in the '60s, Alexandre Grothendieck wanted to undertake a vast *programme of* reciprocal *'translation'* between Algebra and Geometry, implying a rupture with traditional intuition and the intervention of new techniques which seem 'abstract', but which however are revealed to be the most adapted to this 'translation', as his introduction to the language of diagrams (*schémas*) emphasises: 'as in many of the parts of modern mathematics, the first intuition moves further and further away, in appearance, from the language appropriate to express it in all the desired precision and generality' (Grothendieck and Dieudonné 1960).

Grothendieck, in his writing, sometimes refers to Galois' theory, as an example still to be reconsidered of what one could call *a pure algebraic concretism*, as much to the work on classical theories such as those of Lie and Sylow groups, as on the most recent theories. One can understand Galois' theory as *training*: that of the progressive discernment of roots, the formal conditions of such a discernment paying attention to the *sequences of reduction,* while the explicit formulae of resolution become subsidiary. To bring to bear all the effort of research on the sequences of groups, fibres, 'bundles', etc., to be able to grasp in flight *the very gesture of learning. . .,* such would be, according to Grothendieck, Galois' unforgettable lesson.

This algebraic concretism can then espouse another concretism, that of Geometry, which is that of *learning the gestures for grasping Space*. We would like to analyse in detail some examples of this 'geometrico-algebraic' concretism in the work of Grothendieck. His conception of the point is particularly enlightening: the 'classical' point of Geometry is simply the trace in space-time of the act of designation of *this point here*, while the point conceived by Grothendieck (and now by all contemporary geometers-algebraists) is an operation, an infinite panoply of virtualities, whose designation would be the most trivial, a *monadic point* able 'to concentrate in one point' all that previously claimed to hold separately the attention of the mathematician. The modern point is re-knotted with this 'evidence' that was always anticipated, but never grasped before Grothendieck as mathematical evidence, making available a new operative power (*puissance*) and above all a formidable allusive power: the 'concentration in one point' is the complete opposite of a subsidence in one point but appears, on the contrary, as an *operation of liberation and amplification of geometrical virtualities.*

Grothendieck saw clearly that mathematics never succumbs to an abstraction deprived of all the richness of determination: the 'generalisations' of mathematics are never confused with inoffensive generalities: there is definitely an audacity specific to mathematicians, certainly associated with a strict discipline of verification, but above all permitting access to a field where yet unclarified virtual determinations emerge. Installation in such a field possesses all the character of a *diagnosis* that operates in a decisive way, well before any exhaustive analysis: the most flagrant example is that of the attack on such and such a conjecture by a stronger – and thus *a priori* more difficult to deduce – conjecture which completely displaces a problem and reveals the old conjecture to be a poorly posed problem. To confuse mathematics with simple deductive chains is to be unaware of the crucial character of the sense of the 'good conjecture' – of that which we have called the *diagnostic of a mathematician*: this is why Grothendieck's 'evidence' is not related to the proximity of two terms in a deductive chain, but to the 'natural' effect related to the abolition of the space between the symbol which captures and the gesture which is captured.

Statements of the type 'Let us consider such a point. . ., such a subset. . . Let us extend this segment of line. . .' are frequent during a demonstration.[11] They are certainly inserted in the deductive chain but have to some extent a *strategic character* appreciated by experts, and they surreptitiously introduce another rhythm. 'It is here that something happens. It was necessary to have the idea to consider this or that. These elements, these statements or these constructions were quite available, but asleep and seem to become animated abruptly by virtue of this 'let us consider', which puts all the attention on what becomes a *pivotal element,* expressing a type of concretism which is much more intense than the 'concrete' allegedly encountered at street-corners by the naïve empiricist.

We are at the antipodes of the 'abstraction' which always results from the violent deduction of a part, and thus of a mutilation, whereas while the 'lever' does not subtract anything and acts like certain fragments of a *puzzle* which, from the outset, emerge and impose or dictate the solution: to be absolutely concrete is to persevere to some extent in *a kind of tangential approach of thought which grasps its own movement.*

Grothendieck's undertaking – like any translation – is not content to define *a simple bank of reciprocal references* between 'purely algebraic' and 'purely geometrical' concepts that are left intact. The theory almost forcefully dislodges the attention of the mathematician from 'points' and fixed sets towards arrows (these morphisms) and makes it possible to understand algebraically geometrical syntheses.[12] One almost wants to say that, thanks to the introduction of topology, the structures of commutative algebra themselves fabricate an 'environment' without remaining under the supervision of co-ordinates. We would like to show that one may understand the programme of translation as crowning a *tradition of discovery and development of analogies* between Number Theory and Geometry initiated by Kronecker and developed by Weil (the analogy between bodies of algebraic numbers and 'paths' of coverings of an algebraic curve, etc.).

Translated by Simon Duffy

Notes

* The notes between square brackets are mine [CA]; the others are Gilles' The parentheses within the text are also mine [CA]

6 See in particular, ch. 3.4, 'Indifference centres and knots of ambiguity, fulcra of the balances of Being', p. 88 ff; and above all, ch. 5, 'Electrogeometric Space', § 5, 'The electrogeometric experiment as square root', C, 'The screw as bold metaphor', p. 176 ff.

7 The last paragraph of Châtelet 2000, ch. 5.6, entitled: 'Towards the knot as secularisation of the invisible' (p. 183–6). The following new material is an extension of this work.

8 In an astonishing way, Gilles Châtelet no longer speaks here of 'kno (*nœuds*) (cf. Châtelet 2000, 183 ff), but of '*nœufs*'(sic), without any expla tion. The mathematical context in which the author intercedes refers to Kau man, *Knots and Physics* (1991). In this fundamental work, Gilles Châte notes, by hand, above the first two diagrams on the 'Trifoil' and the Yan Baxter equation (p. 108): '*nœuf*' and 'inverted *nœuf*' This syntagm is th not a typographical error, and the author obviously had an idea in mind. I a therefore reduced by this to propose a 'conjecture' about a possible li between the mathematical concept of knot and (perhaps) the number ni (*neuf*), the idea of a reference to the egg (*œuf*) appearing to me at the ve least inconsistent. The syntagm *nœuf* (for *nœud* or knot) could then be relat to the fact that, in the use of the knot diagram, considered as an 'Abstra Tensor Diagram' for the Yang-Baxter 'nodal' solution, Kauffman establish a link to a list of 9–tuples from which this equation can be read (Kauffma 1991, 318).

9 One of the best examples is that of Yang-Baxter's formula which connects relation of matrix commutation to a knot (*nœuf*) diagram deformation.

10 We can appreciate here the whole path traversed since the work of Yukaw Heisenberg still anxious to illustrate, endeavouring to fix diagrammaticall the concept of 'particle of exchange' But this still remained 'to the side' calculations and too captive to a relation of *illustration* and similitude wi chemical imagery. Contemporary diagrams do not draw their force fro similitude but from the capacity of their new indexations to ensure *a co-pe etration of the image and the calculation.*

11 The example of the demonstrations and constructions of so-called elementa Geometry is very enlightening: it is enough to think of the proof of the poin wise convergent character of the sides of a triangle, transformed by astu extensions into the perpendicular bisector of another triangle. There is a 'effect of synthesis' caused by certain points or remarkable constructions a by no means given by a simple representation on a figure: *the figure becom diagram* because it suggests a dotted line.

12 Two central intuitions traverse Grothendieck's work:

a) the substitution of the point of view of the arrow for the point of view excessively fixed sets: the arrow deposes sources and targets;

b) the grasp of the point as capable at the same time of *condensation* (th most sophisticated structures could become 'points'): this is the case fo vectorial fibre classes on X seen as points of K(X), and also of multipli cation of geometrical virtualities (this is the case with singular points Let us take some very simple examples:

1) that of the notion of the ideal; one should not consider it as a simple 'gen eralisation' of the multiples of arithmetic, but as an autonomous entity, *point* which has its place and which holds at the same time to the set (and to the element – this is the point of view of the *spectrum* (*spectre*).

2) that of an *A-modulus M* of finite type: it seems more complicated to define it by the existence of exact sequences of the type $A^p \to M \to O$, than in the usual way. It is however this type of definition which encourages *operating on blocks* – exact sequences which appear just as condensed as a geometrical point, and which are found at the core of the development of Bundle Theory.

4
Mathematics, metaphysics, philosophy

Jean-Michel Salanskis

The idea of a close, intimate connection between mathematics and philosophy has been rejected, either in a conscious and intentional way or not, by the two major schools of philosophy who fought an unequal battle throughout the twentieth century: analytic philosophy, being the dominant school, and phenomenology, the surviving challenger. This rejection, it has to be observed, took very different forms in each case.

The founding fathers of analytic philosophy, Frege and Russell, were deeply concerned with mathematics, the former having even held a position as professor of mathematics. But their scientific endeavour primarily addressed the field of logic, which was, so to say, re-invented by them, as they built for the first time what we now call first order logic, and discussed the new and central topic of 'foundations of mathematics' on the basis of the new logical language they initiated. They also paved the way for doing philosophy using the same first order logic as its unique reference tool: by analysing the logical structure of sentences and investigating the problems revealed by such an analysis. Their contribution, for these reasons, resulted in a shift of interest from mathematics to logic. Logic, improved and raised to the level of formal rigour, appeared to inspire directly some philosophy, and thereby to enjoy the kind of generality that is needed to deserve the name of philosophy; and even when philosophy was supposed to deal specifically with mathematics, it was understood that the only task was to come to terms with the foundational issue, and it was recognised that the only rational way to do so was to put things in the framework of contemporary logic. More than a hundred years later, the resulting eclipse of mathematics within the field of philosophy, at least as a living literature to be addressed, is absolutely clear. Who, in the context of analytic philosophy, could imagine for example anything comparable to the long remark about infinitesimal calculus made by Hegel in the middle of the first book of his *Science of Logic* (1969)? If philosophy would welcome technical formalities, it would only be logical developments, as is the case with Kripke or Hintikka.

The pioneering author of phenomenology, Husserl, was also originally a mathematician. He wrote his dissertation under the direction of Weierstrass, and held his first university position as a mathematician. And all the way through the astonishing philosophical life he lived, writing every minute he could, building and re-building phenomenology as a systematic whole without losing hope or energy, he maintained the idea of an intimate connection between philosophy and science, seeing phenomenology as a new science deeply analogous, because of its *a priori* eidetic character, to mathematics. But all Husserl's followers, perhaps under the influence of the first of them, Heidegger, divorced themselves from such an orientation, and recommended rather a way of doing phenomenology which would be free from the objective stance of all scientific discourses, and which would rather look for guiding patterns on the side of poetry or literature. Again, one hundred years later, we must recognise that Ricoeur, Derrida, Levinas and Gadamer, to quote great names, did not move any meaningful distance from the post-Husserlian rejection of the affinity between mathematics and philosophy.

Faced with such a situation, we now have to explain and justify why philosophy should think of a new companionship with mathematics. But we also have to try and distinguish between different ways of realising such an alliance.

Justifying the partnership: back to Plato

As far as the 'why?' question is concerned, the job is rather easy. All we have to do is go back to the beginnings of philosophy and its only universally acknowledged birth: the Platonic one. It was Plato who defended the essential partnership between mathematics and philosophy, which we still keep in mind today thanks to the motto written on the front door of the Academy ('The non geometer does not enter') and to the mathematical education every future leader should receive in order to become a philosopher, as is explicitly stated in The Republic (1987b). Let us focus our attention on one main point: Plato conceives of philosophical questioning itself as arising 'from' mathematics. Once again in *The Republic*, he tells us that the future city-keeper has to be taught to seek the Idea, to look for it beyond given sensations. And Plato argues that this cannot happen if sensation seems certain, if we believe ourselves to possess some object through it. The only chance for us to seek the transcendent, never changing idea, is to experience perception as self-contradictory. And this happens, for Plato, precisely with and because of mathematics. He claims that we perceive through our senses the big at the same time as the small, and the one at the same time as the infinite multiplicity. In this last case, and if I am right in following the hints of ancient philosophy scholars, what Plato has in mind is that once we have settled some unit on the straight line, we can still re-define this unit as an arbitrarily small part of the former unit, making the former *one* become an arbitrarily large, or big, multiple. In other words, we experience that any sensitive assignment of the one as a reference unit may be changed. And such an experience teaches us that the 'real' *one* lies only beyond any such assignment, within the ideal realm. We can easily understand how the example of the big and the small has something in common with the first example: what is said to be big can truly be said to be small if we compare it to something much bigger. Every judgment of the big and the small, then, depends on the choice of some sensitive standard of the 'medium size'. Notice that the example can be further enriched thanks to the works of twentieth-century non-standard analysis (Cf., for an overview, Salanskis 1999): the puzzle of the big and the small remained the same throughout the ideal framework of mathematics, and only contemporary non-standard theories have approached a solution for the first time. But let us leave this point aside and return to the first, more typical example, which proves, for Plato, that perception is contradictory when mathematics is involved. But there seems to be a gap in the proof he gives: we could ask why it seems possible to us to swap the sensitive incarnation of the one, the unit, for another one. And the answer could only be: because we feel it arbitrarily. And it can only be so because we have some kind of pre-understanding of the one as an ideal entity beyond any sensitive incarnation. In a way, the whole argument seems circular: we are led to the idea beyond sensation only because our preunderstanding of what the idea is in itself leads us to experience a contradiction in the realm of perception.

But I believe that we must agree on the argument exactly along these lines. What Plato explains is that if we are already mathematicians, even only unaccomplished ones, we are, so to say, compelled by our relation to the mathematical idea, to see the world in a contradictory way, even if this relation is not mastered explicitly. And I would add that here lies precisely for him the origin of philosophical questioning. To question in a philosophical way every kind of data is something that we do if we are inspired by ideality, and ideality is originally revealed to us by mathematics, or better: it is originally experienced as something towards which mathematics sends us, as something which becomes required for the mathematically-minded individual.

And if we read Plato, we understand that his philosophical approach is always the same: we ask for a definition of virtue that would not be attached to any particular virtue, that would not be formulated in terms of the virtue of the doctor, the shoemaker, the painter or the warrior. And we raise such a question because we are following the model of the general definitions of mathematics, such as Theodore's classification of irrational numbers (1987a, 146-8).

So Plato's teaching is that mathematics lies at the core of philosophical questioning, that it remains its basic resource. To do philosophy is to try to shed, in the realm of ever-changing beings, some universal light, of the kind that we enjoy inside the mathematical realm. And this attempt always originates in some question addressed to the world, this question being itself connected with some aporia brought about by the mathematical perspective.

We can state, at this stage, that philosophy is originally guilty of the accusation recently formulated by Sokal and Bricmont (2003). By its very essence, it does not pay respect to mathematical truth by keeping it within the frontiers of mathematics, such that it remains unproblematic, proved and always illustrated by mathematical objects. Philosophy tries to transfer insights, concepts and maybe truths which are only designed for the mathematical world into the non-mathematical world. And this is, I believe, the reason why we can never reduce philosophy to some pure and secure pattern of rationality.

Let us go back now to the rest of my argument. I will now try to describe the way in which the old Platonic partnership can be renewed. And I will begin by evoking the example of Deleuze, who I will consider as a representative of the attempt to restore this partnership in a metaphysical way.

Mathematics through metaphysics

Metaphysics should neither be understood here in the Nietzschean sense nor in the Heideggerian one, but simply in the Kantian sense, that is, what is called 'dogmatic metaphysics' in the first *Critique* (1998), and which is exemplified for Kant by Leibniz and Wolff. 'Dogmatic metaphysics' is defined as the attempt to draw some substantial conclusions at the level of absolutely general concepts, which pertain only to being as such, with no other tool than logic. Following Kant, the true statements that we are in a position to form at this absolutely general level are only analytic, which then convey no information. There is indeed a valid 'metaphysics of nature' if we now understand metaphysics as meaning *a priori* synthetic knowledge about every natural being that we know as true before having to encounter any such being: but this knowledge explicates the determinations that may be bestowed on natural beings by virtue of their coming to our mind through such and such 'forms of presentation' (space and time) and their having to be judged, in order to count as genuine objects, with such and such pure concepts (the twelve categories). Metaphysics, when it is not explication of conditions that we are imposing on beings of some specific area, that is when it claims to be the logical deduction of significant properties of beings as such, is condemned to be empty, or self-contradictory. The relevance of *a priori* knowledge depends on the reference to transcendental structures which anticipate such knowledge by the very fact that they explicate our conditions for 'meeting' and 'judging' beings of such and such a kind. This is one of the ways 'finitude' in the Kantian sense may be asserted: there is no logical clue which would open for us the realm of being 'in itself', the 'noumenal' truth lies by definition beyond our discursive logical power ; we do not know in the same manner as God, who understands beings by the very gesture through which he created them, but we are knowing beings insofar as they are given to us and able to be judged by us in such and such a setting.

Plato, in the *Timaeus* (1977), was at least considering the possibility of a mathematical description of everything exactly as it was generated from the 'chora' Such a knowledge would be, if I understand him well, a metaphysical knowledge of the 'noumenal' genesis of the world, not of the beings in the way they appear and are constituted by us, but of the beings as they are, truly individuated in the general process of Being and Becoming.

In *Difference and Repetition* (1994), Deleuze gives us, in the chapter 'Ideas and the Synthesis of Difference', a general description of individuation. He first starts with the 'problematic idea', as he describes it. The idea, for him, is characterised as including a *problem*, the authenticity of the idea is nothing but the unfolding of a problem. That is why the idea has an internal temporality. This temporality originates in the *question*, which is compared to the throwing of a dice, but with a imperative character. The question as imperative throwing of a dice frees a plural game of singularities, which generates a proper space of the problem, an internal topology of the virtual, through a process which Deleuze called *differentiation*. But this process cannot avoid simultaneously being the process of the concrete formation of real individuals, and the design of actual areas in the world, occupied by these individuals: such an effective and actual process Deleuze calls *differenciation*. So the real integrated process of individuation is termed 'different/ciation'

This description, or, better said, this matrix-story, is supposed to hold for the individuation of every being in the universe: it is, so to say, 'cosmological'. In particular, Deleuze uses this framework in the natural realm (of *Naturwissenschaften*) and in the cultural realm (of *Geisteswissenschaften*) at the same time. He tries to understand in terms of different/ciation the development of an egg, as well as the development from the social Idea (whose anger is the revolution) of some social and political configuration.

I claim that the discourse of different/ciation is a metaphysical one in the Kantian sense, and that it is built on some precise mathematical references. In the way in which I have explicated it, it is possible to miss that point, because I insisted on the Platonic analogy with the *Timaeus* on the one hand, and the biological metaphor on the other. When Deleuze speaks of differenciation, we cannot help thinking of the distinction between parts of a material whole, parts which assume some function with respect to some individual enclosing them, we cannot help imagining the arising of the individual as the birth of this particular individual, and this individual itself as a living one. We are led to such a conception by the very reference to the guiding role of the idea and the unfolding of its problem: this seems to introduce finality, which we attribute more willingly to living beings.

But Deleuze does not want to be understood that way; he wants his dramatic picture of individuation to hold beyond the distinction between animate and inanimate nature. He wants this picture to be absolutely universal, metaphysically universal: and this is why, I think, he explains different/ciation in mathematical terms.

We have seen that individuation is described by him as a drama, whose scenes would be: question, problem, singularities, unfolding and reciprocal determination, parts and individuals. The first act of this drama has to be named 'idea'. But we must insist on two essential points. First, Deleuze calls for the Kantian motto of the idea. He begins his exposition of the drama of individuation by reminding us that ideas, in Kant, are problematic, in such a way that the content of our first act receives so Kantian colour. And secondly, he gives us some mathematical hi about how we should understand different/ciation: the word *differen tion*, itself, is nothing but the mathematical one, which used to refe the pre-Cantorian mathematical framing, to the operation of wri the infinitesimal variation of some variable, expressed with respect some others (let us say that if $y=x^2+y^3$, we differentiate by writ $dy=2xdx+3y^2dy$). Very likely, Deleuze is thinking here of what He tells us in the *Science of Logic* (1969) concerning the relation which fr itself from the effectiveness of the relata: the differential ratio *dy/dx* for Hegel, an important example of something which is determined a relation without its relata having to be assigned any actual value. Deleu sees differentiation mathematically as the process of the self-determi tion of the problem at its specific level, a determination which plac some constraints on the actual values, places shapes and properties what counts as a solution to the problem. Following very clearly (a explicitly) the inspiration of Lautman in his *Essai sur l'unité des math matiques* (1938), he insists upon two mathematical models for the unde standing of such a process: the model of differential calculus a dynamical systems, and the model of the Galois theory of polynom equations.

The first model is based on the idea that singularities of vec fields – as nodes, focuses and centres – strongly determine the lo topological behaviour of trajectories. These singularities may described purely in terms of the given 'problem' – that is to say the vec field – even if they also organise the actual shapes and places of the so tions (the trajectories): the internal analysis of the problem (differen tion) leads again to the actual assignment of solutions in the space trajectories. So, if Deleuze was right in understanding differential cal lus essentially as a calculus of problems, it seems that the theory dynamical systems, interpreted as the qualitative and topological theo of problems connected in the most natural way with differential calcu (differential equations), is able to support and to illustrate his fundam tal intuition of different/ciation.

The second model concerns the process of specification and det mination of the roots of some polynomial over a field k. Generally spe ing, the roots that one seeks cannot be found in the field k itself, appear in some extension $k \rightarrow k'$ of k. If we choose for k′ some algeb closure K of k, then we are sure to obtain all the possible roots. Galois' idea was to focus not on the roots themselves in order to comp them, but rather on the substitutions acting upon the set of roots. This leads to the definition of the Galois group of any extension: if $k \rightarrow k'$ is such an extension, $Gal(k'/k)$ is the group of field automorphisms of k' which keep all elements of k fixed. The main theorem of the now classical Galois theory asserts the exact correspondence between subgroups of the Galois group of a finite Galois extension and intermediate fields of the extension. We can then interpret, as Deleuze suggests, that there is a process of enrichment of the field, from k to k', yielding progressively the required roots of the polynomial: each step corresponds to a subgroup of the Galois group $Gal(k'/k)$, and to a certain range of roots and to substitutions defined on them. If we understand the Galois group, its subgroups and quotient groups as expressing the 'problem' we can see the lattice structure of groups and subgroups as the 'differentiation' connected with the development and 'differenciation' of roots: the bigger the subgroup, the smaller the subfield of its fixed points, the fewer roots we have.

Such examples are suggestive, but I think their role goes a little further. In the quoted chapter, Deleuze uses these examples in order to explain individuation: the general drama of the problematic idea. He defends a strongly dynamical picture of individuation, and a mathematical setting and discourse is needed in order to describe and explain such a process in an absolutely general way. Mathematical objects have been used in physics – that is to say, by the dominant philosophy of nature – in order to make sense of the basic notions of change and movement. When Deleuze tries to advocate a sophisticated and universal conception of individuation, it is not surprising that he comes to mathematical language. This is first shown by the very choice of the word *differentiation*, which belongs to mathematics, and whose Deleuzian use is totally motivated by this belonging. But I think that the reference to a Kantian framework indirectly plays the same role: if ideas are to be considered as the resource of individuation for any being, this will be at the level of what Kant called the 'metaphysics of nature' And for Kant, as we know, the only relevant and true metaphysics is relative to space and time as forms of our intuition, and as such requires a mathematical setting.

But here comes the difference: Deleuzian metaphysics is supposed to explain how any individual arises through different/ciation without any regard to the forms of sensibility. Deleuze's insight reaches the heart of becoming without any reference to phenomena as our way of receiving data. The individuation of social, artistic or cultural beings whose presentation owes nothing to space or time, or is only connected in some very indirect way to them, will still have to be thought in terms of different/ciation. For these reasons, the mathematics of different/ciation is, in the Deleuzian construction, a direct and essential key to the becoming process, shedding some decisive light on the relation between the actual and the virtual. Mathematics plays the central part in the new philosophy of nature, which is at the same time and in the same way philosophy of culture, but it does so in the qualitative, absolute and immediate manner which characterises for Kant dogmatic metaphysics: metaphysics which claims to grasp being and becoming as such with purely conceptual tools. The only difference, but it is an important one, is that the conceptual key is mathematical, and not logical.

Since mathematics is not logic, the status of this new Deleuzian metaphysics remains controversial and can be read in at least two ways.

Either we insist on the fact that the mathematical theory or theories about individuation, as theories of different/ciation, involving two spaces, on the one hand, the virtual – of singularities and of the problem – and the other, the actual – of the parts and of the individuals – count for Deleuze as metaphysical theories telling the truth about individuation by mathematical speculation without any reference to conditions or forms of presentation of underlying phenomena. This is exactly the way in which we just explicated Deleuze's conception.

Or, we argue that the reference to dynamical systems is at least based, in an implicit way, on the theoretical role of space and time in contemporary physics. It is only because the physics of the XIXth and XXth centuries ultimately picture change in matter in terms of dynamical systems that Deleuze can imagine and sustain his general conception of individuation. Following this alternative view, Deleuze's conception is no longer purely metaphysical, it refers to our best transcendentally-based knowledge and to its mathematical apparatus. The encompassing metaphysical value arises only when Deleuze generalises his view to cultural, political or social domains for which corresponding conditions on the presentation of data do not hold. But we should add that in doing so Deleuze is playing the same metaphysical game as René Thom (1977) when he comes to assert the universal value of 'Catastrophe Theory'. Up to a certain point, as Jean Petitot (1985) saw very well, the two philosophies or theories express and do the same thing.

Non-analytic and non-metaphysical partnership for philosophy and mathematics

But I would like to come now to a more personal question: is it possible to reinforce the connection, affinity, and even partnership between mathematics and philosophy without simply bestowing on mathematics the metaphysical power that contemporary analytic philosophy is only ready to admit to logic? More precisely, is it possible to do it in such a way that we would remain faithful to an extent to Kant's teaching about metaphysics?

I will now try to define such an alternative path, aiming for the rejection of the analytic and post-Heideggerian dominance of logic and/or language, as well as of metaphysics in the Kantian sense.

I think we can bring new life to the ancient affinity between mathematics and philosophy in three ways. Two of these are old ones, even if we need some effort in order to redirect them towards this affinity. The third seems to me to be new, connected with my personal programme.

The first two are simply those of the philosophy of mathematics and the transcendental philosophy of science.

Philosophy of mathematics today

Contemporary analytic philosophy sees the basic problem of the philosophy of mathematics as settled by the so-called Benaceraff's dilemma. In short, the dilemma asks whether we can conceive of mathematical truths as implicitly governed by the same semantic framework that analytic philosophy considers to be the only possible one to assess the objective truth of any knowledge. Put in such a way, the major aim of the philosophy of mathematics becomes the solving of a logical problem, or at least of a problem that is necessarily expressed in logical terms: even if some particular answer begins with non-technical terms, in the end it will have to be formulated in such a way as to be expressed in some (modified) Tarskian scheme. In opposition to such a conception, I see the philosophy of mathematics more as an attempt to grasp and describe the specificity of mathematics among other human enterprises. The philosophy of mathematics, as I hope it should be worked, would accept as its main concern the identity of mathematics. I contend that this identity can be approached through five typical questions, which I regard as the basic questions of the philosophy of mathematics (Cf. Salanskis 2002, 81–106):

1) What is the relationship between mathematics and philosophy, in what way can we say that philosophy shares something with mathematics and remains indebted to it, and in what way does philosophy depart from mathematics and go beyond it?
2) What is the status of the mathematical object, and, in particular, in what sense is it an ideal one, considering the old problem of the ontological validity of the idea? (this question is asked because we have every reason to suspect that the identity of mathematics is strongly dependent on the status of the mathematical object, on the way in which it refers to what are its recognised objects, that mathematics is, up to a certain point, this discourse that manages to deal with an ideal object.)
3) How can we define and understand the frontier between logic and mathematics? (this question is asked in the context of the big change undergone at the beginning of the XXth century, following which mathematics and logic are now explicated in the same formal and symbolic way, and give birth to the same kind of theoretical developments: in such a situation, if we cannot distinguish mathematics from logic, we do not know what mathematics is).
4) How can we understand the historicity of mathematics, the fact that new fields, new objects, new structures, new questions and new truths are and have been all the time introduced, without ever having disposed of any mathematical content recognised as such? It seems that in this very peculiar way of changing and not changing through time, something essential about the identity of mathematics appears.
5) How can we understand the geography of mathematics: that it divides itself in a manifold of branches, whose names and total number have undergone various changes through history, from the ancient division between arithmetic and geometry to the contemporary one involving dynamical systems, arithmetical geometry or finite group theory? These divisions contribute to defining the epistemological image of mathematics at every particular epistemological state, and therefore are essential for its identity.

Each of these five questions requires from the philosopher both an inside and an outside position with respect to mathematics. They have to inhabit mathematics from the inside in order to evaluate its identity from any one of the five points of view: they should know what is important and what is not, what contributes normatively to the identity of mathematics and what does not. But they also have to speak and understand from the outside in order to settle this typically epistemological problem which is the problem of the identity of a discipline. Criteria for distinctions, concepts for understanding the identity cannot but come from philosophy, which remains the only discipline in charge of a common viewpoint including at least mathematics, logic and humanities, or able to formulate in adequately general terms the questions about time and space that are perhaps in the background of the last two questions, referring to the historicity and geographicity of mathematics.

So, the philosophy of mathematics, understood in this way, makes the old connection between philosophy and mathematics, not only by giving philosophy a mathematical theme, but also insofar as it forces the philosopher to assume the typically inside/outside attitude of epistemology, that J.-T. Desanti describes in his well-known paper 'Qu'est-ce que l'épistémologie?'(1975, 110-132).

Transcendental philosophy of science today

This is the attitude to be found also in the transcendental philosophy of science. I define it as the project of understanding the different sciences in terms of the way they imagine and anticipatively picture a world, within which they plan to understand evolutions from a necessary perspective. The aim of the transcendental philosophy of science is not to account for the notion of truth in science by equating it with the commonsense truth supposed to be governed by external reality, nor is it primarily concerned with the notion of guaranteeing scientific truth, formulating its conditions of possibility or evaluating or criticising its value. It is rather concerned with what science leads us to envision, what kind of world it brings through its theoretical apparatus. In the case of physics, such a world is determined by a geometrical setting, that of a differential manifold in the case of general relativity and that of a Hilbert space in the case of elementary quantum theory (but the world to which we are introduced, in that second case, is the world of virtuality). The identity of every science, and even more the identity of every theory lies in such a mathematical setting, simply because mathematics is the language of the imagination of worlds. So if we want to understand the specific gesture of some important theory within a certain science, if we want to share the perspective on reality that this theory makes possible, then we have to analyse its mathematical setting. For this is how we can understand the basic setting in which this theory sees evolution and change, how it defines the thing undergoing them (point mass or state vector, for example), and in terms of what kind of functional mathematical objects the 'trajectories' of the change are expressed.

The programme of such a philosophy of science is simply the Kantian programme. This programme focuses on the 'aesthetic' moment or strata[1], where some intuitive framework accounting for phenomena in the area under consideration is assumed, and some mathematical interpretation of this framework decided (in the classical case of Newtonian mechanics, the intuitive framework is space and time as a priori intuitive forms, in the case of the theory of general relativity, the intuitive framework is the same, but it receives another mathematical interpretation, to give two easy and clear examples). The philosophy of science does not jump immediately to the categories and possible statements whose truth value would have to be evaluated. Kantian epistemology holds that the locus of the biggest creativity and of the biggest events in contemporary science is this aesthetic locus, which the logical empiricist epistemology of the Vienna circle wanted to eliminate. But what good can come from such an epistemology which is neither aiming at understanding the specificity of theories of relativity or of elementary quantum theory, nor interested in quantum electrodynamics or contemporary attempts to unify field theory? How could it possibly be that epistemology abandoned altogether the aesthetic perspective precisely in a century when everything new in science happened on that very level?

Kantian epistemology – it has to be conceded – is guilty of not being easily applied to sciences that do not yet mathematically interpret their intuitive framework. So, it is not adapted to account for history or sociology, at least at first glance (perhaps, in such cases, we should complete Kantian epistemology with some Husserlian supplement). But it may be successfully used in the case of the epistemology of cognitive science: it is not the case, contrary to what Quine's famous paper (1969) seems to announce, that cognitive science took the place of classical epistemology. To support this point of view, it should be emphasised that the classical Kantian epistemology illuminates very well the conflict between the computationalist, the morpho-dynamicist and the constructivist paradigms, when the choice of some image of thinking is at stake: it is either interpreted within the framework of representation processing, associated with logical-computational finite and discrete mathematics, or within the concurrent framework of stabilisation within a continuous process of adaptation, associated with the mathematics of dynamical systems.

Therefore, if the classical Kantian stance is put into practice in the philosophy of science, it can preserve the strong historical relationship between philosophy and science, based on a fine understanding of mathematics as the art of imagining worlds: philosophy has to comprehend this essential capacity of mathematics and to make it its own, up to a certain point at least, in order to clarify the role played by mathematics in the context of each particular scientific achievement.

Mathematics in the context of the 'etho-analysis' programme

I now come to the third possibility: it is opened by my etho-analysis programme, which I explicated for the first time at the end of *Sens et philosophie du sens* (2001). The idea is that something should be preserved of what Kant rejected under the name of dogmatic metaphysics, something that we should try to regain after the 'Copernician revolution' Classical metaphysics should not be viewed only as the purely *a priori* theory of what is, on the sole hypothesis that it is. It was not only the all-encompassing science of which every science was a limitation. Before the Kantian revolution, metaphysics was also the language capable of saying in a philosophical way how we were to understand, live, practice and receive the various kinds of human affairs. The over-arching concepts of metaphysics, the *One*, the *Other*, the *Same, Substance*, and so on, were used in order to formulate such speculative descriptions of various aspects of what we see as our reality. In other words, the idea would be now to keep on writing philosophy using the logical generality of the concepts of metaphysics, but in a non-dogmatic way.

The etho-analysis programme, as I understand it and try to follow it, has to contribute to this renewal of metaphysics. At first sight, this may appear strange, because etho-analysis is defined as the result of shifting the major interest of philosophy from being to sense: etho-analysis is the typical investigation of what I call 'philosophy of sense' To my mind, philosophy should depart from science as knowledge of the world as it had to depart from religion, which means that it should not keep on thinking that it has the ability and the methodological right to say the truth. The best possible truth is the scientific one, and science has no direct need for philosophy in its truth-telling task. But philosophy has another responsibility, which is to clarify the sense underlying any kind of human affair. This was already the Kantian perspective, insofar as Kant, in his three *Critiques* (1998, 1997, 1987), did not try to decide the truth about nature, or the good with respect to ethics, or the beautiful with respect to aesthetics, but rather to evaluate under what normative conditions the truth, the good and the beautiful could make sense for us. I try to radicalise and generalise this philosophical point of view by introducing a new understanding of the notion of sense: roughly speaking, sense has to be considered not as a way of relating to objectivity, as it happens in Frege as well as in Husserl, but it has to be connected with what I call the 'plot of sense' Some message reaches me, I am the addressee of this message, and this message asks me to understand it: to throw it back faithfully with respect to the asking that it bears. I underline three moments of this plot: envelopment of sense (the sense appears as enveloped in itself), relay or switch of sense (sense basically sends us to some *elsewhere*) and directionality of sense (sense gives sense to some *towards*, to some relevant and privileged direction). And I try to describe in a general way how sense gains complexity, and what count as possible coordinates for sense (2001, 136-84). I do not wish to enter into any precise explanation of these ideas and theses, I will only try to give an outline of this new understanding of sense, based on the notion of address: what makes sense addresses me, this is what counts in order for it to qualify as sense. Or, to put it in diagrammatical terms: the classical conception of sense, which I call the *intentional* one, and which would be the conception of phenomenology as well as of analytic philosophy, refers the notion of sense to the arrow going from the subject or the linguistic expression to the object or the denoted; my conception refers sense to the arrow of interpellation, going from the addressor to the addressee.

Let us come back to the etho-analysis programme. It is devoted to the understanding of sense areas, or sense regions: the Husserlian idea of regions is kept, and the human world is seen as encompassing many *sense regions*, inside which human beings are bounded by some sharing, insofar as they address this sense to each other. To be more precise, the local sense, characteristic of the region, is received, but it is received as asking to be faithfully sent back, and this basic experience gives rise to an intersubjective and historical network. We may call such a network a tradition, and I will contend that for every region, we have a tradition, the tradition of the local sense, through which the considered sense goes on being addressed. The task of etho-analysis may be defined as the task of explicating how this sense is typically sent back, how we can faithfully add to the tradition of the considered sense. But this means illuminating the specificity of the considered sense.

In a forthcoming book, I have tried to follow this programme with politics, love and the 'subject' (see also Salanskis 2005). I claim that politics, for example, is not so much a part of reality, than a specific political sense, governing an area of the human world (the area of everything that pertains to politics, be it linguistic expression, practical behaviour or lived experience). I tried to describe how and when we account for something, some circumstance, some event, some phrase, some feeling, some acting as political, and I contend that we always do so when we receive the political sense as addressed. I concentrate therefore on making explicit what the political asks of us: it is only when we feel that what happens answers the asking of political sense that we declare the event or circumstance political. I also tried to carry out such a job for *love*, or for the *subject*: like in the case of politics, I do not see love, for example, as the logical name for every loving object, labelling a class of specific items of the world, love counts rather for me as a sense, love has to be understood in reference to the tradition of love.

In doing etho-analysis, we are not far from classical metaphysics. When classical metaphysicians asked 'What is love?', for example, it is not clear that they were really aiming at defining some part of reality. More probably, they wanted to achieve some conceptual clarification of the specificity of love, they wanted to capture in very general terms what I would call the 'salt' of love. It can be argued, at least, that the classical metaphysical essentialist questions of the form 'What is. .?' did not really ask for a conceptual key to a class of beings, but for a conceptual basis – called *essence* – characteristic of the local sense.

But, as I said before, the classical metaphysicians were doing it by means of a small set of metaphysical concepts designed for a discourse about the general being (the *One*, the *Other*, *Substance*, and so on). So the language of metaphysics, by itself, seems to imply that we refer to a particular class of beings. I think that, when we are performing etho-analytical investigation, we can avoid this unwanted effect by using other concepts, that were discussed by traditional metaphysics, but belong more to mathematics, such as continuum, infinity, space, the discrete, construction, and so on.

Why and how would such concepts help etho-analytical investigation? First of all, because each one, in its own way, introduces the point of view of the *infinite*. For etho-analysis, finiteness is not the right atmosphere for description of human affairs. For sure, finiteness has a privilege with respect to being, because nothing is ever at hand, available in our experience, but finite data. But etho-analysis describes local senses, which refer to the 'other than being' of asking and addressing, rather than to being. We try to understand what is asked, what is ordered, what the programme is or what the stakes are. And here the infinite becomes relevant, as Emmanuel Levinas demonstrates when describing labyrinth of moral responsibility for the other man: I am not only res sible for him, but also for the consequences on him of my first c responsibility as it is performed by me, and so on, (and this labyr also grows along intersecting lines, because, for example, I – let us me A – am responsible for the way my responsibility for the other m let us call him B – affects his responsibility for some third man – le call him C). In that case the infinite progression of responsibility for other is shown with the help of what we would call in contempo terms the concept of *construction* (building items on the basis of gi primitive items and given building rules).

I think that, more generally, we may use concepts like continu infinity, space, the discrete, and construction, to quote again the m familiar examples, in order to explain how regional senses shape world, our behaviour, the linguistic topics we address: only th concepts can describe not what the case is, but the element in wh everything has to be thought, following the governing sense.

I will only try to give another example, by referring to what I wr about action, in *Modèles et pensées de l'action* (2000). In this book, aim is to analyse action in a transcendental way: I try to bring to the fo the basic meaning of the features in the absence of which we would unable to call something an action. The conceptual *a priori* image action offered in the book is supposed to be the anticipated characteris tion of something which will be counted as reality, as in the Kantian ca for transcendental analysis: put in different words, the sense of actio which I claim to clarify is an objective, intentional sense, and not th more radical figure of a sense 'as addressed', that interests etho-analysi My more specific result is that we know three models of action, all them satisfying the 'general definition' of action previously introduce These models are strongly different from one another with respect to th continuum and the discrete. The first, the dynamical model, defines th result of an action as a stabilisation of some trajectory inside some attra tor of the dynamic, in such a way that it depends heavily on the contin uum (action is *a priori* seen as the reaching of some topological limit b some continuous process). The two other models, that of speech acts an of mathematical construction are rather *a priori* locating action in some discrete world: the world of linguistic statements and behind them th oriented graphs of the minimal societies they involve (for example: th person uttering the promise and the person to whom the promise i promised), or the world of manipulated distinctive symbols. This transcendental distinction between what I call *models* of action should be seen in a 'scientific' way as the distinction between senses of action which lead to a different theoretical knowledge of action. But, very likely, we have to recognise a *sense region* of action, to consider it in an etho-analytical way. Action also has the status of a sense giving rise to a tradition, of some deontological item. We receive the silent prescription to perform in such a way that we feel entitled to call *action* what results from its connection with the performing. From this point of view, the right question is no longer 'what is action?' but 'what ought action to be?', or more specifically 'how, under what kind of conditions, do we insert new actions in the tradition of action in a faithful way?', and the answer is to be found in the implicit shared decisions of our culture. From such a perspective, the distinction between continuous and discrete action seems very important. It could be that the continuous setting is originally connected with the perspective of objectivation of action, that it asks for integrating action in the dynamical realm of physics as construed by science; in such a way that we would have to dismiss all information coming from that model in our etho-analysis of action. And it could be that, for the same kind of reasons, but considered from the positive side, discreteness is part of what is planned in the programme of action, that discreteness is part of the task of action: to achieve an action is intended to mean, for us, breaking the continuum of being in order to exhibit discrete marks, and action therefore has to occur as a discrete transition with respect to these marks. I have not undertaken the necessary reflection and work which would allow me to make strong assertions of this kind, but I hope the example will be clear enough to illustrate how concepts of the kind I mentioned may interfere with etho-analysis, how they give the relevant language to express in what way local senses 'infinitise' the concerned context of human affairs. Action as a programme reveals, in any case, some infinite horizon of what can become possible, or of the available choices for action, but if we want to say something more specific about this horizon, our concepts are necessary.

It is not to say that etho-analysis will be exclusively focused on these concepts, or that it would be above all concerned with such mathematical framing of horizons. It could be that a great part, or even the major part of etho-analysis does not deal with such qualification problems, or that in many cases all we have to do is to describe the horizon as infinite. But here lies at least the possibility of another kind of continuation of the old partnership between mathematics and philosophy, and I mean nothing more than this.

Conclusion

So far, we have gone through the main issue of this essay, but some brief remarks may be added in conclusion. The question of the connection between mathematics and philosophy is as a matter of fact a burning political question. Many contemporary philosophical trends are ready to 'fight' for mathematics, in order to prove and establish that their use of it is at the same time the most authentic and the most intimate. Even for orientations which refuse the ancient partnership, like the Heideggerian one, such a rejection or denegation becomes a very important part of their message. Altogether, these attitudes, this war, show how deeply philosophy was determined by its Platonic origin. We could formulate it by saying that mathematics is the Jerusalem of the competing philosophical faiths.

Note

1 In the sense of 'transcendental aesthetics'

5

Continental genealogies. Mathematical confrontations in Albert Lautman and Gaston Bachelard

Charles Alunni

À Jacques Derrida, mon ami,
À toi dont l'influence ne se laissera jamais mesurer ni la perte réparer.
In memoriam.
Ulm, le 10 octobre 2004.

I) *Albert Lautman and the interrupted concern for movement.*
  a) *From the Heideggerian seam*
  b) *. to the involutive node of mathematical physics.*
II) *Gaston Bachelard and the 'spectre' of the* ETH.
  a) *Hermann Weyl and the 'phase transition' towards a physical geometry.*
  b) *Wolfgang Pauli or the* 'Schola quantorum.'
    α) *Pauli in 'principle.'*
    β) *Pauli demonstrated by the 'postulate of non-analysis.'*
    γ) *Pauli and his 'metaphysical particle.'*

I) *Albert Lautman and the interrupted concern for movement.*

In October 1984, Bruno Huisman stated with regards to Jean Cavaillès, 'Let us be honest, or at least realistic: today, one can be a professor of philosophy without ever having read a single line of Cavaillès. Often invoked, sometimes quoted, the oeuvre of Cavaillès is little attended for itself' (Huisman 1984).

As for Albert Lautman, it would seem that the situation is even more extreme. In 1994, the publisher Hermann, under the impetus of Bruno Huisman and Georges Canguilhem, collected almost the totality of the Jean Cavaillès papers in one volume (*Oeuvres complètes de* *philosophie des sciences* (Cavaillès 1994)). But, the *Essai sur l'unité des mathématiques et divers écrits* (Lautman 1977), published by the *Union générale d'Éditions* in 1977, had all but disappeared by the early 1980s and yet was never republished! This will remain one of the great indignities of French publishing,[1] for as Jean Petitot rightly affirms: 'Regarded as too speculative, in spite of his exceptional mathematical scholarship and his close connection with Hilbertian axiomatic structuralism, his mathematical philosophy has, until now, been devoid of any particular attention We would like to state clearly from the start, Albert Lautman represents, in our view, without exaggeration, one of the most inspired philosophers of this century' (Petitot 1987, 79–80).

One might make sense, and even and especially 'philosophical' and 'epochal' sense, of such a bleak situation, but I will not do so here. Let us simply recognise that there is a legitimate reputation of difficulty, and even of great austerity, attached to the Cavaillès-Lautman binomial – and it is precisely this that is of value and guarantees their importance, as much for the present as for the future. The constitution of a 'new mathematical philosophy' presupposes a laborious asceticism. The Bachelardian judgement brought to bear on Cavaillès applies (and includes) equally to (and from) Lautman: 'Thus, one can find, in the oeuvre of our friend, no preamble of slow introduction, no outline of facile generality, no elementary psychological preparation. *Reading Cavaillès is work*' (Bachelard 1950, 223). Their logical (and common) concern was always to seize the constructive gestures of effective mathematics in their autonomy. And yet a new philosophical doctrine can only be elaborated from a fresh perspective once one is 'installed *in* the field': 'mathematical knowledge is central to any understanding of what knowledge is' (Cavaillès 1946, 34). For Cavaillès, 'installation' in the field refers to doing the spade work on abstract set theory and the discovery of a 'new world' (the formula is Van der Waerden's), the world of the 'mathematics of the algebraists' who were centred in Göttingen. As for Lautman, the *humus* of his epistemological research would consist not only of the oeuvre of his friend Jacques Herbrand, but also that of Hilbert (his theory of the *body of classes* and spaces of the same name), the *Modern Algebra* of Van der Waerden, the *algebraic topology* developed by Alexandroff, Hopf and Pontrjagin, the *theory of quantum groups* of Hermann Weyl, the work of Élie Cartan (on *symmetrical spaces* and *external algebra*), and finally Nicolas Bourbaki's operation of refoundation (through Claude Chevalley and Henri Cartan).

There is however one exception to the oppressive silence of contemporaries on Lautman's philosophy (outside, of course, philosophers of science, or philosophical scientists, like J.-T. Desanti, Dominique Lambert, Jean Petitot or Gilles Châtelet), namely, Gilles Deleuze[2] in his *Difference and Repetition* from 1968.[3] The incentives to an attentive reading, which emerge from his citations, bear upon the notions of the 'dialectical Idea', of the 'differential' and the theory of the 'problem',[4] of the double aspect of Ideas-problems (transcendence and immanence) through a significant interweaving with the work of George Bouligand (epistemology of the 'problem' and of 'difference' in mathematics), of Louis Rougier (on the concepts of 'intensity' of 'dissymmetry' and again of difference), of Heyting (on distance or 'logical and mathematical difference' according to Griss), or of Paulette Destouches-Février (difference and negation in logic, mathematics and physics). These references would decisively influence the work of Gilles Châtelet, whose philosophical training took place under Deleuze.[5]

a) *From the Heideggerian seam . . .*

Nonetheless, there was at least one major occurrence in the context of contemporary French philosophy which should have served as an injunction to 'professional philosophers' to take a closer look: Lautman's reference to Martin Heidegger. We will try to reconstruct this decidedly economic allusion from his 1939 paper, *Nouvelles recherches sur la structure dialectique des mathématiques*, which appeared in 'Essais philosophiques' (the title of the collection came directly from its director, Cavaillès, [n° 804 of 'Actualités scientifiques et industrielles']). It is worth noting that, in this publication, Heidegger is surrounded on one side by André Weil on the generalisation of Abelian functions and on the other by Hecke on the algebraic theory of numbers!

A philosophical analysis of this paper suggests, in our opinion, a set of questions which are essential to its elucidation:

1) the internal economy of the 'Lautmanian *corpus*';
2) the relation of *concors-discors* to the theoretical operation of Cavaillès;
3) which is not without importance for a detailed analysis of Lautman's too famous Platonism, that which induces in its turn a *feedback* effect on the *totus* of the philosophy of 'philosophers'

There are three elements which we consider significant enough to require a relatively autonomous analytical focus.

1) Contrary to Catherine Chevalley, who, at the end of her article on 'Albert Lautman et le souci logique' (1987), devotes one relatively expeditious line to this Heideggerian appeal, it might be more appropriate to explore the internal requirements of Lautman's text which necessitated the reference to the Freiburg philosopher's *Vom Wesen des Grundes*. For Catherine Chevalley, 'In the 1939 text, he seeks . . in an *imperfectly identified Heidegger*, the idea of a transcendental relation of the domination of Ideas over mathematical theories, which would give an account of their "emanation" by "a kind of procession" We consider this the paradigm of a universal *cliché*!

Reestablishing the context of the 1939 text's theoretical mechanism, we must consider what the established elements of Lautman's philosophy of mathematical unity were, at the moment when the Heideggerian spectre, or one of its variations, appeared.

Lautman, following the Hilbertian axiomatic, associates himself from the outset with *a structural conception*. This conception:

> substitutes for the method of genetic definitions [peculiar for Lautman to the jurassic theories of the nineteenth century] that of axiomatic definitions, and far from wanting to rebuild the whole of mathematics from the starting point of logic [Lautman attacks this time the protocols of *Wiener Kreis*], introduces on the contrary, while passing from logic to arithmetic and from arithmetic to analysis, new variables and new axioms which each time broaden the field of consequences. (p. 26) [6]

Lautman then speaks of a synthesis of the real which 'mixes at once with intelligence and logical rigour, without ever merging completely with one or the other' (p. 26). It is the *structural* and *dynamic conception*,

> the structural conception and the dynamic conception of mathematics seem first of all to be opposed: one indeed tends to understand a mathematical theory as a completed whole, *independent of time*, the other on the contrary does not amend the temporal stages from its development; for the first, theories are like entities (*êtres*) that are qualitatively distinct from one another, while the second sees in each *an infinite power of expansion* outside of its limits and connection with the others, by which the unity of intelligence is affirmed. (p. 27)

Scientific philosophy must take theories, not isolated concepts, as its object. It is as structural (the autonomous and historical movement of theoretical development) that mathematics brings forth *dialectical ideas* which allow it 'to tell another more hidden history, mixed with constructions that are of interest to mathematicians and made for the philosopher' (p. 28).

2) And thus we arrive at the *structural schemas*: 'Partial results, connections halted midway, or attempts which resemble gropings are organised under a thematic unity, and through their movement reveal a connection which forms between certain abstract ideas, that we refer to as *dialectical*' (ibid). 'By Ideas, we do not mean models whose mathematical entities would be only copies, but in the *true* Platonic sense of the term, the *structural schemas* according to which effective theories are organised' (p. 204).[7] Now what is necessary is something to qualify this 'true' Platonism 'differently'; 'mathematical reality does not reside in the differences which would separate finished entities from unfinished entities, perfect entities from imperfect entities; it resides rather in the *possibility* of determining one from the other, that is in the mathematical theory where these connections are affirmed. One sees, then, that the reality which is being considered is not a reality of static entities or *objects of pure contemplation*. If qualitative distinctions exist in mathematics, they characterise the theories rather than the entities' (p. 138). Below we will see the similarity between this 'de-substantialisation', this *an-hypostatic* will, and Heidegger's position.

As for all dialectics, these 'structural schemas' establish specific connections between contrary notions: local/global, intrinsic/extrinsic, essence/existence, continuous/discontinuous, finite/infinite, algebra/analysis, etc. But, at this point, a profound technique of stratification intervenes which has already 'complicated' stirred up, 'clouded' and *undermined* a traditional Platonic dialectical view:

> One can define the nature of mathematical reality from four different points of view: the real is sometimes a mathematical fact and sometimes a mathematical entity; sometimes a theory and sometimes an Idea which dominates these theories. These four notions are hardly opposed, rather they naturally intermix with one another: facts consist of the discovery of new entities, these entities are organised into theories and *the movement* of these theories embodies the schema of the connections of certain Ideas. (p. 135)

Here, the 'structural schemas' constitute, alongside mathematical facts, entities and theories, a *fourth layer* of mathematical reality. Now, a central point, which already constitutes for us a notable *difference* from what we would generally define as Platonism, is that '*comprehension* of the Ideas of this Dialectic *necessarily extends into the genesis of effective mathematical theories*' (p. 203).

> While seeking to determine the nature of mathematical reality, we demonstrated that the theories of mathematics could be viewed as *material*

> ideally designed to give *substance* to a dialectical *ideal*. This dialectic seems to be principally constituted from pairs of opposites, and the ideas of this dialectic present themselves in each case as opposite notions between which relations must be established. *These relations can only be determined in the heart of the domain in which the dialectic takes its form.* (p. 253 [1971, 55–6])

Thus, the dialectic of the concept and the mathematics which give it substance find themselves in a relationship of 'internal exclusion' This relationship has been maintained according to a strange logic which has yet to be reconsidered, that is, 'mathematical theories develop through their own force, in a tight reciprocal interdependence, but without any reference to the Ideas that their movement brings together' (p. 134). It seems to me that we are moving even further away from a simple Platonism in this case and are turning rather towards an operation which Châtelet referred to, for his own purposes, as a 'force of ambiguity' (*force de l'ambiguïté*) and 'dialectical balances' (*balances dialectiques*).[8]

3) Naturally then one is led to a philosophy of 'problems'. Dialectical Ideas are purely problematic and this is why 'logical schemas (the ideas working the theories) *are not anterior to their realisation within a theory*; what we call the extra-mathematical intuition of the urgency of a logical problem lacks, in effect, something to dominate so that the idea of possible relations gives birth to the schema of true relations' (p. 142). A 'problem' only makes sense in a theory; thus, in his analysis of Pierre Boutroux's *L'idéal scientifique des mathématiciens* (1920), Lautman affirms that it is wrong to say that there is an '*independence* of mathematical entities in relation to the theories which define them.' While discussing 'the logical or algebraic clothing in which we seek to represent such an entity', Boutroux presupposes a kind of *neutrality of formalism with regards to meaning*. However, modern algebra shows how the properties of mathematical entities can vary with the domain under which they are considered. The introduction of the axiomatic method in mathematics, on the other hand, makes it absolutely impossible to isolate elementary 'mathematical facts' which would be *like building blocks*. This is sufficient in our opinion, for grasping Lautman's extremely atypical, not to say *atopical*, dialectic of Ideas. We will now consider how this difference tends to offset, in a 'Heideggerian sense', the very concept of metaphysical 'truth'

4) The Heideggerian seam consequently appears as a question of the *passage* from essence to existence. This 'passage' 'the extension of an analysis of essence generating notions relative to the existent,'

(p. 206) – and thus, the transformation of comprehending a *meaning* into *a generation of objects* – takes up, once again, the Heideggerian ontological *difference* between Being and being. For obvious reasons, I will restrict myself to a few initial tracks of this seam which merit excavation. Let us restrict ourselves to the Lautmanian 'concern':

> The order implied by the notion of genesis [the intimate bond between the transcendence of Ideas and the immanence of the logical structure of the solution to a problem within mathematics] is not the order of the logical reconstruction of mathematics, in the sense that the initial axioms of a theory give rise to all the propositions of the theory, because the dialectic is not a part of mathematics, and its notions are without relationship to the primitive notions of a theory. The anteriority of the Dialectic [is] that of the 'concern' or the 'question' in relation to the answer. It happens to be an 'ontological' anteriority to take up the Heideggerian expression again, exactly comparable to that of 'intention' in relation to 'design' (p. 210).

A fact never raised by critics, Lautman already introduced this 'concern' (the Heideggerian *Sorge*) in the conclusion of his *Thesis* published two years earlier in 1937:

> The only *a priori* element that we conceive is given by the experience of this urgency of problems, anterior to the discovery of their solutions
> We understand this *a priori* in a purely relative sense, and in relation to mathematics; it is uniquely *the possibility of experiencing the concern* for a mode of connection between two ideas and of describing this concern phenomenologically, independently of whether the sought-after connection can be operable or not. Some of these logical 'concerns' are found in the history of philosophy, such as for example the concern for the connections between the same and the other, the whole and the part, the continuous and the discontinuous, *essence and existence*.

Heidegger is not named in this passage and yet somehow this passage is already about him, through something which is as though telepathically, that is anticipatively, *promised to him in the name of mathematics*. And this looking forward is conjugated by an unprecedented and radical mode of questioning of Western metaphysics:

> *but the mathematical theories will* conversely *be able to give birth to the idea of new problems which would not have been abstractly formulated beforehand*. Mathematical philosophy, such as we conceive it, thus does not consist so much in *finding a logical problem of traditional metaphysics* within a mathematical theory, than in apprehending overall the structure of this theory in order to extract the logical problem which is at the same time defined and resolved by the very existence of this theory. (pp. 142–143)[9]

Let us return one last time to the text of '39, where Heidegger is discussed *ad nominem* and 'identified' perfectly well.[10] Not only does Lautman reinforce the operation of the Heideggerian 'ontological difference' (of Being and being) by affirming that dialectical Ideas are to mathematical theories what Being and the sense of Being, is to being and the existence of being, but immediately afterwards, and this is a fact which is passed over in complete silence in the commentary, he mobilises the Heideggerian category of the *truth* unfolded as the 'unveiling of Being' ('*devoilement de l'être*'), as this *a-letheia*, that the Freiburg philosopher also translates as *Entbergung* ('unconcealing') (*la 'découverture'*).

> It then happens, and this is for us the fundamental point, that this revealing of the ontological truth of Being cannot be done without the concrete aspects of ontic *existence* taking shape at the same time: 'Characteristic among other degrees is, for example, the project which by outlining the constitution of the being of the existant, marks out at the same time a determined field (Nature, History) as the domain where it will be possible for scientific knowledge to constitute objects' (Heidegger 1969) [*Trans*. translation modified]. One thus sees, in this text, the same activity duplicate itself, or rather acting on two different planes: the constitution of the Being of the existing thing, on the ontological plane, is inseparable from the determination, on the ontic plane, of the existence in fact of a domain where life and matter are the object of a scientific knowledge. The concern with knowing what constitutes the essence of certain concepts is perhaps not directed originally towards the realisation of these concepts, but, as it happens, the conceptual analysis necessarily tends to project the concrete notions in which it is realised or historialised as ahead of the concept (p. 206)

Now, something absolutely remarkable, a few lines later Lautman goes so far as to grasp certain weaknesses in what is referred to as the 'early Heidegger' (the Heidegger of 'existential' analysis) radicalising, *ante litteram*, the very Heideggerian auto-critique of Heidegger: 'The distinction of essence and existence, and especially the consequence of an analysis of essence as generated by notions relative to the existing thing, *are sometimes masked* in the philosophy of Heidegger by the importance of the existential considerations, relative to Being-in-the-world, such as they appear in *Being and Time*' (pp. 206–207). In this case, it would hardly be inappropriate to speak of a remarkably anticipative '*perspicuitas*'

It is this consciousness, certainly sharpened by his own interests, which enabled him to obtain a perfect grasp of the stakes and key concepts in Heidegger. *The Essence of Reasons* (Heidegger 1969) (to which Lautman returns) becomes an epochal principle establishing itself in the course of the successive developments through which new fields of intelligibility have emerged abruptly and spontaneously since the beginning of history: 'Epoch does not mean here a span of time in occurrence, but rather the fundamental characteristic of sending: the actual holding-back of itself in favour of the discernibility of the gift' (Heidegger 1972, 9). The sudden appearance of an epochal principle is nothing other than the arrival of the concrete economy of a metaphysical age. An epochal principle appears in the manner of an address, of a requisition. It requires a community of men searching for a finite way of thinking and acting.

The consequences of this inscription in Heidegger's wake are fundamental for an accurate reconstruction of his thought and philosophy in their most *current* aspects:

1) If, for Lautman, revealing the philosophical meaning of mathematics consists in showing its 'attachment to *a metaphysics* (or Dialectic) of which it is necessarily the consequence' (Letter to the mathematician Fréchet, February 1, 1939), it should be clear by now that we are not speaking of just any 'metaphysics', and certainly not the 'traditional' metaphysics which characterises the well-worn and blind of the 'Vienna Circle' (Cf. the Thesis of '37). Rather he will situate himself on the side of a radical questioning of this 'traditional metaphysics' which was initiated by Heidegger through what he called 'the going beyond' and *Destruktion*;[11]
2) Such an alignment with the Heideggerian interpretation of *A-letheia* as 'revealing of Being' should radically disqualify his all-too famous Platonism – or at least in the sense that it is traditionally, and offhandedly, attributed to him. Thus, Cavaillès himself, in a certain way, provided a poor example: 'In the properly mathematical discussions which took place between advocates of the Vienna School and the Hilbert School, the question was posed as to whether or not there was an ideal realm of objects to which mathematics could refer – this was referred to as a Platonism. I am not totally convinced by the word, *but what's in a word*' (Cavaillès 1939, 603). What offhandedness! Lautman takes up, in Heidegger's theory of truth, the rejection of any conception of truth as *omoiosis* or as *adaequatio rei et intellectu*, the modern and Cartesian version of Platonic Idealism. The question is no longer that of the 'adequacy of the Idea to the

Real,' a point to which Lautman continually returned (Cf. amongst others, his criticism of Boutroux).

3) His engagement with another concept of truth through a 'surpassing' of metaphysics is tied to a *deconstruction of substantialist schemas*. For Heidegger, the metaphysical quest for *first* foundations is, by definition, a search for something which is *below* phenomena, for a *hypokeimenon*, that is a 'substrate.' Lautman, on the other hand, is rigorously engaged in a process of *de-substantialisation* in mathematical philosophy (pp. 95–96), or, more precisely, a displacement and a *complicatio* of the metaphysically foundational relation between *form* and *matter*. I would like to consider a few of the many possible examples.

Herbrand's theorem of 'fields' presented itself as a pure case of the unity between a set of operations formally defined by a system of axioms and the existence of a domain where these operations are realisable. In 1937, Lautman commented on Herbrand's theorem:

> It seems that a certain *restriction* still adheres to this logical schema: the genesis indeed only takes place *in one way*, from the operations to the field. However, if a rigorous appropriation can be established between the field and the definable operations on it, one can attempt to determine the operations starting from the domain as well as the domain starting from the operations Our intention being to show that the internal achievement of an entity is affirmed in its creative power (*pouvoir*), this conception should perhaps logically imply two reciprocal aspects: *the essence of a form being realised within a matter that it would create, the essence of a matter giving birth to the forms that its structure designs....* In fact, the schema of generation that we will describe within more complicated theories, gives up the *too simplistic idea* of concrete domains and abstract operations which would possess in themselves a nature of matter or a nature of form; this conception would indeed tend to stabilize the mathematical entities in certain immutable roles and would be unaware of the fact that the abstract entities which are born from the structure of a more concrete domain can in their turn be used as a base domain for the genesis of other entities. (pp. 95–96)

Here we have the result of a certain philosophical axiomatic and its auto-application through 'reflection' (what Lautman affirms of mathematical logic, of the theorems of existence in the theory of algebraic functions, or of the theory of the representation of groups as different transcendental domains of investigation, is 'turned back' on the philosophical operation itself through symmetry, which, in this initial phase, is located in an operative position–and *conversely*): *it is an extremely powerful operator of dialectical interaction which, starting from the domain of mathematical physics, will be induced from the effects in the field and philosophical activity.*

Destruktion of metaphysics, deconstruction of the Truth conceived as *omoiosis*, Heideggerian desubstantialisation: Heidegger would come to the same conclusions, during the same epoch, in Lautman's preferred domain of mathematical physics. It is the key question for that which might be entitled *Heidegger and Quantum Physics*, linked to the 'Heisenbergian' period of his thought.

b) . . . *to the involutive node of mathematical physics.*

Here, in one of its most impressive manifestations the Lautmanian approach seems quasi 'prophetic' The domain of its greatest philosophical and intuitive inspiration, the dialectical coupling on which all the promises of his mathematical philosophy are polarised seems without question to touch upon the problematic of *symmetry and dissymmetry in mathematics and physics*.[12]

This text is perhaps emblematic of the gap in the Cavaillès-Lautman binomial. If the impact of mathematical physics on his interests had been apparent since his thesis of 1937 (in particular by his attentive study of the papers of Élie Cartan on the 'generalisation of the notion of "space"', 'Absolute parallelism and the unitary theory', or of the analysis of Herman Weyl on 'Riemannian spaces' the work of Eddington, *Espace, temps, gravitation*, or *La structure des nouvelles théories physiques* of Gustave Juvet), it is at the end of his life, before falling in combat, that, thanks to his exceptional mathematical background, he directs all his epistemological activity towards the questions of physics. His very personal and extremely original contribution touches upon questions of envelopment inherent in notions of symmetry and dissymmetry. He sets up his thematic with an analysis of the relationship between the pioneering work of Louis Pasteur and cellular dissymmetry by 'enantiomorphy' 'at the base of all *the structural theories* of modern stereochemistry' (p. 240 [1971, 45]). Then he turns to Pierre Curie's foundational work in the field of physics:

> the mixture of symmetry and dissymmetry becomes for him a necessary condition of physical phenomena in general To each physical phenomenon is linked the idea of a saturation of symmetry, of a maximal symmetry compatible with the existence of the phenomenon and

> characterising it. A phenomenon can only exist in an environment possessing its characteristic symmetry or a lesser symmetry. Thus if the absence of a certain element of symmetry is called an element of dissymmetry, it becomes possible to understand why Pierre Curie wrote: 'Certain elements of symmetry can coexist in certain phenomena, but they are not necessary. What is necessary is that certain elements of symmetry do not exist. It is the dissymmetry that creates the phenomenon. (Curie 1908: 126)[13]

Lautman connects this idea of 'symmetry limits' to Plato's *Timaeus*, and in particular to his theory of the *Chora* as a receptacle which is a 'place' However, and this is the extraordinary speculative force of Lautman, it is not, in this instance, a mere ceremonial reference:

> This reference to Plato enables one to understand that the materials which form the Universe are not so much the atoms and molecules of physical theory as these major pairs of ideal opposites such as the Same and the Other, Symmetry and Dissymmetry, related to one another by the laws of a harmonious mixture   Plato suggests more. The properties of place and substance according to him are not those of sense perception; they are the geometric and physical transposition of a dialectical theory. In the same way, perhaps, the distinction between left and right as observed in the world of sense, is nothing but the transposition to the plane of experience of a dissymmetric symmetry, which is equally a constituent of the abstract reality of mathematics' (p. 241 [46]).

Here, in my opinion, at the peak of his conviction, it is as if Lautman expressed the true brilliance of his thought by doing an about face in the middle of the realm, or the thematic and operational field, of mathematical physics. He, thus, clarifies his thought like a sketch of a thought diagram where the virtual power (*puissance*) of his schemas would come to be visualised in the mind's-eye of the mathematician *and* philosopher. Thought almost allows its own 'visual' perception of itself.[14] It is at that precise moment when he concretely invests in the physico-mathematical domain that the absolute power of his dialectical operation is established. Philosophy reveals its habitation of/in Science, thus revealing their double reciprocal power (*pouvoir*) of suggestion, for here, *science thinks* (and thinks about *itself*), as if it were inhabited by its philosophical spectre. It is in such a moment of suspense in this 'between-two Worlds' that the artificial objections brought to bear on the so-called 'arbitrariness' of its Dialectic themselves come to annul themselves. It is as if the dialectical operation was discovered in one blow, made itself suddenly readable to the eye of the theory by its very 'rise towards the absolute', thus procuring for itself a kind of 'universal surface of coincidence' (*surface universelle de recouvrement*).

From this point on, the development of the Lautmanian argumentation only sharpens this impression. In a final example, Lautman insists by bringing out the mathematical focus of the entire operation: *the operation of involution* as 'universal' operator and core of any dual structure (or principle of 'duality') – this question is approached in an identical way by Herman Weyl for the transformation called *automorphism* for a *zero-dimensional space* (that is, reduced to the structure of the point).

> We must stop a moment to lay stress upon the fashion in which the tinction of left and right in the real world can symbolize the non-commutativity of certain abstract operations of algebra. The fundamental property of symmetry with respect to a plane is applied *once* it gives *a figure distinct in orientation from the original*, and that applied a *second time*, gives *the original figure* again. It is for this reason that the symmetry is called an *involution*. Let us next consider an algebraic operation defined on two quantities X and Y and which we shall write (X,Y); the parenthesis may denote the ordinary product or any other operation defined for the two variables. It is a non-commutative operation if (X,Y) ≠ (Y,X) and the most fecund type of non-commutativity in mathematics is that in which (X,Y) = – (Y,X). The parenthesis (X,Y) is dissymmetric X and Y, but it may be easily verified that it defines an involution, as does ordinary symmetry. The expressions (X,Y) and (Y,X) are called *antisy' metric*, and this word translates well *the mixture* of symmetry and dissymmetry which is thus seen to be deeply embedded in the heart of modern algebra. All of the theory of continuous Lie groups is based on the non-commutativity of the product of two infinitesimal operations of the group. This theory, which is closely associated with the theory of Pfaffian forms, expressions with antisymmetric multiplication, permitted Cartan *to discover a profound anal ogy* between the generalized Riemann spaces which appear in physico-geometric theories of relativity and the space of Lie grou; (p. 48; additional italics)

Professional physicists and mathematicians would be wise to take such text as a model for its 'structural', penetrating and definitional clarity.

Passing to the particle theory of de Brogliean wave mechanics, he postulates that 'antisymmetry seems to play a much more fundamental role in nature than symmetry [it] plays a fundamental role in the study of chemical bonds' (p. 245 [49]). After a precise analysis of the antisymmetry of spin, the folding back of the distinction of wave function: to its mathematical foundation 'of an internal dissymmetry of the group of permutations', he introduces the 'rise towards the absolute', that is towards the 'principle of duality':

> Transposed into more abstract language, this situation is equivalent to the possibility of *distinguishing within a single entity* two distinct entities X and X′, which will be said to be *in a dual* relation, if an inverse orientation or ordering can be found for each of them such that they are inverse to one another, and if, in addition, we can find an involution relating them, that is if X is to X′ as X′ is to X, or if (X′)′ = X. (p. 247 [50])

Note on the notion of duality

Duality (which is a mathematico-physical property), has analogous terms such as those of 'conjugation' or 'reciprocity' (which returns overall to the central idea of 'symmetry').

It is a notion which figures more so in the demonstration of *general constructions* ('duality' of linear spaces, 'adjunct' functors in Category Theory), rather than of particular theorems (Poincare's 'duality' or Artin's laws of 'reciprocity').

All of the varied uses of the term contain the idea of the bilateral symmetry of an object, a construction or a mathematical theory.

In fact, the idea of duality has an extremely *general* character. Among the correlations closest to mathematical duality, one will note, for example:

the principle of classification by 'dichotomy' (in philosophy, the 'dualism' opposed to 'monism' is as old as philosophy itself; it is, for example, what is at stake in the operation of 'deconstruction' at work in the philosophy of Derrida);
the 'oppositions' of *contemporary structuralism*;
the 'complementarity' in quantum physics, modelled by *Heisenberg's Inequalities* – certain couples separated from classical magnitude are found to function here in *combined variables* ('complementarities'). The *Inequalities* then estimate the production of a 'dispersion' around the average value of these two variables, *which no longer commute*.

The 'prototype' of the mathematical duality is without question that of linear spaces. One could even go so far as to say that through it is constituted the ontological status of linearity, of which the physical consequences are 'phenomenal' Within this framework, and as an elementary example, the rise or fall in the indices (as well as contraction) are read as operations of the linear dualisation of tensor operations.[15]

Lastly, glancing back over the fundamental stages in the constitution of 'duality' as a category (Boolean algebra, Poncelet's projective geometry), Lautman concludes with an account of the most recent research undertaken in abstract algebra (Birkhoff, von Neuman, Glivenko and Ore), through Lattice theory (the importance of which is now known for category theory), introducing the notion of the 'dual' (which he calls 'duel') and of anti-isomorphism:

> The general theory of lattices thus is based on the possibility of ordering the same set in two mutually inverse ways. For us it is a result *of fundamental philosophical importance* to see an *internal* duality of two *distinguishable* antisymmetric entities embedded in *one* entity, and to see the duality become the generating principle behind an immense harvest of mathematical results. (p. 250 [53; additional italics])

It is impossible to avoid thinking of the shadow cast by *Récoltes et semailles* (*Harvests and Sowing*) of one Alexandre Grothendieck.[16]

To conclude, by opening, the whole cardinal importance of the questions and techniques investigated by Lautman has since been revealed: the degree to which the 'non-commutative' has lived up to its promise (one has only to think of Alain Conne's work on *non-commutative geometry*); the degree to which the concept of 'duality' and 'dual' structure has become the very heart of the *most current* physico-mathematical problems. A concept that Lautman had tackled, as early as 1937 in his thesis by cultivating the 'theorems of duality' defined as 'structural schemas' within the already well-known framework of 'intrinsic properties and induced properties' The importance of 'theorems of duality' in the characterisation of *Hopf Algebra* has been shown, for example, in the case of the development of *quantum groups*. Might I add that the 'ontological difference' itself is defined by Heidegger as 'Duality' (*Zwiespalt*)!

And yet, the ideas of Lautman and Weyl on this point would be almost entirely ignored until 1956.

With regards to the 'local-global' antagonism, he had called upon the work of Élie Cartan as well, whose value had elicited very little interest before 1935, and whose central place in mathematics has since been universally recognised.

And that is to say nothing of what already constituted his interest in what later became the fascinating *category theory*.

It was a highly-respected mathematician who affirmed: 'We now see that Lautman had foreseen this extraordinary development of mathematics, which destiny did not allow him to witness; he filled it with enthusiasm, as much by the unequalled harvest of new theories and solutions to ancient problems, as by the eminently aesthetic character which now offers (to those who, like Lautman, seek to understand them) the central parts of this immense edifice' (Jean Dieudonné, 'Foreword' to *Essai* (Lautman 1977)).

Lastly, I would add that what sets Lautman apart, in my view, from Jean Cavaillès (inversely proportional to their notoriety in the philosophical environment, I might add. But, of course, this is rather a question of personal sensitivity to research approaches, and thus a question of style) is Lautman's creation of a philosophy open to the promise of what was then the future of philosophy, of mathematics, of physics, and of mathematical-physics, that is, nothing less than *our present*.

II) *Gaston Bachelard and the 'spectre' of the* ETH.

I will pose from the start the syntagm 'School of the ETH'[17] as a spectral marker of a constellation of thoughts that are singular, and thus unique, but nevertheless articulated (and not isolated in a solipsism). 'Spectral' is to be taken here first as a 'discrete' operator of declension of the philosophical singularities expressed in each referred corpus, like zones of 'interference' of domains of explanation and pronominalisation; the model of this 'spectral operator' in this 'magical' connection takes after the *Janus* of mathematico-physics. It is the *purely mathematical* 'spectral' characteristic of the abstract space of Hilbert (developed by him in 1910), which will later allow Heisenberg to induce in a brilliant way what constitutes in reality the possible legibility of 'forms' appearing on the frequency spectrum of a body (on the basis of the principle of combination or 'law of composition' of Ritz-Rydberg).[18] Henceforth, any physical body finds its signature in its quantum spectrum.[19] Note in passing that the 'practical' and paradigmatic example of 'mixed' mathematics in Albert Lautman is nothing other than this same Hilbert space: *continuous* for the topology of its elements; *discontinuous* for its structural decomposition.

But 'spectral' will also relate us back to these 'phantomnal' presences which haunt the great work of Gaston Bachelard, in the form of this epochal constellation that our friend Mario Castellana qualifies as 'Italo-Francophone "neo-rationalism"'.[20] As the place where an immense epistemological tradition was instituted, our present should start by taking its measure. In an historically exemplary way, one of the highly symbolic referents of this constellation was represented by the group theoretically opposed to the 'Wiener Kreis' at the time of the *Descartes Congress* in 1937 (Bachelard, 'silent guest' of the *Congress*, Federigo Enriques the Italian, Ferdinand Gonseth the Swiss, Jean Cavaillès and Albert Lautman the two French. One could add to this the 'anonymous protagonists' such as Paulette Destouches-Février, Jean-Louis Destouches, André Lalande, etc.).

It just goes to show that the mobilisation by Bachelard of these different mathematical, physical and philosophical systems is no more occasional than vague. Because it does not (as is the rule with a number of his contemporary philosophers and of ours), happen to portray them caught up in pure auto-justification or illustrative references to a philosophy already closed-in on its own presuppositions. What is at stake is rather to inhabit them in an *active* and 'open' way, to accompany them in the asceticism of their specific techniques, to make them foster *in situ* and *in actu* this speculative power (*pouvoir*), always 'available', committed in their gestures of thought: 'no spectrum is more extended than the spectrum that helps classify the philosophemes of the physical sciences. It is moreover well understood that all the parts of a science are not at the same point of philosophical maturity. It is thus always by way of experience and of well defined problems that it is necessary to determine the philosophical values of science' (Bachelard 1949, 7).

It is thus insofar as it is located in *topoi* and on perfectly identifiable textual nodes that we must try to retrace the spectral presence of the 'School of the *ETH*' (the nature of its theoretical, at the same time scientific and philosophical – not to say 'metaphysical' – stakes, its induced solidarity, its produced potentialities). The limited choice of certain 'loci' of the Bachelardian texts must be met by a choice of signatures limited to the representatives of this School whose neighbourhood of 'proximity' was not limited to the sole company of the same *elite Schule*.

We will turn our attention exclusively towards certain names of the *ETH*, the heart of which served to recharge Bachelard on regular occasions, diffusing and extending their reflexive work in his own way: Herman Weyl, Wolfgang Pauli and, *last but not least*, Gustave Juvet. It would obviously be necessary to thematise the relations of complicity of the Bachelardian texts with other eminent representatives of this School: Albert Einstein or Ferdinand Gonseth. The first has already been explored elsewhere.[21]

a) *Hermann Weyl and the 'phase transition' towards a physical geometry.*

Hermann Weyl appears from the start as an 'initiatory' presence in the Bachelardian oeuvre. He is called upon as much in his *Essai sur la connaissance approchée* which constituted the *principal Thesis* for his *Doctorate* presented in front of the *Faculty of Arts at the University of Paris*, May 23, 1927, as in his *complementary Thesis* presented 28 March of the same year, under the title *Étude sur l'évolution d'un problème de physique. La propagation thermique dans les solides*.[22]

Weyl opens and closes the *Essai* on two important occasions, first on page 82, opening the chapter on 'The formulae of dimension'. Let us begin then by opening the contextual folds of this inaugural reference. What is Bachelard's fundamental idea in this chapter? It is the philosophical interrogation of the 'new metrology' vis a vis absolute (of unity)/arbitrary (of measurement) *duality*:

> By absolute measurement, one should not understand a measurement carried out with a particular precision, nor by absolute unity a unit of perfect construction; in other words, by making use of the words absolute measurement or absolute unity, one doesn't mean that the measurements made or the units of measurement are absolutely perfect, but only that these measurements, in place of being established by a simple comparison of the quantity to be measured with a quantity of the same kind, are related to fundamental units of which the notion is admitted as an axiom.
>
> Thus *metrology* is itself also *preceded* by a true axiomatic since it has as its base the elements of a perfect and arbitrarily posed purity. These elements, as axioms, will only be constrained to *form a system that is coherent*, irreducible and to be independent. Lastly, just as diverse geometries derive from different sets of postulates, in the same way different fundamental systems are offered to support all of the measurements of physics. (Bachelard 1927, 85)

The scene is set and would already permit him alone to read just beneath the surface a whole avenue of *connected* problems *to come*, as well as other bundles of quotations. How does the first mobilisation of Weyl's *Espace, temps, matière*, in its Juvet-Leroy translation of 1922, intervene here?

> Thus [in connection with the arbitrary, *masked* out of habit] it is believed that the arbitrary is eliminated from the definition of the unity of volume as soon as this unit is tied to the unity of length by choosing the cube as standard volume? The *memory* is obviously relieved since it follows the direction of traditional elementary *geometry* But there are points of

> view which would perhaps become clearer with *another choice*. Thus the sphere has in certain regards undeniable *rational advantages*. It is the volume of minimum definition, *its symmetry* is of an inexhaustible richness. . In the same way still in a physically anisotropic space, it may be of interest to dilate or contract certain co-ordinates following more or less complicated functions. It is an artifice often employed in new generalised spaces. One can always have the units joined together in complexes to cut the numerical coefficients off from various geometrical measurements – or at least to reduce all the coefficients to unity preceded by the + sign or the – sign. In a quadratic form, only the numbers of the + signs and – signs remain *invariant characteristics* [Cf. Weyl 1922, pp. 24–7]. (Bachelard 1927, 82)

This citation refers us to the first chapter of *Espace, temps, matière*, 'Euclidean Space. Its mathematical form and its role in physics' §4 'Foundations of metrical geometry' Weyl confronts there the *invariant* conditions of orthogonal linear transformations into Cartesian co-ordinates. In technical terms, it is the crossing point from a *theory of invariance* for linear transformations with *conditions of orthogonality*, to a theory of 'generalised invariance' known as *tensor calculus*, the mathematical body of General Relativity. Weyl concludes this §4 with his programme: 'We shall here develop the Theory of Invariance along lines which will enable us to express in a convenient mathematical form, not only geometrical laws, but also all physical laws.' Bachelard will resume and very precisely follow the thread of this Weylian program two years later, in 1929, in the *Valeur inductive de la relativité*.[23] The conclusion *pro domo* that the author of the *Essai* draws from this chapter, after this first passage through Weyl, is the following:[24] 'It seems that by going *from measurements to ideas*, knowledge is quickly lost in logicism [not likely to give rise to experience]. It is by another route, by returning *from the mind towards things*, that knowledge will still be able to be mobilised and be given *the flexibility sufficient to touch the real*' (Bachelard 1927, 92). The point of contact is already largely a shared trajectory.

It is page 282 of the concluding chapter of the *Essai*, entitled 'Correction and Reality', that will deploy the whole power of connection and fibration of the Weylian and Bachelardian approaches. This is the reason why we are going to reconstruct more patiently what is implied by the context.

> A geometrisation of the material cannot be a starting point, *it is a schema*, it is a goal, in short a late discovery. In fact, in contemporary science, extension conceived *a priori* as a uniform and general quality has taken the

> place of an extension loaded with character and *seized by its differential side*. And it is now the differential element that determines the 'explication'
>
> It is perhaps the most striking feature of the new physics. Riemann's idea to define the mathematical function by its infinitesimal variations has just penetrated physics itself. And by a singular reversal of principles which will involve a real upheaval of epistemology, it is the integral law which, in principle, becomes the simple consequence of the differential relation. The '*laws of infinitely near action* [shall be regarded] as the true expression of the uniformity of action in nature.' (Weyl 1922, 55 [1952: 66]. The quotation is drawn from Ch. I, § 9, 'Stationary Electromagnetic Field')

And the quotation continues:

> 'The principle of gaining knowledge of the external world from the behaviour of its infinitesimal parts is the mainspring of the theory of knowledge in infinitesimal physics as in Riemann's geometry.' (Weyl 1922, 79 [1952: 65]. The quotation is drawn from Ch. II, 'The Metric Continuum', §11 'Riemann's Geometry')
>
> In the first text, Weyl compares 'Coulomb's law as the law of '*action at a distance*,' [which] expresses that the field at one point depends on the charges located in all the other points, near or far, in space,' to the *laws of infinitely near action*, 'which are far simpler ..., as a knowledge of the values of a function in an arbitrarily small region surrounding a point is sufficient to determine the differential quotient of the function at the point. the values of **r** [density of charge] and **e** [the vector field] at a point and in its immediate neighbourhood are brought into connection with one another by [the equations] (51) [*rot* **e** = 0; *div* **e** = r]; ... we look upon [equation] (49) [$\mathbf{e} = -\int \rho \mathbf{r}/4\pi r^3 dV$] merely as a mathematical result following logically from [equations (51)]. In the light of the laws expressed by (51), which have such a simple intuitive significance we believe that we *understand* the source of Coulomb's Law. In doing this we do indeed bow to *an epistemological constraint*.' [Weyl 1952, 66, *Trans.* translation modified]

The research of this 'epistemological constraint' is obviously Bachelard's purpose; and it is in Riemann that he will seek its original axiomatic. It is also in the Riemann-Weyl relation that Bachelard exposes to some extent the 'figure' of this geometrical *revolution* in the physics actually imposed by Weyl, and particularly within the framework of General Relativity. It is the geometrico-physics interface, ensured a promising future, which is thus exposed, with its fundamental *displacement* of the *a priori - a posteriori* relation:

> From the simple differential laws *rot* **e** and *div* **e** = r which express that the rotation of the electric field **e** is null and that its divergence at any point is equal to the electrical density at this point, one *deduces* Coulomb's law according to which electrified bodies are attracted by a force inversely proportional to the square of the distance. The general law thus ceases to be *a priori*, in simple agreement with a system of categories, related [the law] to logical principles, very close to intellectual intuition. It is, in every sense of the word, the consequence of a fact, rather than of an extraordinary number of facts. But it does not summarise them, because it is burdened with constants of integration. (pp. 282–283)

There are no less than seven Riemannian occurrences in the *Essai*. This is not by chance, and hardly accidental, but it rather results from a lucid awareness of the connection with Weyl. Let us partially rebuild the epistemological spectrum of these Riemannian references:

> The definition of the [Riemannian] function by simple correspondence still has a very different flexibility. 'This definition', says Riemann, 'does not stipulate any law between values isolated from the function, because when this function is mapped over a determined interval, the mode of its extension outside of this interval remains completely arbitrary.' Thus the perfect knowledge of an *analytic entity* in a determined domain no longer implies the least knowledge outside of this field. The *entity*, in Analysis, thus seems to us the result of a construction which, in principle, if not always in fact, is *a free construction*.

> In analysis as in geometry, the restrictive conditions which fix the rules of construction do not ruin the hypothetical character of the defined analytic element. Thus, in a curious analogy, to define a transcendent, one finds the same types of conditional relations as the axioms of geometry. 'As founding principle in the study of a transcendent,' writes Riemann, 'it is, first of all, necessary to establish a system of conditions, independent amongst themselves, sufficient to determine this function.' The transcendent thus only establishes between its elements the sole connections which are specified by the system of conditions. It does not have reality outside this system which must be, like a system of postulates, complete and fundamental (pp. 184–185).

This is the pedestal from which an entire problematic concerning the categories 'reality', the 'possible' and the 'virtual' open up, categories at work in the entire Bachelardian oeuvre, and particularly thematised in the *Valeur inductive de la relativité* of 1929. And it is here that Bachelard points to a 'functional constructivism' in Riemann, which could be extended to an 'ontology necessarily projected' in the 'metaphorical existence' attributed to the mathematical entity: a 'constructive ontology [which] is never at its end since it corresponds to an action rather than a lucky find, relating to a 'second order reality' (Bachelard 1929, 186). But what also interests Bachelard is a point which Albert Lautman will later make in his own *Thesis* of 1937, namely the fact that Riemann will *have been the one who crossed over from the genetic thought of laws to that of structures*: 'mathematics has incontestably a *structural conception*'; 'Certainly, in Riemann's sense, *which is the major sense*, the [mathematical] function *only* translates the idea of correspondence' (p. 201).

Let us return to Weyl, 'geometer of matter', and to the importance of Riemann's laws *of infinitely near action* which Bachelard brings forth. A form of spectral community will be seen to emerge as a result of certain epistemological consequences. The Riemann-Weyl bond is due to the fact that, from the point of view of the theory of knowledge, Riemann understood that imagining the infinitely small contained much more essential natural information, than imagining the infinitely big. He thinks of a kind of profound unity, noncontingent upon the mathematical models of the infinitely small and the physical laws according to which this infinitely small expresses itself and appears in the nature of phenomena. What Weyl found to be at the base of Riemann's new differential geometry were the same theoretical principles which animated the new physics of infinitely near action. From these followed the possibility of establishing a parallelism between the geometry of Riemann and the physics of Maxwell, which Bachelard, in turn, marked as the turning point of 'new physics' In an important article (contemporaneous with the first edition of *RZM*, 1918) where Weyl undertakes the project of elaborating a 'pure infinitesimal geometry' (*eine Infinitesimalgeometrie*) and where, in doing so, he pursues the construction of a generalisable purely geometrical theory of physical phenomena, he affirms:

> the *general theory of relativity* admits, in accordance with the spirit of modern physics, of infinitely near action, only what has validity in the infinitely small [that is locally], and with regard to the metric of the Universe (*Weltmetrik*) it calls upon the general concept of metrical determination founded on a quadratic *differential form*, proposed by Riemann in his *Habilitationsvortrag* of 1854. But the truly important element of this innovation is the view according to which the metric is not a property of the Universe in itself (*an sich*); but rather, as a form of phenomenon, space-time is a completely amorphous four-dimensional continuum, in the sense of the *Analysis situs*, with the metric expressing however something of

> reality which has an existence in the Universe, exerting effects on matter by means of centrifugal and gravitational forces and whose state, inversely, is also conditioned, according to natural laws (*naturgesetzlich*), by the distribution and the constitution of matter. (Weyl 1918, 384)

Finally Weyl concludes:

> According to this theory [the 'pure infinitesimal geometry'], all that is real, that is all that exists in the Universe, is a manifestation of the metric of the Universe; the concepts of physics are not something other than those of geometry (*die physikalischen Begriffe sind keine andern als die geometrischen*). The only difference between geometry and physics lies in what geometry probes in a general way of the essential nature of metric concepts, but it is physics which, for its part, inquires into the law in virtue of which the real Universe is distinguished from all possible four-dimensional metric spaces, according to their geometry. (Weyl 1918, 411)

Exploring Riemann's thought coherently and profoundly, Weyl develops the philosophical idea that the metric expresses both an *a priori* and *a posteriori element* of space. It is thus seen that 'the Riemannian conception does not neglect the existence of an *a priori* element in the structure of space; only, *the border between the a priori and the a posteriori is displaced*' I will not comment on all that this displacement implies in relation to Kantianism. But what does Bachelard draw for himself from the Riemannian change operated by the differential relation of the law of infinitely near action?

> [Coulomb's] general law thus stops being *a priori* It will be objected that the general is tangent to the particular, that Euclidean spaces are the first simplification of the given infinitesimal itself. But a Euclidean system of reference that must little by little be transported in an all-in-all non-Euclidean manner in order to follow the pseudo-generality, does it really have the Euclidean value that is attributed to it? The on-the-spot description could perhaps be put within the framework *Euclidean at first approximation*. But it happens to be an essentially relative *description*, that is, which must serve elsewhere and in another time, which must bind with thought the successive and immediate states of reality. The descriptive movement must thus fold under the curve of the Universe. From it will result an *a posteriori* post-experimental geometry, which will not have the value of prediction that is allotted to an *a priori* informative geometry, but which, *in exchange*, will be ready to record the discontinuous of the future and of being . Matter thus appears to us in the form of a contingency that is to some extent foliated. (Bachelard 1927, 283)

I will qualify this potentialising echo to Hermann Weyl as the typical (and topical) mark of a great capacity for philosophical 'auscultation' responding to the 'methods of auscultation used by the mathematician,' as he will formulate it in his conclusion (p. 295). What is 'auscultated' here of the Riemann-Weyl relation, 'is the rectifying allure of a thought. Nothing is more clear and more captivating than the *junction* of the ancient and the new. The rectification is a reality, better, is the true epistemological reality (the 'constraint' of Weyl), since it is thought in action, in its major dynamism' (p. 300).

It would be worth taking the time to analyse once again the recurring presence of Hermann Weyl in *La valeur inductive de la relativité* of 1929. It is the Weyl who was creating a 'true geometry of electrical character ., in reciprocal relation with the purely mechanical character of General Relativity' This is what he qualifies as 'Weyl's fusion', as the attempted assimilation of the electrical with dynamics. Once again he draws from this an epistemological lesson: 'Weyl's method consists essentially in a widening of the axiomatic' Then, in solidarity, Bachelard points to the 'geometry of gauges':

> Before the work of Weyl, the unity of length held the same value after a closed cycle of transformations in space. That the postulate of the integrability of length is now abandoned, and in the so constituted pangeometry ('the geometry of gauges') it will be realised that the electromagnetic field is entirely definable *by algebraic means*. (p. 136–137)

While he is conscious of the difficulties, he concludes nonetheless 'that the sense of the attempt by Weyl *must retain the attention of the epistemologist*. This attempt is appropriate, believe me, to prepare *this conclusion*: the mathematical unity that is constituted in an axiomatic of Physics entirely orders the unity of the phenomenon' (p. 137). It is in an effort to further defend Weyl in a debate with Zaremba, relating precisely to the 'axiomatisation' and to the definition of a rigid body in General Relativity, that he insists positively on Weyl's definition, 'all in virtuality' (pp. 175–175). Finally he approaches the 'generalities/specifications' dialectic in his unitary theory (p. 207), to finish with Weyl's 'axiomatic fusing', an 'axiomatic' in which he discovered 'the trace of electrical potentials' (again his gauge theory). A reproduction of this 'divided up' economy would be meaningful in this case, but I cannot give it here.

b) *Wolfgang Pauli or the* 'Schola quantorum.'[25]

One can hardly speak of the place of Pauli in Bachelard's writing without pointing out that he was a man of wit and of *Witz*, that is someone of great 'spirit.' During a seminar, von Neumann demonstrated a theorem at the blackboard. Pauli cut him short, stating, 'If doing physics was simply a question of demonstrating theorems, you would be a great physicist.' It was his way of gently terrorising physicists all over the world, including Louis de Broglie or Heisenberg who literally fled from the conferences which Pauli attended. I have always thought that, in the context of philosophy conferences, there was a little of this type of behaviour in Gilles Châtelet. Those who knew him will understand.

α) *Pauli in 'principle'*

For Bachelard, Pauli belongs to the men of 'Provisional Ontology', that is the ontology of the physical entities of our time. In 1932, Pauli arrived on the scene, the scene of 'spectres'. Book III, chapter XII of *Pluralisme cohérent de la chimie moderne*, entitled 'From Location to Measurement: From Measurement to Mathematical Harmony in the Problems of Spectral Analysis', constitutes the necessary threshold for understanding the decisive contribution of the Pauli Principle which is treated in detail in chapter XIII, 'Quantum Description' Bachelard argues that 'the fundamental discovery of theoretical spectroscopy is the fact that the frequency of a line is presented mathematically in the form of a *difference of two terms*. This theorem cannot be contemplated enough' (Bachelard 1932, 202). Indeed, by clinging to the philosophy of science emerging out of the work of Fresnel and Maxwell, 'it was not clear how one could test the necessity of forming a substructure from the notion of frequency' (ibid.). It is an 'algebraic *symmetry*' which confirms the brilliant intuition of Walter Ritz: frequency is not the fundamental concept since it is a notion that can be *constructed* from terms that we will find *thereafter* inscribed in the nature of things. And yet, the number of necessary terms is smaller than the number of lines:

> *every procedure that restricts the means of explanation puts us on the track of the rationalisation of the experiment*. There again, we see the pluralism of phenomena ordering itself while tending systematically towards its minimum. This *systematic economy* – quite distant in content from the occasional and pragmatic economy advocated by Mach – sanctions the notion of the term which appears thus, in contemporary spectroscopy, as a notion that is both *primordial* and *organic*. . It is with *terms* that the concept derived from frequency, considered for such a long time as a quasi

> immediate notion, will be constructed. These are the *terms* that we will have to illuminate *by schemas*. It is the founding of their reality that will occupy a whole generation of physicists. (pp. 204–205)

This will pass by the (incomplete) 'construction of Bohr' and by the (relativistic) 'rectification of Sommerfield': it is the passage from the terms of Balmer to the terms of Rydberg.

Pauli will appear as the great organiser on the path of the 'spectra' He will to some extent bring coherency to the field. What opens up with the discovery of the eponymous principle is an *arithmetic* system of description which would impose itself little by little on modern chemistry. This general description is coordinated by a new principle, the Pauli principle, 'which suddenly illuminates with *rational light* the table of elementary substances. The Pauli Principle creates an *arithmetic coherence of the diverse*' (p. 215). And from the outset Bachelard situates his action simultaneously within the framework of the 'Schola quantorum' and the 'quantum arithmetic' Indeed, Bohr's formula, which condenses the principles of quantisation relative to electrons, will be oriented towards a thought 'in tabular form' and to a summary formula explaining square formations. A cardinal enumeration of quanta is put in close connection with the ordinal numeration of the lines of the spectrum. '"We see clearly," as Eugène Bloch (1930) points out, that quantum arithmetic "can provide a guide for the theoretical interpretation of tabular formations [electron shells and maximum number of electrons]"' (p. 220).

Here, Bachelard draws three lessons directly from the work of Pauli:

1) The necessity and urgency to build a *quantum metaphysics* of a non-realist type (in the sense of the 'philosophies of no'):

> It could be objected that Bohr's rule maintains a certain arbitrariness and that it rediscovers in an *artificial* way the correspondence between the number of electrons and the number of elements in the diverse periods. It is difficult to see by what characteristic the electrons fix quanta. But what brings this objection is the persistence of *the false idea of the realist* that the electron is directly *qualified* by quanta. *We suffer from a deficit of metaphysical thought*. Indeed, it appears that we still lack a type of thought which would explain, by a kind of group attribution, the distribution of quanta across different electrons. In other words, the plurality of electrons and the plurality of quanta should be put in immediate correspondence. *It is in this way that we could perhaps interpret Pauli's rule directly*. (p. 220)

2) The reinforcement of the 'rationalisations *of the possible*':

> This rule [of Pauli's] suitably interpreted leads to Bohr's restrictive formula. It measures the real by fixing the impossible. This rule will not be illuminated by contemplating the particular nature of the electron; on the contrary it will be affirmed *mathematically* by considerations of general suitability, *in a thought which encloses and systematises a plurality of conditions*.
>
> Once again, the rationalisation of the possible *preceded and prepared* the rationalisation of the real. (pp. 220–221)

This is the extension, in quantum physics, of the previous conclusions of 'relativistic gravitation's' detailed analysis. The Anti-substantialism and rationally constructive power (*puissance*) of a preparation of possibles which take the form of 'mathematical harmonisation' through the organization of virtual operations.

3) finally, opening on the 'Philosophical problem of substantial harmony' (the title of the work's conclusion):

> It seems to us that we might, in a certain way, consider the *experimental reasoning* which is confirmed by means of a harmony [rational and no longer factual] as an *extension* of inductive reasoning. It happens indeed to be an extension that *overcomes classes*, which postulates from one quality to another quality, which is entrusted to a *homography* of substances. Different elements, integrated in a series, receive this series *like the reflection of an ideal unity*. 'The seriality can be considered as a particular case of continuity' ., to the extent that we might speak of the continuity of a well-ordered discontinuous. Here, as in mathematics, the law of series precedes the structure of the elements, or at least, one only retains of the elements' structure that which clarifies the construction of a general law facilitating the most audacious inductions. It can really be said that inductive thought passes from the phenomenon to the noumenon; in other words, there exists the impression of having found the reason of the induction [Cf. Cassirer 1910, 290 & 292 [1923.]] Just as we've defined a complete induction, so too perhaps we might speak in this case of a *complete construction*. (p. 227 and p. 230)

Here, the 'rational' deciphering of the Pauli principle, by means of its 'epistemological substitution' of a harmony of substances as *reasoning*, with a harmony as 'fact', still potentialises the categorical mechanism of the 'inductive', of 'constructivity' and 'noumenology' (of the non-Kantian type) elaborated previously in the field of General Relativity or through the analysis of 'approximate knowledge'

Two years later, in 1934, the pressing presence of Pauli became literally 'spectro- *scopic*' during the time of the *The New Scientific Spirit*. Bachelard radicalises the quantum 'cut', and thus specifies and enriches its consequences for philosophical thought:

> Instead of ascribing properties and forces directly to electrons, physicists assigned quantum numbers and from the distribution of these numbers *deduced* the (orbital) locations of the electrons within atoms and molecules. Notice, here, how realism *suddenly evaporates: Number becomes an attribute, a predicate, of substance* [B.'s *italics*]. Four quantum numbers are all that is needed to identify an individual electron. The *mathematics respects* this individuality, moreover. In any given atom, no electron is allowed to take on exactly the same set of quantum numbers as any other electron. Two different electrons in the same atom must differ in at least one of the four quantum numbers. *This is the philosophical meaning of the Pauli exclusion principle*. It is clear that *this principle contradicts any attempt to argue that the quantum properties are substantial*, deeply ingrained *in* the substance of the electrons themselves, for the numbers are in a sense *attributes in extension*. (Bachelard 1934, 79 [1984, 81; additional italics])

But the Pauli principle must be extended beyond the molecule, 'to all forms of matter': 'it follows that the organisation of matter is in a sense synonymous with the *quantum principle of individuation* of its constituent elements. Wherever there is true organisation, it is appropriate to bring the Pauli principle into play. In philosophical terms, this principle entails systematic exclusion of the *same*, and systematic appeal to the *other*' (p. 80 [82]). But it is the categorical characterisation of chemical bodies which will consequently find another status: 'Thus there occurs a surreptitious transition from the *corps chimique* or chemical substance to the *corps arithmétique* in the technical sense of the term.[3] A chemical substance or *corps chimique* is therefore a *corpus* of laws, an enumeration of numerical characters' (ibid.).

The affinities with the future Lautmanian approach appear to us here to be particularly 'elective': 'This marks the first step in the *transition* from materialist realism to mathematical realism' (ibid.).

Finally, the last great lesson for the philosopher of modern chemistry, and the new task of the detailed study of quantum radicality:

> The attribution of the four quantum numbers to each electron has to be desusbstantialised still further, however. The crucial point is that this attribution is actually statistical in nature; the need for a statistical justification

of the Pauli principle is fairly clearly understood. Thus quantum arithmetic turns step by step into a kind of statistical arithmetic. (p. 81 [82–3])

In fact, it is the 'metaphysical abyss' between the mind and the external world which appears less formidable:

> It is even possible to conceive of a veritable displacement of the real, a purification of realism, a metaphysical sublimation of matter. The procedure is as follows: First transform reality into mathematical realism, then dissolve mathematical realism in the new statistical realism of quantum mechanics. . . Let me therefore sum up the supremacy of number over the thing and of probability over number in the following polemical formula: *A chemical substance is but the shadow of a number (l'ombre d'un nombre)*. (p. 81 [84; additional italics])

Such were the three discursive and principal stages of the *schola quantorum*.

β) *Pauli demonstrated by the 'postulate of non-analysis.*

In 1937, the year of Lautman's *Thesis*, Bachelard further extends Wolfgang Pauli's 'initiatory' gesture, in the heart of a work which is central for any detailed comprehension of Quantum Mechanics: *L'expérience de l'espace dans la physique contemporaine* (1951). The aim of the work is to understand categorically (understood in the sense of a Kantian 'categorical imperative') – then *to implement in thought* – Heisenberg's inequalities. For Bachelard it is a question of performing an effective *experiment* (understood in this case as a permanent 'thought experiment') in the new quantum space, this is the programmatic meaning of the title. This is a cathartic and 'non-Cartesian' imperative that must absolutely be reckoned with:

> Let us get a firm idea of this thought: what renders the description of the atomic field in the usual terms of space and time inadequate is that we neglect *the correlation* between geometrical and dynamic uncertainties. To neglect this correlation is to accept *the Cartesian postulate* of an exhaustive spatial analysis capable of achieving localisation at a point
> We call the *postulate of non-analysis* the fundamental postulate of this non-Cartesian physics. (Bachelard 1951, 42)

Three years later, Bachelard further refines his definition:

> we made full use of the principle of Heisenberg, under the name of the *postulate of non-analysis*, the generalized function of which is to forbid the separation of spatial from dynamic qualities in the determination of the micro-object. In accord with this principle, the micro-object is presented

> as an *object* with two species. Correlatively, if we consider such *dual specificity*, we come to understand that an object statically localized by ordinary intuition is wrongly specified, if one were desirous for knowledge of the second degree of approximation. To state it even more differently, its entirely local specification is a distortion of the dual specification which is henceforth indispensable for the organization of micro-physics ... In other words, *the space of ordinary intuition in which objects find themselves is only a degeneration of the functional space in which the phenomena are produced.* (Bachelard 1940, 109 [1968,93–4])

Following now the account of F.A. Lindemann, *The Physical Significance of the Quantum Theory* (1932), Bachelard announces an 'extension' of the exclusion principle: 'we will show ... that the application of the Pauli principle necessarily follows from the postulate of non-analysis. This demonstration will bring a true *coherence* with quantum pluralism' (Bachelard 1951, 61). What follows is a rigorous demonstration, which proves to be *foundational*, for a simplified case (a set of electrons brought together and localised on a straight line segment, *L*) of the Pauli exclusion principle. I will leave the reader with the speculative joy of discovering it in the text, pp. 60–65. What is the immediate and *rational* speculative consequence of this demonstration?

> We witness ... *the birth of an ordinal arithmetic* that no longer has quite the same properties as ordinary cardinal arithmetic. This ordinal arithmetic designates objects as fundamentally different from the sole fact that they do not appear in *identical* experiments ... For us, the electron is only a summary of experiments ... The sense of the exploration appears to us . undeniable: it is necessary *to go from the method to the entity*, contrary to realist instruction. (Bachelard 1951, 65–66)

What is found here is the idea of a kind of 'induced' and 'provisional' ontology.

γ) *Pauli & his 'metaphysical particle.*

The referential and 'spectral' returns of Pauli come to an end in 1951 with *L'activité rationaliste de la physique contemporaine*, an ending marked by the arrival of a 'phantom'

Let us recall a bit of history. As of 1930, Bothe and Becker bombard beryllium with *alpha* particles (already identified by Rutherford) issuing from polonium and discover that a very energetic neutral radiation is emitted. Two years later, Frederic Joliot and Irene Curie do the experiment again and are able to specify that this radiation is composed of *neutral particles* capable of forcing protons out of the nuclei of paraffin. The same year (1932), Chadwick identified the nuclei as practically identical to the proton and called them *neutrons* in memory of Rutherford, who had predicted their existence.

Protons and neutrons were then identified as two different states of the same particle, the *nucleon*, which appeared sometimes with a positive charge (the proton), sometimes without charge (the neutron). Its great characteristic, which was new at the time, was its *instability*. This hypothesis, due to Chadwick, was proposed as of 1935, but was only proven in 1948 by Snell, and was made definitive in 1950 by Robson. It could then be identified with the β–disintegration of nuclei discovered in 1900, and considered the result of an interaction (different from the electromagnetic interaction) for which Fermi had provided the theoretical foundations in 1934. Now, in the meantime, Pauli examined the dynamics of β–disintegration and noted that neither energy, nor momentum were conserved. It then seemed probable to him that a *neutral and massless hypothetical particle* must carry with it, after the reaction, the missing quantities. That was the theoretical hypothesis of the *neutrino*, ν, which would only be discovered 21 years later, in 1956, by Reines and Cowan.

In 1951, Bachelard thus raises with new freshness his approximalist and metaphysical rectification program:

> This corpuscle appears highly appropriate *to sensitise* the philosophical nuances of applied rationalism. In this connection, indeed, many philosophical *questions* can be posed: Does the neutrino correspond to a simple working hypothesis? Is this a convenient entity, a convention useful to the expression of the facts? If it is a convention, why is it so generally accepted? Or, can it be hoped that a new type of experiment, an increased sensitivity in detectors will bring evidence of its reality? The philosopher must find there true lessons for the *spirit of metaphysical finesse*. (p. 118)

Wolfgang Pauli is again at the centre of an improvement, an elaboration of philosophical thought, the intensifying consequence of a similar gesture inaugurated in the protocols of his principle and continued in the following words: 'corpuscular philosophy [takes on] then a great variety which will be approached by the avenue of theories or by the examination of experiments The corpuscle thus seems to us *the very being of an applied rationalism*. Corpuscular philosophy cannot be understood without an essentially transactional philosophy, without a philosophy of two movements' (p. 127). Pauli made it possible 'to specify how a particle that is not yet *physical*, a particle that the realist must hold as *metaphysical*, is however a particle indispensible to the rational organisation of experimental thought' (p.118). Since this time, Pauli has been one of the great 'constructors' of quantum mechanics. He will ultimately take part in the 'interpretation of the notion of spin into an organisation of operators' (p. 178). His mechanics will have started 'by restarting in *another algebrism*, in an algebrism that, this time, is in search of its reality' (p. 176). Quantum chemist, traditional *alchemist*, he will have taken part 'in opening a rationalism that is multiple, beyond the rationalism of identity.'[28]

Let us finish with a wish, the surreal wish of Gaston Bachelard:

> [W]e should like to give the impression that it is in this area of dialectical surrationalism that the scientific mind *dreams*. It is here and not elsewhere that anagogical dreaming comes into being, dreaming which ventures into thought, dreaming which thinks while it ventures, dreaming which seeks an illumination of thought by thought, which finds a sudden intuition beyond the veils of informed thought.
>
> In our estimation, anagogical dreaming under its present scientific impulse, is essentially mathematical. It aspires to more mathematics and to more numerous, more complex, mathematical functions. When one follows the efforts of contemporary thought to understand the atom, one comes close to believing that the fundamental role of the atom is to oblige men to do mathematics. *De la mathématique avant toute chose . . Et pour cela préfère l'impair* In short, the *ars poetica* of physics is done with numbers, with groups, with spins, to the exclusion of monotonous distributions, repetitive quanta, and without obstacle to the working out of any process whatever. What poet will arise to sing of this panpythagorism, this synthetic arithmetic which begins by giving its four quanta, its four figure number to every existing thing as if the simplest, the poorest, the most abstract of electrons already had, of necessity, more than thousand faces.
>
> The atom is a mathematical society which has not yet told us its secret. (Bachelard 1940, 49–50 [1968, 32–3])

Referential continuities which take part in the power of the continuum of thought. A *single* Bachelard, but a prismatic Bachelard, for whom scientific, epistemological and metaphysical perspectivism (the place of his *Nietzschean* affinities) constitutes the *differential unity* of a thought on the move. *To surprise the sciences in their successive approximations that is what makes the jubilatory surprise of thought.*

Translated by Simon Duffy and Stephen W. Sawyer

Notes

1 Quotation from the private correspondence of Jacques Lautman addressed to us, as acknowledgement of the constraint.

2 And obviously of Alain Badiou! 'I must declare here that the writings of Lautman are well and truly admirable, and that *what I owe them, even in the foundational intuitions of this book, is without measure*' [Additional italics] (Badiou 1988, 522, note to pages 18 and 19). It is appropriate to add the following occurrences: Badiou 1992, 158 n. 26; Badiou 1989, 83–84 [1999, 100–101]; Badiou 1997, 144 [2000, 98]; Badiou 1998a, 13, 14, 17; Badiou 1998b, 56. [*Trans.* reference to the English translation editions are included, where available, in square brackets after the French.]

3 Deleuze 1968, 212–213, 230–238 [1994, 163–4, 178–83, 323–4]. Actually, Lautmanian references traverse the whole Deleuzian oeuvre. For example: Deleuze 1969, 32, 69, 127 [1990a, 54, 337, 339]; Deleuze and Guattari 1980, 462, 606–607 [1987, 485, 556 n.39]; Deleuze 1986b, 85 [1988b, 78]; Deleuze 1988a, 136 [1993, 57].

4 For a clarifying reference to the concept of 'problem' in Deleuze, particularly in the relation of 'analogy in [the two] tasks' of science and philosophy, cf. Deleuze and Guattari 1991, 126 [1994, 133].

5 In connection with George Bouligand, this is what I will readily qualify as an 'incidental revelation' of his links with Jean Cavaillès. In the reproduction of a course manuscript, which have been thoroughly undervalued, the author announces in the *foreword*: 'This booklet joins together my notes, drawn up in preparation for six conferences, held in March-April-May 1943 with the students of the Faculty of Arts in Paris, on the invitation of Professor Emile Bréhier, Director of Studies in Philosophy   Some students will certainly feel the desire to go into the questions [treated here] in depth.   The theses of Cavaillès will bring to them, in addition to their rich substance, very broad bibliographical information. *Note*. – the study of this text must precede that of the theses of J. Cavaillès. I took advantage in the drafting of this booklet to specify, on some points, my background research on the structure of theories' (Bouligand 1948).

6 [*Trans*. all citations from Lautman are from Lautman 1977, unless otherwise indicated.]

7 Here, what Lautman resumes with the term 'Idea' that is, in a better translation of the Greek, with the concept of 'Form' should rather signal the operational, 'schematising' and 'structuring' concept of *differential form* according to the so-called theory of '*p-form*', rather than an 'Idea', scholarly, traditionally and falsely understood as a Platonic hypostasis.

8 Châtelet 1993 [2000]. Cf. in particular, Chapter 4 'Geometry and Dialectic' § 3 'To articulate and generate'; § 5 'The intensive/extensive dialectic' On Lautman, see p. 27 [6].

9 Let us express here provisionally the hypothesis of a link to the *homology*

cohomology relation, and in particular to Whitney's theorem showing the equivalence or the 'morphism' of two theories (or operations).

10 Note that he will assert this 'identification' until the end, particularly in relation to the philosopher: 'Hyppolite says to me that to pose a problem is to conceive of nothing; I respond to him, after Heidegger, that it is to delimit the field of the existant' – Response to Hyppolite at the meeting of the *Société française de philosophie*, 4 February, 1939.

11 It is necessary here to compare the metaphysics of another mathematician, that proclaimed and shared by his friend André Weil: 'Nothing is more fertile, as all mathematicians know, than these obscure analogies, these confusing reflections of one theory on another, these furtive caresses, these inexplicable disagreements; nothing else gives quite as much pleasure to the researcher. The day will come when illusion will dissipate; the presentiment change into certainty; the twin theories reveal their common source before disappearing; as the *Gita* teaches, one reaches knowledge and indifference at the same time. *Metaphysics became mathematical*, ready to form matter from a treatise whose cold beauty could no longer move us' (1979, 408).

12 This is the title of one of Lautman's very last papers [1942], first printed in a separate booklet in the 'Actualités Scientifiques et Industrielles' in 1946, before joining other contributions in the project initiated in 1942 by François le Lyonnais, *Les grands courants de la pensée mathématique*, in 1948 (Le Lyonnais 1971) [New edition thanks to the initiative of our friend, the mathematician Bernard Teissier, Paris: Hermann, 1998].

13 'Each physicist makes daily use, in a more or less explicit way, of these notions of symmetry; so we are very much surprised not to see them stated in any treatise on Physics. These notions are however fundamental and, stated from the very beginning, they facilitate much of the pupils' comprehension of phenomena. Many demonstrations are indeed immediately simplified when one utilises the concepts of symmetry' (Curie 1908, 150).

14 On this status of the 'literal metaphor' (*métaphore à la lettre*) cf. Alunni 2001, 154–172.

15 Cf. on this essential (and transdisciplinary) category the remarkable article of Gel'fand and Manin 1978, 126–178. This text will be published in a collection that I devote to Manin in my series 'Pensée des sciences', Editions-Rue d'Ulm.

16 [*Trans.* Grothendieck 1986. The first quarter of the second unpublished manuscript (1500 pages), in circulation in 1986, is available at http://mapage.noos.fr/recoltesetsemailles.]

17 *ETH* = *Eidgenössische Technische Hochschule* or *École Polytechnique Fédérale* of Zürich (Swiss).

18 Walter Ritz, the Swiss physicist who died at 32 years of age (1878–1909), and who taught in Zürich and Göttingen, is quoted by Bachelard from 1931, as a herald of the 'new physics': 'Here it is that contemporary Physics brings us

messages from an unknown world. These messages are written in 'hieroglyphics', following the expression of Walter Ritz' (Bachelard 1970, 12).

19 I return here to the dialogue between Marc Schützenberger and Alain Connes in Connes, Lichnérowitz and Schützenberger 2000. Connes, in his recent conferences, has consistently insisted on the 'magic' of this link between the 'spectral' characteristic of an abstract space (*Hilbert Space*) and concrete 'spectroscopy' in Physics. He returns this paradigmatic node of contemporary physico-mathematics to the 'phantom' of Riemann (to his 'spectre' to some extent) and to his revolution of a *geometry* literally 'made' for physics. We will see how Bachelard very precisely pointed to this in Riemann and Hermann Weyl.

20 Cf. on this point, Castellana 2005, in Alunni 2005, treating the *School of the ETH and the surrationalist programme*.

21 Cf. Alunni 1999, 73–110.

22 Bachelard 1927 [Reprinted 1969]. Here, in connection with this founding work of the whole oeuvre, is what for example Gilles Deleuze says: 'Gaston Bachelard's book, *Essai sur la connaissance approchée* (Paris: Vrin, 1927) remains the best study of the steps and procedures constituting a whole rigour of the inexact, and of their creative role in science' (Deleuze and Guattari 1980, 455 n. 27 [1987, 555 n. 33]).

23 Cf. Alunni 1999, *passim*.

24 On all these Weylian questions, cf. the now 'classic' work of Erhard Scholz, and in particular Scholz 2001. See also Ria 2005, in Alunni 2005, treating the *School of the ETH and the surrationalist program*.

25 'the discipline of quantum theory – the *schola quantorum*' (Bachelard 1984, 84).

26 [*Trans*. *corps arithmétique* translates into English as 'field of rational numbers', which unfortunately ruins the play on words.]

27 On the Bachelardian 'anticipation' of 'Feynman's paths or diagrams' (*Trans*. Feynman's path integrals) starting from the Buhlian concept of the 'non-analytic trajectory', cf. Alunni 2001, 168–169.

28 Bachelard 1953, 224, last sentence of the text.

6
Cavaillès and the historical a priori in Foucault

David Webb

In the Preface to *The Order of Things*, Foucault recalls his laughter on reading Borges' description of a Chinese encyclopedia in which animals were classified according to whether they looked like flies when seen from a distance, were embalmed, innumerable, tame, sirens, or had just broken the water pitcher (Foucault 1970, xv). He also recalls that his laughter was tinged with unease, which he attributes to his inability to imagine how anyone could arrive at such a classification and to what sort of world it could belong. Laughter is provoked not by the juxtaposition of unlikely classes in itself, but by the fact that they exist in the absence of any possible ground for their appearance. There is no space in which animals that are 'frenzied' and 'drawn with a fine camelhair brush' can co-exist; and the impossibility of their co-existence in the absence of such a space draws our attention to its presence in the overwhelming majority of cases; usually, things find their places in a common space of classification. But where do such spaces of classification come from? As Foucault points out, when we say that a cat and a dog resemble each other less than two greyhounds do, we appeal to a certain coherence. Generally, the origin of this order has been sought either directly in the things perceived, or in an *a priori* scheme that passes itself off as necessary. Borges reminds us that neither option is satisfactory; order, for that is what Foucault is considering here, is neither simply given along with things, nor is it a fixed ideal form that makes experience possible.[1] Its independence from both perception and the transcendental a priori grants order the possibility of a history, and it is this history that *The Order of Things* sets out to examine. However, its independence from perception and the transcendental a priori also raises the question of how we are to understand order, and above all how we are to conceive of the dimension in which it occurs. Insofar as order is closely aligned with the historical a priori, the question can be reformulated in terms of history. If order is to be discovered neither in things nor in the transcendental a priori, any appeal either to history or to the a priori must be problematic. To speak of the historical a priori therefore seems like a deliberate provocation: how can there be an a priori that is not transcendental, and a history that is not of things and wordly events?

Foucault clarifies the conception of order that interests him by contrasting it to two 'regions' (Foucault 1970, xx). On the one hand, there are the 'ordering codes' that determine how those within a given culture see and speak, and what values they share. These codes 'establish for every man, from the very first, the empirical orders with which he will be dealing and within which he will be at home' (p. xx). On the other, there are scientific and theoretical reflections on these codes by which we come to ground them and to account for them. When Foucault proposes that neither of these two regions, and the reflections on them, are as fundamental as they presume, it would be easy to misconstrue order as occupying a higher level again; as if it were a code that somehow stands above both critical philosophy and the human sciences. But, as we shall see, this would replicate the problem of foundation that Foucault wishes to address, leaving the relation between philosophy and the human sciences untouched and merely re-inscribing them within a more general or higher order discourse that would ground them both. Foucault's intent is rather to disrupt their relation in order to account for a variability of the codes that neither discipline can adequately describe.

Insofar as order lies 'between the already "encoded" eye and reflexive knowledge' (Foucault 1970, xxi), it plays a dual role with respect to the empirical codes and the theoretical reflections on them; at once fundamental and transformative. However, as Foucault notes, historically speaking the ordering codes to which he refers have been traced back either to the empirical order of things or to the transcendental structures of consciousness. If order falls between codes and the reflection on them, it therefore works a twofold disruption; mediating both between the empirical order of things and the human sciences, and between the transcendental structures of consciousness and transcendental or critical philosophy. Each form of reflection is distanced from its object by this intermediary domain whose very existence remains unknown to them, and which exposes the codes and their respective forms of determination to a variation for which they cannot account. When a culture finds itself 'imperceptibly deviating' from its current empirical orders, it is confronted with the realisation that the orders it lives by are neither fixed in things nor determined by a general principle that can be brought into view in a theoretical reflection. History springs up by slipping the bonds on either side. And with this history comes the possibility of critique as a creative engagement that opens a future that cannot be anticipated by a reflection on the existing state of things and the codes by which we know them.

It is important to recognise that order in this sense is not a strategic invention on Foucault's part. He clearly states that 'between the use of what one might call the ordering codes and reflections upon order itself, there is the pure experience of order and of its modes of being' (Foucault 1970, xxi). These lines might be taken to imply that Foucault is providing an ontological account of order here, as though there were order as such over and above its particular configurations. However, one should be wary of such a reading. It is hard to substantiate elsewhere in Foucault's work, and there is no real suggestion anywhere of just how such an ontology of order would work. Indeed, given that order does not provide the condition for the possibility of experience as such, but rather the condition for the actuality of this or that particular experience, one must be cautious of any supposition that order can be given as such or as a whole. As I will show in this essay, for Foucault, the experience of order must be an experience of the specific transformations that characterise order in its historicity.[2] To substantiate this claim will involve setting his interpretation of the historical a priori and of archaeology apart from phenomenology (at least in its Husserlian and Heideggerian forms). This is not to deny the importance of phenomenology for the development of this aspect of Foucault's work. As several recent studies have pointed out, Foucault's debt to phenomenology is considerable, not least for the term 'archaeology' itself.[3] One might also suggest the importance for Foucault of Husserl's demonstration that a formal ontology can be characterised by an historical development that is entirely non-dialectical. This aspect of Husserl's understanding of formal logic reflects his interest in mathematics and science, which underwent radical change in the late nineteenth and early twentieth centuries, and it is from another philosopher of mathematics, Jean Cavaillès, that Foucault draws most inspiration. In works written during a short and intense period prior to his death in 1944, Cavaillès developed a singular interpretation of the historical character of mathematical thought. This included a trenchant critique of Husserl, in which he argued that no philosophy still rooted in subjective consciousness could adequately account for the structure and movement of modern mathematics.[4] It is his description of mathematics as a formal discipline whose development, independently of both empirical reality and transcendental consciousness, is driven by the internal demands of a given problematic that most decisively prefigures Foucault's work. In a parallel sense, Foucault regarded phenomenology as unable to account for the condition of thought operative in contemporary culture. Nor could it underpin the critical approach of archaeology that Foucault developed to address order and the historical a priori as he understood them. To grasp Foucault's deployment of these ideas, we therefore have to clarify the sense in which order is an 'intermediary' region between the ordering codes and the reflection on them. We must also understand the form of development that characterises order and the nature of what Foucault calls the experience of 'pure order' This will allow us to see why the introduction of order and the historical a priori does not leave Foucault caught up in the same problems relating to the determination of experience that he sought to avoid by following Cavaillès' break with phenomenology in the first place. To approach these questions, let us go back to *The Order of Things*, but this time to the chapter 'Man and his Doubles'

The analytic of finitude

When natural history, the analysis of wealth and the reflection on language became biology, economics and philology respectively, man took up an ambiguous position as both an object of knowledge and the subject that knows (Foucault 1970, 312). In their early phases, these new sciences sought to mine the truth of their object of study from its own depths; life was to be defined from itself, labour to illuminate the meaning and conditions of exchange, profit and production, and language to yield up the conditions of grammar and discourse (p. 312). Order belonged to the things themselves. Yet as the laws of life, production and language close in upon themselves, the figure of man is revealed in all its ambiguity. It is only in terms of his body, works and his language that he can be known at all, yet the sciences on which his disclosure depends themselves rely on man as a living being, and above all as the one whose labour is exchanged for profit and whose desires and thoughts language expresses. At the very point where the laws of life, production and language seemed to exclude man, he reappears at their heart; and at the very point where man seemed most fully determined by these laws, he is revealed as their condition. On the one hand, the articulation of man inevitably has recourse to knowledge and language that precedes his existence, whereupon he is reduced to an anterior exteriority (a body, a capacity for labour, a language) and the growth of the positive sciences brought this finitude into ever sharper focus. But on the other hand, these very determinations of man's finitude themselves depend on the figure of man, and above all on the finitude of this figure. Empirical positivities rest on the finitude of man understood not as a limitation, but as 'a fundamental finitude which rests on nothing but its own existence as a fact, and opens upon the positivity of all concrete limitation' (p. 315). The determination of this fundamental finitude calls for an analytic of man's mode of being. Here, man is revealed as a transcendental-empirical doublet, 'since he is a being such that knowledge will be attained in him of what renders all knowledge possible' (p. 318). Empirical sciences, such as neurophysiology, history and linguistics, depend on the human as an object of study. In this sense, they presuppose the existence of a truth to be discovered. But they also presuppose that discourse involves a commensurate truth, such that it can effectively communicate the truth of what it describes. Again, Foucault traces the dilemma faced by modern thought: either the truth of the object determines the truth of the discourse that describes it, leading to positivism, or the truth of the philosophical discourse constitutes the truth of the phenomenon, leading to a form of discourse that Foucault calls 'eschatological'; that is, 'the truth of the discourse constitutes the truth in formation' (p. 320). Far from setting the two modes of thought apart, Foucault presents them as indissociable from one another. Each alone is incomplete and calls forth the other, yet any attempt to combine them condemns discourse to a pre-critical naiveté. Modern thought has sought to break open this bond by discovering 'a discourse whose tension would keep separate the empirical and the transcendental, while being directed at both' (p. 320). Such a discourse would have to illuminate the ground of both the empirical human condition and the capacity of the human for knowledge. Foucault sees this complex role as having been performed by 'the analysis of actual experience' (p. 321), which is situated in a Kantian framework as an intermediary between a 'quasi-aesthetics' and a 'quasi-dialectics' and grounds each of them in a theory of the subject that is recognisably phenomenological.

Actual experience is, he continues, 'both the space in which all empirical contents are given to experience and the original form that makes them possible in general and designates their primary roots' (Foucault 1970, 321). In spite of, or perhaps because of, the closeness with which the analysis of actual experience traces the contours of man as a transcendental-empirical doublet, it appears to contest positivism and eschatology, to suppress the naïvete of empirical discourse and 'restore the forgotten dimension of the transcendental' (p. 321). It can do this only insofar as, beneath the division between positivism and eschatology, it limits actual experience as a third alternative, 'an ambiguous stratum, concrete enough for it to be possible to apply to it a meticulous and descriptive language, yet sufficiently removed from the positivity of things for it to be possible, from that starting point, to escape from that naiveté, to contest it and seek foundations for it' (p. 321). In this way, it opens up communication between the body and culture, between nature and history, but only 'on condition that the body, and, through it nature, should first be posited in the experience of an irreducible spatiality, and that culture, the carrier of history, should be experienced first of all in the immediacy of its sedimented significations' (p. 321). What concerns Foucault is that the analysis of actual experience, or phenomenology, is in this way privileged twice over; the *irreducibility* of space secures its radicality, and the *immediacy* of experience safeguards its evidential basis. As a consequence, while the analysis of actual experience succeeds in bringing to light the dimension underpinning both positivism and eschatology, its own claim to be fundamental exerts a conservative influence on the very structure it apparently calls into question. Foucault recognises that in order for thought to shake off this constraint and move freely beyond the division between transcendental and empirical forms of inquiry, the irreducibility of the space of actual experience and the immediacy of its evidence must both be called into question.

In *The Order of Things*, Foucault provisionally suggests that linguistics can take on this role. Alongside the other human sciences, it is first amongst equals, as it were, by virtue of the fact that it deals with the common medium through which they are all articulated. Moreover, as a study of 'pure language' (Foucault 1970, 381) independently of the speaking subject, it engages in an order of positivities exterior to the human, thereby opening the question of finitude *without* turning to the depths of the human. Linguistics suggests the possibility of re-writing the analytic of finitude without the human. Rather than grounding both the transcendental and the empirical in a fundamental articulation of human finitude, it permits an analysis of the rules of formation that make the positive sciences what they are *in their specificity*, with regard both to their objects and to the discourses that deal with them. In this way, the framework that led modernity back to the analysis of actual experience is broken down, and the demand for an analysis that establishes the conditions under which knowledge becomes possible is satisfied in a quite different way; not by a determination of the conditions for the possibility of knowledge as such, but by an identification of the conditions under which specific concepts, objects and discourses arise. However, linguistics alone cannot supply all that Foucault perceives as necessary to overcome the framework of thought secured by the figure of the human. Its historical concern with the possibility of formal languages reflects its particular interest in dealing with the content of the positive sciences, but in pursuing such an interest it has tended to neglect the question of language *as such*, which would be the true analogue of the displaced inquiry into the human. In this respect, literature, and in particular the tradition of French modernism from Mallarmé to Artaud, Roussel and Blanchot, has been far more successful in raising the question of the being of language. Foucault therefore needs to couch his analysis in terms of a combination of linguistics and literature; themes that do indeed feature prominently in his work. But he turned to language in order to establish, and displace, the conditions under which the framework of knowledge in modernity first arose and still continues to hold. In addition, he sought to determine a method for a critical discourse capable of dealing with such conditions as they are modified in the density of historical and theoretical practice. This is where we meet the limit of what can be achieved by the discipline of linguistics, even when supplemented by literature. For in its deployment of formalism, linguistics is indebted to conceptions of form and the formal to which it contributes, but which it did not conceive alone. When Foucault writes that the return of language is itself encompassed within the modern *episteme* that emerged at the beginning of the nineteenth century, this is only in part to say that language was always important and had merely been awaiting its moment. It is also to recognise that the new sciences of biology, economics and philology all harboured a nascent formalism. So too, of course, does mathematics, which is particularly significant here, not through any naïve sense of its being a fundamental science, but simply because it went further than any other discipline in the articulation of formalism, the nature of its objects, its historical development and the mode of thought appropriate to it. This is why Foucault is so interested in the relation between the human sciences and mathematics, which extends the promise of 'a second critique of pure reason on the basis of new forms of the mathematical *a priori*' (Foucault 1970, 383). It is clear that Foucault believes this mathematical a priori may provide the tools to develop an analytic to supplant the analysis of actual experience in its relation to what Foucault called the 'quasi-aesthetics' of empirical knowledge of the human and a 'quasi-dialectics' of what makes it possible (p. 320–1). This confidence in mathematics may seem at first to conflict with Foucault's turn to linguistics and literature, but any such conflict is in fact superficial. Each move arises in response to the same impasse in the determination of the conditions of knowledge and Foucault draws on both mathematics and literature in describing the real character of the substratum between positivism and eschatology.

Not only did mathematics blaze the trail with regard to the development of formalism in the nineteenth century, it did so in a way that precisely matches Foucault's reading of the state of knowledge in modernity. The advent of non-Euclidean geometry and the emergence of mathematical analysis both lent weight to the argument that mathematics could be grounded neither in empirical experience nor in the transcendental structures of consciousness. As a consequence, mathematics could follow neither of the routes that Foucault identifies as open to the human sciences in their own deployment of formalism; namely, to discover the 'law' of things in the things themselves or in the transcendental structures of consciousness. In this way, the framework of Kantian thought, within which the human sciences had themselves taken shape, was subjected to a serious challenge. Moreover, the nature of this challenge was itself subjected to scrutiny in the early twentieth century when the foundation of mathematics became so problematic that it demanded a response. This is the context in which Cavaillès worked and in which he came to a singular view on the formal character of mathematics. This view will help us to see how Foucault addresses the role of the analytic of finitude, departs from phenomenology and sets in place a formalism indebted to the philosophy of mathematics.

Cavaillès

Cavaillès's work can be understood as a response to the question, first raised by Poincaré, of how it is that mathematics can exist at all as a deductive science (Poincaré 1952, 1). If the force of its reasoning were derived from an order embedded in the world, then mathematics would be an empirical science, not a deductive one. Such a view was generally discounted, not least because the development of non-Euclidean geometry and mathematical analysis had split mathematical concepts and objects away from the world of our experience. On the other hand, if mathematics were reducible to a set of a priori laws, then it would amount to little more than a tautology. Moreover, in order not to lose its purchase on the natural world, it would either have to claim that its basic rules enjoyed a fundamental and invariable status, or that they were in turn grounded in the structure of consciousness. Either way, it would be hard to explain the capacity of mathematical thought to create new concepts and objects. His recognition of the urgent need to account for this creativity led Cavaillès to reformulate Poincaré's question. For Cavaillès, one had to explain how mathematics could be at once a deductive science *and historical*. In a series of closely argued analyses, Cavaillès examined the available alternatives in the philosophy of mathematics, and rejected each of them in turn. Logicism, intuitionism, the Kantian solution and ultimately phenomenology, all seek a foundation for mathematics outside the discipline itself, in logic or the transcendental structure of consciousness. For Cavaillès, such appeals not only undermined the autonomy of mathematics, they also failed to deliver what they promised, claiming as a secure foundation what in Cavaillès' view remained no more than a contingent point of departure. For example, logicism exhibits a naive realism in its definition of a set of elementary signs and the rules governing their organisation, and phenomenology appeals to an ideal of evidence that Cavaillès describes as merely the point at which thought abandons its effort. He was more sympathetic to the idea of mathematics as a practice of demonstration; that is, to the break with intuition initiated by Bolzano and continued in Hilbert's conception of an axiomatic foundation. Treating mathematics as a conceptual practice that does not depend on the conditions of consciousness avoided the appeal to a foundation external to mathematics. Moreover, the Hilbertian conception of the sign as a constructed object without any further representative function underpinned the independence of mathematical activity from empirical reality (a point to which we shall return shortly). However, Cavaillès believed that the legitimacy of an axiomatic basis was fatally damaged by Gödel's theorem of incompleteness. It would be fair to say that in keeping the problem of foundation alive, this served to reinforce Cavaillès' conviction that mathematics could not be reduced to a single fixed foundation without compromising its capacity to generate new concepts and objects. The challenge he faced was therefore to allow for this capacity without compromising the necessity of mathematics as a deductive science. In order to explain the nature of his response to this challenge, and its relevance to Foucault, I shall look at his essay 'On Logic and Theory of Science' (Cavaillès 1970).

Cavaillès begins 'On Logic and Theory of Science' by recalling Kant's observation that to look for the laws of logic in psychology would be no better than drawing morality from life. Instead, Kant proposes that the rules governing the thought be found in the understanding, whereupon the task is to determine how the understanding itself proceeds. In this respect, Foucault's identification of the role played by the analytic of finitude in modern thought is no more than an acknowledgement of the influence of Kant, not least on phenomenology in both its Husserlian and Heideggerian forms. Cavaillès himself agrees with Kant's rejection of a general ontology abstracted from experience, but he also refuses to accept Kant's alternative that the rules of thought must therefore be written in consciousness. The demand that one find the rules either in experience, thereby surrendering any notion of necessity, or in consciousness, is in Cavaillès' view based on a false analysis of the event of synthesis. It is a mistake, he argues, to suppose that synthesis can work on a pre-given multiplicity. However, neither should we assume that this unity pre-exists the act of synthesis. In other words, Cavaillès thinks Kant was wrong to regard the ground of the formal unity of our experience as pre-existing the act through which that experience is constituted. The tendency to do so arises, he suggests, from the separation between form and matter in the light of which the question is addressed. This separation encourages a conception of formal logic as an 'inner structure' that remains true for each and every possible experience (Cavaillès 1970, 361). Analysis thus falls apart into two halves: on one hand, it addresses itself to the 'object as such' (no longer abstracted from experience as in general logic); on the other, it leads to an endless and empty repetition of the necessity for thought to agree with itself (p. 361). Either way, all positive content is lost as priority is given to the formal conditions of the unity of experience, whether these be the forms of time and space in the transcendental aesthetic, the formal characteristics of judgement, or ultimately the transcendental unity of apperception. For Cavaillès, such an account of the unity of experience can only stand in the way of an understanding of science as at once necessary and dynamic.

The challenge is in effect to determine the rules that govern how the act of synthesis occurs. In this respect, Cavaillès' comments on Brunschvicg are instructive. Like Cavaillès, Brunschvicg rejects the idea that rationality can be determined extrinsically and maintains that thought is capable of more than can be determined in advance at any given moment. However, he conceives of the historical development of thought as a linkage of ideas in consciousness. For Cavaillès, the rationality of such linkage could only be secured by appealing to some *further* condition of rationality, thereby drawing the account of mathematics back to the choice between an absolute ground and the conditions of transcendental consciousness. Both cases, he writes, fail to distinguish adequately between mathematics and other sciences, leaving the door open to a dependence on the external world. This surrenders the creative autonomy of mathematics – while also confronting thought with the necessity of an ontological inquiry into the being of both external reality and of the subject (Cavaillès 1970, 369). His own preference is to maintain the focus on what in Kantian terms we have called the work of synthesis, or the act of constructive thought itself. The rules governing construction determine the act from which arises the historical dimension that Cavaillès recognises as essential to scientific thought. Our attention is thereby drawn to the historical character of these rules. For Cavaillès, synthesis is not governed by rules grounded in some more fundamental condition. However, he is not, one might have expected, left struggling to account for their nature and provenance, precisely because his understanding of mathematical thought does not involve bringing a manifold of intuitions under the unity of a concept. In his view, there is no division between 'form' and 'content' in mathematical thought and its formal aspect does not constitute our empirical experience; and so the problem does not arise. The object of mathematical thought is not a synthesis of intuitions; indeed concepts become objects in an ongoing series of operations that work on the state of mathematical formalism at any given time. With the collapse of this division, a crucial change occurs to the constraints on thought that previously arose from the tension between formal rules (e.g. the determination of forms of judgement in formal apophantics) and the object they disclose (formal ontology), each demanding a unity whose contours will change only so long as there is a discrepancy between them. As long as this division is in place, thought will encounter its limits either in the transcendental subject or in the material condition of what is there to be thought (represented). But once the division gives way, the constraints on thought can only come from thought itself. In this way, the immediacy of evidence that we saw Foucault identify as a problem in the analysis of actual experience is eliminated and the possibility of further change and development necessarily kept open.[5] This by no means entails that 'anything goes' On the contrary, Cavaillès himself is concerned with how deductivity can survive within an historically developing discipline.

The elimination of transcendental subjectivity, as well as any fixed axiomatic basis, from the account raises the related question of the unity of mathematics; that it must have such a unity, Cavaillès readily concedes, otherwise its status as a science would be in jeopardy. Instead of looking outside mathematics for the ground of this unity, Cavaillès proposes that we find it in the very movement of the historical development of mathematics, which he describes as 'a conceptual becoming that cannot be stopped' (Cavaillès 1970, 376). Mathematics therefore has no need for any transcendental condition or metaphysical ground to determine the proper bounds of its legitimacy, since it performs these functions itself. If mathematics has an essential nature, it is determined by its own development, by the continuity spanning its history, with all its bifurcations and revisions. The unity of this movement also determines the historically specific limits of what is possible at any time. And yet in doing so it is already indicating the direction in which thought must, with a necessity that may only reveal itself retrospectively, step into the unknown and undertake what existing rules may tell us is impossible. Each step in a problematic is prepared by the last, even as its construction modifies the conditions that determine the stage from which it emerges. The truth embodied in mathematics at any stage is therefore the truth embodied in its own movement and is for this reason historically specific; the rules governing the formation of concepts and objects at any given time do not pretend to legislate for the future. Indeed, the situation is almost the reverse; the rules at any given time call forth a future they at once demand and yet exclude. To see the development of a mathematical problematic in this way as a perpetually re-opened breach into the future is quite different to seeing it as the adjustment of a structure towards an equilibrium with itself, and the difference springs from the recognition that the structure is itself the vehicle of change. As Cavaillès writes, 'there is in reality no essential distinction between the hardened rings which seem to mark the terms and the movement that traverses them' (Cavaillès 1970, 373).

Movement, then, is always of the whole of mathematics (Cavaillès 1970, 372), and is to be understood as the revision of everything that has come before (p. 409). In spite of the apparent discrepancy, this is what Cavaillès describes when he writes that 'Progress is materialist, or between singular essences, its driving force the overcoming of each in turn' (p. 409). To call the movement of a purely formal discipline such as mathematics 'materialist' is certainly striking. First and foremost, we can say that it is to underline the absence of any division between the form or concept and the content; in short, it proposes a view of form as itself the outcome of a process whose development is governed by rules determined over again at each new stage. Even if we think we are looking at a change in a higher order determination of mathematical concepts and objects, in fact we are always dealing with a specific change that has emerged from a well defined series of antecedant operations, and which is thereby enmeshed in the body of mathematics as a whole. It is precisely because progress is 'materialist, or between singular essences' that Cavaillès can describe it as a revision of everything that has come before (p. 409).

Mathematics, literature, order

Conceiving of the formal in line with Cavaillès' description of mathematics takes thought out of the subject, which no longer conceals the ground on the basis of which inquiry supposedly rests. In place of the figure of man, described by Foucault as an empirical and transcendental doublet combining the object of knowledge and the condition of its possibility, there is now the open and autonomous development of order as a formal system with its own historicity. As a result, both empirical inquiry and the reflection on transcendental conditions are revealed not only as partial approaches unaware of their dependence on one another, but more seriously as incapable of recognising the existence of a strata that conditions the knowledge that each of them would call its own. The analytic of finitude that was to provide them both with a secure basis no longer has anything on which to work. The site previously occupied by phenomenology has been transformed and the figure of man erased. To the extent that Cavaillès indeed influences the course of Foucault's thinking here, mathematics provides a model for overcoming the philosophy of the subject. What is not yet clear is how its contribution leads to an understanding of the historical a priori. To approach this question, we can return to the proximity between mathematics and literature to which Foucault alludes in *The Order of Things*. This will make it possible to outline two interpretations of what Foucault describes as an experience of pure order.

Writing about Blanchot, Foucault departs from the view that modern literature is characterised by a 'doubling back' that enables it to refer only to itself, and thereby to open up an extreme interiority (Foucault 1986, 11–12). On the contrary, he continues, literature's break with representation has left it free to develop from itself in a 'network in which each point is distinct, distant from even its closest neighbours, and has a position in relation to every other position in a space that simultaneously holds and separates them all' (p. 12). The disappearance of the speaking subject clears a neutral space in which, far from closing in on itself, language is exposed to an exteriority more radical than anything the subject can experience, since it is the emptying out of the dimension within which the subject can call an experience its own. Unable to take up a position *before* language, the subject must relinquish the privilege of being present to itself. Only language speaks. As Foucault concludes, this may be why Western thought held back from thinking the being of language; 'as if it had a premonition of the danger that the naked experience of language poses for the self-evidence of the "I think"' (p. 13). One could begin to pull on the thread that leads from here back to Cavaillès' critique of necessity in phenomenology. However, it is important to see how clearly Foucault concedes that a reflection on the being of language has taken over from the analytic of finitude. In part, perhaps, because the text from which these lines are drawn is dedicated to Blanchot, the reflection on the being of language is characterised by Foucault not as a representation of any kind, but as an experience of the outside, as the space of subjectivity gives way to the 'naked experience of language' Foucault seems to be writing about an experience of language 'as such' which usually conveys the mastery of the subject in whom the field in question finds its formal or transcendental ground. Yet *this* experience of language, without a subject, is the undoing of any thought that aspires to such mastery. It is an experience of the outside arising from the effacement of the subject in the neutrality of order independent of all representation. In this way, the Being of language comes to the fore with the disappearance of the human subject, for whom the experience of language as such is indissociable from its own perpetual dissolution and therefore the *impossibility* of determining language as such. Contrary to what one might expect, to speak of language in this way is still to concede a determining role to the subject in shaping the structure of thought, even as the subject is overcome. The conservative influence of phenomenology is thereby only partially reversed. Where Nietzsche, having remarked on the death of God, observed that we shall continue to live in his shadow for a hundred years, so Foucault, in welcoming the disappearance of man, risks committing us to live in *his* shadow – unless he can distance himself more fully from the pursuit of the 'as such' that characterises phenomenology and still haunts the experience of pure language conceived in terms of impossibility and the dissolution of the subject. Cavaillès is decisive here, insofar as his approach to mathematics steadfastly avoids any idea that it can be determined or experienced 'as such' While it is true that he insists on the autonomy of mathematics from other disciplines, it is *not* the case that this independence is established formally and once and for all on the basis of a condition that stands apart from mathematics itself. Mathematics cannot be given as such, it can only be experienced in and through its actual practice. Similarly, the experience of mathematical objects is not given *to* a subject that brings a manifold under a form of unity, and so such experience does not point back to the ground of that unity in a determination of human finitude (such as the analytic). Cavaillès approaches mathematical thought as a series of combinatorial gestures or acts determined by rules that are immanent to the problematic in which they are lodged. Even the forms of time and space themselves are composed through the combinations and operations of mathematical thought as it takes each new step.[6] What that step will be depends on the configuration of the problematic that has led up to the point at which the question is asked, but each step will be necessary, constrained ultimately by the unity of the movement by which mathematics develops as a whole.

This departure from the consideration of any formal order *as such* already takes us some way towards Foucault, but there is at least one further step that must still be made. Since it does not concern Cavaillès directly, it will be outlined only briefly here. Cavaillès insisted on the necessity of mathematical reason and derived that necessity from the overall unity of mathematics (even if this is not evident to the mathematician engaged in a given problem). Foucault, of course, breaks with this condition of unity and necessity, and does so at least in part under the influence of Michel Serres, whose early work describes a model of rational systems in which points are linked by multiple paths in such a way that both relations and points vary in a complex pattern of reciprocal modification (Serres 1966). Serres takes this model as indicative of how thought operates in general, beyond the bounds of mathematics itself and his work explores similarities in the patterns of relation found between different disciplines, including those of science and literature. Although this is not the place to develop this point, there is a resonance between Serres' discussion of networks and Foucault's description of literary speech.

From the perspective of Cavaillès' critique of the philosophy of the subject and of the conception of experience as a synthesis of form and a manifold of intuitions, reinforced by a rejection that formal systems are characterised by unity and necessity at all, there can therefore be no encounter with either mathematics or language *as such*; neither on the basis of a totalising condition, nor even, as in Blanchot, as the being of language is disclosed by the withdrawal of its expressive and representative functions. Because there is no underlying logic defining its limits, and because its formal and material aspects are already combined in the concepts and objects it constructs, the experience of mathematics is *always specific*. With regard to language, experience is the relation we strike up with the constructed forms that language bears, *and* it is the experience of this construction itself. Order, then, does not lie beneath formal thought and language, as though it were a stable ground whose determination would legitimise the former's role and activity. The conditions for what is actual at a particular moment lie *within* the system whose development they determine. They cannot be disclosed as determining of a local set of phenomena or acts, since it would be impossible to formulate them without changing the system they describe. The conditions that Foucault is looking for are the actual regularities occurring both within a given discourse and across discourses. Foucault can therefore describe these conditions as an *a priori* insofar as they determine actuality without themselves appearing as such (they are not even possible objects of experience). At the same time, they are *historical* because the pattern of conditions that determines the effect they have at any given time is mutable.

Conditions understood in this way will necessarily be missed by the human sciences, which continue to search for the law of phenomena in things themselves and trace their development as a defined set of phenomena. They will also necessarily be missed by a reflection on the transcendental conditions of experience. Each approach fails to treat the conditions that lie concealed by their very distribution across a complex open system. This is why Foucault can say that order slips *between* the codes and the reflections on them. Order itself changes and the changes in the empirical sciences and transcendental philosophy are merely symptoms of this change. As such, not only are they not fundamental; they cannot even account adequately for transformations within their own discipline. The experience of order is therefore *not* the experience of a object, actual or formal. It is the experience of specific and unpredictable transformations in the codes of our experience. As Foucault writes, it is because there is order that a culture may find itself 'imperceptibly deviating from the empirical orders prescribed for it by its primary codes' (Foucault 1970, xx).

Foucault is therefore happy to accept the idea that mathematical reason can set the conditions for a 'second critique of pure reason' (Foucault 1970, 383), as long as it is based on a conception of the 'mathematical a priori' understood via Cavaillès. Moreover, Foucault's deployment of the historical a priori does not leave him having to account for the determination of experience by this new a priori, any more than Cavaillès continued to confront the problem of synthesis in Kant. For Foucault, there is neither actual experience, nor a sovereign subject to which such experience belongs. Phenomenological analysis gives way to an archaeology of the conditions of knowledge. Against expectations, perhaps, this gives a new lease of life to the question of the subject, which is released from the constraints of its role as a foundation for forms of knowledge ostensibly dedicated to its liberation. What interests Foucault is how subjects form by thinking, speaking and acting in the currency of such knowledge. Nearly fifteen years after the publication of *Les mots et les choses*, Foucault described critique as a reflection on limits, not in order to trace a possible metaphysics, and thereby secure the proper bounds of rational knowledge, but as a means of answering the following question: 'In what is given to us as universal, necessary and obligatory, what place is occupied by whatever is singular, contingent and the product of arbitrary constraints?' (Foucault 1994, 315). This conception of critique as an experimental and creative engagement with the conditions of what one can see, say, know and do has its beginning in Foucault's reading of Cavaillès.

Notes

1 For Foucault, the condition for the possibility of experience as such no longer coincides with the condition for the possibility of *this* object. This releases the conditions of *this* experience from the need to ground experience as such, and thereby allows the conditions of particular forms of experience to become historical.

2 Borges induced us to have a particularly extreme experience of this kind, but one that is nonetheless continually repeated in more discreet ways.

3 See Han 2002; Hyder 2003; elden 2001.

4 Cavaillès 1970. For the original French edition of this essay, see Cavaillès 1994, 473–560.

5 For Cavaillès, and subsequently for Foucault, the problematic character of the immediacy of evidence in phenomenology is associated with the way phenomenological understanding aims to reach acts and content that no longer refer to anything. 'On the one hand, there is nothing to question

beyond the act or the content in their immediate presence. On the other hand, the higher authority is this very presence or rather the impossibility of dissociating a single part or character from it through variation without losing everything. The foundation of all necessity is this "I cannot do otherwise" of eidetic variation which however legitimate it may be, is an abdication of all thought' (Cavaillès 1970, 408).

6 Cf. Cassou-Noguès 2001b: 'Puisque le temps ne préexiste pas à l'enchaînement des constructions, il doit être tiré de ces constructions mêmes. En réalité, le temps n'est que le rythme de la synthése, un ordre dans le geste' (p.4). Cf. also Cassou-Noguès 2001a, 174ff.

7
The mathematics of Deleuze's differential logic and metaphysics[1]

Simon Duffy

In *Difference and Repetition*, Deleuze explores the manner by means of which concepts are implicated in the problematic Idea by using a mathematics problem as an example, the elements of which are the differentials of the differential calculus. What I would like to offer in this essay is a historical account of the mathematical problematic that Deleuze deploys in his philosophy, and an introduction to the role played by this problematic in the development of his philosophy of difference. One of the points of departure that I will take from the history of mathematics is the theme of 'power series' (Deleuze 1994, 114), which will involve a detailed elaboration of the mechanism through which power series operate in the differential calculus deployed by Deleuze in *Difference and Repetition*. Deleuze actually constructs an alternative history of mathematics that establishes a historical continuity between the differential point of view of the infinitesimal calculus and modern theories of the differential calculus. It is in relation to this differential point of view that Deleuze determines a differential logic which he deploys, in the form of a logic of different/ciation, in the development of his project of constructing a philosophy of difference.

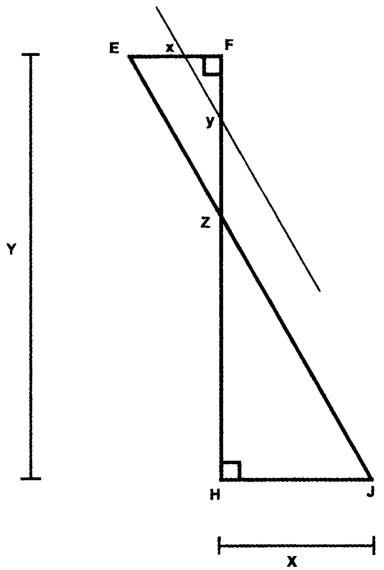
The differential point of view of the infinitesimal calculus.

The concept of the differential was introduced by developments in the infinitesimal calculus during the later part of the seventeenth century. Carl Boyer, in *The History of the Calculus and its Conceptual Development*, describes the early stages of this development as being 'bound up with concepts of geometry . . . and with explanations of . . . the infinitely small' (1959, 11). Boyer presents the infinitesimal calculus as dealing with 'the infinite sequences . . . obtained by continuing . . . to diminish ad infinitum the intervals between the values of the independent variable. . By means of [these] successive subdivisions the smallest possible intervals or differentials [are obtained]' (p. 12). The differential can therefore be understood to be the infinitesimal difference between consecutive values of a continuously diminishing quantity. Boyer refers to this early form of the infinitesimal calculus as the infinitesimal calculus from 'the differential point of view' (p. 12). From this point of view, Boyer argues that, 'the derivative would be defined as the quotient of two such differentials, and the integral would then be the sum of a number (perhaps finite, perhaps infinite) of such differentials' (p. 12).

The infinitesimal calculus consists of two branches which are inverse operations: differential calculus, which is concerned with calculating derivatives, or differential relations; and integral calculus, which is concerned with integration, or the calculation of the infinite sum of the differentials. The derivative, from the differential point of view of the infinitesimal calculus, is the quotient of two differentials, that is, a differential relation, of the type dy/dx. The differential, dy, is an infinitely small quantity, or what Deleuze describes as 'a vanishing quantity' (1981): a quantity smaller than any given or givable quantity. Therefore, as a vanishing quantity, dy, in relation to y, is, strictly speaking, equal to zero. In the same way, dx, in relation to x, is, strictly speaking, equal to zero, that is, dx is the vanishing quantity of x. Given that y is a quantity of the abscissa, and that x is a quantity of the ordinate, $dy = 0$ in relation to the abscissa, and $dx = 0$ in relation to the ordinate. The differential relation can therefore be written as $dy/dx = 0/0$. However, although dy is nothing in relation to y, and dx is nothing in relation to x, dy over dx does not cancel out, that is, dy/dx is not equal to zero. When the differentials are represented as being equal to zero, the relation can no longer be said to exist since the relation between two zeros is zero, that is $0/0 = 0$; there is no relation between two things which do not exist. However, the differentials do actually exist.[2] They exist as vanishing quantities insofar as they continue to vanish as quantities rather than having already vanished as quantities. Therefore, despite the fact that, strictly speaking, they equal zero, they are still not yet, or not quite equal to, zero. The relation between these two differentials, dy/dx, therefore does not equal zero, $dy/dx \neq 0$, despite the fact that $dy/dx = 0/0$. Instead, the differential relation itself, dy/dx, subsists as a relation. 'What subsists when dy and dx cancel out under the form of vanishing quantities is the relation dy/dx itself' (1981). Despite the fact that its terms vanish, the relation itself is nonetheless real. It is here that Deleuze considers seventeenth century logic to have made 'a fundamental leap', by determining 'a logic of relations' (1981). He argues that 'under this form of infinitesimal calculus is discovered a domain where the relations no longer depend on their terms' (1981). The concept of the infinitely small as vanishing quantities allows the determination of relations independently of their terms. 'The differential relation presents itself as the subsistence of the relation when the terms vanish' (1981). According to Deleuze, 'the terms between which the relation establishes itself are neither determined, nor determinable. Only the relation between its terms is determined' (1981). This is the logic of relations that Deleuze locates in the infinitesimal calculus of the seventeenth century.

The differential relation, which Deleuze characterises as a 'pure relation' (1981) because it is independent of its terms, and which subsists insofar as $dy/dx \neq 0$, has a perfectly expressible finite quantity designated by a third term, z, such that dy/dx equals z. Deleuze argues that 'when you have a [differential] relation derived from a circle, this relation doesn't involve the circle at all but refers [rather] to what is called a tangent' (1981). A tangent is a straight line that touches a circle or curve at only one point. The gradient of a tangent indicates the rate of change of the curve at that point, that is, the rate at which the curve changes on the y-axis relative to the x-axis. The differential relation therefore serves in the determination of this third term, z, the value of which is the gradient of the tangent to the circle or curve.

When referring to the geometrical study of curves in his early mathematical manuscripts, Leibniz writes that 'the differential calculus could be employed with diagrams in an even more wonderfully simple manner than it was with numbers' (Leibniz 1920, 53). Leibniz presents one such diagram in a paper entitled 'Justification of the Infinitesimal Calculus by That of Ordinary Algebra', when he offers an example of what had already been established of the infinitesimal calculus in relation to particular problems before the greater generality of its methods were developed (Leibniz 1969, 545). An outline of the example that Leibniz gives is as follows:



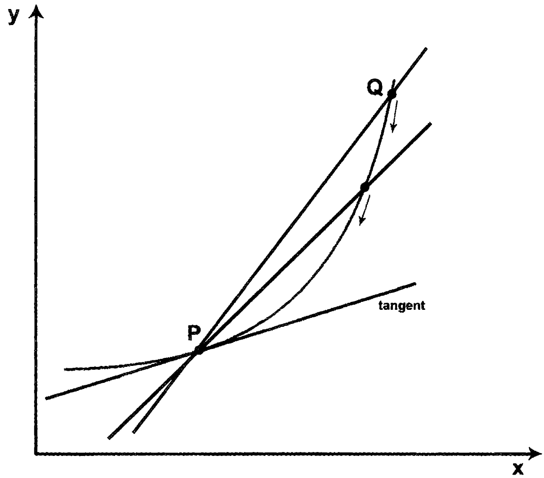
since the two right triangles, ZFE and ZHJ, that meet at their apex, point Z, are similar, it follows that the ratio *y/x* is equal to $(Y - y)/X$. As the straight line EJ approaches point F, maintaining the same angle at the variable point Z, the lengths of the straight lines FZ and FE, or *y* and *x*, steadily diminish, yet the ratio of *y* to *x* remains constant. When the straight line EJ passes through F, the points E and Z coincide with F, and the straight lines, *y* and *x*, vanish. Yet *y* and *x* will not be absolutely nothing since they preserve the ratio of ZH to HJ, represented by the proportion $(Y - y)/X$, which in this case reduces to *Y/X*, and obviously does not equal zero. The relation *y/x* continues to exist even though the terms have vanished since the relation is determinable as equal to *Y/X*. In this algebraic calculus, the vanished lines *x* and *y* are not taken for zeros since they still have an algebraic relation to each other. 'And so,' Leibniz argues, 'they are treated as infinitesimals, exactly as one of the elements which differential calculus recognises in the ordinates of curves for momentary increments and decrements' (1969, 545). That is, the vanished lines *x* and *y* are determinable in relation to each other only insofar as they can be replaced by the infinitesimals *dy* and *dx*, by making the supposition that the ratio *y/x* is equal to the ratio of the infinitesimals, *dy/dx*. In the first published account of the calculus, Leibniz defines the ratio of infinitesimals as the quotient of first-order differentials, or the differential relation. He says that 'the differential *dx* of the abscissa *x* is an arbitrary quantity, and that the differential *dy* of the ordinate *y* is defined as the quantity which is to *dx* as the ratio of the ordinate to the subtangent' (Boyer 1959, 210). Leibniz considers differentials to be the fundamental concepts of the infinitesimal calculus, the differential relation being defined in terms of these differentials.

A new theory of relations

Leibniz recognised integration to be a process not only of summation, but also of the inverse transformation of differentiation, so the integral is not only the sum of differentials, but also the inverse of the differential relation. In the early nineteenth century, the process of integration as a summation was overlooked by most mathematicians in favour of determining integration, instead, as the inverse transformation of differentiation. The main reason for this was that by extending sums to an infinite number of terms, problems began to emerge if the series did not converge. The value or sum of an infinite series is only determinable if the series converges. Divergent series have no sum. It was considered that reckoning with divergent series would therefore lead to false results. The problem of integration as a process of summation from the differential point of view of the infinitesimal calculus did, however, continue to be explored. It was Augustin Cauchy (b.1789 – 1857) who first introduced specific tests for the convergence of series, so that divergent series could henceforth be excluded from being used to try to solve problems of integration because of their propensity to lead to false results (Boyer 1959, 287).

The object of the process of integration in general is to determine from the coefficients of the given function of the differential relation the original function from which they were derived. Put simply, given a relation between two differentials, *dy/dx*, the problem of integration is to find a relation between the quantities *y* and *x,* themselves. This problem corresponds to the geometrical method of finding the function of a curve characterised by a given property of its tangent. The differential relation is thought of as another function which describes, at each point on an original function, the gradient of the line tangent to the curve at that point. The value of this 'gradient' indicates a specific quality of the original function; its rate of change at that point. The differential relation therefore indicates the specific qualitative nature of the original function at different points on the curve.

The inverse process to integration is differentiation, which, in geometrical terms, determines the differential relation as the function of the line tangent to a given curve. Put simply, to determine the tangent to a curve at a specified point, a second point that satisfies the function of the curve is selected, and the gradient of the line that runs through both of these points is calculated. As the second point approaches the point of tangency, the gradient of the line between the two points approaches the gradient of the tangent. The gradient of the tangent is, therefore, the limit of the gradient of the line between the two points as the points become infinitesimally close to one another.



It was Newton who first came up with this concept of a limit. He conceptualised the tangent geometrically, as the limit of a sequence of lines between two points, P and Q, on a curve, which he called a secant. As the distance between the points approached zero, the secants became progressively smaller, however they always retained 'a real length' The secant therefore approached the tangent without reaching it. When this distance 'got arbitrarily small (but remained a real number)' (Lakoff and Núñez 2000, 224), it was considered insignificant for practical purposes, and was ignored. What is different in Leibniz's method is that he 'hypothesized infinitely small numbers – infinitesimals – to designate the size of infinitely small intervals' (p. 224). For Newton, on the contrary, these intervals remained only small, and therefore real. When performing calculations, however, both approaches yielded the same results. But they differed ontologically, because Leibniz had hypothesised a new kind of number, a number Newton did not need, since 'his secants always had a real length, while Leibniz's had an infinitesimal length' (p. 224).

For the next two hundred years, various attempts were made to find a rigorous arithmetic foundation for the calculus. One that relied on neither the mathematical intuition of geometry, with its tangents and secants, which was perceived as imprecise because its conception of limits was not properly understood; nor the vagaries of the infinitesimal, which made many mathematicians wary, so much so that they refused the hypothesis outright, despite the fact that Leibniz 'could do calculus using arithmetic without geometry – by using infinitesimal numbers' (pp. 224–5).

What is at stake in the debate on the legitimacy of the infinitesimal is 'the integration of the infinitesimal into the register of quantity' (Salanskis 1996, 71), that is, of the infinite in the finite, which comes down to the alternative between infinite and finite representations. This is precisely what is at issue in what Deleuze describes as 'the "metaphysics" of the calculus' (Deleuze 1994, 176). Throughout the eighteenth century, there was disagreement as to the particular kind of 'metaphysics' by which 'to rescue the procedures of the calculus' from the vagaries of the infinitesimal. In speaking of the history of the differential calculus, Giorello argues that 'it was indeed a matter of rival metaphysical frameworks that provided the basis for widely differing programs' (1992, 160).

It was not until the late nineteenth century, that an adequate solution to this problem of rigour was posed. It was Karl Weierstrass (b. 1815 – 1897) who 'developed a pure nongeometric arithmetization for Newtonian calculus' (Lakoff and Núñez 2000, 230), which provided the rigour that had been lacking. The Weierstrassian program was determined by the following question: 'is the fate of calculus tied to infinitesimals, or must it not be given a rigorous status from the point of view of finite representations?' (Deleuze 1994, 177). 'Weierstrass' theory was an updated version of Cauchy's earlier account' (Lakoff and Núñez 2000, 309), which had also experienced problems conceptualising limits. Cauchy actually begs the question of the concept of limit in his proof.[3] In order to overcome this problem of conceptualising limits, Weierstrass 'sought to eliminate all geometry from the study of derivatives and integrals in calculus' (p. 309). In order to characterise calculus purely in terms of arithmetic, it was necessary for the idea of a function, as a curve in the Cartesian plane defined in terms of the motion of a point, to be completely replaced with the idea of a function that is, rather, a set of ordered pairs of real numbers. The geometric idea of 'approaching a limit' had to be replaced by an arithmetized concept of limit that relied on static logical constraints on numbers alone. This approach is commonly referred to as the epsilon-delta method. Deleuze argues that 'It is Weierstrass who bypasses all the interpretations of the differential calculus from Leibniz to Lagrange, by saying that it has nothing to do with a process Weierstrass gives an interpretation of the differential and infinitesimal calculus which he himself calls static, where there is no longer fluctuation towards a limit, nor any idea of threshold' (Deleuze 1972). The calculus was thereby reformulated without either geometric secants and tangents or infinitesimals; only the real numbers were used.

Because there is no reference to infinitesimals in this Weierstrassian definition of the calculus, the designation 'the infinitesimal calculus' was considered to be 'inappropriate' (Boyer 1959, 287). Weierstrass' work not only effectively removed any remnants of geometry from what was now referred to as the differential calculus, but it eliminated the use of Leibnizian-inspired infinitesimal arithmetic in doing the calculus for over half a century. It was not until the late 1960's, with the development of the controversial axioms of non-standard analysis by Abraham Robinson (b. 1918 – 1974), that the infinitesimal was given a rigorous foundation (See Bell 1998), and a formal theory of the infinitesimal calculus was constructed, thus allowing Leibniz's ideas to be 'fully vindicated' (Robinson 1996, 2), as Newton's had been thanks to Weierstrass.

As far as Deleuze is concerned, it is no longer a question of reluctantly tolerating the 'inexactitude' of the infinitesimal. They must rather be 'separated from [their] infinitesimal matrix' (Deleuze 1994, 171), that is, from their static representation as numbers, which eludes even the axioms of non-standard analysis, and this is effected by means of their implication in differential relations according to the logic of the differential from the differential point of view of the infinitesimal calculus. The undetermined differentials, or infinitesimals, dy or dx, are only determinable insofar as each is involved in a differential relation with another, that is, in reciprocal relation to one another, dy/dx. What counts is that it is within the differential relation itself that the differential possesses rigour and coherence; that the undetermined are determinable, by a process of reciprocal determination.

The Deleuzian solution to the debate over the legitimacy of the infinitesimal distinguishes itself from the Weierstrassian solution insofar as it is not resolved according to the program of discretization. Rather than representing one by means of the other, Deleuze argues that the alternative between infinite and finite representations, and therefore the metaphysics of the calculus, are 'strictly immanent to the techniques of the calculus itself' (Deleuze 1994, 176).

It is specifically in relation to these developments that Deleuze contends that, when understood from the differential point of view of that infinitesimal calculus, the value of z, which was determined by Leibniz in relation to the differential relation, dy/dx, as the gradient of the tangent, functions as a limit. When the relation establishes itself between infinitely small terms, it does not cancel itself out with its terms, but rather tends towards a limit. In other words, when the terms of the differential relation vanish, the relation subsists because it tends towards a limit, ¿. Since the differential relation approaches more closely to its limit as the differentials decrease in size, or approach zero, the limit of the relation is represented by the relation between the infinitely small. Of course, despite the geometrical nature of the idea of a variable and a limit, where variables 'decrease in size' or 'approach zero' and the differential relation 'approaches' or 'tends towards' a limit, they are not essentially dynamic, but involve purely static considerations, that is, they are rather 'to be taken automatically as a kind of shorthand for the corresponding developments of the epsilon-delta approach' (Lakoff and Núñez 2000, 277). It is in this sense that the differential relation between the infinitely small refers to something finite. Or, as Deleuze suggests, it is in the finite itself that there is the 'mutual immanence' (1981) of the relation and the infinitely small.

Given that the method of integration provides a way of working back from the differential relation, the problem of integration is, therefore, how to reverse this process of differentiation. This can be solved by determining the inverse of the given differential relation according to the inverse transformation of differentiation. Or, a solution can be determined from the differential point of view of the infinitesimal calculus by considering integration as a process of summation in the form of a series, according to which, given the specific qualitative nature of a tangent at a point, the problem becomes that of finding, not just one other point determinative of the differential relation, but a sequence of points, all of which together satisfy, or generate, a curve and therefore a function in the neighbourhood of the given point of tangency, which therefore functions as the limit of the function.

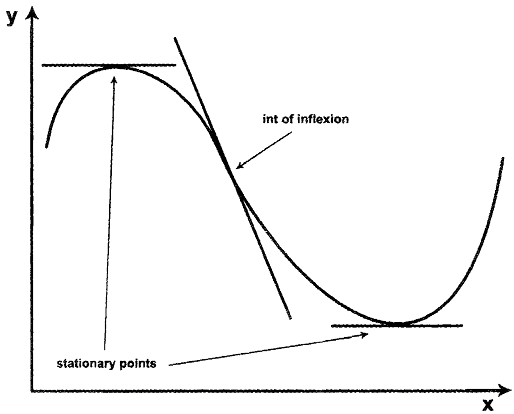
Deleuze considers this to be the base of the infinitesimal calculus as understood or interpreted in the seventeenth century. The formula for the problem of the infinite that Deleuze extracts from this seventeenth century understanding of the infinitesimal calculus, is that 'something finite consists of an infinity under a certain relation' (1981). Deleuze considers this formula to mark 'an equilibrium point, for seventeenth-century thought, between the infinite and the finite, by means of a new theory of relations' (1981). It is the logic of this theory of relations that provides a starting point for the investigation into the logic that Deleuze deploys in *Difference and Repetition* as part of his project of constructing a philosophy of difference.

The logic of the differential.

Having located the logic of the differential from the differential point of view of the infinitesimal calculus in the work of Leibniz, the subsequent developments that this logic undergoes will now be examined in relation to the work of some of the key figures in the history of this branch of the infinitesimal calculus. These figures are implicated in an alternative lineage in the history of mathematics by means of which the differential point of view of the infinitesimal calculus is aligned with the differential calculus of contemporary mathematics. The logic of the differential from the differential point of view of the infinitesimal calculus is then implicated in the development of Deleuze's project of constructing a philosophy of difference. The manner by means of which these figures are implicated in an alternative lineage in the history of mathematics will now be examined.

Ironically, one of the mathematicians who contributed to the development of the differential point of view is Karl Weierstrass, who considers the differential relation to be logically prior to the function in the process of determination associated with the infinitesimal calculus, that is, rather than determining the differential relation from a given function, the kinds of mathematical problems with which Weierstrass dealt involved investigating how to generate a function from a given differential relation. Weierstrass develops a theory of integration as the approximation of functions from differential relations according to a process of summation in the form of series. Despite Weierstrass having eliminated both geometry and the infinitesimal from the calculus, Deleuze recovers this theory in order to restore the Leibnizian perspective of the differential, as the genetic force of the differential relation, to the differential point of view of the infinitesimal calculus, by means of the infinitesimal axioms of non-standard analysis.

According to Deleuze's reading of the infinitesimal calculus from the differential point of view, a function does not precede the differential relation, but is rather determined by the differential relation. The differential relation is used to determine the overall shape of the curve of a function primarily by determining the number and distribution of its distinctive points, which are points of articulation where the nature of the curve changes or the function alters its behaviour. For example, in geometrical terms, when the differential relation is zero, the gradient of the tangent at that point is horizontal, indicating that the curve peaks or dips, determining therefore a maximum or minimum at that point. These distinctive points are known as stationary or turning points.



The differential relation characterises or qualifies not only the distinctive points which it determines, but also the nature of the regular points in the immediate neighbourhood of these points, that is, the shape of the branches of the curve between each distinctive point. Where the differential relation gives the value of the gradient at the distinctive point, the value of the derivative of the differential relation, that is, the second derivative, indicates the rate at which the gradient is changing at that point, which allows a more accurate approximation of the nature of the function in the neighbourhood of that point. The value of the third derivative indicates the rate at which the second derivative is changing at that point. In fact, the more successive derivatives that can be evaluated at the distinctive point, the more accurate will be the approximation of the function in the immediate neighbourhood of that point.

This method of approximation using successive derivatives is formalised in the calculus according to Weierstrass' theory by a Taylor series or power series expansion. A power series expansion can be written as a polynomial, the coefficients of each of its terms being the successive derivatives evaluated at the distinctive point. The sum of such a series represents the expanded function provided that any remainder approaches zero as the number of terms becomes infinite; the polynomial then becomes an infinite series which converges with the function in the neighbourhood of the distinctive point.[4] This criterion of convergence repeats Cauchy's earlier exclusion of divergent series from the calculus. A power series operates at each distinctive point by successively determining the specific qualitative nature of the function at that point. The power series determines not only the specific qualitative nature of the function at the point in question, but also the specific qualitative nature of all of the regular points in the neighbourhood of that distinctive point, such that the specific qualitative nature of a function in the neighbourhood of a distinctive point insists in that one point. By examining the relation between the differently distributed distinctive points determined by the differential relation, the regular points which are continuous between the distinctive points, that is, in geometrical terms, the branches of the curve, can be determined. In general, the power series converges with a function by generating a continuous branch of a curve in the neighbourhood of a distinctive point. To the extent that all of the regular points are continuous across all of the different branches generated by the power series of the distinctive points, the entire complex curve or the whole analytic function is generated.

So, according to Deleuze's reading of the infinitesimal calculus, the differential relation is generated by differentials and the power series are generated in a process involving the repeated differentiation of the differential relation. It is due to these processes that a function is generated in the first place. The mathematical elements of this interpretation are most clearly developed by Weierstrassian analysis, according to the theorem on the approximation of analytic functions. An analytic function, being secondary to the differential relation, is differentiable, and therefore continuous, at each point of its domain. According to Weierstrass, for any continuous function on a given interval, or domain, there exists a power series expansion which uniformly converges to this function on the given domain. Given that a power series approximates a function in such a restricted domain, the task is then to determine other power series expansions that approximate the same function in other domains. An analytic function is differentiable at each point of its domain, and is essentially defined for Weierstrass from the neighbourhood of a distinctive point by a power series expansion which is convergent with a 'circle of convergence' around that point. A power series expansion that is convergent in such a circle represents a function that is analytic at each point in the circle. By taking a point interior to the first circle as a new centre, and by determining the values of the coefficients of this new series using the function generated by the first series, a new series and a new centre of convergence are obtained, whose circle of convergence overlaps the first. The new series is continuous with the first if the values of the function coincide in the common part of the two circles. This method of 'analytic continuity' allows the gradual construction of a whole domain over which the generated function is continuous. At the points of the new circle of convergence which are exterior to, or extend outside the first, the function represented by the second series is then the analytic continuation of the function defined by the first series; what Weierstrass defines as the analytic continuation of a power series expansion outside its circle of convergence. The domain of the function is extended by the successive adjunction of more and more circles of convergence. Each series expansion which determines a circle of convergence is called an element of the function (Kline 1972, 643–4). In this way, given an element of an analytic function, by analytic continuation one can obtain the entire analytic function over an extended domain. The analytic continuation of power series expansions can be continued in this way in all directions up to the points in the immediate neighbourhood exterior to the circles of convergence where the series obtained diverge.

Power series expansions diverge at specific 'singular points' or 'singularities' that may arise in the process of analytic continuity. A singular point or singularity of an analytic function is any point which is not a regular or ordinary point of the function. They are points which exhibit distinctive properties and thereby have a dominating and exceptional role in the determination of the characteristics of the function.[5] The distinctive points of a function, which include the turning points, where $dy/dx = 0$, and points of inflection, where $d^2y/dx^2 = 0$, are 'removable singular points', since the power series at these points converge with the function. A removable singular point is uniformly determined by the function and therefore redefinable as a distinctive point of the function, such that the function is analytic or continuous at that point. The specific singularities of an analytic function where the series obtained diverge are called 'poles' Singularities of this kind are those points where the function no longer satisfies the conditions of regularity which assure its local continuity, such that the rule of analytic continuity breaks down. They are therefore points of discontinuity. A singularity is called a pole of a function when the values of the differential relation, that is, the gradients of the tangents to the points of the function, approach infinity as the function approaches the pole. The function is said to be asymptotic to the pole, it is therefore no longer differentiable at that point, but rather remains undefined, or vanishes. A pole is therefore the limit point of a function, and is referred to as an accumulation point or point of condensation. A pole can also be referred to as a jump discontinuity in relation to a finite discontinuous interval both within the same function, for example periodic functions, and between neighbouring analytic functions. Deleuze writes that 'a singularity is the point of departure for a series which extends over all the ordinary points of the system, as far as the region of another singularity which itself gives rise to another series which may either converge or diverge from the first' (Deleuze 1994, 278). The singularities whose series converge are removable singular points, and those whose series diverge are poles.

The singularities, or poles, that arise in the process of analytic continuity necessarily lie on the boundaries of the circles of convergence of power series. In the neighbourhood of a pole, a circle of convergence extends as far as the pole in order to avoid including it, and the poles of any neighbouring functions, within its domain. The effective domain of an analytic function determined by the process of the analytic continuation of power series expansions is therefore limited to that between its poles. With this method the domain is not circumscribed in advance, but results rather from the succession of local operations.

Power series can be used in this way to solve differential relations by determining the analytic function into which they can be expanded. Weierstrass developed his theory alongside the integral conception of Cauchy, which further developed the inverse relation between the differential and the integral calculus as the fundamental theorem of the calculus. The fundamental theorem maintains that differentiation and integration are inverse operations, such that integrals are computed by finding anti-derivatives, which are otherwise known as primitive functions. There are a large number of rules, or algorithms, according to which this reversal is effected.

Deleuze presents Weierstrass' theorem of approximation as an effective method for determining the characteristics of a function from the differential point of view of the infinitesimal calculus. The mathematician Albert Lautman (b. 1908 – 1944) refers to this process as integration from 'the local point of view', or simply as 'local integration'

(Lautman 1938a, 38). This form of integration does not involve the determination of the primitive function, which is generated by exercising the inverse operation of integration. The development of a local point of view, rather, requires the analysis of the characteristics of a function at its singular points. The passage from the analytic function defined in the neighbourhood of a singular point, to the analytic function defined in each ordinary point is made according to the ideas of Weierstrass by analytic continuity. This method was eventually deduced from the Cauchy point of view, such that the Weierstrassian approach was no longer emphasised. The unification of both of these points of view, however, was achieved at the beginning of the twentieth century when the rigour of Cauchy's ideas, which were then fused with those of Georg Riemann (b.1826 – 1866), the other major contributor to the development of the theory of functions, was improved. Deleuze is therefore able to cite the contribution of Weierstrass' theorem of approximation in the development of the differential point of view of the infinitesimal calculus as an alternative point of view of the differential calculus to that developed by Cauchy, and thereby establish a historical continuity between Leibniz's differential point of view of the infinitesimal calculus and the differential calculus of contemporary mathematics, thanks to the axioms of non-standard analysis which allow the inclusion of the infinitesimal in its arithmetization.

The development of a differential philosophy

While Deleuze draws inspiration and guidance from Salomon Maïmon (b. 1753 – 1800), who 'sought to ground post Kantianism upon a Leibnizian reinterpretation of the calculus' (Deleuze 1994, 170), and 'who proposes a fundamental reformation of the *Critique* and an overcoming of the Kantian duality between concept and intuition' (Deleuze 1994, 173), it is in the work of Höené Wronski (b.1778 – 1853) that Deleuze finds the established expression of the first principle of the differential philosophy. Wronski was 'an eager devotee of the differential method of Leibniz and of the transcendental philosophy of Kant' (Boyer 1959, 261). Wronski made a transcendental distinction between the finite and the infinitesimal, determined by the two heterogeneous functions of knowledge, understanding and reason. He argued that 'finite quantities bear upon the objects of our knowledge, and infinitesimal quantities on the very generation of this knowledge; such that each of these two classes of knowledge must have laws proper [to themselves], and it is in the distinction between these laws that the major thesis of the metaphysics of infinitesimal quantities is to be found' (Wronski 1814, 35; Blay 1998, 158). It is imperative not to confuse 'the objective laws of finite quantities with the purely subjective laws of infinitesimal quantities' (p. 36; 158). He claims that it is this 'confusion that is the source of the inexactitude that is felt to be attached to the infinitesimal Calculus   This is also [why] geometers, especially those of the present day, consider the infinitesimal Calculus, which nonetheless they concede always gives true results, to be only an indirect or artificial procedure' (p. 36; 159). Wronski is referring here to the work of Joseph-Louis Lagrange (b. 1736 – 1813) and Lazarre Carnot (b. 1753 – 1823), two of the major figures in the history of the differential calculus, whose attempts to provide a rigorous foundation for the differential calculus involved the elimination of the infinitesimal from all calculations, or as Wronski argued, involved confusing objective and subjective laws in favour of finite quantities (See Blay 1998, 159). Both of these figures count as precursors to the work of Cauchy and Weierstrass. Wronski argued that the differential calculus constituted 'a *primitive algorithm* governing the *generation* of quantities, rather than the laws of quantities *already formed*' (Boyer 1959, 262). According to Wronski, the differential should be interpreted 'as having an a priori metaphysical reality associated with the generation of magnitude' (p. 262). The differential is therefore expressed as a pure element of quantitability; insofar as it prepares for the determination of quantity. The work of Wronski represents an extreme example of the differential point of view of the infinitesimal calculus which recurs throughout the nineteenth century.

Another significant figure in this alternative history of mathematics that is constructed by Deleuze is Jean Baptiste Bordas-Demoulin (b. 1798 – 1859), who also champions the infinitesimal against those who consider that infinitesimals had to be eliminated in favour of finite quantities. Bordas-Demoulin does not absolve the differential calculus of the accusation of error, but rather considers the differential calculus to have this error as its principle. According to Bordas-Demoulin, the minimal error of the infinitesimal 'finds itself compensated by reference to an error active in the contrary sense.   It is in all necessity that the errors are mutually compensated' (Bordas-Demoulin 1874, 414; my translation). The consequence of this mutual compensation 'is that one differential is only exact after having been combined with another' (p. 414). Deleuze repeats these arguments of Wronski and Bordas-Demoulin when he maintains that it is in the differential relation that the differential is realised as a pure element of quantitability. Each term of the relation, that is, each differential, each pure element of quantitability, therefore 'exists absolutely only in its relation to the other' (Deleuze 1994, 172), that is, only insofar as it is reciprocally determined in relation to another.

The question for Deleuze then becomes: 'in what form is the differential relation determinable?' (Deleuze 1994, 172) He argues that it is determinable primarily in qualitative form, insofar as it is the reciprocal relation between differentials, and then secondarily, insofar as it is the function of a tangent whose values give the gradient of the line tangent to a curve, or the specific qualitative nature of this curve, at a point. As the function of a tangent, the differential relation 'expresses a function which differs in kind from the so-called primitive function' (p. 172). Whereas the primitive function, when differentiated, expresses the whole curve directly,[6] the differential relation, when differentiated, expresses rather the further qualification of the nature of the function at, or in the immediate neighbourhood of, a specific point. The primitive function is the integral of the function determined by the inverse transformation of differentiation, according to the differential calculus. From the differential point of view of the infinitesimal calculus, the differential relation, as the function of the tangent, determines the existence and distribution of the distinctive points of a function, thus preparing for its further qualification. Unlike the primitive function, the differential relation remains tied to the specific qualitative nature of the function at those distinctive points, and, as the function of the tangent, it 'is therefore differentiable in turn' (p. 172). When the differential relation is repeatedly differentiated at a distinctive point generating a power series expansion, what is increasingly specified is the qualitative nature of the function in the immediate neighbourhood of that point. Deleuze argues that this convergence of a power series with an analytic function, in its immediate neighbourhood, satisfies 'the minimal conditions of an integral' (p. 174), and characterises what is for Deleuze the process of 'differen*t*iation' (p. 209).

The differential relation expresses the qualitative relation between, not only curves and straight lines, but also between linear dimensions and their functions, and plane or surface dimensions and their functions. The domain of the successive adjunction of circles of convergence, as determined by analytic continuity, actually has the structure of a surface. This surface is constituted by the points of the domain and the direction attached to each point in the domain, that is, the tangents to the curve at each point and the direction of the curve at that point. Such a surface can be described as a field of directions or a vector field. A vector is a quantity having both magnitude and direction. The point of departure of the local genesis of functions is from the point of view of the structure of such a surface as a vector field. It is within this context that the example of a jump discontinuity in relation to a finite discontinuous interval between neighbouring analytic or local functions is developed by Deleuze, in order to characterise the generation of another function which extends beyond the points of discontinuity which determine the limits of these local functions. Such a function would characterise the relation between the different domains of different local functions. The genesis of such a function from the local point of view is initially determined by taking any two points on the surface of a vector field, such that each point is a pole of a local function determined independently by the point-wise operations of Weierstrassian analysis. The so-determined local functions, which have no common distinctive points or poles in the domain, are discontinuous with each other; each pole being a point of discontinuity, or limit point, for its respective local function. Rather than simply being considered as the unchanging limits of local functions generated by analytic continuity, the limit points of each local function can be considered in relation to each other, within the context of the generation of a new function which encompasses the limit points of each local function and the discontinuity that extends between them. Such a function can initially be understood to be a potential function, which is determined as a line of discontinuity between the poles of the two local functions on the surface of the vector field. The potential function admits these two points as the poles of its domain. However, the domain of the potential function is on a scalar field, which is distinct from the vector field insofar as it is composed of points (scalars) which are non-directional; scalar points are the points onto which a vector field is mapped. The potential function can be defined by the succession of points (scalars) which stretch between the two poles. The scalar field of the potential function is distinct from the vector field of the local functions insofar as, mathematically speaking, it is 'cut' from the surface of the vector field. Deleuze argues that 'the limit must be conceived not as the limit of a [local] function but as a genuine cut [*coupure*], a border between the changeable and the unchangeable within the function itself. . . . the limit no longer presupposes the ideas of a continuous variable and infinite approximation. On the contrary, the concept of limit grounds a new, static and purely ideal definition' (Deleuze 1994, 172); that of the potential function. To cut the surface from one of these poles to the next is to generate such a potential function. The poles of the potential function determine the limits of the discontinuous domain, or scalar field, which is cut from the surface of the vector field. The 'cut' of the surface in this theory renders the structure of the potential function 'apt to a creation' (Lautman 1938a, 8). The precise moment of production, or genesis, resides in the act by which the cut renders the variables of certain functional expressions able to 'jump' from pole to pole across the cut. When the variable jumps across this cut, the domain of the potential function is no longer uniformly discontinuous. With each 'jump', the poles which determine the domain of discontinuity, represented by the potential function sustained across the cut, seem to have been removed. The less the cut separates the potential function on the scalar field from the surface of the vector field, the more the poles seem to have been removed, and the more the potential function seems to be continuous with the local functions across the whole surface of the vector field. It is only insofar as this interpretation is conferred on the structure of the potential function that a new function can be understood to have been generated on the surface. A potential function is only generated when there is potential for the creation of a new function between the poles of two local functions. The potential function is therefore always apt to the creation of a new function. This new function, which encompasses the limit points of each local function and the discontinuity that extends between them, is continuous across the structure of the potential function; it completes the structure of the potential function, as what can be referred to as a 'composite function' The connection between the structural completion of the potential function and the generation of the corresponding composite function is the act by which the variable jumps from pole to pole. When the variable jumps across the cut, the value of the composite function sustains a fixed increase. Although the increase seems to be sustained by the potential function, it is this increase which actually registers the generation or complete determination of the composite function.

The complete determination of a composite function by the structural completion of the potential function is not determined by Weierstrass' theory of analytic continuity. A function is able to be determined as continuous by analytic continuity across singular points which are removable, but not across singular points which are non-removable. The poles that determine the parameters of the domain of the potential function are non-removable, thus analytic continuity between the two functions, across the cut, is not able to be established. Weierstrass, however, recognised a means of solving this problem by extending his analysis to meromorphic functions.[7] A function is said to be meromorphic in a domain if it is analytic in the domain determined by the poles of analytic functions. A meromorphic function is determined by the quotient of two arbitrary analytic functions, which have been determined independently on the same surface by the point-wise operations of Weierstrassian analysis. Such a function is defined by the differential relation:

$$\frac{dy}{dx} = \frac{Y}{X}$$

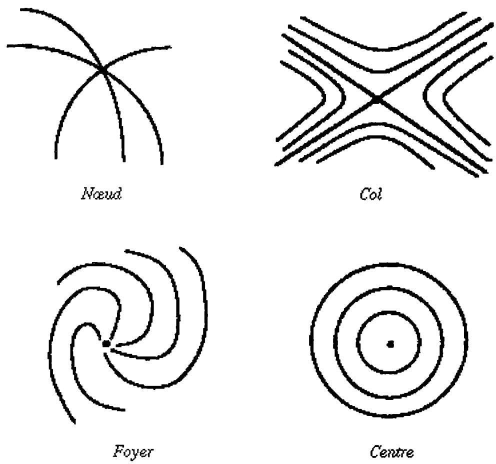
where X and Y are the polynomials, or power series of the two local functions. The meromorphic function, as the function of a differential relation, is just the kind of function which can be understood to have been generated by the structural completion of the potential function. The meromorphic function is therefore the differential relation of the composite function. The expansion of the power series determined by the repeated differentiation of the meromorphic function should generate a function which converges with a composite function. The graph of a composite function, however, consists of curves with infinite branches, because the series generated by the expansion of the meromorphic function is divergent. The representation of such curves posed a problem for Weierstrass, which he was unable to resolve, because divergent series fall outside the parameters of the differential calculus, as determined by the epsilon-delta approach, since they defy the criterion of convergence.

The qualitative theory of differential equations.

Henri Poincaré (b.1854 – 1912) took up this problem of the representation of composite functions, by extending the Weierstrass theory of meromorphic functions into what was called 'the qualitative theory of differential equations' (Kline 1972, 732). In place of studying the properties of complex functions in the neighbourhood of their singularities, Poincaré was primarily occupied with determining the properties of complex functions in the whole plane, that is, the properties of the entire curve. This qualitative method involved initial investigation of the geometrical form of the curves of functions with infinite branches; only then was numerical determination of the values of the function able to be made. While such divergent series do not converge, in the Weierstrassian sense, to a function, they may indeed furnish a useful approximation to a function if they can be said to represent the function asymptotically. When such a series is asymptotic to the function, it can represent an analytic or composite function even though the series is divergent.

When this geometrical interpretation was applied to composite functions, Poincaré found the values of the composite function around the singularity produced by the function to be undetermined and irregular. The singularity of a composite function would be the point at which both the numerator and denominator of the quotient of the meromorphic function determinative of the composite function vanish (or equal zero). The peculiarity of the meromorphic function is that the numerator and denominator do not vanish at the same point on the surface of the domain. The points at which the two local functions of the quotient vanish are at their respective poles. The determination of a composite function therefore requires the determination of a new singularity in relation to the poles of the local functions of which it is composed. Poincaré called this new kind of singularity an essential singularity. Observing that the values of a composite function very close to an essential singularity fluctuate through a range of different possibilities without stabilising, Poincaré distinguished four types of essential singularity, which he classified according to the behaviour of the function and the geometrical appearance of the solution curves in the neighbourhood of these points. The first type of singularity is the saddle point or dip (*col*), through which only two solution curves pass, acting as asymptotes for neighbouring curves. A saddle point is neither a maximum nor minimum, since the value of the function either increases or decreases depending on the direction of movement away from it. The second kind of singularity is the node (*nœud*), which is a point through which an infinite number of curves pass. The third type of singularity is the point of focus (*foyer*), around which the solution curves turn and towards which they approach in the same way as logarithmic spirals. And the fourth, called a centre, is a point around which the curves are closed, concentric with one another and the centre.

The type of essential singularity is determined by the form of the constitutive curves of the meromorphic function. Whereas the potential function remains discontinuous with the other functions on the surface from which it is cut, thereby representing a discontinuous group of functions, the composite function, on the contrary, overcomes this discontinuity insofar as it is continuous in the domain which extends across the whole surface of the discontinuous group of functions. The existence of such a continuous function, however, does not express any less the



properties of the domain of discontinuity which serves to define it. The discontinuous group of local functions and the continuous composite function attached to this group exist alongside each other, the transformation from one to the other being determined by the process of the generation and expansion of the meromorphic function. The potential function is actualised in the composite function when the variable jumps from one pole to the other. Its trajectory, in the form of a solution curve, is determined by the type of essential singularity created by the meromorphic function. The essential singularity determines the behaviour of the composite function, or the appearance of the solution curve, in its immediate neighbourhood by acting as an *attractor* for the trajectory of the variable across its domain. It is the value of this function which sustains a determined increase with each jump of the variable. Insofar as the trajectory of each variable is attracted to the same final state represented by each of the different essential singularities, these essential singularities can be understood to represent what Manuel DeLanda describes as the 'inherent or intrinsic *long-term tendencies* of a system, the states which the system will spontaneously tend to adopt in the long run as long as it is not constrained by other forces' (2002, 15).

Deleuze distinguishes this differential point of view of the infinitesimal calculus from the Weierstrassian theory of approximation when he writes that: 'No doubt the specification of the singular points (for example, dips, nodes, focal points, centres) is undertaken by means of the form of integral curves, which refer back to the solutions for the differential equations. There is nevertheless a complete determination with regard to the existence and distribution of these points which depends upon a completely different instance – namely, the field of vectors defined by the equation itself. The complementarity of these two aspects does not obscure their difference in kind – on the contrary' (Deleuze 1994, 177). The equation to which Deleuze refers is the meromorphic function, which is a differential equation or function of a differential relation determined according to the Weierstrassian approach, from which the essential singularity and therefore the composite function are determined according to Poincaré's qualitative approach. This form of integration is again characterised from the local point of view, by what Deleuze describes as 'an original process of differenciation' (p. 209). Differenciation is the complete determination of the composite function from the reciprocally-determined local functions or the structural completion of the potential function. It is the process whereby a potential function is actualised as a composite function.

Deleuze states that 'actualisation or differenciation is always a genuine creation', and that to be actualised is 'to create divergent lines' (Deleuze 1994, 212). The expanded power series of a meromorphic function is actualised in the composite function insofar as it converges with, or creates, the divergent lines of the composite function. Differenciation, therefore, creates an essential singularity, whose divergent lines actualise the specific qualitative nature of the poles of the group of discontinuous local functions, represented by a potential function, in the form of a composite function. These complex functions can be understood to be what Poincaré called 'Fuschian functions', which, as Georges Valiron points out, 'are more often called automorphic functions' (Valiron 1971, 171). The discontinuous group of local functions can therefore also be understood to be Fuschian groups. Poincaré's pioneering work in this area eventually lead to the definitive founding of the geometric theory of analytic functions, the study of which 'has not yet been completely carried out' (p. 173), but continues to be developed with the assistance of computers.

Benoit Mandelbrot (b. 1924) considers Poincaré, with his concept of essential singularities, to be 'the first student of fractal ('strange') attractors', that is, of the kinds of attractors operative in fractals which occur in mathematics, and cites certain theories of Poincaré as having 'led [him] to new lines of research', specifically 'the theory of automorphic functions' which made Poincaré and Felix Klein (b. 1849 – 1925) famous (1982, 414).[8]

Deleuze does not consider this process of differen*c*iation to be arrested with the generation of a composite function, but rather it continues, generating those functions which actualise the relations between different composite functions, and those functions which actualise the relations between these functions, and so on. The conception of differen*c*iation is extended in this way when Deleuze states that 'there is a differenciation of differenciation which integrates and welds together the differenciated' (Deleuze 1994, 217); each differen*c*iation is simultaneously 'a local integration', which then connects with others, according to the same logic, in what is characterised as a 'global integration' (p. 211).

The logic of the differential, as determined according to both differen*t*iation and differen*c*iation, designates a process of production, or genesis, which has, for Deleuze, the value of introducing a general theory of relations which unites the Weierstrassian structural considerations of the differential calculus with the concept of 'the generation of quantities' (Deleuze 1994, 175). 'In order to designate the integrity or the integrality of the object', when considered as a composite function from the differential point of view of the infinitesimal calculus, Deleuze argues that, 'we require the complex concept of differen*t*/*c*iation. The *t* and the *c* here are the distinctive feature or the phonological relation of difference in person' (p. 209). Deleuze argues that differen*c*iation is 'the second part of difference' (p. 209), the first being expressed by the logic of the differential in differen*t*iation. Where the logic of differen*t*iation characterises a differential philosophy, the complex concept of the logic of differen*t*/*c*iation characterises Deleuze's 'philosophy of difference'

The differential point of view of the infinitesimal calculus represents an opening, providing a trajectory for the construction of an alternative history of mathematics; it actually anticipates the return of the infinitesimal in the differential calculus of contemporary mathematics, thanks to the axioms of non-standard analysis. This is the interpretation of the differential calculus to which Deleuze is referring when he appeals to the 'barbaric or pre-scientific interpretations of the differential calculus' (Deleuze 1994, 171). Deleuze thereby establishes a historical continuity between the differential point of view of the infinitesimal calculus and modern theories of the differential calculus which surpasses the methods of the differential calculus which Weierstrass uses in the epsilon-delta approach to support the development of a rigorous foundation for the calculus. While Weierstrass is interested in making advances in mathematics to secure the development of a rigorous foundation for the differential calculus, Deleuze is interested in using mathematics to problematise the reduction of the differential calculus to set theory, by determining an alternative trajectory in the history of mathematics, one that retrospectively allows the reintroduction of the infinitesimal into an understanding of the operation of the calculus. According to Deleuze, the 'finitist interpretations' of the calculus given in modern set-theoretical mathematics – which are congruent with 'Cantorian finitism' (Maddy 1988, 488), that is, 'the idea that infinite entities are considered to be finite within set theory' (Salanskis 1996, 66) – betray the nature of the differential no less than Weierstrass, since they 'both fail to capture the extra-propositional or sub-representative source from which calculus draws its power' (Deleuze 1994, 264). He maintains that 'the derivative and the integral have become ordinal rather than quantitative concepts' (p. 176).

In constructing this theory of relations characteristic of a philosophy of difference, Deleuze draws significantly from the work of Albert Lautman, who refers to this whole process as 'the metaphysics of logic' (Lautman 1938a, 3). It is in *Difference and Repetition* that Deleuze formulates a 'metaphysics of logic' that corresponds to the logic of the differential from the differential point of view of the infinitesimal calculus. However, he argues that 'we should speak of a dialectics of the calculus rather than a metaphysics' (Deleuze 1994, 178), since 'each engendered domain, in which dialectical Ideas of this or that order are incarnated, possesses its own calculus. It is not mathematics which is applied to other domains but the dialectic which establishes the direct differential calculus corresponding or appropriate to the domain under consideration' (p. 181). Just as he argued that mathematics 'does not include only solutions to problems; it also includes the expression of problems relative to the field of solvability which they define. That is why the differential calculus belongs to mathematics, even at the very moment when it finds its sense in the revelation of a dialectic which points beyond mathematics' (p. 179). It is in the differential point of view of the infinitesimal calculus that Deleuze finds a form of the differential calculus appropriate to the determination of a differential logic. This logic is deployed by Deleuze, in the form of the logic of differen*t*/*c*iation, in the development of his project of constructing a philosophy of difference.

The relation between the finite and the infinitesimal is determined according to what Lautman describes as 'the logical schemas which preside over the organisation of their edifices' (Lautman 1938b, 58). Lautman argues that 'it is possible to recover within mathematical theories, logical Ideas incarnated in the same movement of these theories' (p. 58). The logical Ideas to which Lautman refers include the relations of expression between the finite and the infinitesimal. He argues that these logical Ideas 'have no other purpose than to contribute to the illumination of logical schemas within mathematics, which are only knowable through the mathematics themselves' (p. 58). The project of the present essay has been to locate these 'logical Ideas' in the mathematical theory of the infinitesimal calculus from the differential point of view, in order then to determine how Deleuze uses these 'logical Ideas' to develop the logical schema of a theory of relations characteristic of a philosophy of difference.

Notes

1 This essay draws on earlier work that was published in *Angelaki* 9.3 (2004). See Duffy 2004b.

2 Deleuze acknowledges that 'the interpretation of the differential calculus has indeed taken the form of asking whether infinitesimals are real or fictive' (Deleuze 1994, 177). However, for Deleuze, the question is rhetorical, for it is of little importance whether the infinitely small are real, and if they are not this does not signify the contemptible fictive character of their position. What is at stake in the debate on the legitimacy of the infinitesimal is 'the integration of the infinitesimal into the register of quantity' (Salanskis 1996, 71), that is, of the infinite in the finite, which comes down to the choice between infinite and finite representations. This problem is taken up in the next section of this essay entitled 'A new theory of relations'

3 For a thorough analysis of this problem with limits in Cauchy, see Boyer 1959, 281.

4 Given a function, $f(x)$, having derivatives of all orders, the Taylor series of the function is given by

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

where $f^{(k)}(a)$ is the *kth* derivative of f at a. A function is equal to its Taylor series if and only if its error term R_n can be made arbitrarily small, where

$$R_n = \left| f(x) - \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \right|$$

The Taylor series of a function can be represented in the form of a power series, which is given by

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdot \quad \cdot + a_n x^n +$$

where each a is a distinct constant. It can be shown that any such series either converges at $x = 0$, or for all real x, or for all x with $-R < x < R$ for some positive real R. The interval $(-R, R)$ is called the circle of convergence, or neighbourhood of the distinctive point. This series should be thought of as a function in x for all x in the circle of convergence. Where defined, this function has derivatives of all orders. See Reinhardt 1985.

5 Deleuze argues that 'It was a great day for philosophy when . Leibniz proposed . that there is no reason for you simply to oppose the singular to the universal. It's much more interesting if you listen to what mathematicians say, who for their own reasons think of "singular" not in relation to "universal", but in relation to "ordinary" or "regular"' (Deleuze 1980).

6 Note: the primitive function $\int f(x)dx$, expresses the whole curve $f(x)$.

7 It was Charles A. A. Briot (b. 1817 – 1882) and Jean-Claude Bouquet (b. 1819 – 1885) who introduced the term 'meromorphic' for a function which possessed just poles in that domain (Kline 1972, 642).

8 Mandelbrot qualifies these statements when he says of Poincaré that 'nothing I know of his work makes him even a distant precursor of the fractal geometry of the visible facets of Nature'(1982, 414).

8

Axiomatics and problematics as two modes of formalisation: Deleuze's epistemology of mathematics*

Daniel W. Smith

1. Introduction: problematics, dialectics, and ideas

Throughout his work, Gilles Deleuze has developed a distinction between two modes of formalisation, in mathematics and elsewhere, which he terms, respectively, 'axiomatics' and 'problematics.'[1] The axiomatic (or 'theorematic') method of formalisation is a familiar one, already having a long history in mathematics, philosophy, and logic, from Euclid's geometry to Spinoza's philosophy to the formalised systems of modern symbolic logic. Although problematics has had an equally determinate trajectory in the history of mathematics, it is a more subterranean and less visible trajectory, but one that has increasingly become the object of study in contemporary philosophy of mathematics. Deleuze argues that the recognition of the irreducibility of problems and their genetic role in mathematics is 'one of the most original characteristics of modern epistemology,' as exemplified in the otherwise diverse work of thinkers such as Canguilhem, Bouligand, Vuillemin, and Lautman.[2]

Deleuze's contribution to these debates has been to take up the mathematical concept of problematics and to give it an unparalleled status in philosophy. The fundamental difference between these two modes of formalisation can be seen in their differing methods of deduction: in axiomatics, a deduction moves from axioms to the theorems that are derived from it, whereas in problematics a deduction moves from the problem to the ideal accidents and events that condition the problem and form the cases that resolve it. More generally, Deleuze characterises axiomatics as belonging to a 'major' or royal form of science, which constantly attempts to effect a reduction or repression (or more accurately, an arithmetic conversion) of the problematic pole of mathematics, itself wedded to a 'minor' or nomadic conception of science. 'What we have are two formally different conceptions of science, and ontologically *a single field of interaction* in which royal science [e.g., axiomatics] continually appropriates the contents of vague or nomad science [problematics], while nomad science continually cuts the contents of royal science loose' (Deleuze and Guattari 1987, 362, 367. Emphasis added). Many of the most important concepts in Deleuze's own philosophy – such as multiplicity, the differential, singularity, series, zones of indiscernibility, and so on – were adopted from this problematic pole of mathematics, and particularly from the history of the calculus. My primary purpose in this essay will be to elucidate the epistemological differences between problematics and axiomatics.

We should note, however, that Deleuze's interest in the mathematics of problematics is not purely epistemological, but stems from his more general interest in the status of problems in philosophy. The activity of thinking has often been conceived of as the search for solutions to problems, but this is a prejudice whose roots, Deleuze suggests, are both social and pedagogical: in the classroom, it is the mathematics teacher who poses the problems, the pupil's task being to discover the correct solution. What the notions of 'true' and 'false' serve to qualify are precisely the responses or solutions that are given to these already-formulated questions or problems. Yet at the same time, everyone recognises that problems are never given ready-made, but must be constructed or constituted – hence the scandal when a 'false' or badly-formulated problem is set in an examination. 'While it is relatively easy to define the true and the false in relation to solutions whose problems are already stated,' Deleuze writes in *Bergsonism*, '*it is much more difficult to say what the true and false consist of when they are applied directly to problems themselves*' (1988, 16–17). In fact, philosophy is concerned less with the solution to eternal problems than the constitution of problems themselves, and the means for distinguishing between legitimate and illegitimate problems, that is, between true and false problems.

In the history of philosophy, the science of problems has always had a precise name: *dialectics*. In Plato's dialectic, for instance, the appeal to a foundational realm of essence (Ideas) first appeared as the response to a particular way of posing problems, a particular form of the question – namely the question 'What is. .?' [*ti estin*?]. 'The idea, writes Deleuze, 'the discovery of the Idea, is not separable from a certain type of question. The Idea is first of all an 'objectity' [*objectité*] that corresponds, as such, to a way of posing questions' (Deleuze 2004, 95. Translation modified). The question 'What is. .?' thus presupposes a particular way of thinking that points one in the direction of essence: for Plato, it was *the* question of essence, the *only* question capable of discovering the Idea. Aristotle in turn defined dialectics as the art of posing problems as the subject of a syllogism, while analytics gives us the means of resolving the problem by leading the syllogism to its necessary conclusion. Deleuze's own dialectic, however, is indexed primarily on Kant's 'Transcendental Dialectic' in the *Critique of Pure Reason*. Against Plato, Kant attempted to provide a purely *immanent* conception of Ideas that exposed the illusion of assigning to Ideas a transcendent object (such as the Soul, the World, or God). If the Ideas of reason give rise to illusion and lead us into false problems, Kant argued, this is first of all because reason is the faculty of posing problems in general: the object of the Idea, since it lies outside of experience, can neither be given nor known, but must be represented in a *problematic* form, without being determined. But this does not mean that Ideas have no real object; more profoundly, it means that *problems as problems are the real objects of Ideas*.[3]

This is the source of the link one finds in Deleuze's work between dialectics, problematics, and Ideas: dialectics is the science of problems, but problems themselves are Ideas. Deleuze has often been characterised (wrongly) as an anti-dialectical thinker, but it would be more accurate to say that *Difference and Repetition* (especially its fifth chapter, 'Ideas and the Synthesis of Difference') is a book that proposes *a new concept of dialectics*, one that both is indebted to but breaks with the work of the great dialectical thinkers such as Plato, Kant, and Hegel. This is where Deleuze's interest in mathematical problematics intervenes: it provides him with a model for this new conception of dialectics. If Plato found his model in Euclidian geometry, and contemporary philosophers tend to turn toward set theory and axiomatics, Deleuze has found his model for dialectical Ideas in problematics and the history of the calculus. The discussion that follows focuses primarily on the mathematical origins of Deleuze's conception of dialectics, and the problematic/axiomatic distinction that lies at its core. It will examine, in turn, the historical background of Deleuze's notion of problematics, the precise nature of the relation between axiomatics and problematics, and finally, the means by which Deleuze has attempted to provide a *formalisation* of problematics in his theory of multiplicities.

2. Problematics versus axiomatics: historical background

Although Deleuze formulates the problematic-axiomatic distinction in his own manner, it in fact reflects a fairly familiar tension within the history of mathematics, which can be illustrated by means of three historical examples.

1. The first example comes from the Greeks. Proclus, in his *Commentary of the First Book of Euclid's Elements*, had already formulated a distinction, within Greek geometry, between problems and theorems.[4] Theorems concern the demonstration, from axioms or postulates, of the inherent properties belonging to a figure, whereas problems concern the actual construction of figures, usually using a straightedge and compass. From this viewpoint, determining a triangle the sum of whose angles is 180 degrees is theorematic, since the angles of every triangle will total 180 degrees. By contrast, constructing an equilateral triangle on a given finite straight line is problematic, since we could also construct a non-equilateral triangle or a non-triangular figure on the line (moreover, the construction of an equilateral triangle must first pass through the construction of two circles). Classical geometers struggled for centuries with the three great unresolved 'problems' of antiquity – trisecting an angle, constructing a square equal to a circle, and constructing a cube having double the volume of a given cube – although only in 1882 was it proved (theorematically) that none of these problems was solvable using only a straightedge and compass.[5]

But this is why theorematics and problematics involve two different conceptions of deduction: if in theorematics a deduction moves from axioms to theorems, in problematics a deduction moves from the problem to the ideal *events* that condition it and form the *cases* of solution that resolve it. In theorematics, for instance, a figure is defined statically, in Platonic fashion, in terms of its essence and its derived properties: Euclidean geometry defines the essence of the line in purely static terms that eliminate any reference to the curvilinear ('a line which lies evenly with the points on itself').[6] Problematics, by contrast, found its classical expression in the 'operative' geometry of Archimedes, in which the straight line is characterised dynamically as 'the shortest distance between two points.' Here, the problem (How to construct a line between two points?), with its determinate conditions, has an infinite set of possible solutions (curves, loops, etc.), and the straight line is simply the case that constitutes the 'shortest' solution. Similarly, in the theory of conic sections, the ellipse, hyperbola, parabola, straight lines, and the point are all 'cases' of the projection of a circle onto secant planes in relation to the apex of a cone. If Archimedean geometry (especially the Archimedes of 'On the Method') can be said to be an operative geometry, it is because it defines the line less as an essence than as a continuous operation or *process* of 'alignment', the circle as a continuous process of 'rounding' the square as the process of 'quadrature', and so on. In problematics, a figure is defined dynamically by its *capacity to be affected* – that is, by the ideal accidents and events that can befall the figure (sectioning, cutting, projecting, folding, bending, stretching, reflecting, rotating, and so on). As a theorematic figure, a circle may indeed be an organic and fixed essence, but the morphological variations of the circle (figures that are 'lens-shaped', 'umbelliform' 'indented', etc.) form problematic figures that are, in Husserl's words, 'vague yet rigorous' 'essentially and not accidentally inexact.'[7]

Greek thought nonetheless set a precedent that would be followed by later mathematicians and philosophers: Proclus had already pointed to (and defended) the relative triumph, in Greek geometry, of the theorematic over the problematic. The reason: to the Greeks, 'problems concern only events and affects which show evidence of a *deterioration* or a projection of essences in the imagination,' and theorematics could thus present itself as a necessary 'rectification' of thought.[8] This 'rectification' must be understood, in a literal sense, as a triumph of the rectilinear over the curvilinear. In the 'minor' geometry of problematics, figures are inseparable from their inherent variations, affections, and events (the straight line being a simple case of the curve). The explicit aim of 'major' theorematics is 'to uproot variables from their state of continuous variation in order to extract from them fixed points and constant relations,' thereby setting geometry on the 'royal' road of theorematic deduction and proof (Deleuze and Guattari 1987, 408–409).

2. For our second example, we jump ahead two millennia. By the seventeenth-century, the tension between problems and theorems, which was internal to Greek geometry, had shifted to a more general tension between geometry itself, on the one hand, and algebra and arithmetic on the other. Desargues' *projective geometry*, for instance, which was a qualitative and 'minor' geometry centred on problems-events (as developed, most famously, in Desargues' *Draft Project of an Attempt to Treat the Events of the Encounters of a Cone and a Plane*), was quickly opposed in favour of the *analytic geometry* of Fermat and Descartes – a quantitative and 'major' geometry that translated geometric relations

into arithmetic relations that could be expressed in algebraic equations (Cartesian coordinates) (Boyer 1968, 393). 'Royal' science, in other words, now entailed an *arithmetisation* of geometry itself. 'There is a correlation,' Deleuze writes, 'between geometry and arithmetic, geometry and algebra that is constitutive of major science.'[9] Descartes was dismayed when he heard that Desargues' *Draft Project* treated conic sections without the use of algebra, since to him 'it did not seem possible to say anything about conics that could not more easily be expressed with algebra than without.'[10] As a result, Desargues' methods were repudiated as dangerous and unsound, and his practices of perspective banned. Theorematics (in the form of algebra) once again triumphed, and brought about an arithmetic conversion of a problematic field.

This triumph of theorematics can be said to have reached its greatest philosophical expression in Spinoza's *Ethics*, which assumes a purely theorematic or axiomatic form of argumentation and deduction. 'In Spinoza,' Deleuze complains, '*the use of the geometric method involves no 'problems' at all*' (1994, 323, n. 21). Indeed, with regard to problematics, Deleuze suggests that in fact Descartes actually went further than Spinoza, and that Descartes the geometer went further than Descartes the philosopher. The 'Cartesian method' (the search for the clear and distinct) is a method for solving problems, but the analytic procedure that Descartes presents in his *Geometry* is focused on the constitution of problems as such ('Cartesian coordinates' appear nowhere in the *Geometry*).[11] The *Geometry* does not move from axioms to theorems, but rather starts with a problem and 'analyses' it to find a solution. 'With the [analytic] method I use,' Descartes wrote, 'everything falling under the geometers consideration can be reduced to as single class of *problem* namely, that of looking for the value of the roots of a certain equation. Nonetheless, one of the most significant innovations of Deleuze's reading of Spinoza is to have presented a *problematic* reading of the *Ethics*, which operates alongside and within Spinoza's explicit demonstrative apparatus. Rather than beginning with the axioms and following Spinoza's theorematic deductions, Deleuze starts his analysis 'in the middle', that is, with the problematic composition of finite modes and the *affections* that befall them, and undertakes a problematic deduction of the concept. Human modes of existence have affections just as geometrical figures. 'The relation between mathematics and humanity,' Deleuze writes in *Logic of Sense*, 'may thus be conceived in a new way: the question is not that of quantifying or measuring human properties, but rather, on the one hand, that of problematising human events and, on the other, that of developing as various human events the conditions of a problem' (p. 55). Spinoza's work is thus susceptible to two kinds of reading: a *conceptual* (theorematic) reading and an *affective* (problematic) reading. This is why, in his analysis of the *Ethics*, Deleuze consistently emphasised the role of the scholia (which are the only elements of the *Ethics* that fall outside the axiomatic deductions, and develop the theme of 'affections') and the fifth book (which introduces problematic hiatuses and contractions into the deductive exposition itself).[13] Pierre Macherey has complained that Deleuze, in approaching Spinoza's thought in such a manner, is attempting to introduce a new version of Spinozism that is at variance, if not completely at odds, with the model of 'demonstrative rationality' explicitly adopted by Spinoza himself.[14] But it should be clear that Deleuze's approach to Spinoza is itself a 'case' of his broader approach to philosophy from the viewpoint of problematics. 'The whole problem of reason,' Deleuze has suggested elsewhere, 'will be converted by Spinoza into a special case of the more general problem of the affects' (Deleuze 1980c).

The attempt to 'arithmetise' geometry would continue well into the nineteenth-century, when Desargues' projective geometry was revived in the work of Monge, the inventor of descriptive geometry, and Poncelet, who formulated the 'principle of continuity,' which led to developments in *analysis situs* and topology. Topology (so-called 'rubber-band geometry') was initially a problematic science that concerned the property of geometric figures that remain invariant under transformations such as bending or stretching. Under such transformations, figures that are theorematically distinct in Euclidean geometry – such as a triangle, a square, or a circle – can be seen as one and the same 'homeomorphic' figure, since they can be continuously transformed into one another. This entailed an extension of geometric 'intuitions' far beyond the limits of empirical or sensible perception (à la Kant). 'With Monge, and especially Poncelet,' writes Deleuze, commenting on Léon Brunschvicg's work, 'the limits of sensible, or even spatial, representation (striated space) are indeed surpassed, but less in the direction of a symbolic power of abstraction [i.e., theorematics] than toward a trans-spatial imagination, or a trans-intuition (continuity).'[15] In the twentieth-century, computers have extended the reach of this 'trans-intuition' even further, provoking renewed interest in qualitative geometry, and allowing mathematicians to 'see' hitherto unimagined objects such as the Mandelbrot set and the Lorenz attractor, which have become the poster children of the new sciences of chaos and complexity. 'Seeing, seeing what happens,' continues Deleuze, 'has always had an essential importance, greater than demonstrations, even in pure mathematics, which can be called visual, figural, independently of its applications: many mathematicians nowadays think that a computer is more precious than an axiomatic' (Deleuze and Guattari 1994, 128. Translation modified). But already in the early nineteenth-century, there was a renewed attempt to turn projective geometry into a mere practical dependency on analysis, or so-called higher geometry (the debate between Poncelet and Cauchy).[16] The development of the theory of functions would eventually eliminate the appeal to the principle of continuity, substituting for the geometrical idea of smoothness of variation the arithmetic idea of 'mapping' or a one-to-one correspondence of points (point-set topology). Theorematics would once again triumph over problematics.

3. Finally, this double movement of major science toward theorematisation and arithmetisation would reach its full flowering in the late nineteenth-century, primarily in response to problems posed by the invention of the *calculus*. In its origins, the calculus was tied to problematics in a double sense. The first refers to the problems that the calculus confronted: the differential calculus addressed the problematic of *tangents* (how to determine the tangent lines to a given curve), while the integral calculus addressed the problematic of *quadrature* (how to determine the area within a given curve). The greatness of Leibniz and Newton was to have recognised the intimate connection between these two problematics (the problem of finding areas is the inverse of determining tangents to curves), and to have developed a symbolism to link them together and resolve them. The calculus quickly became the primary mathematical engine of what we call the 'scientific revolution' Yet for two centuries, the calculus, not unlike Archimedean geometry, itself maintained a problematic status in a second sense: it was allotted a para-scientific status, labelled a 'barbaric' or 'Gothic' hypothesis, or at best a convenient convention or well-grounded fiction. In its early formulations, the calculus was shot through with dynamic notions such as infinitesimals, fluxions and fluents, thresholds, passages to the limit, continuous variation – all of which presumed a *geometrical* conception of the continuum, in other words, the idea of a process. For most mathematicians, these were considered to be 'metaphysical' ideas that lay beyond the realm of mathematical definition. Berkeley famously ridiculed infinitesimals as 'the ghosts of departed quantities'; D'Alembert famously responded by telling his students, *Allez en avant, et la foi vous viendra* ('Go forward, and faith will come to you').[17]

The calculus would not have been invented without these notions, yet they remained problematic, lacking an adequate mathematical ground.

For a long period of time, the enormous success of the calculus in solving physical problems delayed research into its logical foundations. It was not until the end of the nineteenth-century that the calculus would receive a 'rigorous' foundation through the development of the 'limit-concept.' 'Rigour' meant that the calculus had to be separated from its problematic origins in geometrical conceptions or intuitions, and reconceptualised in purely arithmetic terms (the loaded term 'intuition' here having little to do with empirical perception, but rather the ideal geometrical notion of continuous movement and space).[18] This 'arithmetisation of analysis', as Félix Klein called it,[19] was achieved by Karl Weierstrass, one of Husserl's teachers, in the wake of work done by Cauchy (leading Guilio Giorello to dub Weierstrass and his followers the 'ghostbusters').[20] Analysis (the study of infinite processes) was concerned with *continuous* magnitudes, whereas arithmetic had as its domain the *discrete* set of numbers. The aim of Weierstrass' 'discretisation' programme was to separate the calculus from the geometry of continuity and base it on the concept of number alone. Geometrical notions were thus reconceptualised in terms of sets of discrete points, which in turn were conceptualised in terms of number: points on a line as individual numbers, points on a plane as ordered pairs of numbers, points in n-dimensional space as n-tuples of numbers. As a result, the concept of the variable was given a *static* (arithmetic) rather than a *dynamic* (geometrical) interpretation. Early interpreters had tended to appeal to the geometrical intuition of continuous motion when they said that a variable x 'approaches' a limit (e.g., the circle defined as the limit of a polygon). Weierstrass' innovation was to reinterpret this variable x arithmetically as simply designating any one of a collection of numerical values (the theory of functions), thereby eliminating any dynamism or 'continuous variation' from the notion of continuity, and any interpretation of the operation of differentiation as a process. In Weierstrass' limit-concept, in short, the geometric idea of 'approaching a limit' was arithmetised, and replaced by static constraints on discrete numbers alone (the episilon-delta method). Dedekind took this arithmetisation a step further by rigorously defining the continuity of the real numbers in terms of a 'cut': 'it is the cut which constitutes. .the ideal cause of continuity or the pure element of quantitativity' (Deleuze 1994, 172). Cantor's set theory, finally, gave a discrete interpretation of the notion of infinity itself, treating infinite sets like finite sets (the power set axiom) –

or rather, treating all sets, whether finite or infinite, as mathematical objects (the axiom of infinity).[21]

Weierstrass, Dedekind, and Cantor thus form the great triumvirate of the programme of discretisation and the development of the 'arithmetic' continuum (the redefinition of continuity as a function of sets over discrete numbers). In their wake, the basic concepts of the calculus – function, continuity, limit, convergence, infinity, and so on – were progressively 'clarified' and 'refined,' and ultimately given a set theoretical foundation. The assumptions of Weierstrass' discretisation problem that only arithmetic is rigorous, and that geometric notions are unsuitable for secure foundations – are now largely identified with the 'orthodox' or 'major' view of the history of mathematics as a progression toward ever more 'well-founded' positions.[22] This contemporary orthodoxy has often been characterised as an 'ontological reductionism'; as Penelope Maddy describes it, 'mathematical objects and structures are identified with or instantiated by set theorematic surrogates, and the classical theorems about them proved from the axioms of set theory.'[23] Reuben Hersh gives it a more idiomatic and constructivist characterisation: 'Starting from the empty set, perform a few operations, like forming the set of all subsets. Before long you have a magnificent structure in which you can embed the real numbers, complex numbers, quaterions, Hilbert spaces, infinite-dimensional differentiable manifolds, and anything else you like' (1997, 13). The programme would pass through two further developments. The contradictions generated by set theory brought on a sense of a 'crisis' in the foundations, which Hilbert's formalist (or formalisation) programme attempted to repair through *axiomatisation*, that is, by attempting to show that set theory could be derived from a finite set of axioms, which were later codified by Zermelo-Fraenkel (given his theological leanings, even Cantor needed a dose of axiomatic rigor). Gödel and Cohen, finally, in their famous theorems, would eventually expose the internal limits of axiomatisation (incompleteness, undecidability), demonstrating that there is a variety of mathematical forms in 'infinite excess' over our ability to formalise them consistently. Deleuze, for his part, fully recognises the position of the orthodox programme: 'Modern mathematics is regarded as based upon the theory of groups or set theory rather than on the differential calculus' (1994, 180). Nonetheless, he insists that the fundamental difference in kind between problematics and axiomatics remains, even in contemporary mathematics: 'Modern mathematics also leaves us in a state of antinomy, since the strict finite interpretation that it gives of the calculus nevertheless presupposes an axiom of infinity in the set theoretical foundation, even though this axiom finds no illustration in the calculus. What is still missing is the extra-propositional and sub-representative element expressed in the Idea by the differential, *precisely in the form of a problem*' (p. 178).

A final example can help serve to illustrate the ongoing tension between problematics and axiomatics, even in contemporary mathematics. Even after Weierstrass' work, mathematicians using the calculus continued to obtain accurate results and make new discoveries by using infinitesimals in their reasoning, their mathematical conscience assuaged by the (often unchecked) supposition that infinitesimals could be replaced by Weierstrassian methods. Despite its supposed 'elimination' as an impure and muddled metaphysical concept, the ghostly concept of infinitesimals continued to play a positive role in mathematics *as* a problematic concept, reliably producing correct solutions. 'Even now,' wrote Abraham Robinson in 1966, 'there are many classical results in differential geometry which have never been established in any other way [than through the use of infinitesimals], the assumption being that somehow the rigorous but less intuitive ε, δ method would lead to the same result.'[24] In response to this situation, Robinson developed his *non-standard analysis*, which proposed an axiomatisation of infinitesimals themselves, at last granting mathematicians the 'right' to use them in proofs. Using the theory of formal languages, he added to the ordinary theory of numbers a new symbol (which we can call i for infinitesimal), and posited axioms saying that i was smaller than any finite number $1/n$ and yet not zero; he then showed that this enriched theory of numbers is consistent, assuming the consistency of the ordinary theory of numbers. The resulting axiomatic model is described as 'non-standard' in that it contains, in addition to the 'standard' finite and transfinite numbers, non-standard numbers such as hyperreals and infinitesimals. In the non-standard model, there is a cluster of infinitesimals around every real number r, which Robinson, in a nod to Leibniz, termed a 'monad' (the monad is the 'infinitesimal neighbourhood' of r). Transfinites and infinitesimals are two types of infinite number, which characterise degrees of infinity in different fashions. In effect, this means that contemporary mathematics has 'two distinct rigorous formulations of the calculus': that of Weierstrass and Cantor, who eliminated infinitesimals, and that of Robinson, who rehabilitated and legitimised them.[25] Both these theorematic endeavours, however, had their genesis in the imposition of the notion of infinitesimals *as* a problematic concept, which in turn gave rise to differing but related axiomatisations. Deleuze's claim is that the ontology of mathematics is poorly understood if it does not take into account the specificity and irreducibility of problematics.

3. The relation between problematics and axiomatics

With these historical examples in hand, we can now make several summary points concerning the relation between the problematic and axiomatic poles of mathematics, or more broadly, the relation between minor and major science. First, according to Deleuze, mathematics is constantly producing notions that have an objectively problematic status; the role of axiomatics (or its precursors) is to codify and solidify these problematic notions, providing them with a theorematic ground or rigorous foundation. Axiomaticians, one might say, are the 'law and order' types in mathematics: 'Hilbert and de Broglie were as much politicians as scientists: they reestablished order' (Deleuze and Guattari 1987, 144). In this sense, as Jean Dieudonné suggests, axiomatics is a foundational but *secondary* enterprise in mathematics, dependent for its very existence on problematics: 'In periods of expansion, when new notions are introduced, it is often very difficult to exactly delimit the conditions of their deployment, and one must admit that one can only reasonably do so once one has acquired a rather long practice in these notions, which necessitates a more or less extended period of cultivation [*défrichement*], during which incertitude and controversy dominates. Once the heroic age of pioneers passes, the following generation can then codify their work, getting rid of the superfluous, solidifying the bases – in short, putting the house in order. At this moment, the axiomatic method reigns anew, until the next overturning [*bouleversement*] that brings a new idea.'[26] Nicholas Bourbaki puts the point even more strongly, noting that 'the axiomatic method is nothing but the "Taylor System" – the "scientific management" – of mathematics' (1971, 31). Deleuze has adopted a similar historical thesis, noting that the push toward axiomatics at the end of the nineteenth-century arose at the same time that Taylorism arose in capitalism: axiomatics does for mathematics what Taylorism does for 'work'.[27]

Second, problematic concepts often (though not always) have their source in what Deleuze terms the 'ambulatory' sciences, which includes sciences such as metallurgy, surveying, stonecutting, and perspective. (One need only think of the mathematical problems encountered by Archimedes in his work on military installations, Desargues on the techniques of perspective, Monge on the transportation of earth, and so on.)

The nature of such domains, however, is that they do not allow science to assume an autonomous power. The reason, according to Deleuze, is that the ambulatory sciences 'subordinate all their operations to the sensible conditions of intuition and construction – *following* the flow of matter, *drawing and linking up* smooth space. Everything is situated in the objective zone of fluctuation that is coextensive with reality itself. However refined or rigorous, "approximate knowledge" is still dependent upon sensitive and sensible evaluations that pose more problems than they solve: problematics is still its only mode' (Deleuze and Guattari 1987, 373). Such sciences are linked to notions – such as heterogeneity, dynamism, continuous variation, flows, and so on – that are barred or banned from the requirements of axiomatics, and consequently they tend to appear in history as that which was superseded or left behind. By contrast, what is proper to royal science, to its theorematic or axiomatic power, is 'to isolate all operations from the conditions of intuition, making them true intrinsic concepts, or 'categories' .Without this categorical, apodictic apparatus, the differential operations would be constrained to follow the evolution of a phenomenon'(p. 373–374). In the ontological field of interaction between minor and major science, in other words, 'the ambulant sciences confine themselves to *inventing problems* whose solution is tied to a whole set of collective, nonscientific activities but whose *scientific solution* depends, on the contrary, on royal science and the way it has transformed the problem by introducing it into its theorematic apparatus and its organisation of work. This is somewhat like intuition and intelligence in Bergson, where only intelligence has the scientific means to solve formally the problems posed by intuition' (p. 374).

Third, what is crucial in the interaction between the two poles are thus the processes of translation that take place between them – for instance, in Descartes and Fermat, an algebraic translation of the geometrical; in Weierstrass, a static translation of the dynamic; in Dedekind, a discrete translation of the continuous. The 'richness and necessity of translations,' writes Deleuze, 'include as many opportunities for openings as risks of closure or stoppage' (Deleuze and Guattari 1987, 486). In general, Deleuze's work in mathematical epistemology tends to focus on the reduction of the problematic to the axiomatic, the intensive to the extensive, the continuous to the discrete, the nonmetric to the metric, the nondenumerable to the denumerable, the rhizomatic to the arborescent, the smooth to the striated. Not all these reductions, to be sure, are equivalent, and Deleuze (following Lautman) analyses each on its own account. Deleuze himself highlights two of them. The first is 'the complexity of the means by which one translates intensities into extensive quantities, or more generally, multiplicities of distance into systems of magnitudes that measure and striate them (the role of logarithms in this connection)'; the second, 'the delicacy and complexity of the means by which Riemannian patches of smooth space receive a Euclidean conjunction (the role of the parallelism of vectors in striating the infinitesimal)' (p. 486). At times, Deleuze suggests, axiomatics can possess a deliberate will to halt problematics: 'State science retains of nomad science only what it can appropriate; it turns the rest into a set of strictly limited formulas without any real scientific status, or else simply represses and bans it' (p. 362; cf. p. 144). But despite its best efforts, axiomatics can never have done with problematics, which maintains its own ontological status and rigor. 'Minor science is continually enriching major science, communicating its intuitions to it, its way of proceeding, its itinerancy, its sense of and taste for matter, singularity, variation, intuitionist geometry and the numbering number. Major science has a perpetual need for the inspiration of the minor; but the minor would be nothing if it did not confront and conform to the highest scientific requirements' (p. 485–6). In Deleuzian terms, one might say that while 'progress' can be made at the level of theorematics and axiomatics, all 'becoming' occurs at the level of problematics.

Fourth, this means that axiomatics, no less than problematics, is itself an inventive and creative activity. One might be tempted to follow Poincaré in identifying problematics as a 'method of discovery' (Riemann) and axiomatics as a 'method of demonstration' (Weierstrass).[28] But just as problematics has its own modes of formalisation and deduction, so axiomatics has its own modes of intuition and discovery (axioms are not chosen arbitrarily, for instance, but in accordance with specific problems and intuitions).[29] 'In science an axiomatic is not at all a transcendent, autonomous, and decision-making power opposed to experimentation and intuition. On the one hand, it has its own gropings in the dark, experimentations, modes of intuition. Axioms being independent of each other, can they be added, and up to what point (a saturated system)? Can they be withdrawn (a 'weakened' system)? On the other hand, it is of the nature of axiomatics to come up against so-called *undecidable propositions*, to confront *necessarily higher powers* that it cannot master. Finally, axiomatics does not constitute the cutting edge of science; it is much more a stopping point, a reordering that prevents decoded flows in physics and mathematics [= problematics] from escaping in all directions. The great axiomaticians are the men of State within science, who seal off the lines of flight that are so frequent in mathematics, who would impose a new *nexum*, if only a temporary one, and who lay down the official policies of science. They are the heirs of the theorematic conception of geometry' (Deleuze and Guattari 1987, 461). For all these reasons, problematics is, by its very nature, 'a kind of science, or treatment of science, that seems very difficult to classify, whose history is even difficult to follow' (p. 361).[30]

4. *The formalisation of problematics: Deleuze's theory of multiplicities*

One of the aims of Deleuze's new concept of dialectics is to provide a *formalisation* of problematics that would constitute the basis for the theory of Ideas – a parallel to the formalisation that long ago took place in axiomatics. The difficulties of such a task, however, should be evident from the remarks above. The formalisation of theorematics has had a long history in mathematics and philosophy, and the theory of extensive multiplicities (Cantor's set theory) and its rigorous axiomatisation (Zermelo-Fraenkel, et al.) is one of the great achievements of modern mathematics. Deleuze, by contrast, is proposing to construct a hitherto non-existent (philosophical) formalisation of problematic multiplicities that are, by his own account, selected against by 'major' mathematics. In this regard, Deleuze's relation to the history of mathematics is similar to his relation to the history of philosophy: even in canonical figures there is something that 'escapes' the official histories of mathematics.[31] Nonetheless, there were a number of important precursors in mathematics who were working in this direction: Abel, Galois, Riemann, and Poincaré are among the great names in the history of problematics, just as Weierstrass, Dedekind, and Cantor are the great names in the discretisation programme, and Hilbert, Zermelo, Frankel, Gödel, and Cohen the great names in the movement toward formalisation and axiomatisation. We can therefore highlight at least three mathematical domains that have served as precursors in formalising the theory of problems in mathematics, and which Deleuze appealed to in formulating his own concept of problems as multiplicities.[32]

1. The first domain is the theory of *groups*, which initially arose from questions concerning the solvability of certain *algebraic* (rather than differential) equations. There are two kinds of solutions to algebraic equations, particular and general. Whereas a *particular* solution is given

by numerical values ($x^2 + 3x - 4 = 0$ has as its solution $x = 1$), a *general* solution provides the global pattern of all particular solutions to an algebraic equation (the above equation, generalised as $x^2 + ax - b = 0$, has the solution $x = \sqrt{a^2/2 + b} - a/2$). But such solutions, writes Deleuze, 'whether general or particular, find their sense only in the subjacent problem which inspires them' (1994, 162). By the sixteenth century, it had been proved (Tataglia-Cardan) that *general* solvability was possible with squared, cubic, and quartic equations. But equations raised to the fifth power and higher refused to yield to the previous method (via radicals), and the puzzle of the 'quintic' remained unresolved for more than two centuries, until the work of Lagrange, Abel, and Galois in the nineteenth-century. In 1824, Abel proved the startling result that the quintic was in fact *unsolvable*, but the method he used was as important as the result: Abel recognised that there was a pattern to the solutions of the first four cases, and that it was this pattern that held the key to understanding the recalcitrance of the fifth. Abel showed that the question of 'solvability' had to be determined internally by the *intrinsic* conditions of the problem itself, which then progressively specifies its own 'fields' of solvability.

Building on Abel's work, Evariste Galois developed a way to approach the study of this pattern, using the technique now known as *group theory*. Put simply, Galois 'showed that equations that can be solved by a formula must have groups of a particular type, and that the quintic had the wrong sort of group' (Stewart and Golubitsky 1992, 42). The 'group' of an equation captures the conditions of the problem; on the basis of certain substitutions within the group, solutions can be shown to be indistinguishable insofar as the validity of the equation is concerned.[33] In particular, Deleuze emphasises the fundamental procedure of *adjunction* in Galois: 'Starting from a basic 'field' R, successive adjunctions to this field (R', R", R'" ...) allow a progressively more precise distinction of the roots of an equation, by the progressive limitation of possible substitutions. There is thus a succession of 'partial resolvants' or an embedding of 'groups' which make the solution follow from the very conditions of the problem' (1994, 180). In other words, the group of an equation does not tell us what we know about its roots, but rather, as George Verriest remarks, 'the objectivity of what we do *not* know about them.'[34] As Galois himself wrote, 'in these two memoirs, and especially in the second, one often finds the formula, *I don't know*. '[35] This non-knowledge is not a negative or an insufficiency, but rather a rule or something to be learned that corresponds to an *objective* dimension of the problem. What Deleuze finds in Abel and Galois, following the exemplary analyses of Jules Vuillemin in his *Philosophy of Algebra*, is 'a radical reversal of the problem-solution relation, a more considerable revolution than the Copernican.'[36] In a sense, one could say that 'unsolvability' plays a role in problematics similar to that played by 'undecidability' in axiomatics.

2. The second domain Deleuze utilises is the calculus itself, and on this score Deleuze's analyses are based to a large extent on the interpretation proposed by Albert Lautman in his *Essay on the Notions of Structure and Existence in Mathematics* (1938). Lautman's work is based on the idea of a fundamental difference in kind between a problem and its solution, a distinction that is attested to by the existence of problems *without* solution. Leibniz, Deleuze notes, 'had already shown that the calculus. .expressed problems that could not hitherto be solved, or indeed, even posed' (1994, 177). In turn Lautman establishes a link between the theory of differential equations and the theory of singularities, since it was the latter that provided the key to understanding the nature of *nonlinear* differential equations, which could not be solved because their series diverged. As determined by the equation, singular points are distinguished from the ordinary points of a curve: the singularities mark the points where the curve changes direction (inflections, cusps, etc.), and thus can be used to distinguish between different *types* of curves. In the late 1800's, Henri Poincaré, using a simple nonlinear equation, was able to identify four types of singular points that corresponded to the equation (foci, saddle points, knots, and centres) and to demonstrate the topological behaviour of the solutions in the neighbourhood of such points (the integral curves).[37] On the basis of Poincaré's work, Lautman was able to specify the nature of the difference in kind between problems and solutions. The conditions of the *problem* posed by the equation is determined by the existence and distribution of singular points in a differen*t*iated topological field (a field of vectors), where each singularity is inseparable from a zone of objective indetermination (the ordinary points that surround it). In turn, the *solution* to the equation will only appear with the integral curves that are constituted in the neighbourhood of these singularities, which mark the beginnings of the differen*c*iation (or actualisation) of the problematic field. In this way, the ontological status of the problem as such is detached from its solutions: in itself, the problem is a multiplicity of singularities, a nested field of directional vectors which define the 'virtual' trajectories of the curves in the solution, not all of which can be actualised. Non-linear equations can thus be used to model objectively problematic (or indeterminate)

physical systems, such as the weather (Lorenz): the equations can define the virtual 'attractors' of the system (the intrinsic singularities toward which the trajectories will tend in the long-term), but they cannot say in advance which trajectory will be actualised (the equation cannot be solved), making accurate prediction impossible. A problem, in other words, has an objectively determined structure (virtuality), apart from its solutions (actuality).[38]

3. But 'there is no revolution,' in the problem-solution reversal, continues Deleuze, 'as long as we remain tied to Euclidean geometry: we must move to a geometry of sufficient reason, a Riemannian-type differential geometry which tends to give rise to discontinuity on the basis of continuity, or to ground solutions in the conditions of the problems' (1994, 162). This leads to Deleuze's third mathematical resource, the *differential geometry* of Gauss and Riemann. Gauss had realised that the utilisation of the differential calculus allowed for the study of curves and surfaces in a purely intrinsic and 'local' manner; that is, without any reference to a 'global' embedding space (such as the Cartesian coordinates of analytic geometry).[39] Riemann's achievement, in turn, was to have used Gauss's differential geometry to launch a reconsideration of the entire approach to the study of space by analysing the general problem of *n-dimensional* curved surfaces. He developed a non-Euclidean geometry (showing that Euclid's axioms were not self-evident truths) of a multidimensional, non-metric, and non-intuitable 'any-space-whatever,' which he termed a pure 'multiplicity' or 'manifold' [*Mannigfaltigkeit*]. He began by defining the distance between two points whose corresponding coordinates differ only by infinitesimal amounts, and defined the curvature of the multiplicity in terms of the *accumulation* of neighbourhoods, which alone determine its connections.[40] For our purposes, the two important features of a Riemannian manifold are its variable number of dimensions (its *n*-dimensionality), and the absence of any supplementary dimension which would impose on it extrinsically defined coordinates or unity.[41] As Deleuze writes, a Riemannian multiplicity is 'an n-dimensional, continuous, defined multiplicity... By *dimensions*, we mean the variables or coordinates upon which a phenomenon depends; by *continuity*, we mean the set of [differential] relations between changes in these variables – for example, a quadratic form of the differentials of the co-ordinates; by *definition*, we mean the elements reciprocally determined by these relations, elements which cannot change unless the multiplicity changes its order and its metric' (1994, 182).

In *Difference and Repetition*, Deleuze draws upon all these resources to develop his general theory of problematic or differential multiplicities. The fifth chapter of *Difference and Repetition* ('Ideas and the Synthesis of Difference') draws on all these resources in order to present a theory of Ideas *as* problematic (problems *are* Ideas), which in effect presents Deleuze's new concept of dialectics. The formal conditions of a problematic Idea can be briefly summarised as follows. (1) The elements of the multiplicity are merely 'determinable', their nature is not determined in advance by either a defining property or an axiom (e.g., extensionality). Rather, they are pure virtualities that have neither identity, nor sensible form, nor conceptual signification, nor assignable function (principle of determinability). (2) They are nonetheless determined reciprocally as singularities in the differential relation, a 'non-localisable ideal connection' that provides a purely intrinsic definition of the multiplicity as 'problematic'; the differential relation is not only *external* to its terms, but *constitutive* of its terms (principle of reciprocal determination). (3) The values of these relations define the complete determination of the problem, that is, 'the existence, the number, and the distribution of the determinant points that precisely provide its conditions' *as* a problem (principle of complete determination).[42] These three aspects of sufficient reason, finally, find their unity in the temporal principle of progressive determination, through which, as we have seen in the work of Abel and Galois, the problem is resolved (adjunction, etc.) (1994, 210).

The strength of Deleuze's project, with regard to problematics, is that, in a certain sense, it parallels the movement toward 'rigour' that was made in axiomatics: it presents a formalisation of the theory of problems, freed from the conditions of geometric intuition and solvability, and existing only in pure thought (even though Deleuze presents his theory in a purely philosophical manner, and explicitly refuses to assign a scientific status to his conclusions).[43] In undertaking this project, he had few philosophical precursors (Lautman, Vuillemin), and the degree to which he succeeded in the effort no doubt remains an open question. Manuel DeLanda, in a recent work, has proposed several refinements in Deleuze's formalisation, drawn from contemporary science: certain types of singularities are now recognisable as 'strange attractors'; the resolution of a problematic field (the movement from the virtual to the actual) can now be described in terms of a series of spatio-temporal 'symmetry-breaking cascades' and so on.[44] But as DeLanda insists, despite his own modifications to Deleuze's theory, Deleuze himself 'should get the credit for having adequately *posed the problem*' of problematics (2002, 102).

Notes

This essay draws on earlier work that was published in the *Southern Journal of Philosophy* 41.3 (2003). See Smith 2003.

1 See Deleuze 1994, 323 n. 22: Given the irreducibility of 'problems' in his thought, Deleuze writes that 'the use of the word "problematic" as a substantive seems to us an indispensable neologism.'

2 Deleuze 1994, 323 n. 22. Deleuze is referring to the distinction between 'problem' and 'theory' in Canguilhem 1978; the distinction between the 'problem-element' and the 'global synthesis element' in Bouligand 1949; and the distinction between 'problem' and 'solution' in Lautman 1939, discussed below. All these thinkers insist on the double irreducibility of problems: problems should not be evaluated extrinsically in terms of their 'solvability' (the philosophical illusion), nor should problems be envisioned merely as the conflict between two opposing or contradictory propositions (the natural illusion) (see Deleuze 1994, 161). On this score, Deleuze largely follows Lautman's thesis that mathematics participates in a *dialectic* that points beyond itself to a meta-mathematical power – that is, to a general theory of problems and their ideal synthesis – which accounts for the genesis of mathematics itself. See Lautman 1939, particularly the section entitled 'The Genesis of Mathematics from the Dialectic': 'The order implied by the notion of genesis is no longer of the order of logical reconstruction in mathematics, in the sense that from the initial axioms of a theory flow all the propositions of the theory, for the dialectic is not a part of mathematics, and its notions have no relation to the primitive notions of a theory' (p. 13–14).

3 When Kant says that Ideas are 'problems to which there is no solution' (Kant 1998, 319, A328/B384), he does not mean that they are necessarily false problems, and therefore insoluble; on the contrary, this means that *true problems are Ideas*, and that these Ideas do not disappear with their solutions, since they are the indispensable condition without which no solution would ever exist. See Deleuze 1994, 168.

4 Proclus 1970, 63–67, as cited in Deleuze 1994, 163; Deleuze 1987, 554 n. 21; and Deleuze 1990a, 54. See also Deleuze's comments in Deleuze 1989, 174: theorems and problems are are 'two mathematical instances which constantly refer to each other, the one enveloping the second, the second sliding into the first, but both very different in spite of their union.' On the two types of deduction, see 185.

5 See E. T. Bell's comments on this issue in Bell 1937, 31–32.

6 See Deleuze 1994, 174: 'The mathematician Houël remarked that the shortest distance was not a Euclidean notion at all, but an Archimedean one, more

physical than mathematical; that it was inseparable from a method of exhaustion; and that it served less to determine the straight line than to determine the length of a curve by means of a straight line – "integral calculus performed unknowingly"' (citing Houël 1867, 3, 75). Carl B. Boyer makes a similar point in his 1968: 'Greek mathematics sometimes has been described as essentially static, with little regard for the notion of variability; but Archimedes, in his study of the spiral, seems to have found the tangent to the curve through kinematic considerations akin to the differential calculus' (p. 41).

7 Husserl 1931, 208, §74. Whereas Husserl saw problematics as 'protogeometry,' Deleuze sees it as a fully autonomous dimension of geometry, but one he identifies as a 'minor' science; it is a 'proto'-geometry only from the viewpoint of the 'major' or 'royal' conception of geometry, which attempts to eliminate these dynamic events or variations by subjecting them to a theorematic treatment.

8 Deleuze 1994, 160. Emphasis added. Deleuze continues: 'As a result [of using *reductio ad absurdum* proofs], however, the *genetic* point of view is forcibly relegated to an inferior rank: proof is given that something cannot not be, rather than *that* it is and *why* it is (hence the frequency in Euclid of negative, indirect and *reductio* arguments, which serve to keep geometry under the domination of the principle of identity and prevent it from becoming a geometry of sufficient reason).'

9 Deleuze and Guattari 1987, 484. On the relation between Greek theorematics and seventeenth-century algebra and arithmetic as instances of 'major' mathematics, see Deleuze 1994, 160–161.

10 Boyer 1968, 394. Deleuze writes that 'Cartesian coordinates appear to me to be an attempt at reterritorialization' (Deleuze 1972).

11 See Deleuze 1994, 161 and 323 n. 21. See also Reuben Hersh's comments on Descartes in Hersh 1997, 112–113: 'Euclidean certainty boldly advertised in the *Method* and shamelessly ditched in the *Geometry*.'

12 Descartes, as cited in Hersh1997, 113.

13 For the role of the scholia, see Deleuze 1992, 342–350 (the appendix on the scholia); for the uniqueness of the fifth book, see 'Spinoza and the Three Ethics', in Deleuze 1997, 149.

14 See Macherey 1996, 143. For a discussion of these issues, see Duffy 2006, 155–158.

15 Deleuze and Guattari 1987, 554 n. 23, commenting on Brunschvicg 1972. Deleuze also appeals to a text by Michel Chasles (1837), which establishes a continuity between Desargues, Monge, and Poncelet as the 'founders of a modern geometry' (Deleuze and Guattari 1987, 554 n. 28).

16 See Brunschvicg 1972, 327–331.

17 See Boyer 1959, 267. Deleuze praises Boyer's book as 'the best study of the history of the differential calculus and its modern structural interpretation' (1990a, 339).

18 For a discussion of the various uses of the term 'intuition' in mathematics, see the chapters on 'Intuition' and 'Four-Dimensional Intuition' in Davis and Hersch 1981, 391–405; as well as Hans Hahn's classic article 'The Crisis in Intuition', in Newman 1956, 1956–1976.

19 Boyer 1968, ch. 25, 'The Arithmetization of Analysis' (p. 598–619).

20 Giorello 1992, 135. I thank Andrew Murphie for this reference.

21 See Maddy 1997, 51–52, for a discussion of Cantorian 'finitism'

22 For a useful discussion of Weierstrass's 'discretisation program' (albeit written from the viewpoint of cognitive science), see Lakoff and Núñez 2000, 257–324.

23 Maddy 1997, 28. Reuben Hersh gives this a more idiomatic and constructivist characterization: 'Starting from the empty set, perform a few operations, like forming the set of all subsets. Before long you have a magnificent structure in which you can embed the real numbers, complex numbers, quaterions, Hilbert spaces, infinite-dimensional differentiable manifolds, and anything else you like' (1997, 13).

24 Robinson 1966, 83. See also p. 277: 'With the spread of Weierstrass' ideas, arguments involving infinitesimal increments, which survived particularly in differential geometry and in several branches of applied mathematics, began to be taken automatically as a kind of shorthand for corresponding developments by means of the ε, δ approach.'

25 Hersh 1997, 289. For discussions of Robinson's achievement, see Jim Holt's useful review, 'Infinitesimally Yours', in *The New York Review of Books*, 20 May 1999, as well as the chapter on 'Non-standard Analysis' in Davis and Hersh 1981, 237–254. The latter note that 'Robinson has in a sense vindicated the reckless abandon of eighteenth-century mathematics against the straight-laced rigour of the nineteenth-century, adding a new chapter in the never ending war between the finite and the infinite, the continuous and the discrete' (p. 238).

26 Jean Dieudonné, *L'Axiomatique dans les mathématiques modernes*, 47–48, as cited in Blanché 1955, 91.

27 See Deleuze 1972: 'The idea of a scientific task that no longer passes through codes but rather through an axiomatic first took place in mathematics toward the end of the nineteenth-century. .One finds this well-formed only in the capitalism of the nineteenth-century.' Deleuze's political philosophy is itself based in part on the axiomatic-problematic distinction: 'Our use of the word "axiomatic" is far from a metaphor; we find *literally* the same theoretical problems that are posed by the models in an axiomatic repeated in relation to the State' (Deleuze and Guattari 1987, 455).

28 Poincaré 1898–1899, 1–18, as cited in Boyer 1968, 601. Boyer notes that one finds in Riemann 'a strongly intuitive and geometrical background in analysis that contrasts sharply with the arithmetizing tendencies of the Weierstrassian school' (p. 601).

29 See Deleuze 1988a, 64: 'axioms concern problems, and escape demonstration.'

30 This section of the 'Treatise on Nomadology' (p. 361–374) develops in detail the distinction between 'major' and 'minor' science.

31 At one point, he even provides a list of 'problematic' figures from the history of science and mathematics: 'Democritus, Menaechmus, Archimedes, Vauban, Desargues, Bernouilli, Monge, Carnot, Poncelet, Perronet, etc.: in each case a monograph would be necessary to take into account the special situation of these scientists whom State science used only after restraining or disciplining them, after repressing their social or political conceptions.' Deleuze and Guattari 1987, 363. See Deleuze's well-known comments on his relation to the history of philosophy in 'Letter to a Harsh Critic', in Deleuze 1995, 5–6. The best general works on the history of mathematics are Boyer 1968 and Kline 1972.

32 For analyses of Deleuze's theory of multiplicities, see Durie 2002, 1–29; Ansell-Pearson 2002; and DeLanda 2002.

33 See Kline 1972, 759: 'The group of an equation is a key to its solvability because the group expresses the degree of indistinguishability of the roots. It tells us what we do *not* know about the roots.'

34 Deleuze 1994, 180, citing Verriest 1951, 41.

35 Deleuze 1997, 149, citing a text by Galois in Dalmas 1956, 132.

36 Deleuze 1994, 170. Deleuze is referring to Vuillemin 1962.

37 For discussions of Poincaré, see Kline 1972, 732–738; Lautman1946, 41–43; and Deleuze 1980b. Such singularities are now termed 'attractors': using the language of physics, attractors govern 'basins of attraction' that define the trajectories of the curves that fall within their 'sphere of influence'

38 For this reason, Deleuze's work has been seen to anticipate certain developments in complexity theory and chaos theory. Delanda in particular has emphasized this link in his 2002 (see n. 78). For a presentation of the mathematics of chaos theory, see Stewart 1989, 95–144.

39 See Lautman 1938a, 43: 'The constitution, by Gauss and Riemann, of a differential geometry that studies the intrinsic properties of a variety, independent of any space into which this variety would be plunged, eliminates any reference to a universal container or to a center of privileged coordinates.'

40 See Lautman 1938a, 23–24: 'Riemannian spaces are devoid of any kind of homogeneity. Each is characterized by the form of the expression that defines the square of the distance between two infinitely proximate points. .It follows that "two neighboring observers in a Riemannian space can locate the points in their immediate vicinity, but cannot locate their spaces in relation to each other without a new convention." Each vicinity is like a shred of Euclidean space, *but the linkage between one vicinity and the next is not defined and can be effected in an infinite number of ways. Riemannian space at its most*

general thus presents itself as an amorphous collection of pieces that are juxtaposed but not attached to each other.'

41 See Deleuze 1994, 183, 181: A Riemannian multiplicity 'is intrinsically defined, without external reference or recourse to a uniform space in which it would be submerged. .It has no need whatsoever of unity to form a system.'

42 See, in particular Deleuze 1994, 183, although the entirety of the fifth chapter is an elaboration of Deleuze's theory of multiplicities.

43 See Deleuze 1994, xxi: 'We are well aware. .that we have spoken of science in a manner which was not scientific.'

44 See Delanda 2002, 15 (on attractors), and chapters 2 and 3 (on symmetry-breaking cascades).

9 Problems in the relation between maths and philosophy

Robin Durie

How should the relation between Deleuze's philosophy and mathematics be characterised? Does his ontology 'depend' upon maths? Does it 'reduce' to maths? Does maths 'illustrate', or provide 'metaphors' for, his ontology? Yet perhaps these are all ill-conceived questions, or problems, in so far as they trade in traditional modes of philosophical accounting, the very sorts of accounting that Deleuze's philosophy seeks to call into question. Perhaps, therefore, we should reformulate our initial question in the following way: what is the *problem* in the relation between mathematics and ontology, or metaphysics, in Deleuze's philosophy?

In a brief essay entitled 'A Philosophical Concept. , Deleuze argues that concepts function within 'fields of thought' which are defined by internal variables, but which are also subject to the effects of external variables. He then writes:

> This means that a concept does not die simply when one wants it to, but only when new functions in new fields discharge it. This is also why it is never very interesting to criticise a concept: it is better to build the new functions and discover the new fields that make it useless or inadequate. (Cadava, Connor and Nancy 1991, 94)

An example of such a discharge would be the way in which Bergson discovered a new field of thought in *Matter and Memory* within which the concept of the virtual functioned, on the basis of which discovery he was subsequently able to discharge the classical Aristotelian concept of potentiality. But how might it be possible to discover such new fields, and so build new functions that would discharge old concepts?

Deleuze consistently expressed his discomfort with the Platonic formula of the One and the Multiple, and its subsequent ramification into the formulae of the universal and the singular or the general and the particular. But must the singular remain within the field of thought defined by classical philosophical variables? In the third of a delightful series of seminars on Leibniz dating from 1980, Deleuze speaks of the 'great mathematical discovery [. . .] that singularity is no longer thought in relation to the universal, but is thought rather in relation to the ordinary or to the regular' (Deleuze 1980b). Notice here that a new field of thought is signalled, to the extent that the variables defining the field do not consist in the relations between singular and universal but rather singular and ordinary, or regular. That is to say, not only do the notions of regular and ordinary differ from that of universal, but the *relations* between regular or ordinary and singular differ from the relation subsisting between singular and universal. As a consequence, the concept of the singular itself must differ in kind in this new field. Deleuze continues: 'The singular is what exceeds the ordinary and the regular. And saying that already takes us a great distance since saying it indicates that, henceforth, we wish to make singularity into a philosophical concept, even if it means finding reasons to do so in a favourable domain, namely mathematics.'

Here then is one possible answer to our question regarding the relation of mathematics and philosophy in Deleuze's work – the field of mathematics can yield concepts whose functions can reveal the inadequacy of traditional philosophical concepts, at the very moment that they displace these concepts. In so 'discharging' the old concepts, new fields of philosophical thought are necessarily defined. But this answer provokes a further set of questions: What reasons might emerge within the field of mathematics that might motivate us to make of singularity a philosophical concept? And what might the relations be between the new mathematical field within which the concept functions, and the philosophical field, such that the concept that functions in the new mathematical field can map on to a new philosophical field in which traditional concepts have been discharged?

In order to begin to develop responses to these questions, we must bear in mind Deleuze's stipulation that 'the creation of a concept always occurs as the function of a problem.'[1] As he writes in *What is Philosophy?*: 'All concepts are connected to problems without which they would have no meaning and which can themselves only be isolated or understood as their solution emerges. Of course, everything changes if we think that we discover another problem.'[2] What is of the utmost significance for Deleuze in making these claims is the principle, derived from

Lautman, that problems (or questions) differ in kind from solutions (or answers).[3]

Deleuze's discussion of problems in *Difference and Repetition* has both an epistemological and an ontological dimension. We shall concentrate on the section in chapter three of that book which for the most part deals with the former dimension, although we shall see that this discussion is not isolated from the ontological dimension.[4] As is well known, chapter three of *Difference and Repetition* develops a series of eight postulates which, Deleuze argues, define the traditional image of philosophical thought – Deleuze's seventh postulate, on the 'modality of solutions', pertains to 'responses and solutions according to which truth and falsehood only begin with solutions or only qualify responses' (Deleuze 1994, 158).[5] The assumption underlying this traditional philosophical position is the belief that 'problems are given ready-made, and that they disappear in the responses or solution' (p. 158). The problem is little more than a trigger for the all-important solution, from which it differs only in degree, to the extent, that is, that the problem simply contains less of the solution than the solution does. Furthermore, this difference in degree is itself a mere effect of 'negative empirical conditions imposed upon the knowing subject' (p. 159). We catch a glimpse of this postulate in the infamous cases the philosophical tradition affords us as examples of thought 'going wrong', such as mistaking Theodorus for Theaetetus.

Why should such a mistake – doubtless the effect of poor light, or looking from a distance – be of any philosophical value? They are examples of, as Plato already emphasised, failures of recognition, just as a failure to arrive at the 'right solution' – in the terms of the 7th postulate – would be no more than a failure to recognise this solution in a given problem. Objects of recognition do not '*force* us to think' (Deleuze 1994, 138). What forces us to think is the object which is *encountered*, and encountered objects are precisely not recognised but first and foremost *sensed* (p. 139). This sense constitutes, in effect, a sign of the event of the faculty of sensibility or intuition 'finding itself before its own limit' (p. 140). This limit of the sensible is experienced because sensation is unable to perform its regular function, whereby it presents a content to the understanding which in turn applies a concept to this content, by which the content is recognised. What can only be sensed 'moves the soul, "perplexes" it – in other words, forces it to pose a problem, as though the object of encounter, the sign, were the bearer of a problem – as though it were a problem' (p. 140; referring to Plato, *Republic* 524a) It is this perplexity in the encounter, forcing the 'soul' to 'pose a problem', which lies behind Deleuze's claim that 'recognition' of the fact of the 7th Postulate is not sufficient – rather it is a discovery which:

> must be raised to the transcendental level, and problems must be considered not as 'givens' (data) but as ideal 'objecticities' [*objectités*] possessing their own sufficiency and implying acts of constitution and investment in their respective symbolic fields. Far from being concerned with solutions, truth and falsehood primarily affect problems. A solution always has the truth it deserves according to the problem to which it is a response, and the problem always has the solution it deserves in proportion to *its own* truth or falsity – in other words, in proportion to its sense.
>
> (Deleuze 1994, 159)

Deleuze reminds us that the distinction in Aristotle between Analytics and Dialectics represents the division between the techniques involved in solving a problem already given and those involved in posing problems legitimately, how to engender the element of a syllogism. But the history of mathematics tended to steer clear of dialectics, preferring to concentrate on the method by which solutions are deduced from theorems. 'The reason is that theorems seem to express and to develop properties of simple essences, whereas problems concern only events and affections. As a result, the genetic point of view is forcibly relegated to an inferior rank' (Deleuze 1994, 160). Mathematics thus develops methods 'for solving supposedly given problems, [rather than] method[s] of invention appropriate to the constitution of problems' (p. 161).

Thus both philosophy and mathematics are determined by the principle whereby problems are evaluated 'according to their solvability [*résolubilité*]' (Deleuze 1994, 161). However, 'solvability' is an *extrinsic* value, an external condition. There is no intrinsic reason why a good problem should be solvable, nor a bad problem insoluble – quite the contrary.[6] What the traditions of philosophy and mathematics miss in their insistence on this external ground 'is the internal character of the problem as such, the imperative internal element which decides in the first place its truth or falsity and measures its intrinsic genetic power: that is, the very object of the dialectic or combinatory, the "differential" Problems are the differential elements in thought, the genetic elements in the true' (p. 161–2).

Let us emphasise what is at stake in this distinction between external ground and internal genetic power, for it is a distinction which lies at the heart of Deleuze's attempt to develop a philosophy of immanence. As we have seen, the defining characteristic of the traditional conception of the relation between problem and solution lies in the reducibility of the problem, its 'subservience' to the solution. The solution is, from this perspective, *indifferent* to the problem – it remains unaffected by the problem as such, since the problem is merely the way to the solution. This indifference is a defining characteristic of all philosophies of transcendence – thus, Platonic Forms are paradigmatically indifferent to their particular instantiations. A key imperative for Deleuze's philosophy of immanence is therefore to demonstrate that, for instance, the virtual does not remain indifferent to its actualisation, that the actual can affect the virtual. In the same way, an intrinsic, or immanent, relation between problems and solutions must entail that 'the true and the false do not suffer the indifference of the conditioned with regard to its condition, nor does the condition remain indifferent with regard to what it renders possible' (Deleuze 1994, 161)

In order to begin to see how the relation between problem and solution can call into question the traditional relation between One and multiple or universal and particular, we must inquire into the conditions of problems, into the sufficient reasons for problems, an inquiry which, Deleuze suggests, is akin to that of Riemannian differential geometry which, he claims, 'tends to give rise to discontinuity on the basis of continuity, or to ground solutions in the conditions of the problems' (Deleuze 1994, 162).

The key to understanding the nature of problems lies in understanding what Deleuze means when he writes that 'problems are Ideas themselves' (Deleuze 1994, 162). Ideas are, as Deleuze will go on to explain in chapter four of *Difference and Repetition*, indeterminate, and it is in this indeterminacy that their problematicity resides. Propositions by themselves, on the other hand, are always determinate, and in this way particular. Problems thus differ in kind from propositions. To be sure, certain propositions generate general solutions, but this is only because of 'the subjacent problem which inspires them' (p. 162). It is problems, Deleuze claims, which are strictly universal, and propositions which are particular. To understand or constitute a problem, therefore, must consist in determining 'the conditions under which the problem acquires a maximum of comprehension and extension. conditions capable of communicating to a given case of solution the ideal continuity appropriate to it' (p. 161). Just as forgetfulness of the problems which generate concepts in the history of philosophy renders these concepts abstract, so also forgetfulness of the problem leaves us with merely abstract general solutions. Deleuze has Bergson in mind when he writes that such abstract solutions are the effect of separating the continuity which functions as the problematic Idea from the discontinuity which is produced by the solving of the problem. It is such abstract discontinuity which leads to the conception of solutions as particular cases, which have 'the status of particular propositions whose sole value is designatory' (p. 163). The fundamental task in maintaining the relation between the continuity of the Idea with the discontinuity of its solution, therefore, is to regain the genuine sense of both universal and singular:

> for the problem or the Idea is a concrete singularity no less than a true universal. Corresponding to the relations which constitute the universality of the problem is the distribution of singular and distinctive points which determine the conditions of the problem. (p. 163)

In order that propositions do not become mere abstract particulars, they must maintain their relation with what Leibniz characterised as the continuity of the events, the 'how and the circumstances' which give propositions their sense.

Deleuze characterises such events as 'ideal', in distinction to the 'real events which they determine in the order of solutions' (Deleuze 1994, 163). The sense of the *ideal* of the problem here is to be distinguished from the traditional notion of essences, consisting instead in 'multiplicities or complexes of relations and corresponding singularities' (p. 163). Thus, rather than a relation of instantiation between essence or universal and particular (or indeed a relation between essence and accident), we must first of all think of the problem as a multiplicity. Now, the key difference between a multiplicity and a set is that sets still maintain a relation with a certain external principle (or essence) which determines the elements of the set (the set of whole numbers, for instance), whereas the only determining principle of a multiplicity is that certain relations are possible between the elements of the multiplicity – whatever these elements might be – and that there are certain fundamental principles or laws which determine the form of these possible relations, these laws being the only determinate aspect of the multiplicity. The objects of the multiplicity are themselves determined solely by the relations into which they can or do enter – while remaining wholly indeterminate with respect to their own form or matter.[7] We must therefore understand the problem as distributing the elements of the multiplicity according to the relations by which they are determined. Singular elements are then singular in relation to the distribution of ordinary or regular elements 'in the neighbourhood' of them, and in this way owe their singularity to these elements. But equally, the distribution of the continuity of regular elements is itself determined by the singular elements. A simple representation of this relation between the regular and the singular would be that of the singular point at which the direction of a curve changes, and the continua of regular points constituting the curve on either side of the singular point. In this case, singularities would represent the maxima and minima of curves. Another way of conceiving singularities derives from Poincaré, who developed the resources of topology to understand the dynamic behaviour of systems. In this context, a singularity can function as an 'attractor' or 'basin of attraction', in such a way that, if, at a given point, the trajectory of the system falls within the sphere of influence of the basin of attraction, then it will inevitably tend towards the attractor (just as a marble spun around the rim of a bowl will have a trajectory the end point of which is the centre of the bowl).[8]

What then is the relation between the continuity of events, the distribution of singularities which constitute the conditions of the problem, and the determined particulars that constitute the solution of the problem, from this new perspective of the problem which emerges from the critique of the 7th Postulate? Deleuze emphasises that while problems differ in kind from solutions, they nevertheless do not exist apart from their solutions. Rather, the problem 'insists and persists in the solutions':

> A problem is determined at the same time that it is solved, but its determination is not the same as its solution: the two elements differ in kind, the determination amounting to the genesis of the concomitant solution. (In this manner the distribution of singularities belongs entirely to the conditions of the problem, while their specification already refers to solutions constructed under these conditions.) The problem is at once both transcendent and immanent to its solutions. Transcendent, because it consists in a system of ideal liaisons or differential relations between genetic elements. Immanent, because these liaisons or relations are incarnated in the actual relations which do not resemble them and are defined by the field of the solution. (Deleuze 1994, 163)

There are a number of different courses which an exegetical reading of Deleuze could follow at this point, including, among others, a discussion of the differential element of problems, or the relation of reciprocal determination between problems and solutions which foreshadows the account of the relation between the virtual and the actual to be developed in the succeeding chapters of *Difference and Repetition*.[9] However, we wish to return to our initial questions regarding the 'problem' of the relation between mathematics and philosophy. We have followed the development of a new way of conceiving the problem as such for philosophy. It would be possible to trace a certain mathematical lineage, or better, field, within which this concept of the problem developed, such that it could come, in Deleuze's philosophy, to discharge the traditional philosophical concept of the problem. Rather than do this explicitly, however, we propose to outline a series of developments in mathematics, developments which determine the field from within which a number of concepts that will come to play a decisive role in Deleuze's philosophy emerge. These remarks will in turn pave the way for some more or less speculative reflections in which we will seek to address the question of the problem that the relation with mathematics in general, and the particular field with which we will be interested, poses philosophy. The spirit of this gesture is Bergsonian, since it turns on a certain conviction regarding the complementarity of developments in science and philosophy, a complementarity which entails reciprocal implication between developments in both fields.

It is well known that the primary mathematical influence on Deleuze's philosophy is that of the calculus. It is worth recalling the motivation for the calculus. Seventeenth century science was characterised first and foremost by its concentration on problems of motion. This work led to the development of the concept of the *function*, which expressed symbolically the relation between variables. The calculus constituted a response to a series of diverse needs, primarily that of calculating the velocity and acceleration of a moving body at an instant, given the relation between variables which established that the distance covered by a moving body is a function of time. The difficulty presented is that velocity and acceleration vary from instant to instant. This problem requires a method for finding the instantaneous rate of change of one variable with respect to another. In his later work, Newton conceived of the variables in the function as generated by the continuous motion of points, lines and planes, calling such variable quantities 'fluents', and their rates of change 'fluxions' (what is now called the 'derivative' of the function). Thus, given a relation between two fluents (which change with time), the task for the calculus consists in determining the relation between their fluxions (or *vice versa*). Whereas Newton used infinitely small increments in the variables as a means of determining the fluxion, Leibniz dealt directly with the infinitely small increments in the variables, that is, with *differentials*, and sought to determine the relation between these.[10]

From the founding work of Newton and Leibniz on the calculus stemmed a number of new areas of mathematical inquiry, including differential geometry and the calculus of variations (dealing with the problem of discovering the maximum and minimum values of a function). Differential geometry responded to the need for greater understanding of curves and surfaces, to the extent that the paths described by moving objects are curves, while the objects that are moving are bounded by surfaces. The recourse to the method of the calculus in differential geometry enabled the study of curves and surfaces which vary from point to point, just as the velocity and acceleration of moving bodies vary from instant to instant. The main advance in differential geometry consisted in the proposal that a surface can be conceived as a space in itself, rather than being embedded within a higher-dimensional space (in classical geometry, a surface is studied on the basis that it is a figure lying within three-dimensional space). The geometry of a surface conceived as a space in itself will be 'intrinsic' to the surface, rather than deriving from the space which is assumed to surround the surface.[11] Thus, if a surface is conceived as a space in itself, then it will have a geometry which may well turn out to be non-Euclidean. Riemann was the first to generalise this notion, developing an intrinsic geometry for any space (whatsoever), characterising such 'n-dimensional spaces' as 'multiplicities of n-dimensions'.[12] A point in such a multiplicity is determined by the assignation of values in n variable parameters, and the multiplicity consists in the aggregate of all such possible points.[13]

Because there is no extrinsic determination of the nature of the points comprising the multiplicity, and because the space varies from point to point, Riemann proceeded from the principle that it is only ever possible to have local knowledge of a space. On the other hand, Riemann also sought to determine the notion of a 'curvature of a multiplicity', on the basis of which it becomes possible to determine those spaces on which figures can be moved without changing shape or magnitude. In this case, however, since the 'curvature of the multiplicity is defined in terms of quantities determinable on the multiplicity itself', then curvature must be understood as 'a property of the metric imposed on the multiplicity rather than of the multiplicity itself' (Kline 1972, 891–2). Riemann had earlier distinguished between continuous and discrete multiplicities according to whether the ground of the metric relations which can subsist between the elements of the multiplicity are an intrinsic part of the multiplicity or need to be imposed from outside the multiplicity. This notion was to prove fundamental in its influence on general relativity, since, to the extent that a multiplicity – in this case, our own space to which Euclidean geometry has traditionally been applied – is subject to binding physical forces which act upon it, then not only does space vary from point to point, but, as matter moves through space (understood as the continual passing of a curvature or distortion of space from one portion of that space to another in the manner of a wave) so space also varies from time to time, or instant to instant. Thus, just as the principle of differential geometry determines surfaces as spaces in themselves rather than being embedded in a containing space, so also this latter principle demonstrates the fact that any understanding of the nature of physical space depends upon the interrelation between matter and space. In both of these ways, therefore, it is clear how traditional philosophical conceptions of space, or more strictly, the coherent field of concepts from which the philosophical understanding of space derives, is not compatible with Riemannian geometry (nor will it be with Einstein's general theory of relativity as a consequence).

This last point offers a brief glimpse of the extent of the displacement of philosophical conceptuality which Riemann's thought might be enabled to effect in Deleuze's philosophy, and what in turn is the extent of the conceptual implications involved in his recourse to such Riemannian notions as n-dimensional space and multiplicities. If we now briefly turn to the mathematical theory of singularities, we find that it consists first and foremost in a new course of development for the theory of ordinary differential equations. In general, work had progressed on the presupposition that differential equations contain continuous functions in the domains in which solutions were to be considered. In the middle of the nineteenth century, again based in part on the work of Riemann, focus was turned on differential equations which, 'when expressed so that the coefficient of the second derivative is unity, have coefficients that are singular, and the form of the series solutions in the neighbourhood of the singular points particularly that of the second solution, is peculiar.' Such singular points are to be distinguished from *ordinary* points, points at which all the coefficients are continuous (Kline 1972, 721). Since the solutions in the neighbourhood of singular points are series, the form of the series has to be determined before these solutions can be calculated. Such knowledge can only be obtained from the differential equation, prompting the mathematician Fuchs to write that 'the problem of the theory of differential equations' has become that of 'deducing from the equation itself the behaviour of its integrals at all points of the plane.' This work was primarily undertaken by Riemann and Fuchs himself.

Riemann's approach consisted in assuming that certain relevant functions within a linear equation are single-valued continuous (in fact, analytic) functions except at isolated singular points. The task consisted in accessing the behaviour of analytic solutions in the neighbourhood of such isolated singular points, where these solutions, despite the functions being single-valued, are themselves not in general single-valued over the entire domain of possible values. What Riemann showed was that, if we 'trace the behaviour of an analytic [solution] along a closed path enclosing a singular point,' then that solution 'will change its value to another branch of the same function, though it remains a solution of the differential equation.' The general solution formed from the linear combination of *n* such particular solutions thus consists in a group, whereby each particular solution 'undergo[es] a certain linear transformation when each is carried around a closed path enclosing a singular point. Such a transformation arises for any closed path around each of the singular points or combination of singular points. The set of transformations forms a "group"' (in the technical sense deriving from the work of Galois and Abel) (Kline 1972, 722–3).

The work on the theory of singularities was further advanced by Poincaré's 'qualitative' theory of non-linear differential equations, which he applied to the problem of the stability of planetary motion. In order to determine whether a planet's orbit is stable, Poincaré inquired into whether the trajectory of a moving point describes a closed curve, and whether or not it remains within the interior of a certain portion of a plane. Poincaré began with the simplest form of non-linear differential equation appropriate to his inquiry, and found that singular points of the equation played a key role. Poincaré qualitatively distinguished four types of singular points – focus (or origin), saddle point, node and centre, and described the behaviour of solutions of the equation around these points. In the case of the focus, the solution spirals around and approaches the origin, whereas in the case of the saddle point, solutions approach and then depart from the single point. At a node, an infinity of solutions cross, while a centre is a point around which closed trajectories exist, one enclosing another and all enclosing the centre (Kline 1972, 733).

Poincaré's qualitative theory of differential equations is essentially topological, since it is concerned with the *form* of integral curves and the *nature* of singular points. Like Riemannian geometry, topology deals with the surfaces of figures as spaces in themselves, rather than from the perspective of the space within which they might be embedded. As a consequence, the properties of figures relevant to topology differ radically from those relevant to traditional geometry. For instance, there is no topological difference between a circle, a triangle or a square, nor is there a difference, in the infamous example, between a doughnut and a coffee-cup (both of the latter being *tori* – i.e., surfaces which 'have' a hole). Furthermore, the relevant 'domain' in topology is the local rather than the global, since topology is basically concerned with the properties of geometric figures which remain invariant when these figures are transformed – where what is meant by invariance is that no new points are created in the transformation, there is a one-to-one correspondence between the points of the original figure and the transformed figure, and where the transformation carries locally nearby points into locally nearby points. It is this latter property which is called *continuity* in topology, and the requirement is that the transformation and its inverse both be continuous, in which case the transformation is called a *homeomorphism* (Stewart 1995, 144ff).

Topological research has divided into two branches – 'point set topology' which is concerned with geometrical figures regarded as collections of points; and 'combinatorial topology', which regards geometrical figures as aggregates of smaller building blocks. This latter branch owes its provenance to Leibniz, who as early as his *Characteristica Geometrica* (1679) had sought to identify basic geometric properties of geometric figures, and to combine these properties, utilising a novel symbolic representation, to produce new properties, a technique he called *analysis situs*. The aim of this approach was to move away from the concern with magnitude typical of Cartesian coordinate geometry towards a form of geometric analysis which focused on location [*situs*] directly, just as algebra dealt directly with magnitude.[15] Once more, it was Riemann who was to make the decisive contribution to the development of combinatorial topology, by way of his work in complex function theory, in which later he found it necessary to introduce the principle of the 'connectivity' of surfaces.[16] On the basis of this principle, Riemann was able to classify surfaces according to their connectivity, an intrinsic topological property. The combinatorial topology of closed spaces initiated by Riemann was subsequently generalised by Poincaré. The problem motivating Poincaré's work was that of determining the structure of four-dimensional 'surfaces' prompting him into a systematic study of n-dimensional spaces in general. He developed a purely geometric theory of multiplicities, conceived as generalisations of Riemannian surfaces.

This brief survey has traced certain strains of thought, from calculus through differential geometry and theory of singularities to topology, emphasising the roles played in particular by Leibniz, Riemann and Poincaré. We have occasionally noted themes and concepts which have been influential for Deleuze. Before returning to our opening questions, and the concluding, speculative, remarks towards which we gestured, there is one final mathematical field to which we wish to draw attention. Throughout, we have seen that the significant issues in these various developments have tended to involve the principles of continuity and localness, as we might have assumed having begun from the calculus. However, there is one recent development in topology which, while maintaining the concentration on the local, has introduced the principle of *discontinuous processes*, namely, the catastrophe theory of René Thom. Catastrophe theory involves the topological description of systems which display discontinuous changes. Crucially, such discontinuous changes can themselves be the effects of continuous processes – the motion of a light switch is continuous, but there is a discontinuous change between the two states of the light bulb. One of the most important areas of application for catastrophe theory is biological morphogenesis, the basis of which is the discontinuous process of cell division. The process of evolution, of course, consists in lines of divergence or division, a theme at the heart of Deleuze's response to Bergson's work in *Creative Evolution* (1911). Indeed, a key question here is how evolution, conceived as a continuous process, 'causes' the discontinuous changes involved in such lines of divergence.

In Thom's analyses, we find that continuous changes to surfaces can lead to discontinuous changes in the trajectories of moving bodies on these surfaces. For instance, such a change can occur when a continuous change in the surface leads to the disappearance of a minimum – if the object were initially located at that minimum, then the disappearance of the minimum will 'cause' the object to 'jump', even though its trajectory had been expressive of the object's tendency to move continuously. Thom defines the space of the surface, in this example, as the 'control space' and the space of the moving object as the behaviour space, and specifies that the behaviour of the system is governed by the 'potential', or energy, in the system. Given that all events in the physical world are determined by four variables (three of space and one of time), Thom restricts his analyses to four dimensions of control. He then demonstrates that given four dimensions of control, there are precisely seven topologically distinct kinds of discontinuity which can occur in a dynamical system.[17]

Thom characterises what he calls the 'morphology of a process' by initially defining a domain on which the process takes place, and allowing an (ideal) observer the ability to investigate the neighbourhoods of any point x in that domain. Then if the observer 'can see nothing remarkable' in the neighbourhood of x, that is, 'if x does not differ in kind from its neighbouring points, then x is a *regular point* of the process.' These regular points form an 'open set' in the domain, and the complementary 'closed set' is the set of *catastrophe points*, 'the points with some discontinuity in every neighbourhood' This closed set, 'and the description of the singularities at each of its points, constitute the morphology of the process' (Thom 1975, 38).

With this brief discussion of Thom's work, we return to the guiding principle for Deleuze's discussion of the problem, namely the relation between the singular and the ordinary (or generic), based on a comparison of neighbouring things, that is, from the perspective of localness. This represents an underlying shift of theoretical attention, from things (more or less in isolation) to neighbourhoods, and the relations and singularities that define these neighbourhoods within multiplicities.[18]

As we have indicated, our survey of certain developments in mathematics is suggestive of the coherence of a mathematical field of research. In Deleuze's philosophy, it is this field which has enabled various 'functions' to displace a series of traditional philosophical concepts, and, more importantly, the philosophical field from which they emerged, and to begin to develop a new philosophical ontology, of multiplicities, the defining characteristics of which are differential relations and the distribution of singularities. It is an immanent ontology, in keeping with Riemann's shift from surfaces as spaces embedded in higher dimensional (and hence transcendent) spaces to surfaces as spaces in themselves, and the focus of the work it opens will be fundamentally local.

Can we now then offer a response to our initial question concerning the *problem* of the relation between maths and philosophy? It seems as if one principle underpinning the mathematical work that we have been considering is that it embodies a fundamentally *relational* approach. There have been occasional relational approaches at the margins of philosophy, such as Saussurean structuralism, but these have for the most part been transient. Philosophy remains essentially object-focussed, as Heidegger has already demonstrated. On the basis of this immanent critique of the history of philosophy, and of the underlying principle of the mathematical field of enquiry we have been outlining, we may say that a delineation of the relation between maths and philosophy in the case of the singularity that is Deleuze precisely poses philosophy a *problem* – the problem of how to *become relational*. Furthermore, if we follow Deleuze, then this relationality will be *differential*. If philosophy is able to become relational, and differential, then, we believe, the possibility is opened up for a new relation to be formed with such emerging sciences as those of complexity, which are themselves founded in the principle of relationality, but which, as yet, have not been able to develop a coherent ontology for their research. It is in the iteration of such relations that new problems will inevitably emerge for philosophy.

Notes

1 Deleuze and Parnet 1996, 'H as in "History of Philosophy"' (overview by C.J. Stivale, available online at http://www.langlab.wayne.edu/CStivale/D-G/ABC2.html#anchor700599). This is among the richest of the discussions with Parnet in the *Abécédaire*. Deleuze insists that if one doesn't discover the problem to which a concept corresponds, then all philosophical work remains abstract. To discover the problem is to render a concept concrete. One reason, we might suggest, why Deleuze's works in the 'history of philosophy' remain so engaging today is evident from his claim that to 'do' history of philosophy 'is to restore problems and, through this, to discover what is innovative' in the concepts that philosophers create. For example – what is Hume's problem? The traditional discussions of impressions and ideas, or of causation, remain abstract because they do not confront this issue. Deleuze argues that Hume's problem is how a mind becomes a subject, or human nature. It is only in the context of this problem that Hume's concept of *belief* becomes concrete, or that his new logic of (external) relations becomes concrete.

2 Deleuze and Guattari 1994, 16. Dan Smith argues convincingly that the root of Badiou's failure to engage with Deleuze's philosophy, and the ill-formed critique which follows as a consequence, stems from his initial failure to appreciate the true differend for Deleuze's thought, which is located, as Smith shows, 'in the difference between axiomatics and problematics, major and minor science' (Smith 2003, 434). This distinction, between axiomatics and problematics, was developed as early as Proclus' *Commentary on the First Book of Euclid's Elements* – see Deleuze 1990a, 9th Series Of the Problematic – and has, Deleuze suggests, a lineage which can be traced through the history of mathematics. One manifestation of this lineage is the role which Poincaré accords to 'intuition' in mathematical creativity (see Poincaré 1952, 'On the Nature of Mathematical Reasoning'; and Poincaré 1958, 'Intuition and Logic in Mathematics'). The relation between creative intuition and

organisational axiomatics echoes that between intuition and intelligence in Bergson, whereby 'only intelligence has the scientific means to *solve* formally the *problems* [emphases added] posed by intuition' Deleuze then comments in a footnote: 'In Bergson, the relations between intuition and intelligence are very complex, and they are in perpetual interaction' (Deleuze and Guattari 1987, 374, 556 n. 40). It is the nature of this interactivity with which we are ultimately concerned, whether it be between intuition and intelligence, problematics and axiomatics, problem and solution, or the mathematical field and the philosophical field.

3 Deleuze 1994, 108. Dan Smith confirms Deleuze's derivation of this principle from Lautman. See Smith 2003.

4 Problems and singularities also play a significant role in Deleuze's *Logic of Sense* (1990a) – see in particular the 9th and 15th Series.

5 Deleuze's 'alternative' theory of the faculties, developed towards the end of his discussion of the fourth postulate, derives from the problem of how the faculties are able to enter into harmonious accord – his 'solution', namely the notion of a discordant, or ungoverned, accord itself remains, of course, provocatively problematical. A concept from earlier in the book which returns in the context of this problem is that of the 'dark precursor' [*precursor sombre*], which enables a differential communication between the faculties, which is, as Deleuze argues, 'sufficient to enable the different as such to communicate, and to make it communicate with difference' (Deleuze 1994, 145). This most obscure of concepts is amongst the most significant in *Difference and Repetition*.

6 In a similar way, Poincaré was given to arguing that an uninteresting true theory was of substantially less value than an interesting theory which nevertheless remained unproven.

7 Thus, for a multiplicity of any elements whatsoever, we may specify that a possible relation _ subsists between them, where $\ddagger$ is commutative and associative, such that $x \ddagger y = y \ddagger x$, and $x \ddagger (y \ddagger z) = (x \ddagger y) \ddagger z$. Addition and multiplication would be two possible instantiations of this purely formal relation.

8 A key aspect of problem constitution thus lies in the way in which the differential relation between singular and ordinary elements in the multiplicity is determined. Consider, for example, an argument about the effect on the environment of driving sports utility vehicles (SUVs) – in which I found myself uncomprehendingly participating recently! While I was denouncing them for their gas-guzzling tendencies, my interlocutor defended them as being relatively efficient. However, for him, their efficiency was relative to other, less efficient cars, whereas from my perspective, the contrast was with forms of public transport such as trains or buses. The failure of our argument was an effect of

the different problems which distributed the singular element (SUVs) and the ordinary elements from which it was differentiated. In other words, we were working on the basis of a poorly defined problem.

9 The reader interested in such issues, and their relation to recent developments in science which can be seen to have had a more or less direct influence on Deleuze's thinking, could do little better than consult Manuel DeLanda's excellent *Intensive Science & Virtual Philosophy* (2002).

10 See Boyer 1959, especially Chapter V.

11 The difference between intrinsic and extrinsic features of the surface is fundamental. Two surfaces which have equivalent intrinsic features may turn out not to be equivalent surfaces, when viewed within their embedding space, that is, from the perspective of their extrinsic features. The surface of a sphere differs intrinsically from the surface of a plane. The surface of a plane, on the other hand, does not differ intrinsically from that of a cylinder; it does, however, differ extrinsically, that is, when viewed from the perspective of its embedding 3–dimensional space. A further distinction can also be made between the local and the global with respect to surfaces. *Locally*, it is impossible to determine any difference in the features at any given point on a plane or the surface of a cylinder. *Globally*, however, a geodesic on a plane will extend infinitely, whereas a geodesic on the surface of a cylinder will return to its starting point. Thus, the intrinsic similarity of plane and cylinder surface is a local, rather than a global, matter – their global difference is, ultimately, an effect of the *topological* structures of the two surfaces. See Sklar 1977, 40–42.

12 The concept of a curve, or space, which is not embedded in a higher dimensional space, as well as that of the n-dimensional multiplicity, were, of course, to have a profound impact on Deleuze's thinking. It would be possible to trace a conceptual lineage from Leibniz to Einstein on the basis of their shared rejection of the notion of an absolute, or embedding, space, by way of Riemann's differential geometry, which latter was to provide the mathematical means for expressing Einstein's general theory of relativity.

13 Returning to our earlier discussion of multiplicities, we can specify that what distinguishes a mere collection of points from a space is that in a space, there is some form of relation which 'binds' the points together, and it is on this basis that we may talk of a multiplicity as a space.

14 Cited in Kline 1972, 721.

15 Letter to Huygens (1679), cited in Kline 1972, 1163.

16 Riemann defines the connectivity of a surface in the following way: 'If upon a surface *F* [with boundaries] there can be drawn *n* closed curves a_1, a_2,

a_n which neither individually nor in combination completely bound a part of this surface F, but with whose aid every other closed curve forms the complete boundary of a part of F, the surface is said to be $(n + 1)$-fold connected.' (Riemann's 1851 Thesis on Complex Functions, cited in Kline 1972, 1166.)

17 The seven types of discontinuity are: fold, cusp, swallow's tail, butterfly, hyperbolic umbilic, elliptic umbilic and parabolic umbilic. See Thom 1975, Chapter 5.

18 This same shift informs the work in theoretical biology of both Robert Rosen and Brian Goodwin. See Rosen 1991and 2000; and Goodwin 1994.

10
Manifolds: on the concept of space in Riemann and Deleuze

Arkady Plotnitsky

1. Concepts, spaces, and sets

Bernhard Riemann is arguably the most significant mathematical presence in and influence upon Gilles Deleuze's work. I would like, in this essay, to explore some of the reasons for this influence and for the conjunction of Riemann's mathematics and Deleuze's philosophy, a conjunction that, I would argue, has important implications for our understanding of the relationships between mathematics and Deleuze's philosophy.

The argument of the essay arises from a view of Riemann's mathematics as *conceptual* mathematics (which gives primacy to thinking in mathematical concepts rather than formulas or, as I shall explain, sets) and from the view of Deleuze's philosophy as *conceptual* philosophy. The latter view itself derives from Deleuze's understanding of philosophy as the invention of concepts, an understanding developed especially in his and Félix Guattari's *What is Philosophy?* (1994). In both cases at stake is also, and, I shall argue, correlatively, a conceptual mathematics and a conceptual philosophy of spatiality, especially continuous spatiality – phenomenal, mathematical, or physical, and, in the case of Deleuze's philosophy, cultural and political. Accordingly, I argue that what most essentially links Riemann and Deleuze are conceptuality and spatiality, especially, again, continuous spatiality, and the relationships between them, ultimately coupled to the problematic of materiality. My argument is also framed by the work of Leibniz (a key figure for both Riemann and Deleuze), at one end, and, at the other, some of the key works in twentieth-century mathematics, extending Riemann's ideas, which shape some of the deepest and most significant developments in modern mathematics.

Riemann's conceptual mathematics may be *contrasted* to set-theoretical mathematics. (I am not saying simply opposed, and both could often be translated into one another.) The concept of set was introduced by Georg Cantor shortly after Riemann's death (in 1866) and has shaped foundational thinking in mathematics from then on.[1] Concomitant with this difference in Riemann's case is, I argue, Riemann's grounding of mathematical thinking in a new *concept* of space, as against that of Euclidean geometry and even preceding versions of non-Euclidean geometry, through the concept of manifold or manifoldness [*Mannigfaltigkeit*]. (I shall explain this concept and its Deleuzian counterpart, the concept of smooth space, below, noting for the moment that both are primarily defined by their constitution as conglomerates of local spaces and multiple transitions between them.) At least such is the case when Riemann deals with *space*, whether in corresponding disciplinary domains, such as geometry or topology, or elsewhere in mathematics, such as analysis. This qualification is not as strange as it might appear. Apart from the fact that not all of Riemann's mathematics is spatial or concerns space, the most crucial point here is how one conceives of space, what is one's concept of space, which may, as a *concept*, have a complex structure or architecture. In particular, the question is whether in a study of space one considers space as a primary, grounding concept or whether, with Cantor and most (but not all) subsequent mathematics, one considers it as derived from the concept of set, say, by considering a given space as a particular set of points. I suspend for the moment the difficulties (such as those of famous paradoxes) of the concept of set and shall use a 'naïve' definition, given by Pierre Cartier, via Bourbaki: 'A *set* is composed of *elements* capable of having certain *properties* and certain *relations* among themselves or with elements of other sets' (2001, 393). More recently, roughly from the 1950s on, the so-called category theory may be argued to offer an approach closer to that of Riemann, especially through its connections to such mathematical fields as topology and algebraic geometry, fields in turn developed following and shaped by Riemann's work.[2] In this view, too, spatiality, at least a certain conception, topos, of a spatial type, developed in the so-called 'topos theory' introduced by Alexandre Grothendieck, becomes a primary concept, more primary than set. Topos theory also allows for such esoteric constructions as spaces consisting of a single point or spaces without points, sometimes slyly referred to by mathematicians as 'pointless topology' Philosophically, however, this notion is far from 'pointless', for it suggests that space or, again, at least something spatial in character, is a more primary object than point, or again, a set of points. By the same token, space also becomes a Leibnizian, monadological concept, for example, insofar as points in such a space (when it has points) are themselves better seen as monads, that is, as certain elemental but structured spaces, rather than structure-less entities (classical points).

As will be seen, however, the monadology of this type of concept or of Riemann's concept of space, to begin with, is extended and radicalized by its nomadology, that is, by mathematical practice in a smooth and, thus, itself Riemannian, space of 'minor' or 'nomadic' mathematics, as Deleuze and Guattari call it. Indeed such a concept is more likely to arise, and Riemann's had, in such a space, thus making the concept and the mode of its production reflect each other or even *on* each other in a kind, to use Charles Baudelaire's and Paul de Man's term, of *dedoublement* (which often defines the concepts of smooth space). By contrast the concept of set or the concepts of space based on it are more likely to function and have functioned as a major or state concept. Mathematics and specifically geometry, too, may be a state, major science, as in Gaspard Monge, or a nomadic, minor science, as in Gérard Desargues, or later in non-Euclidean geometry culminating in Riemann's geometry, and use a given mathematical concept accordingly (Deleuze and Guattari 1987, 362–65).[3] Cantor's set theory, too, started very much as a minor science. Besides, Deleuze and Guattari rightly argue, major and minor science usually co-exist in complex relationships. Always? At least major science would not be possible without minor or something that is, in its structure and functioning, minor (1987, 484–86). On the other hand, a science that is strictly minor is conceivable, although it could only be a *minor* science as against one *major* science or another.

Deleuze sees the primary significance or even the very nature of philosophy in its invention of new *concepts*, or, as Deleuze and Guattari argue more strongly, concepts 'that are always new' (1994, 5). This view is underlined by a different understanding or concept of philosophical concept, which may be summarized as follows. According to Deleuze and Guattari, a philosophical concept is not an entity established by a generalization from particulars or 'any general or abstract idea' (p. 11–12, 24), but always has a complex multi-layered structure. As they state, 'there are no simple concepts. Every concept has components and is defined by them. It therefore has a combination [*shiffre*]. It is a multiplicity [manifold(ness)?]. There is no concept with only one component' (p. 16). Each concept is a multi-component conglomerate of concepts (in their conventional senses), figures, metaphors, particular elements, and so forth, which may or may not form a unity. This concept of concept is traceable to much earlier texts, in particular, *Difference and*

Repetition (Deleuze 1994), and may indeed be seen as defining most of Deleuze's philosophical work. At the same time, the argument of *What is Philosophy?* (Deleuze and Guattari 1994) is shaped by spatial thinking, beginning with linking the very invention of philosophical concepts to a spatial and indeed Riemannian concept, the plane of immanence, thus making the space of a given concept a Riemannian, smooth space. The concept finds its predecessors and avatars, such as the plane of consistency, throughout Deleuze's work, which may indeed be seen as a kind of topo-philosophy.

As the invention and construction of new concepts, philosophy may even need to be primarily spatial, at least insofar as it is a minor, nomadic science, a science traversing (again, often *dedoublement*-like) smooth Riemannian spaces – phenomenal, cultural, political, or geographical, or, when we move with Deleuze and Guattari to 'geophilosophy,' geopolitical (1994, 85–113). Geophilosophy lives in geo-smooth spaces, which enables it to resist and overcome all *state* (in whatever sense) borders and striations.

2. *Continuums, manifolds, and smooth spaces*

I would like to begin my argument by way of a caution, always necessary in dealing with mathematics outside its proper disciplinary sphere, with Deleuze's comment in *Cinema 2* (1989), which, like *Cinema 1* (1986) (largely guided by Bergson's philosophy), has deep connections to Riemannian spaces, motivating this comment itself. Deleuze writes:

> *Of course*, we realize the danger of citing scientific propositions outside their own sphere. It is the danger of arbitrary metaphor or of forced application. But *perhaps* these dangers are averted if we restrict ourselves to taking from scientific operators a particular conceptualizable character which itself refers to non-scientific areas, and converge with science without applying it or making it [simply] a metaphor. (Deleuze 1989, 129; emphasis added)

Deleuze amplifies this point in one of his interviews, and in the process justifies philosophy's claim upon the exploration of scientific concepts and gives science itself (obviously including mathematics) a philosophical and artistic conceptual dimension. He says:

> There are two sorts of scientific concepts. Even though they get mixed up in particular cases. There are concepts that are exact in nature, quantitative, defined by equations, and whose very meaning lies in their exactness:

> a philosopher or writer can use these only metaphorically, and that's quite wrong, because they belong to exact science. But there are also essentially inexact yet completely rigorous concepts that scientists can't do without, which belong equally to scientists, philosophers, and artists. They have to be made rigorous in a way that's not directly scientific, so that when a scientist manages to do this he becomes a philosopher, an artist, too. This sort of concept's not unspecific because something's missing but because of its nature and content. (Deleuze 1995, 29; translation modified)

I am not altogether sure whether one should even use the word 'inexact' here. Deleuze's appeal to the quantitative, numerical side of mathematical and scientific concepts is essentially right, and need not be seen as reducing the disciplinary complexity and richness of mathematics and science. It also carries a certain conceptual specificity and significance for Deleuze, as is clear from Deleuze and Guattari's discussion, via Riemann, on the one hand, and Bergson, on the other, of a juxtaposition between the (qualitative) concept of distance and the (quantitative) concept of magnitude. This juxtaposition becomes correlative to that between the smooth and the striated spaces in *A Thousand Plateaus* (Deleuze and Guattari 1987, 483–84). Bergson's duration may be seen (causalities are more complex) as an extraction or distillation of an inexact, qualitative, non-numerical concept of multiplicity or manifoldness from Riemann's concept of manifold, a concept that is juxtaposed to 'metric manifoldness or the manifoldness of magnitude, a (mathematically) exact, numerical counterpart of it in Riemann's overall conceptual architecture of manifold (p. 483; translation modified).[4] Mathematics and specifically topology, as a mathematical discipline, can and must give 'smooth spaces' (or those mathematical objects upon which smooth spaces could be modelled) mathematically exact, numerical features. Once there is mathematics, including that of space, there is always a number somewhere. This need not be the case in philosophy, as, for example, Bergson's or Deleuze's deployment of Riemann's ideas, or those of Einstein's relativity theory, would show (p. 483–85).

The ancient Greeks might be argued to have developed a complex *philosophical* topology (as in Plato's concept of *khora* in *Timaeus*), but they did not have a mathematical discipline of topology, and their only mathematical, exact and quantifiable, science of space was geometry. Geometry and topology, while both concerned with space, are distinguished by their different mathematical provenances. Geometry (geo-*metry*) has to do with measurement, while topology disregards measurement and scale, and deals only with the structure of space qua space and with the essential shapes of figures. Such figures are themselves usually seen as spaces, continuous spaces, as topology is primarily a science of continuity. Insofar as one deforms a given figure continuously (i.e. insofar as one does not separate points previously connected and, conversely, does not connect points previously separated) the resulting figure is considered the same. Thus, all spheres, of whatever size and however deformed, are topologically equivalent. They are, however, topologically distinct from tori. Spheres and tori cannot be converted into each other without disjoining their connected points or joining the disconnected ones. The holes in tori make this impossible. Such properties can be related to certain algebraic and numerical properties associated with topological spaces, through, in particular, the so-called cohomology theory, one of the great achievements of modern mathematics, extending in part from Riemann's work. Anticipated by Leibniz's '*analysis situs*,' these ideas were gradually developed in the late eighteenth century and then in the nineteenth century in the works of (in addition to Riemann) Leonard Euler, Karl Friedrich Gauss, Henri Poincaré, and others, establishing topology as a mathematical discipline by the twentieth-century.

The relationships between our mathematical and philosophical or, to begin with, phenomenological intuition, especially that of continuity, are a delicate and difficult matter. As Hermann Weyl astutely observed in *The Continuum* [1917]: 'The conceptual world of mathematics is so foreign to what [phenomenal] intuitive continuum presents to us that the demand for coincidence between the two must be dismissed as absurd. Nevertheless, those abstract schemata supplied us by mathematics must underlie the exact sciences of domains of objects in which continua play a role' (Weyl 1994, 108). It is of some interest and significance in the present context that Weyl refers on this point to Bergson's *Creative Evolution*: 'It is to the credit of Bergson's philosophy to have pointed out forcefully this deep division between the world of mathematical concepts and the immediate experience of continuity of phenomenal time (*la durée*)' (Weyl 1994, 90). This 'deep division' notwithstanding, Bergson's ideas concerning continuity may have a Riemannian genealogy, in accordance with Deleuze and Guattari's argument, mentioned above. This is not inconsistent. For specific or specifically (disciplinarily) refined forms of intuition and conceptuality can also traffic between different domains, disciplinary or other, for example between Riemann and Deleuze, as mathematics and its exact concepts enter new rhizomatic networks. There may be differences between mathematics (or science) and the human sciences (or literature and art) as concerns what Deleuze calls inexact rigour, rather than only in terms of the exact (such as numerical) rigour of one and the inexact rigour of the other. As such, however, mathematics can provide ideas that could be developed elsewhere.

Weyl's insight concerning the difference in the mathematical and phenomenal intuition of continuity occurs primarily in the context of Cantor's work, which adopted and made famous the term 'continuum' and which led to, for our phenomenal intuition, highly counterintuitive conceptions. The concept of continuity is, however, central to Riemann's thinking, mathematically different from Cantor's, thinking that is generally conceptual rather than set-theoretical, that is, thinking in terms of various concepts rather than grounding his mathematics in the concept of set. (The concept of set does of course have its philosophical dimensions as well, including in Cantor.) Weyl's view itself could be traced to Riemann's comments on his concept of manifold(ness). Weyl discusses Riemann's work in detail in his classic *Space Time Matter* [1918] (1952), written just after *The Continuum*, and in his book *The Concept of Riemann Surface* [1913] (1955). Riemann writes in his famous *habilitation* lecture, 'On the Hypotheses which Lie at the Bases of Geometry' (Riemann 1873), which introduced the idea of the Riemannian manifold and Riemannian geometry:

> The concepts of magnitude are only possible where there is an antecedent general concept which admits of different specialisations. According as there exists among these specialisations a continuous path from one to another or not, they form a *continuous* or *discrete* manifoldness [*Mannigfaltigkeit*]; the individual specialisations are called in the first case points, in the second case elements, of the manifoldness. Concepts whose specialisations form a *discrete* manifoldness are so common that at least in the cultivated languages any things being given it is always possible to find a concept in which they are included. (Hence mathematicians might unhesitatingly found the theory of discrete magnitudes upon the postulate that certain given things are to be regarded as equivalent.) On the other hand, so few and far between are the occasions for forming concepts whose specialisations make up a *continuous* manifoldness, that the only simple concepts whose specialisations form a multiply extended manifoldness are the positions of perceived objects and colours. More frequent occasions for the creation and development of these concepts occur first in the higher mathematics. (sec. 1, pt. 1; translation modified)[5]

Riemann, thus, thought mathematics a primary, perhaps *the* primary, source of the concept of 'continuous manifold(ness)' (similarly to the case of Cantor's concept of set), at least as concerns simple concepts. The statement leaves space to complex concepts of everyday life (or those of philosophy) whose mode of determination is analogous to that of continuous manifold.

As this description suggests and as his overall discussion makes clear, Riemann defines mathematical objects not in terms of ontologically pre-given assemblies ('sets') of points, which are then given a certain set of relations between them, but in terms of concepts. Each concept has a particular mode of determination, such as discrete vs. continuous manifold, whose elements, such as points, are related through a given determination. Thus, beyond giving an essential priority to thinking and specifically to thinking in concepts over calculational or algorithmic approaches, Riemann's mathematics is *structurally* conceptual. It is based on specifically determined concepts, as against the set-theoretical mathematics that followed him or the mathematics of formulae that preceded him.[6] In other words, continuous and discrete manifolds are subject to a different conceptual determination and shaped by a different conceptual architecture, and thus are, in effect, *different* concepts, which brings Riemann's conceptuality of manifolds close to Deleuze and Guattari's sense of philosophical concepts.

It is significant that Riemann speaks of 'points' only in the case of continuous manifolds, and in the cases of discrete manifolds uses the term 'elements' for the simplest constitutive entities comprising them. This is astute, since, phenomenologically, points qua points only appear as such in relation to some continuous space, ambient or background, present or implied, such as a line or a plane (although mathematically, especially set-theoretically, the situation involves considerable complexities). Riemann allows for a possibility that discrete manifolds may function mathematically as spaces, or that space in nature (or what appears to us as space in nature) may be a discrete manifold. He does, however, primarily pursue a conception of space as a continuous (three-dimensional) manifold. A (continuous) manifold is a conglomerate of (local) spaces, each of which can be mapped by a (flat) Euclidean or Cartesian, coordinate map, without allowing for a global Euclidean structure or a single coordinate system for the whole, except in the limited case of a Euclidean homogeneous space itself. That is, every point has a small neighbourhood that can be treated as Euclidean, while the manifold as a whole in general cannot. The idea itself of mapping a given space with local spaces is, however, significant for our understanding of Euclidean spaces as well, beginning with the straight line, in part in contrast to the set-theoretical view, for the following reasons.

Weyl's concept of continuum, as presented in *The Continuum* (1994), is only partially indebted to Cantor. More significantly, it extends the intuitionist ideas of Luitzen E.J. Brouwer, further shaped or at least coloured by the phenomenology of Franz Brentano, Edmund Husserl, and Henri Bergson, as well as by earlier ideas of such figures as Fichte (all mentioned in the book), a philosophical tradition, extending from Kant, significant to intuitionist mathematics. Topology describes a space not so much by its points but by the class of its so-called *open* sets, the concept that underlies Riemann's concept of the manifold and that ultimately allows for a very general mathematical definition. For the present purposes, one can conceive of such sets on the model of open intervals of the line, say, all points between 1/4 and 3/4, except these two points themselves, which are boundaries. A closed interval will include its boundaries. Open or closed intervals can be thought of alternatively as spaces or sets, or both. The problem, mathematical and philosophical, of the continuum, is how a given continuum is constituted (as a set) by its points, and, in particular, whether we can exhaust the straight line by a set of real numbers; the problem known as Cantor's continuum hypothesis. The answer to the latter question is a complex issue (even though the problem is generally considered solved in mathematics) and cannot be addressed here. In any event, the question, mathematical and philosophical, of the *constitution* of the continuum is a separate, if related, question.

Brouwer questioned the set-theoretical concept of the continuum of the straight line as constituted by real numbers or even by points (i.e. point by point) as, in principle, inaccessible to human intuition, general or even mathematical, and he was more reluctant to dissociate them than Weyl. As Cartier notes, 'Brouwer criticized the possibility of affirming the equality of two numbers [since this presumes that one can verify an, in general, infinite number of equalities among the decimal digits comprising such numbers, which is not possible], but he held that the notion of the open interval]1/4 3/4[was legitimate, that is, that it is possible to verify the inequalities $1/4 < x < 3/4$ (if they hold) by a *finite* process' (2001, 395). Accordingly, a continuous space, say, a straight line (or what appears to us as such intuitively), may now be described and even defined not by (the set of) its points, to begin with, but by a class of its open subspaces, *covering* it (this last term is actually used in mathematics), which may but need not be seen as sets. Such subspaces may overlap, as for example would]1/4 3/4[and]1/2 1[, generating new open subspaces, in this case,]1/4 1/2[, in the overall covering atlas. This construction of space is an essential, grounding idea of topology as a mathematical discipline. Any such open interval or set containing a given point is also called a neighbourhood of this point. Thus both]1/4 3/4[and]1/4 1/2[would be a neighbourhood of 1/3, and the first of these neighbourhoods will contain the second, or will overlap with a neighbourhood such as]1/6 2/5[. Topologically, all such intervals are equivalent, and]0 1[would represent any one of them.

The procedure just sketched can be used to define the topology of curves or higher-dimensional spaces, flat or curved. It enables Riemann to define manifolds of any dimensions, even infinite-dimensional spaces. This also allows one, with Gauss and Riemann, to define a space, say that of a curved surface or (a more difficult problem, solved by Riemann) a manifold of dimensions three and higher, in terms of its inner properties rather than in relation to the ambient Euclidean space, where such a surface could be placed. The infinitesimal flatness of such spaces does not prevent them from having a curvature at any given point.

It is true that, if one appeals, as is usual, to open sets, this concept of the line retains the concept of set as a primitive concept. This approach, however, offers one a general structure (that of defining space as comprised or, again, covered by other spaces) which allows one to use this structure as a primitive one by replacing the covering of a space by 'open sets' with its covering by 'open spaces', such as open intervals. A general topological space is defined as (covered by) a collection of such open spaces as sub-spaces of the initial spaces by providing certain (algebraic) rules for the relationships between these subsets. These ideas ultimately extend to Grothendieck's topos theory, which is almost prohibitively difficult in view of its abstractness and mathematical (exact) rigour. The essential philosophical ideas involved may, however, be sketched as follows.

What one needs to enact this program is a certain primitive space, which could be any space, or in Deleuze's language and following his concept, via Bergson, in *Cinema 1* (1986), 'any space whatever,' which would extend an open interval in the case of the real (straight) line. Such a primitive space or indeed all spaces considered may, at least initially, be left unspecified, and could thus indeed be, in a certain sense, any spaces whatever. What would be specified, for example, in terms of sets (sets of relations, as explained above), would be the relationships, such as between spaces, such as mapping or covering one or a portion of one, by another. We can call this structure the arrow structure $Y \rightarrow X$ (X is the main space), where the arrow designates the relationship(s) in question, and such spaces *arrow spaces* (The notation itself is used in mathematics). This procedure enables one to specify, to give a space-like structure to, a given object not in terms of its intrinsic structure (e.g. a set of points with relations among them) but, in Yuri I. Manin's terms, 'sociologically', throughout its relationships with other spaces of the same category, say that of Riemannian spaces as manifolds (2002, 7). On this view, one does not have to start with a Euclidean space, whether seen in terms of sets of points or otherwise. Instead the latter is just one, specifiable object of a large categorical multiplicity, and possibly marked, for example, in the case of the category of Riemannian spaces, by virtue of a particularly simple way we can measure the distance between any two points. Most crucial, however, is that any given space (it may be a point, for example) is defined in terms of its relations to other spaces, which may but need not necessarily be subspaces of a given space, or, as in the case of Riemann's manifolds, spaces mapping subspaces of a given space. This view may be seen as, at least for now, the ultimate extension of Riemann's philosophy of space. One can generalize the notion of neighbourhood in this way as well, by defining it as a relation between a given point and space associated with it, which, it may be noted, is one of the reasons why the (inexact) concept of neighbourhood is so important for Deleuze and his philosophical topology.

One of the starting points of Riemann's reflection on space was the possibility of non-Euclidean geometry, which also led him to a particular new type of the non-Euclidean geometry, that of positive curvature. This also means that there are no parallel shortest or, as they are called, geodesic lines crossing any point external to a given geodesic. In Euclidean geometry, where geodesics are straight lines, there is only one such a parallel line, and in non-Euclidean geometry of negative curvature or hyperbolic geometry of Gauss, Johann Bolyai, and Nikolai I. Lobachevsky, the first non-Euclidean geometry discovered, there are infinitely many such lines. Riemannian geometry encompasses all of these as special cases and, as will be seen, allows for still others. Significant as the discovery of non-Euclidean geometry was for the history of mathematics or even intellectual history, it was only a small part and, as Weyl says, in retrospect 'a somewhat accidental point of departure' for Riemann's radical rethinking of the nature of spatiality and the developments that followed his ideas (1952, 92).

Riemann's concept of manifold is, arguably, a uniquely crucial feature of this rethinking, and Riemannian geometry would more properly refer to the study and the very definition of space in terms of manifold; the approach that makes both Euclidean and non-Euclidean spaces only particular cases of this general understanding of space, specifically as a continuous manifold(ness). Riemann, however, also considers manifolds of higher dimensions and even of infinite dimensions, or of course of lower dimensions, such as one-dimensional straight lines or curves or two-dimensional (Euclidean and non-Euclidean) planes and surfaces. As we have seen, he also considers discrete manifolds (which mathematically have the dimension zero), formed by isolated, rather than continuously connected, points or, in his terminology, *elements*, and the concept of discrete manifold becomes important for his view of space in physics. We recall, however, that, according to Riemann, discrete manifolds obey a different principle of conceptual determination from that of continuous manifolds, as Deleuze and Guattari indeed note (1987, 32).

In modern usage the term manifold is reserved primarily for continuous manifolds, Riemann's most significant contribution to modern mathematics and to our understanding of space in general.[7] The conceptual architecture of continuous manifolds, I argue, also provides the primary mathematical model of smooth space for Deleuze and Guattari, although they acknowledge the role of discrete manifolds in Riemann, and their significance or the significance of still other, such as porous, spaces in mathematics and elsewhere. Weyl goes so far as to speak of Riemann's mathematics of manifolds as 'a true *geometry*': 'This theory is a true *geometry*, a doctrine of *space itself* and not merely like Euclid, and almost everything else that has been done under the name of geometry, a doctrine of the configurations that are possible in space' (1952, 102). Deleuze and Guattari agree and take the point further, and give it a philosophical inflection, including the sense of creating new concepts. They state emphatically, in 'The Smooth and the Striated' in *A Thousand Plateaus*: 'It was a decisive event when the mathematician Riemann uprooted the multiple (manifold) from its predicate state and made in into a noun, "manifold"[*multiplicité*]' (p. 482–83; translation modified). They describe Riemannian spaces themselves, as (now clearly continuous) manifolds, as follows:

> But so far we have only considered the first aspect of smooth and nonmetric multiplicities [manifolds], as opposed to metric multiplicities: how the situation of one determination can make it part of another without our being able either to assign that situation an exact magnitude or common unit, or to discount it. This is the enveloping or enveloped character of smooth space. But there is a second, more important, aspect: when the

situation of the two determinations precludes their comparison. As we know, this is the case for Riemannian spaces, or rather, Riemannian patches of space: 'Riemann spaces are devoid of any kind of homogeneity. Each is characterized by the form of the expression that defines the square of the distance between two infinitely proximate points . It follows that two neighbouring observers in a Riemann space can locate the points in their immediate neighbourhood but cannot locate their spaces in relation to each other without a new convention. Each vicinity is therefore like a shred of Euclidean space, *but the linkage between one vicinity and the next is not defined and can be effected in an infinite number of ways. Riemann space at its most general thus presents itself as an amorphous collection of pieces that are juxtaposed but not attached to each other.'* It is possible to define this multiplicity without any reference to a metrical system, in terms of the conditions of frequency, or rather *accumulation*, of a set of neighbourhoods; these conditions are entirely different from those determining metric spaces and their breaks (even though a relation between the two kinds of space necessarily results). In short, if we follow Lautman's fine description, Riemannian space is pure patchwork. It has connections, or tactile relations. It has rhythmic values not found elsewhere, even though they can be translated into a metric space. Heterogeneous, in continuous variation, it is a smooth space, insofar as smooth space is amorphous and not homogeneous. We can thus define two positive characteristics of smooth space in general: when there are determinations that are part of one another and pertain to enveloped distances or ordered differences, independent of magnitude; when, independent of metrics, determinations arise that cannot be part of one another but are connected by processes of frequency or accumulation. These are the two aspects of the *nomos* of smooth space. (1987, 485; translation modified)[8]

The essential underlying mathematical conception of this description must be apparent from the preceding discussions. It may be useful, however, to elaborate on this a bit further, beginning with Riemann's extension of Gauss's ideas concerning the *internal* geometry of curved surfaces, that is, geometry independent of the ambient (three-dimensional) Euclidean space where they could be placed. Riemann's main contribution here was his discovery that Gauss's concept of (internal) curvature could be extended, via the so-called tensor calculus, measurement in curved spaces (of dimension three and higher), which Einstein brilliantly used in general relativity, his non-Newtonian theory of gravitation. For, it is not a matter of curves in a flat space but of the curvature of the space itself, which is why Riemann's concept is so important, given that the space in which we live is of three or possibly even higher dimensions (although one can of course think of spaces of lower dimension as independent rather than embedded). The structure and specifically the curvature of space could also be assumed to vary from point to point, from neighbourhood to neighbourhood. One of Riemann's key contributions (again, crucial to Einstein's work), as against the previous non-Euclidean geometries, was his understanding that the concept of manifold is general enough to allow for such variations. These earlier conceptions retained some Euclideanism by conceiving of the space as globally homogenous, although possibly non-Euclidean, spaces, curved, with the same constant curvature, positive or negative. In Riemann's geometry of manifolds these geometries are, again, merely special cases and spaces of variable curvature are allowed as well.

A related, but separate, feature of new spatiality, crucial to Riemann and then Einstein, extends Leibniz's ideas concerning the relational nature of all spatiality. On this view, the actual space is no longer seen as a given, ambient (flat) Euclidean space or, in Weyl's words, an (infinite) 'residential flat' (flat is a fitting pun here), where (phenomenally) geometrical figures or (physically) material things are put (1952, 98). Instead it emerges as a (continuous) manifold, whose structure, such as curvature, would be determined *internally*, mathematically or materially (for example, by gravity, as in Einstein's general relativity theory, based in part on Riemannian mathematics), rather than in relation to an ambient space, Euclidean or not. I shall return to this subject in the next section. In any event, all spaces, mathematical or physical (or still other), become subject to investigation in their own terms and, essentially, on equal footing, rather than in relation to an ambient or otherwise uniquely primary space. This view, as Deleuze and Guattari indeed suggest, leads to a kind of horizontal rather than vertical, hierarchical science of space as 'a typology and topology of manifolds', which they also associate with the end of dialectic (1987, 483; translation modified).

The internal, constant or variable, structure of a given space may, again, be determined 'sociologically', by the relations between this space and other spaces, 'any spaces whatever' In the case of Riemann's manifold, this relation is defined by the locally Euclidean structure of neighbourhoods, by Euclidean maps locally covering a given manifold, without a global Euclidean map or even in general a single global non-Euclidean map, except in special cases of spaces of constant curvature (zero, positive, or negative). Thus, a manifold is indeed a kind of patchwork of (local) spaces, as Deleuze and Guattari say. This would allow, but importantly will not demand, local striation of a given (smooth) space, but, except in these special cases, will disallow homogeneous global striations. This cartographical terminology and conceptuality, crucial to Deleuze (or Foucault, whom Deleuze discusses from this perspective in his *Foucault* (1988)), are not accidental and have a historical genealogy. Gauss arrived at these ideas in part through his work in land surveying.

The concept of manifold can, as I have indicated, be extended even beyond these limits, that is, beyond assuming their differential nature or even the possibility to define the measurement of distances on them. Indeed, it can be extended still further to general topological spaces that are not manifolds, to 'spaces' that are defined by open neighbourhoods that are not Euclidean. Such neighbourhoods could be 'any (open) spaces whatever', or still more general structures, for example, those defined by a relation, an 'arrow' between two spaces, as considered earlier. In general, such spaces may not be available to our phenomenal spatial (or any) intuition. (Could one still speak of space then?) This general underlying architecture itself is, however, inherent in Riemannian manifolds, from which it was in part developed historically.

Although it is inevitably a matter of interpretation, it appears that the underlying (exact) mathematical model of 'Riemann space at its most general' and hence of smooth space in Deleuze and Guattari is a general topological space rather than a manifold.[9] As indicated, such a space is inherent and underlies any Riemannian (metric or metricizable) space as a manifold, since manifolds are topological spaces in general in the first place. Such spaces may not allow any metric (mathematically) and hence striation, Euclidean or Cartesian (by means of coordinate systems), either global or local. The global one is, again, in general disallowed by Riemannian spaces (apart, again, from special cases such as Euclidean spaces or spaces of constant curvatures), while the second is allowed, but, importantly, is not required. In the case of Riemannian spaces, one might or might not introduce a striation, in contrast to major or state mathematics (or other fields or formations), which would insist on striations. This general structure is, accordingly, important for the first (enveloping) aspect of the smooth space, referred to in Deleuze and Guattari's description of Riemannian spaces cited above, or of certain smooth spaces, if one assumes that certain smooth spaces disable our capacity for striation altogether. The second, the smoothness of the smooth, aspect of the *nomos* (also vs. *logos*) of the smooth space reflects more the interplay of connectivities between neighbourhoods, which also differently (vs. set theoretically, for example) defines continuity. This aspect would be preserved in any topological space, Riemannian manifold or not, even when no striation is possible. This is why it is topology, rather than geometry, that may be the ultimate mathematics of smooth space in Deleuze and Guattari's sense.

4. *Manifolds, matter, and mind*

The discussion so far has primarily concerned the conceptual and phenomenological architecture of Riemannian spatiality and its role in Deleuze's philosophy, and in particular the significance of the concept of smooth space. As must, however, be apparent by now, materiality – physical, aesthetic, cultural, or political, and possibly even mathematical – plays a crucial role in shaping this architecture and in making it possible, and I shall discuss it now in more detail. The role of materiality in our understanding of spatiality also helps us to see a deeper significance of Leibniz for both Riemann and Deleuze, and for the relationships between the thought of all three thinkers and the problematic of space as discussed here.

We recall that, according to Leibniz, space cannot be seen as a primordial ambient given, as a container of material bodies and the background arena of physical processes, along the lines of Newton's concept of absolute space in his *Principia*. Indeed, it is also worth recalling, the concept ultimately troubled Newton as well. I now speak of the *phenomena* of space, or (since there may not even be 'space' as such, only distances, areas, volumes, and other measurable entities) spatiality in physics, although Leibniz's point also impacts on our phenomenological, or mathematical, conceptions of spatiality. It was Einstein who gave a rigorous physical meaning to these ideas and extended them by arguing that space, or time, are not given but arise, are the effects of our instruments, such as rods and clocks, and, one might add, of our perceptual and conceptual interactions with those instruments. Hence, phenomenality continues to play a shaping role in the situation, thus, as will be seen, also making it Leibnizian in yet another sense; that of monadology. Space is, thus, possible as a phenomenon (or concept) by virtue of two factors. The first is the presence of matter and technology, such as rods and clocks (or natural objects that in fact or in effect assume a similar role). The second is the role of our perceptual phenomenal machinery, a role that one might argue, with Kant, to be primary, or (in Kant's language) the condition of the possibility of space, along with time (be these given a priori or not), which, as will be seen presently, may still be due to materiality.

Riemann offers some extraordinary intimations of Einstein's theory, which, when Einstein developed his so-called general relativity, a non-Newtonian theory of gravitation, was indeed based on Riemann's geometry. (Einstein's argument concerning the role of measuring instruments applies in his so-called special relativity as well, essentially a theory of propagation of light in space in the absence of gravity.) In the final section of his lecture, 'Application to Space' (§3), where he refers to Archimedes, Galileo, and Newton, Riemann proceeds from the contrast between discrete and continuous manifolds, and first considers the possibility that the physical reality of space corresponds to a discrete manifold. This may ultimately prove to be the case, although in most physical theories so far (most quantum theories included), space has been viewed as a continuous manifold in Riemann's sense. The main argument in question, however, concerns primarily continuous manifolds and space as a continuous phenomenon, since, according to Riemann, it is only in this case than the nature of the space will be determined by physics rather than mathematics itself (§3, pt. 4). According to Weyl:

> Riemann rejects the opinion that has prevailed up to his own time, namely, that the metrical structure of space is fixed and [is] inherently independent of the physical phenomena for which it serves as a background, and that the real content takes possession of it as of residential flats. *He asserts, on the contrary, that space in itself is nothing more than a three-dimensional manifold devoid of any form; it acquires a definite form only through the advent of the material content filling it and determining its metric relations* (1952, 98; Weyl's emphasis).

It would perhaps be more accurate (and closer to Riemann) to say that space may be given phenomenally at most as a three-dimensional manifold, as a kind of free smooth space with (given its manifold structure) possible striations, and then only phenomenally. Physically, it may be, and on Riemann's and then Einstein's, or Leibniz's view could only be, co-extensive with matter, whether actual bodies, as in Leibniz, or propagating fields, such as electromagnetism or gravity, as in Riemann and Einstein. As just indicated, however, the phenomenal component remains irreducible. Weyl adds: 'Looking back from the stage to which Einstein brought us, we now recognize that these ideas can give rise to a valid [physical] theory only after *time* had been added as a fourth dimension to the three-space dimensions' (p. 101).[10]

In sum, the gravitational field determines the manifold in question and its, in general variable, curvature. The reverse fact, however – the fact that the gravitational field shapes space and, moreover, shapes it as a

Riemannian manifold, as primordially a smooth space (in Deleuze and Guattari's sense) with a potentially infinite (although, again, not, by the same token, unconstrained) multiplicity – still remains crucial. This fact radically transforms our philosophy of space and matter, and of their relationships, at whatever level – mathematical, scientific, philosophical, aesthetic, cultural, political – we use these concepts.

The Riemannian or any other *phenomenality* of space or any phenomenality is, however, only possible by virtue of *materiality*. This materiality is that of the material constitution of the *bodies* we possess and their material history from a certain point of the history, even if not from the very beginning (from and before the Big Bang) of the material universe that we inhabit, and, again, technology which we built or into which we convert what the world offers us. This technology is still enabled by our bodies (and the mind they enable, on this materialist view) and the universe, the ultimate body without organs, which eventually gives rise to the desiring machines of our bodies and the perception, cognition, thinking they enable. One can take advantage here of the diverse concepts designated by the term or signifier 'body', from the ultimate (quantum) constituents of nature, to human bodies, to bodies of stars and galaxies, to the body of the universe itself, or political, textual, and other bodies (which have their own forms of materiality). The ultimate nature of this materiality may be unavailable, not only in practice but more crucially in principle, to our phenomenality or conceptual capacities, to our thinking, unavailable even beyond the way implied by Kant's things-in-themselves, entailing, accordingly, that even such terms as materiality and matter, or body without organs, may be inapplicable. This unavailability is, however, quite different from an unavailability in practice or even unavailability in principle defined by theological or quasi-theological arguments, such as that of Leibniz's monadology, in which the ultimate constitution and architecture of the world is only available to God, and never to any single monads or even any collective of monads (Deleuze 1993, 26). In the present, materialist, view of such unavailability of matter the existence of such unavailable entities would and could, by definition, only be established by virtue of their capacity to affect what is available to us and produce available effects. Our thinking, however, and, accordingly, our bodies, are capacious enough to arrive at the conception of this unavailability concerning material (or possibly certain mental) entities, and, with the help of the universe, to build technologies that establish the existence of such material objects, say, those considered in quantum mechanics, which obey this type of epistemology, at least in certain interpretations.[11]

Hence, the crucial significance of the body, both in its own terms and cum technology (machines) in Deleuze and his other key precursors and influences, such as Nietzsche, or in such contemporary thinkers as Jacques Derrida and Paul de Man, or for that matter Kant. Kant was certainly aware of this situation, even if sometimes against himself, as were (again often against themselves) his key precursors, such as Leibniz, Hume, and even Descartes, and followers, such as Hegel or Husserl.[12] Leibniz's monadology is implied in this argumentation, and Deleuze and Guattari indeed juxtapose 'monads' to 'the unitary Subject of Euclidean space' (1987, 574, *n*. 27).

5. *Monadology, nomadology, and the new baroque*

The Riemannian world (in either sense, spatial and cultural) becomes monadological then? Well not quite, or rather yes and no, or, better, yes and more, as monadology must be made *nomadology* in the new, post-Riemannian, Baroque. Leibniz's monads ultimately interact with each other only through their interaction with the world, whose overall interactive architecture is, in the old Leibnizian Baroque, containable in and converging into a harmony, fully available to or calculable only by God (Deleuze 1993, 26). The divergent harmonies of the new Baroque retain the fold, made mani*fold*, but convert monadology into nomadology, a nomadology containing but not reducible to monadology. In *A Thousand Plateaus* this move is manifest especially in the musical and aesthetical models of the smooth and the striated, in particular in Boulez (who indeed introduced the terms themselves) and Cézanne, although the discussion of both cases is (rhizomatically) interlaced throughout with Riemannian themes, as here considered (1987, 477–78, 493–94). This is how this transition from monadology and nomadology appears in the Cézannean or post-Cézannean, such as cubist, model of smooth space, and of the smooth and the striated (p. 493–94). Certain smooth-space, nomadic features of this model are found much earlier, as early as cave paintings, invoked by Deleuze and Guattari, as well as in such 'marginal' (for whom?) art as carpets or quilts. The image of quilt serves as an epigraph to the chapter, again, framed at the other end by a computer-generated portrait of Einstein, a nomad who, with Riemann and Leibniz, gave us a patchwork of a Riemannian (as against Newtonian) space shaped by gravitation. The portrait may also be seen as an allegory of an underlying, including aesthetic (imagined phenomenal or painted, for example) smooth space, Riemannian or Einsteinian space, and the computer-generated, striated, Cartesian space, superimposed upon it. Also, the same type of model will apply to the music of the new Baroque, from Boulez on, and, at least by implication all other minor practices, certainly literary ones, such as those of Kleist, Kafka, and Woolf, or, again, the mathematical and scientific, and the *dedoublement* of smooth spaces found there (1987, 478–79; Deleuze 1993, 33). Indeed this nomadological extension of monadology is traced by Deleuze and Guattari to Riemannian space:

> All of these points already relate to Riemannian space, with its essential relation to 'monads' (as opposed to the unitary Subject of Euclidean space). . Although the 'monads' are no longer thought to be closed upon themselves, and are postulated to entertain direct step-by-step local [Riemannian-space-type] relations, the purely monadological point of view proves inadequate and should be superseded by a 'nomadology' (the identity of striated spaces versus the realism of smooth space). (1987, 573–74)

I close with Deleuze's conclusion in *The Fold*, which this passage anticipates and which gives this idea the dimension and the harmony – the decentred and divergent harmony – of the new Baroque, now linked to divergent series, on the one hand, and Joycean *chaosmos*, on the other. Infinite series is an area of mathematics (now analysis and algebra, rather than geometry) to which both Leibniz and Riemann made crucial contributions as well, and, as previously mentioned, Riemann's work on the subject led to Cantor's discovery of set mathematics. As in the discussion of the smooth and the striated in *A Thousand Plateaus*, music is germane here. Deleuze writes:

> To the degree that the world is now made up of divergent series (the chaosmos), or that crapshooting replaces the game of Plenitude, the monad is now unable to contain the entire world as if in a closed circle that can be modified by projection. It now opens on a trajectory or a *spiral* in expansion that moves further and further away from a centre. A vertical harmonic can no longer be distinguished from a horizontal harmonic, just like the private condition of a dominant monad that produces its own accords in itself, and the public condition of monads in a crowd that follows the lines of melody. The two begin to fuse on a sort of *diagonal*, where the monads penetrate each other, and modified, inseparable from the groups of prehension that carry them along and make up as many transitory captures.
>
> The question always entails living in the world, but Stockhausen's musical habitat or Dubuffet's plastic habitat does not allow the difference

> of inside and outside, of public and private, to survive. They identify variation and trajectory, and overtake monadology with a 'nomadology.' Music has stayed at home: what has changed now is the organization of the home and its nature. We are all still Leibnizian, although accords no longer convey our world or our text. We are discovering new ways of folding, akin to new envelopments, but we all remain Leibnizian because what always matters is folding, unfolding, refolding. (p. 137; emphasis added)

These processes now take place in spaces that are smooth spaces, *manifolds*, as conceived, at least mathematically, by Riemann, the Leibniz of the new Baroque. The smoothness of these spaces also enables a speedy movement of transition, convergent and divergent, from point to point, from neighbourhood to neighbourhood, from concept to concept, from field to field, from mathematics to philosophy (or vice versa), for example. This movement may and perhaps must be quick but it is not always easy, and sometimes dangerous, as skating on thin ice. But then, as Nietzsche once said, 'thin ice is paradise for those who skate with expertise.'

Notes

1 It is true that Riemann's work precedes set theory, which was, accordingly, not available to him. (Cantor actually discovered set theory in his work on trigonomentric series following Riemann's work on the subject.) The main point here is philosophical, insofar as in question are concepts, mathematical and philosophical, that are, in Deleuze and Guattari's sense, 'always new.'

2 It should be noted that one could see category theory or Grothendieck's work as an extension of rather than in juxtaposition to set theory and Cantor's philosophy, as Yuri I. Manin argues (2002, 8). I might also note (although the subject cannot be addressed here) that the encounter between Deleuze and Alain Badiou (for example, in Badiou 2000), and Badiou's work itself (the work expressly linked to set theory and more recently to category theory) essentially relate to these mathematical problematics, including specifically the idea of continuity and discontinuity, and their relationships.

3 Given my space and focus on Riemann, I will not be able to address this chapter, '12. 1227: Treatise on Nomadology – The War Machine' (p. 351–423), although it is germane to most subjects discussed here.

4 The English translation by Brian Massumi uses 'multiplicity' to render the French '*multiciplité*.' The proper English technical term is manifold, which also translates Riemann's *Mannigfaltigkeit* and preserves both 'fold' and Riemann's apparent suggestion of a folding-unfolding multiplicity of the whole. Riemann was originally trained in theology and the German for trinity is

Dreifaltigkeit. The concept of manifold carries a sense of a multiplicity of points, neighbourhoods, mappings, connections, etc., but is richer.

5 The best access to the English translation, originally published in *Nature*, is found at http://www.maths.tcd.ie/pub/HistMath/People/Riemann/Geom/WKCGeom.html, which I shall cite throughout this article.

6 Cf., Laugwitz's discussion, to which I an indebted here but which takes a more conventional view of Riemann's conceptual mathematics (1999, 303–7).

7 Riemann actually considered the so-called differential or smooth (in the mathematical, rather than Deleuze and Guattari's sense) manifolds, which means that one can define differential calculus on such objects.

8 It is at this juncture that they note that, 'the confrontation between Bergson and Einstein on the topic of Relativity is incomprehensible if one fails to place it in the context of the basic theory of Riemannian manifolds, as modified by Bergson' (1987, 484).

9 Deleuze and Guattari see Benoit Mandelbrot's fractals as moving towards 'a very general definition of smooth space' (1987, 486). I am not altogether certain why fractals (so called because they can, as topological spaces, be given fractional, rather than whole, spatial dimensions) are singled out here, especially given that Riemannian spaces (their dimensions are not fractional) are offered as a/the primary mathematical model of smooth space a bit earlier. There may be, however, aspects of fractal spaces, which indicate that a fractal space 'builds itself' more naturally or more structurally as a smooth space (p. 486–88). This particular point does not appear to me, however, to affect the overall argument of this essay.

10 The resulting smooth spaces are not without interest here, including in the context of the question of temporality in Bergson and then Deleuze himself, as indicated earlier, especially in *The Logic of Sense* (1990).

11 On these issues I permit myself to refer to Arkady Plotnitsky, *The Knowable and the Unknowable: Modern Science, Nonclassical Thought, and the 'Two Cultures'* (2002, 1–107).

12 See Paul de Man's reading of Kant in *Aesthetic Ideology* (1987). The question of materiality and the body in phenomenology of Husserl, Bergson and Maurice Merleau-Ponty, is also an important reference here, which, however, would require a separate analysis.

11
√The Surd

Aden Evens

Those who have exposed themselves to mathematics only in small doses may have the impression that maths is a done deal. In grade school, a maths problem is only ever a contrivance, as the solution is already known and indeed precedes the problem: given 2+5 one must produce 7, and even when the so-called problems become much more sophisticated, they don't really become more problematic; derived from the solution, the problem is only a means of testing the pupil's ability to return to it.

This is the image of mathematics with which most of us are left, where solutions are already contained within their problems, waiting to be teased out by appropriate reductive methods; the answers are *there* already, even if they are as yet hidden from us problem-solvers. But there is another side of mathematics, where the solution is not a foregone conclusion and the problem is worthy of its name. Answers do not follow immediately from questions, but must be secured through resolute and patient labour on the part of the mathematician. This *minor* mathematics – exemplified for instance in the numerous mathematical examples throughout the oeuvre of Gilles Deleuze – contrasts with the static and formalised mathematics that we might call *State* or *royal* mathematics (by analogy with Deleuze and Guattari's notion of State science (1987, 367–74)). Whereas State mathematics comprises formal rules and an established set of symbols that confine problems to the leading edges of research, minor mathematics apprehends maths at its most problematic, at its precarious or uncertain moments, when contingency overwhelms the security of the solutions, and only the urgent intervention of the mathematician can rescue a suitable result from the calculation.

Such active nurturing attends every advance in mathematical research, but certain examples stand out most clearly, when the mathematics ties its fate to that of the mathematician and can be constructed and maintained only through her activity. In such moments, it is as though the mathematics will not solve itself, as though the solution is not yet there in the problem, as though the maths had reached out of the abstract into the empirical, and so could not be taken for granted in its progress. Historically, such moments of minor mathematics are called *constructivist.*

Constructivism underlies each advance in the history of mathematics, but the history books relegate it to a small and somewhat forgotten niche. Like minor science, minor mathematics is overwritten, claimed and formalised by the standard theorems and techniques of State mathematics. If constructivism elevates the problem to the highest position, then its formalisation by major mathematics strips it of its problematic potential and asserts once again the priority of the solution. Of course, State mathematics dominates the history books, so we should not be surprised to find constructivism under-represented there.

For example, Deleuze must reach into the obscure history of mathematics at the outset of Chapter Four of *Difference and Repetition* (1994), to recover mathematical techniques of differential analysis that are ontologically substantive, generative and not merely reductive. Borrowing from a usually forgotten history of the calculus, he presents the differential, *dx*, in a process of *differentiation*, a process that generates the world and the ideas that make sense of it. In the modern calculus, the differential does not perturb the formulae in which it appears; it is largely a placeholder to indicate which variable is to be integrated over or which derivative to take. But for earlier mathematicians – Deleuze draws upon the researches of mathematicians Carnot, Lagrange, and Wronski – the differential had to be wilfully and actively manipulated in its equations; it was an extra term, left over after the rest of the equation had been reduced, and the methods for dealing with it could not be decided in advance. Constructive methods, such as successive approximation, ensured an active role for the mathematician, who reduced the equation bit by bit, sculpted it into a simpler form.

One chapter later, Deleuze offers a prototypical example when he proposes that the universe is the remainder in God's calculations. The remainder mandates a construction, for it is an element left over after the calculations have been performed, a problematic surplus that does not cancel out of the equation and requires contingent methods, techniques developed 'on the ground'

Though constructivism propels even State mathematics, it disagrees with the formal essence of mathematics as defined by the State. Maths is traditionally marked by its universality, its wholesale abstraction, its divorce from contingency, personality, and the empirical. Constructivism, on the contrary, clings to contingency, and develops its methods in a particular context, outside which they may well lose their relevance or their force of solution. Maths claims no politics, but constructivism politicises mathematics, as it is no longer a matter of determinate formalities. In response, royal mathematics elides or cancels out minor mathematics, claiming its results while neutralising or sterilising its activism. The State restores to the mathematics a pure formality, universalism, and objectivity.

In the one-and-a-half examples to follow, this essay attempts to distil the constructivist moment and generalise it, moving from mathematical history to intellectual history: constructivism, and in particular *the surd*, as the motor of becoming. The surd is the anomalous element, the unassimilable, that disrupts the linear progress of history, fractures an established discipline to open the way to new methods and ideas. At its initial appearance, the surd erupts into a constructivist heterodoxy, techniques to be determined in the moment, contextual methods that must be worked out on the spot. Before long, however, constructivism cedes its hold on these methods, as they are formalised and claimed by the State, the revolutionary potential of the surd neutralised.

While the mathematical narrative here is historical, the language of history is ultimately provisional, like the constructivist methods to which it refers. The claims in this paper are not primarily historical but genealogical; that is, they account for events in terms of their motive forces, but the actions of these forces may not be historically distinct from their reclaiming (and official elision) by royal sciences. Thus, as we will see, the forces that employ the surd toward new disruptive methods are the same forces that attempt to corral the surd and restore to mathematics a legitimacy and formality that the surd resists.

Intuitionism and the surd

In mathematics, a surd is an irrational number, a real number that cannot be expressed as the ratio of two integers. It is also the name for the 'root' sign, $\sqrt{}$ These two meanings are not incidentally related: one easy way to generate irrational numbers (surds) is to take roots of rational ones. The square root of two, for example, is a surd.

Typically, traditional (royal) mathematics has little trouble with the surd. In grade school, we just write out the first few digits after the decimal point, and then write an ellipsis to indicate the continuation of the sequence. 'The square root of two is 1.41421. ' The question – practically ignored by State mathematics – is the meaning of this ellipsis. In a rational number, such as one-third, the ellipsis in the decimal representation seems unambiguous; in 0.333. it means 'just keep writing 3's' But in the square root of two, the ellipsis feels somehow inadequate, for it refers to no pattern; the decimal expansion of the square root of two (or of just about any irrational number)[2] is a seemingly haphazard sequence of digits that is fully determined (by the definition of *square root* in this case), but in a strong sense unpredictable. The ellipsis in the representation of an irrational does not succeed in specifying any particular number, so that one must know beforehand what number is being referred to in order to interpret those dots correctly. If the ellipsis means *continue in this manner*, an irrational number does not suggest just which manner is intended.

Luitzen Egbertus Jan Brouwer (1967, 1983) found this ellipsis problematic; even more problematic for Brouwer was the tacit willingness of traditional mathematics to leave the ellipsis unexamined. In response, he founded intuitionism as a branch of mathematics that proceeds first of all from an epistemological (rather than a mathematical) commitment. Amidst the intellectual rubble of post-World War I Europe, Brouwer drew together some strands of nineteenth-century constructivism to forge a mathematics that restores truth and certainty where none had been guaranteed. The principle tenet of intuitionism is this: no mathematical claim is acceptable that cannot be *experienced* as certainly true by the mathematician; allow only what can be grasped in the mathematician's immediate intuition.[3] (We shall see what this means. .)

In intuitionism, the mathematician is no longer the one who works on the mathematics; the mathematician is its vessel and locus. For Brouwer mathematics *is* mental activity, independent of language and independent of the notation used to remember and convey it. Only intuitions are mathematics. Thus, the symbols of maths and other written or verbal communications about it are not the genuine item, but only a means of reminding oneself or others how to 'have' the same mathematics again. Intuitionism insists on the active role of the mathematician in *doing* the mathematics, a constructivism in principle.

But what difference does it make in practice to demand that the truth of the mathematics be experienced? Was there something uncertain about prior mathematics? Were mathematicians allowing claims into mathematics that could not be immediately intuited as true? Brouwer argues that, indeed, many claims were being routinely but illegitimately accepted into the calculus. While claims about finite quantities are, in principle at least, experience-able as true in an immediate intuition, claims about the infinite are another story.[4] In particular, it is *the surd* that fractures mathematics in the crucible of post-World War I maths and logic. The ellipsis in a surd, say the intuitionists, cannot stand for an unwritten infinity of digits, since no such completed infinity can be directly experienced in the mind of the mathematician. Instead, Brouwer offers a definition of number, including irrational number, that is purely finitary, a definition in terms of the *construction* of the number that does not posit some finally constructed infinity of digits.

In Brouwer's words from his early work in 1913, a number is a 'law for the construction of an elementary series of digits after the decimal point, built up by means of a finite number of operations' (1983a, 85). In other words, a number is some procedure by which we can unfailingly specify further digits to arbitrary precision. For instance, start with 0.74, then continue to add digits, alternating between 7 and 4. This would yield a rational number. Another example: start with 1.41421 (the first few digits of the square root of two), then continue to add successive digits, choosing each next digit to be the highest digit such that the square of the resulting rational number is less than 2.[5] This would be the intuitionist definition of the square root of two.

By defining number in terms of a procedure, intuitionism demands that number be not a fixed, eternal entity, existing in some ideal space of number-hood. Rather, an intuitionist number is a rule, a method of construction. Even in this simple definition, we can already see the surd at work, its irrationality calling upon constructive procedures, an engagement by the mathematician. But the surd does not stop at this call for mathematical commitment; its perturbations are more dramatic, as Brouwer illustrates.

He defines a number r, called the *pendulum number*, as the limit of a sequence of elements,

$$c_1, c_2, c_3, ..$$

The sequence is

$$-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, ..$$

It is the geometric sequence defined by the inverse powers of two, where the sign switches back and forth with each element. The general formula is

$$c_n = \left(-\frac{1}{2}\right)^n$$

This sequence bounces back and forth around 0, getting closer to 0 the farther you proceed in the sequence (Figure 1). Its limit is in fact 0, and there is not yet any disagreement between intuitionist and traditional mathematics.

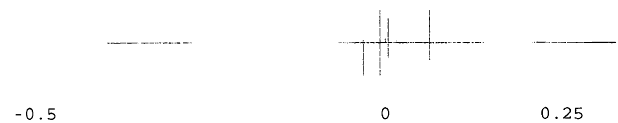


Figure 1: The sequence of c's, bouncing back and forth as it approaches 0.

But Brouwer adds a twist. He alters the definition of the sequence by freezing it beginning at the m^{th} term of the sequence, so that starting at c_m, all the rest of the terms will just be equal to c_m. Now, under this altered definition, the limit of the sequence is no longer 0, but c_m. The question of course is, What is m?

In an original stroke of genius, Brouwer defines m so as to render the limit of the sequence, r, a rather strange number. m is defined as the first place in the decimal expansion of π where the digits 0123456789 occur: Start to write out π, counting its digits. As soon as you reach a place where 0123456789 occurs, note how far along you are in your count. This is m, as illustrated in Figure 2. (Just to avoid confusion, note that π has no prior relationship to r; Brouwer could have chosen any irrational number, and picked π arbitrarily because it is a universally-recognised irrational number. The essential criterion is that π is a surd, whose digits can be *successively* calculated to arbitrary precision, but whose yet-to-be-calculated digits cannot be predicted or patterned in advance of their calculation.)[6]

The thing is, we don't know where such a sequence of digits occurs in the decimal expansion of π. We don't even know *if* such a sequence *ever* occurs in the decimal expansion of π. If such a sequence occurs at an even-numbered place in the decimal expansion of π, for instance, at the 1000th place, then c_m will have an even exponent, it will be equal to

$$c_{1000} = \left(-\frac{1}{2}\right)^{1000}$$

$$\pi = 3.14159265358979323\ldots0123456789\ldots$$

4th digit ↑

m^{th} digit ↑ (at "0123456789")
(may not exist)

Figure 2: The sequence of digits of π, and the (hypothetical) m^{th} position, where 0123456789 occurs.

which is a positive (but small) number. The sequence of c's will 'freeze' at that point, and r, the limit of the sequence, will be equal to this small positive number. If, on the other hand, the sequence 0123456789 occurs at an odd place in π, then c_m will have an odd exponent and r will be a small negative number. But if (on the third hand!) that sequence never occurs in the decimal expansion of π, then there is no m, and the sequence of c's will just keep getting closer to 0, with a limit, r, of 0.

A traditional mathematician, say a Platonist, would not object to this definition of the pendulum number. She would allow that r is a well-defined real number whose value we don't at present know, since we don't know where 0123456789 occurs in π (nor whether it occurs at all). In spite of our ignorance about r, though, she would insist that as a well-defined real number, r is either positive, negative, or equal to zero.

This is where the surd fractures mathematics, splitting intuitionist from traditional mathematics. Brouwer too allows that r is a well-defined, real number: it meets the definition of an intuitionist real number in that we can specify it to arbitrary precision, simply by writing out the digits of π. However, we cannot say of r that it is either positive, negative, or equal to 0. We cannot say this because we cannot immediately experience the truth of any of these three possibilities, and therefore *none of them is true*. It is not true that r is greater than 0, nor is it true that r is less than 0, nor is it true that r is equal to 0. None of the three usual ordering relations holds between r and 0. (In effect, Brouwer's pendulum number instantiates a new logical value: it is not true that $r > 0$, but neither is it false.)[7]

This counterintuitive conclusion of intuitionist analysis deserves reemphasis. It is not a deferral in regard to r, a refusal to answer questions about r due to insufficient data. Rather, r, in all its indefiniteness, enters the mathematics, carrying its ambiguity along with it. It thereby perturbs traditional, even sacred theorems that had held sway for hundreds of years. In this case, r defies the fundamental ordering of the continuum – the property that every real number is either greater than, less than, or equal to any other real number. Here, uncertainty (about the digits of π) is not an extra-mathematical concern that leaves a gap in the maths waiting to be filled, but is incorporated directly into the calculus. Epistemology intrudes upon ontology, to tie the truth of the mathematics to the concrete mental processes of the mathematician. No longer an eternal posit, an ideal relation among ideal objects, mathematical truth is now a contingent event, something that happens to the mathematician and to the mathematics. If at some future moment, the sequence 0123456789 is discovered in the decimal expansion of π, or if it is someday proved that no such sequence can ever occur in the decimal expansion of π, then r will take on a decimal value, and it will obtain one of the three ordered relationships to 0. The future-historical event will change the truth of the mathematics. By virtue of the surd (in this case, π), mathematics bleeds over from the realm of the pure abstract into the empirical. It becomes dependent on human history and human progress. And it insists, above all, on the participation of the mathematician as an essential element of the mathematics.

With the entry of the pendulum number into the mathematics, Brouwer replaces the ordered continuum of real numbers with a disunified set of incommensurables, scattered surds that cannot be adequately compared to other numbers. By definition, the surd is incomparable, singular, and it lends this quality to intuitionist number in general. The surd makes numbers into immeasurable ordinal quantities whose nature can be determined only by pursuing a process that may never yield a certain result. r and 0 cannot be compared adequately to each other, for each is a singular experience, constructed in such a manner that they diverge, r becoming *skew* to 0.

The pendulum number depends for its definition upon what Brouwer terms an *opaque fleeing property*.[8] This is a mathematical property that describes a (hypothetical) mathematical object, such that no object is known that holds the property, and it is also not known that no such object exists. Moreover, there is no finite process that will surely determine whether or not there is such an object. Thus, an opaque fleeing property is a property such as 'the first place in the decimal expansion of π where the digits 0123456789 occur' This opaque fleeing property locates and rarefies the surdity of π, its uncanny simultaneous contingency and necessity. π is contingent for wholly singular, without genus, and governed by no law but its own; nevertheless it is woven into the fabric of the universe, the singular relationship between circumference and diameter of a circle. Its specificity defies its abstraction. It is effectively split between the empirical and the abstract, a pure abstraction with the infinite, irreducible detail that generally accrues only to the concrete. π thus reaches out of the eternal mathematical world and into the empirical human one.[9]

It is a strange and disconcerting move, therefore, when Brouwer, consistent with traditional mathematical practice, abstracts from this particular opaque fleeing property to speak of opaque fleeing properties in general. (For instance, 'as long as a fleeing property exists such that
Or, 'for each fleeing property, f, (1981, 42).) That is, he ceases to specify a particular fleeing property, and instead reifies the concept of opaque fleeing property, using it as a mathematical *effect* in the analysis: 'Consider a number defined by *some* fleeing property, ∂' This is tantamount to taking the form of an opaque fleeing property and voiding it of its content. In this formalisation, the surd, formerly crucial to the fleeing property, is neutralised, drained of its disturbing singularity so that it can serve as a general term.[10] Contingency is no longer the contingency of π, but becomes a purely formal element of the calculus, with rules to govern its manipulations and effects.

When fleeing properties are formalised, the surd is not so much evinced as insisted upon, spontaneously generated in its form but without content. However, a surd without content is impotent. For a surd is precisely an excess of content, an intensity that cannot be contained by a form. Though its results persist in the calculus, they no longer require the engaged attendance of the mathematician. As a formality, intuitionism loses its tie to the human, and becomes just another branch of mathematics, a universal and agreeable formalism. Historically, formalisation of the surdity of intuitionism made it acceptable to other mathematicians. Troelstra (1977) for instance attempts as his chief project to formalise intuitionist mathematics to the point where we no longer need extra-mathematical conceptual apparatus, but can regard intuitionism as a set of rules for manipulating symbols. The triumph of royal mathematics.

Brouwer, on the other hand, wishes to maintain the surdity of his mathematics, for it is just this surdity that attaches the maths firmly to the mental activity of the mathematician. Having lost the opaque fleeing property to formalisation, he needs to conjure another surd, to trump the opaque fleeing property with a more insistent concept. The opaque fleeing property was something of a parlour trick anyway, a game to demonstrate some of the easy and dramatic implications of intuitionist analysis, but not a serious contribution to mathematics.[11] To up the ante, Brouwer introduces the notion of a *free choice sequence*, and a new concept of number to go with it, more radical still than the pendulum number. To generate a free choice sequence, epistemology must be further twisted, the very notion of choice remade as a mathematical concept.

One of the problems for traditional mathematics was to explain the continuum. How is it that points can somehow aggregate to form the continuum, or number line? How do points of dimension 0 fully occupy a line segment, which has a dimension of 1? How can discrete points, that have no length, form a continuous line? Traditional mathematicians posited an axiom of the continuum; they simply asserted as a basic assumption of the mathematics that when you take all of the (infinity of) rational numbers and all of the (even bigger infinity of) irrational numbers, you have covered the continuum to form a continuous line.[12] (As the continuum is most easily conceptualised in geometrical terms, I will henceforth refer interchangeably to *number* or *point*. The same thing is intended by each term.)

This claim, say the intuitionists, is not immediately verifiable as true in the mind of the mathematician. In fact, it is altogether counterintuitive. Intuitively, no matter how many points you place side-by-side, there is no way that they should ever be able to cover a positive linear area, no way that they can ever form a continuous line. To compensate for this *incorrectness* in the calculus – and it is an important claim that underlies a great deal of higher mathematics – the intuitionists offer a further definition of number, one that is intended to overcome this problem by building into number the continuity that number is expected to cover. This new definition of number is based on the idea of a *free choice sequence*, and it once again relies on the surd, which imparts to number a power of the continuum.

The early intuitionist definition, above, makes of number a *process* for determining with arbitrary precision successive digits of a decimal representation. The new definition introduces the innovation of the free choice sequence, which allows a certain freedom or latitude in the process. Now, the rule that determines successive digits need not be wholly determining; rather, the rule for choosing successive digits can allow the mathematician a choice of next digit, and this still defines a particular point (number) in the intuitionist calculus. For example, here (Figure 3) is an intuitionist point: start with 0.31, then choose successive digits so that each next digit is either a 4 or a 2. This definition simply fails to specify a number that a traditional mathematician would recognise as well-defined. For a traditional mathematician, this is not number, but a method for generating different numbers. And while traditionalist would agree that any number generated according to this method will have certain properties (such as the property of being gr than 0.31 and less than 0.32), the existence of these properties does not make the number itself well-defined.

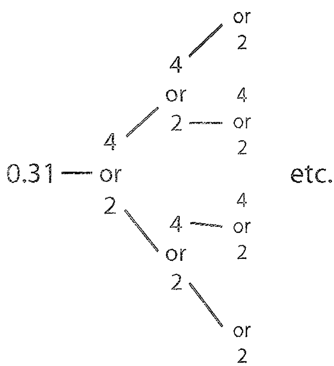


Figure 3: An illustration of a single number, as defined by a free choice sequence.

But the intuitionist understands this definition differently. For Brouwer, this rule defines a point. Though two different mathematicians might make different choices, before any choices have been made or as long as all of the choices made so far coincide, both mathematicians are working with the same point. As soon as they make different choices, they are then no longer working with the same point. A given point, therefore, consists of choices already made plus the possibility of making various choices in the future. As a process of choice, a point includes the potential for any number of points, so that each point contains an internal difference, equal to itself and different from itself.

Unlike a point in traditional mathematics, an intuitionist point, defined in relation to a free choice sequence, is no longer an unimaginable idealisation of dimension 0. On the contrary, using the definition of a free choice sequence, a point is a progressive narrowing of intervals, a process of honing in that keeps open an interval and never finally closes it down to 0. Points thereby become fuzzy, each equal to itself and to its neighbours (but less so). Indeed, in Brouwer's formal treatment of free choice sequences, he does not refer to successive digits of a decimal number; rather he posits a sequence of nested intervals, and the choice is a matter of how each interval is to be fit within its enclosing interval. The first few 'choices' of intervals for a point are illustrated in Figure 4.

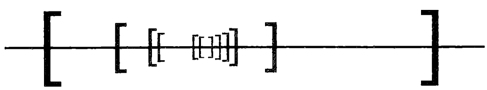


Figure 4: The first few 'choices' of nested intervals that define a point.

Now the problem of the continuum is solved almost by default: the continuum is just the totality of all choices, or the freedom to make choices however one will. Each point, as a process of continual refinement, covers a vanishing but positive linear area. The power of the continuum is built into the points that constitute it, requiring no extra axiom to justify its continuity. The points do not aggregate or bunch together side-by-side to create the continuum. Rather, as choices, the points are in motion, smearing themselves across intervals, blurring their own boundaries to leave the continuum in their wake. Intuitionist mathematics does not exactly promote the line over the point, but, as Deleuze and Guattari recommend, defines both line and point in terms of a motion that generates each of them.

The result of this definition of number in terms of free choice sequences is to impart a maximum of continuity into relations among numbers. But it is a continuity that derives from the fact that the elements, the numbers themselves, incorporate difference into their identity. Because each number has difference within itself – a part still to be determined – the differences between numbers are less dramatic, generating a maximum of continuity among numbers. Every number leaves part of itself not yet determined, and this indeterminacy is a difference that lives within number and forms part of its nature.

The continuity of intuitionist number shows up in the calculus in the intuitionist theorem that *every function of the unit continuum is continuous*. This departs dramatically from the claims of traditional mathematics. Whereas the pendulum number is something of a contrivance, a rather forced example of the oddball cases allowed by the early intuitionist definition of number, this revision of the notion of continuity radically alters the landscape of mathematical analysis and demonstrates the vast distance, opened by the surd, between intuitionist and traditional conclusions.

In traditional mathematics, the continuity of a function is a question of its smoothness. A function is continuous if it has no breaks or gaps, no jumps in it. For instance, the function in Figure 5A, though quite a rollercoaster, is nevertheless continuous: there are no jumps or gaps in the graph of the function; it is 'filled in' from one end to the other. Placing a fingertip at one end of the function, one could trace its entire path without lifting one's finger. The second example, Figure 5B, is a discontinuous function; this is called a jump discontinuity, since the function jumps suddenly at a point from one value to another. Brouwer's radical claim is that, in spite of this jump, the function in 5B is still continuous.

5B represents a continuous intuitionistic function because points (or numbers) are themselves defined so as to be smudges or vanishing intervals rather than finished points of dimension 0. In traditional mathematics, the function in 5B is discontinuous because we can specify a point, l, at which the function suddenly jumps from one value to another. On the left side of this point, the function equals 1, and on the right side, the function equals 2. But for the intuitionists, one cannot specify such a point because there are no absolute points. There are only neighbourhoods, intervals surrounding l, honing in on the classical point l, but never actually reaching it. And any neighbourhood around the classical point l will include space to either side of l. However closely one hones in on l, one still keeps open the possibility of making further choices so as to determine a value of 1 for the function, or of making other choices so as to determine a value of 2. Thus, putting it rather too bluntly, near l the function evaluates to both 1 and 2. Instead of a point at which the function jumps, the intuitionists see a neighbourhood that effectively connects the two sides of the function by tying their ends together in a single point, a point that includes difference within itself. The intuitionist point *l* functions as a kind of wormhole, a black hole or singularity that causes two points otherwise distant in space to overlap each other.

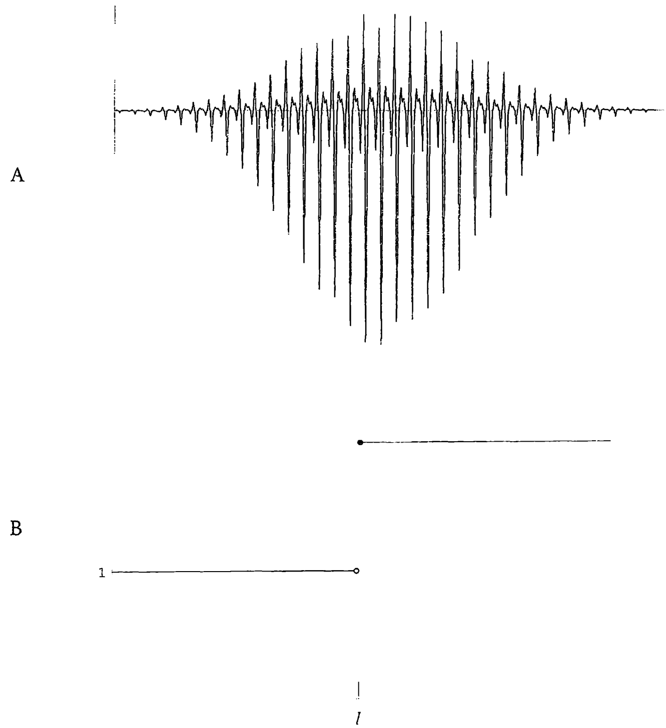


Figure 5: A continuous function and a discontinuous function, at least according to the definition of continuity in traditional mathematics.

The surd appears in this case in the guise of the concept of choice. The intuitionist definition of a point includes within its formalisms an openness, an indeterminacy that, not yet filled in, might be satisfied in any number of ways. As other intuitionists have argued (see Troelstra 1977, 12), choice here is not so much about the actual fact of choosing as about the form of choice, the fact that one can define a point and work with it without having narrowed down all of the intervals that would limit that point to a dimension of 0. Choice is thus simultaneously made and suspended; proofs are carried out as though only so many choices have been made, but the results are valid no matter how many have actually been made. There are always more choices to be made, so that the interval remains open, with a dimension of 1 but ever shrinking towards 0. Thus, in Brouwer's proof of the continuity of every function of the unit continuum, he relies crucially on the fact that, as we hone in on l, selecting increasingly narrow intervals surrounding l, it will always be possible to imagine for the 'next' choice that we are honing in on a point to the left of l *or* that we are honing in on one to the right of l. Once the next choice, say the q^{th} *choice, is actually made, it will undoubtedly cut off certain possibilities that existed prior to choice* q, but it will still be possible to make the next choice so as to aim toward a point to the right or to the left of l. The formal procedure of choice keeps the point from collapsing to zero, and calls once again upon the participation of the mathematician, this time an ideal mathematician, endowed with a power of abstract choice. Choice is the surrogate of the surd.

Though this definition of number is mathematically cumbersome, it is nevertheless intuitively appealing, even 'natural' No longer an idealisation of dimension 0, number is now a process of narrowing, mirroring the physical or mental process we might go through in determining a point, and incorporating the necessarily inexact and provisional endpoint at which we inevitably curtail our process. While empirical pointing happens over a specific time period, intuitionist points only refer to an abstract time, once again drawing upon form (of time) without reference to a specific content.[13]

Corresponding to the notion of an idealised temporality is the positing of an ideal mathematician (or 'creative subject' in some of the literature). Brouwer did not want free choice sequences to subject the mathematics to the caprice of the mathematician, as though the results of the calculus depend on just which choices the mathematician chooses to make. Rather, choice in intuitionism has a Nietzschean resonance, in that *every* choice gets made. Proofs involving free choice sequences do not take for granted any particular choice; they assume only that *some* choice gets made, and that some are still to be made. (Again, it is a matter of idealisation: choice as a form without content.) Choice is affirmed as a principle so that it is not a matter of any particular choice. 'But what is relevant from a mathematical point of view is not any individual choice sequence as such, but the 'mathematical' fact that there exist many perfectly well-defined (lawlike) operations on sequences which can be carried out without assuming the arguments to be determined by a law' (Troelstra 1977, 12). The principle of choice defines number as a *process*, with a maximum of continuity between numbers and a maximum of difference within each number. Every choice is effectively made, every choice is affirmed, and this generates the universe of number (the continuum).

Formalising or idealising the concept of choice freezes a dynamic process into a static element of the mathematics. Whereas traditional mathematics idealises number, ignoring its processual aspect, intuitionism formalises the process *as process*, capturing this motion of numbering *in vivo*, formalising not number but the genesis of number. Intuitionism discovers the essence of number in numbering, naming that essence and giving it a place amongst the symbols of the calculus. Brouwer thus pre-empts the process of numbering, seizes numbering with its virtuality intact, before it has cancelled that virtuality with the actuality of a fixed and determinate value. Notably, the power of the free choice sequence is a power of determination, a finite determination that can always be further determined, but is never finally determined. Intuitionism knows not to explicate too far, as Deleuze puts it. Brouwer thus suspends the mathematics in between virtual and actual, refusing complete determination to promote the process that gives rise to the determinable.

This hold on the vital essence of number could not endure. Much of the language of choice, and the epistemological and ontological commitments associated with Brouwer's intuitionism, were at odds with traditional mathematical standards. Later intuitionists attempted to preserve the results of the intuitionist calculus while discarding the philosophical underpinnings. These intuitionist reformers rejected the 'unmathematical' ideological and epistemological commitments of intuitionism, but wished to retain its formal results; after all, its alterations to traditional numerical analysis and to logic are at least interesting and possibly even useful. Troelstra (1977) and Heyting (1956, 1966, 1983) each formalise significant parts of intuitionism, so that it becomes only another system of symbolic manipulation, a formal alternative to traditional calculus stripped of the surd's original revolutionary power. A term like *free choice sequence* is replaced with the less provocative *infinitely proceeding sequence*, or just a sequence that is *lawless* or even *non-law-like*. The human operator is thereby eliminated from the mathematics, and sterile if productive research can continue without epistemological threats to its universality and objectivity.

Such is the history of the surd, in maths as elsewhere. Its introduction disrupts standard practices, opens a break in the linear progress of a field, inviting markedly new understandings, concepts, and techniques. Initially, these novel elements do not settle comfortably amidst their established cousins, and so demand a real labour and lend the whole radical enterprise an empirical or contingent character. The arrival of the surd involves a struggle that spills over the edges of the discipline, mathematics becoming politics, philosophy, aesthetics, and other concrete and value-laden productions. Eventually – sometimes it takes centuries while other times it is coincident with the appearance of the surd – these results are claimed in the name of State technique, and the surd loses its revolutionary force. The discipline seals off the openings that connect it to contingency and assumes once again an air of self-sufficiency and even self-evidence. Its surdity formalised, intuitionism remains of interest primarily to historians and philosophers of mathematics. The hopes that Brouwer held for practical applications went unrealised, and sometime in the middle of the twentieth century, mathematicians mostly stopped worrying about epistemology, preferring to go about their business as undeclared formalists.

Sound and the surd

With mathematics as a privileged model, my hypothesis is to elevate the surd to a general term: the spur of becoming, the juncture where ideas diverge. The surd inaugurates new ideas, not just in maths but across disciplines – arts, sciences, studies of the spirit. History, or rather genealogy, shocked ahead by the surd.[14] Well short of an adequate demonstration of this hypothesis, this essay will content itself with only one further example.

Aside from its mathematical usage, English has retained another specialised meaning of the term *surd*. Etymologically, *surd* is a Latinate translation of the Greek *alogos*, the irrational or rootless. But *alogos* also refers to what is outside of speech, what cannot be spoken. Linguistics preserves this connotation: in linguistics, the surd is an unvoiced sound or phoneme, that which is not spoken but is nevertheless carried in the speech. (French retains an oddly converse meaning in *sourd*, the word for deaf.) Linguists fail to appreciate the broad scope of this phenomenon and its essential role in making meaning out of speech. The surd is the sonic analogue of the textual *supplément*, the excess of meaning that hides in the pauses between words, the implicit commas, the white noise around sounds that surrounds spoken language and lends itself to the progressive and concrete generation of new meaning. A sentence always says more than its words: the surd, that which cannot finally be treated in words. Deleuze notes in *The Logic of Sense* (1990) that a word can never say its meaning, so that there must always be another word to name the meaning of the first. However, he acknowledges one exception to this rule: the nonsense word, the ab*surd*, the only word that says its own meaning by saying that it does not say. To speak, to sound off, is to draw upon an active if unconscious not-saying, to deploy the surd as the very possibility of initiating meaning in language.

To detect this linguistic phenomenon that generates meaning, examine the edges of spoken language, the thresholds that divide an utterance from the silence that surrounds it. The surd marks that point of fracture, where sound develops from silence and where silence overtakes reverberant sound. Every sound irrupts from silence, beginning with a noisy nonsense, and fades eventually back into silence via a senseless, irregular chaos. Even the formal symbols and numerical indices of acoustics cannot tame sound's ecstatic origins, which rend a hole in the rigid fabric of physics. Two measurable, empirical phenomena evince the effect of the surd in sound: the uncertainty principle and the Gibbs phenomenon. In both cases, it is a matter of suddenness, of stopping or starting, of sharp edges, of singular moments.

Both phenomena relate to the dual nature of oscillating signals such as sounds. A sound (or other signal) can be represented in two complementary manners. Typically, a sound is represented (on a graph) as an amplitude varying over time. Sound is the oscillation of air pressure, and by charting this change in air pressure over time, one represents the sound in all its details. (For comparison, think of a seismogram that is similar but represents the motion of the earth instead of the change in air pressure, or a barometer which also measures changes in air pressure but over a coarser scale of time.) However, instead of showing a sound as a change of pressure over time, one can also represent it as the composite of perfectly regular oscillations. Every sound, no matter how complex, can be constructed by adding together simple sounds (sine waves), and one can therefore identify a sound by noting of which sine waves (frequencies) it is comprised. In fact, while it is clearly significant *when* a sound happens – there is little point in yelling 'Look out!' after your friend has already been crushed by the falling piano – it is perhaps more significant which frequencies contribute to the sound, as its characteristic frequencies determine what it actually sounds like: high or low, harsh or soothing, aaaah or ooooo, bell-like, string-like, or percussive.

The surd at the starts and stops of sound places a limitation on the complementarity of these dual representations. There is an uncertainty principle of acoustics (due to Gabor 1946, 1947) – strictly analogous to

Heisenberg's uncertainty principle for quantum mechanics – which holds that a sound cannot be fully determinate with respect to both frequency and time. The more precisely located is a sound in time, the less precise we can be about its frequency content. And the more precisely we describe its frequencies, the less precise we can be about when the sound occurs. 'A signal can be represented either as a function of time or as a function of frequency (i.e. its spectrum) and as it is compressed in one representation so it expands in the other' (Stuart 1966, 62). Only a sound with no beginning or ending has an exact frequency; every sound with a duration, every sound that starts and stops must include physically inexact frequencies, patches of noise describable by Gaussian distribution functions (bell curves), wherein pitch is defined statistically over a fuzzy range instead of discretely at a specific note. Thus, a singular sound, one that occurs at a particular time, in a particular context, must always begin and end with noise, indeterminacy, the surd. The most sudden events – *transients*, as they are called by engineers and audiophiles – these sudden transitions are inevitably marked by noise that obscures and even distorts them. A sound located at a specific moment loses its definition, becomes a smudge of energy across the frequency spectrum, a pure noise whose meaning is only its temporal singularity but not its (atemporal) timbral characteristics. Conversely, sounds with an exact frequency or set of frequencies cannot be placed in time at all; they are idealisations, omnitemporal sounds that can never begin or end.

If the meaning of a sound is a matter of its location in time and its frequency spectrum, then the surd guarantees that meaning is irreducible, beyond acoustic analysis. Physics cannot account precisely for both components of sound's meaning. The surd is that indeterminate excess of meaning that slurs the edges of sounds and blurs the contributing frequencies, disallowing any absolute distinctions, insisting that every sound must already have begun, since its point of origin is precisely non-localised.

Frequency analysis is the technique by which engineers analyse sound (and many other signals). And the chief method for this analysis is Fourier analysis or the Fourier transform. This is a mathematical technique for taking a signal, represented as a changing amplitude over time, and generating its complementary representation, as a spectrum of frequencies. The claim of the uncertainty principle, manifesting the surd, is that sharp or sudden events in either representation will correlate with broad and smooth events in the other representation. But there is a further wrinkle.

The sharpest or most singular events, those that are most closely anchored to a moment in time, manifest a more severe distortion, a behaviour known as the Gibbs phenomenon. Since Fourier analysis yields a representation of a sound in terms of frequency, one can take this representation as a kind of recipe, such that by recombining these frequencies one can recreate the original sound.[15] But, for signals that have a discontinuity (of the sort discussed in relation to intuitionism above), the Fourier analysis does not yield a strictly accurate representation of the frequencies of the sound. Instead, the recreated signal overshoots the original signal at the point of discontinuity, and this inaccuracy persists, as the recreated signal rises above the original signal then drops below it in a perpetual oscillation. This oscillatory deviation from the original signal near a point of discontinuity is the Gibbs phenomenon (Figure 6).

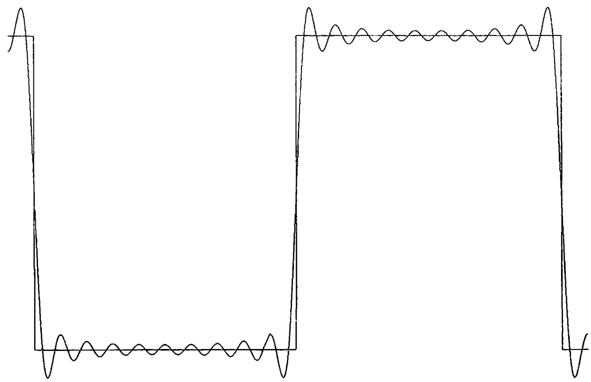


Figure 6: A square wave and an illustration of the Gibbs phenomenon or overshoot that is an artefact of its Fourier transform.

The singular point of discontinuity evades capture by the usual means of analysis, and engineers are forced to alter their methods, tailor their analysis to suit the specific and exceptional case at hand.[16]

There are ways to compensate for this deviation. By modifying the Fourier analysis using a multiplicative factor called the *Lanczos sigma*, one can eliminate the overshoot of the Gibbs phenomenon. Of course, this alteration has its own consequences, as the surd does not simply step aside. For one thing, the introduction of the Lanczos sigma factors causes the entire recreated signal to fall short of the original signal by a small percentage. A second result is an increase in the *rise time* of the recreated signal: eliminating the overshoot causes the recreated signal to take longer to approach the level of the original signal (Figure 7).

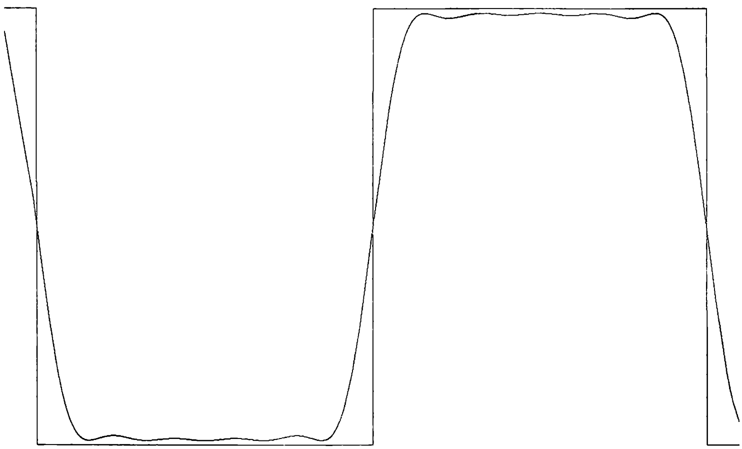


Figure 7: A square wave and its Fourier approximation including compensation with Lanczos sigma factors. Though the overshoot of the Gibbs phenomenon has been eliminated, note the slow rise time and the overall low amplitude of the approximation relative to the original square wave.

The surd – in this case, a discontinuity that represents the specificity, the unique moment of the original signal – ensures that no wholly accurate recreation is possible, that no analysis can do justice to the original signal.

The Lanczos sigma factors, responding to the surd inherent in the discontinuity in the original function, do not succeed in purging the surd from the analysis. Indeed, they reintroduce the surd in another form. They represent the *sinc* function, generally defined as $\frac{\sin x}{x}$, which is effectively a smoothing function; it concentrates its energy at its centre, but gently spreads out from that centre so as not to have any sharp or sudden events (see Figure 8).

Thus, the Lanczos sigma factors make localised, contextual alterations to a function, alterations that soften the function without changing its basic form. They are a means of piecemeal or spot-correction, a *kluge* as it is called in engineering. The Lanczos sigma smoothes the sudden jump at a discontinuity, but otherwise does not alter the overall form of the signal. This smoothing, which is tantamount to spreading the burst of energy at the discontinuity over a larger area, is the reason for the slower rise time (since the suddenness is smoothed into a diagonal rise) as well as the overall shortfall of the function (since the energy required to reach the original amplitude has been spread out slightly).

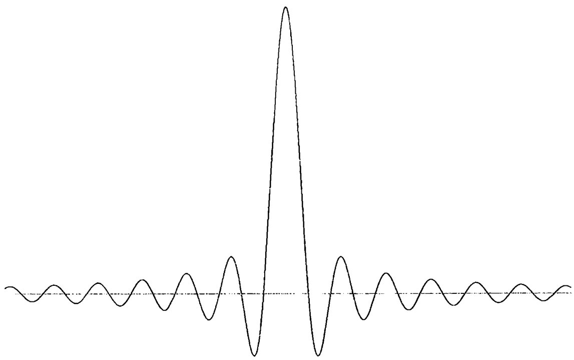


Figure 8: The sinc function, $\frac{\sin x}{x}$, upon which the Lanczos sigma factors are based. The horizontal line is the x-axis, which helps to show how most of the energy of the wave is concentrated at its centre.

Which is to say, the distortion of the Gibbs phenomenon (which tends to sound like ringing in acoustic signals) can be eliminated only by constructive methods that are tailored specifically to the situation at hand. The Lanczos sigma factor is the ultimate local intervention; it is a tool, a magic wand to wave over particular trouble spots, but its effects are mostly local and are designed to tame an otherwise unruly situation. Confronted with the surd in the form of a discontinuity, engineers apply the Lanczos sigma factor, another surd to combat the effects of the first. These are the phenomena that occur at the birth of meaning in sound, the jumps where sound arises out of silence.

Clearly, this theory of the surd needs much further testing. Even within the domains of mathematics and speech, the surd is more promise than result, and there may be better theories of progress, as well as exceptional cases that would call into question the very notion of a general theory of progress within and without these domains. Moreover, the drastic difference in scope between these two examples raises the possibility that the commonality of the term *surd* in each discipline is an accident of history with no further implications. Still, the research thus far is compelling if not decisive, and my initial investigation of other fields, from 'primitive' ethnography to digital technology, discovers the surd there as well at the decisive moments of progress. At least in the digital, the pattern holds true: the digital encounters events or objects that it cannot accommodate, and it must reshape itself in order to make room for these new ideas, but eventually settles back into a placid or rigid formula, neutralising the novelty that challenged it to develop.

Notes

1 *A Thousand Plateaus* perpetuates such constructivist examples, from Riemann spaces that allow no overarching perspective and must be navigated locally and singularly, to fractals that are generally apprehended in the process of construction and not as completed figures, to the numbering number, which could be a general term for the constructive impulse in mathematics, for it is the intensive number, determined always in relation to its context, and insisting upon a uniqueness, atypical of number, that ensures that it can only be understood and manipulated in context.

2 One can construct irrational numbers that have a pattern, and hence are exceptions to the rule. E.g., 0.030030003000030000030000003. . But this is a special, contrived case, and does not typify irrational numbers.

3 The flattening of ontology onto epistemology, the sense of reconstruction on a firm foundation, the reliance on intuition, and the notion of a critique from within all attest to Kant's parentage in Brouwer's work.

4 In the finite realm, intuitionist mathematics is pretty much functionally identical to traditional math. The breaking point lies between countable and uncountable infinities. (Roughly speaking, an infinite number of discrete objects is countable, while a continuum is an uncountable infinity.) For the intuitionists, uncountable infinities were nearly incomprehensible, while countable infinities could be dealt with one discrete element at a time. Discrete objects can be intuited, while a true continuum is beyond intuition.

5 In other words, pick a next digit, say 4, and add it to the end of the digits you have so far: 1.414214. Multiply this number by itself, $1.414214^2 = 2.0000012378$. This is slightly greater than 2 so replace the terminal 4 with a 3 and repeat: $1.414213^2 = 1.99999840937$. 3 is thus the largest digit that can be appended to make the square of the result less than 2. Continue in like manner to generate the next digit and each successive digit ad infinitum.

6 As such, it may be significant that π is not only an irrational number, but also a transcendental one, i.e., a real, irrational number that cannot be expressed as the root of a polynomial. This intensifies the 'surdity' of π, inasmuch as it

has a singular relationship to the universe and does not just express a relationship among numbers. It is as though π reaches beyond mathematics, generating its identity in the empirical domain (in the relationship of a circle's area to its diameter). Other transcendental numbers, such as e, share this property of reaching out of mathematics and into nature. Perhaps this is also true of certain algebraic (non-transcendental) numbers, such as the golden mean, but the relationship between the golden mean and natural phenomena is more approximate, not exact like π.

7 Different versions of intuitionism treat this other truth-value differently. Brouwer adhered to the belief that there are only two truth-values, true and false, so that this value of not-true was not a formal element of the calculus but only a step in thinking about it. Other intuitionists codified the value of not-true, making it a third term alongside true and false.

8 A fleeing property is any property such that (1) it can be determined for each natural number n that either holds or is absurd, (2) no method is known for calculating a number with the property , and (3) the assumption that some number exists with the property is not known to be absurd. The stipulation of opaqueness adds the further condition that (4) the assumption that some number exists with the property is also not known to be non-contradictory.

Verification for '0123456789 in π': (1) For each number, n, we can check whether it is the first place in the decimal expansion of where the series 0123456789 occurs (or not), simply by expanding out to the n+9th place. (2) We have no way of calculating the first place where that series occurs, except by calculating the successive digits of π. (3) We have no reason to believe that the series never occurs in π. Thus, this property is fleeing. (4) We have no reason to believe that the series does occur. Thus, this property is also opaque.

9 In one sense, Brouwer's use of π ties intuitionist mathematics to time and space. The math becomes spatial inasmuch as π is a geometric quantity, a relationship among (abstract or ideal) spatial phenomena. More significantly, though, the math becomes temporal to the extent that π is treated in intuitionism as an object to be discovered, a process for generating digits but a process that necessarily takes its time.

10 Brouwer seems mildly concerned about formalising the opaque fleeing property, as though he suspects that he may be losing the surd so crucial to the radical consequences of his calculus. On occasion, he laments the fact that his proofs (such as those involving the pendulum number) depend on the existence of unsolved mathematical problems (such as the existence of 0123456789 in π). In general, Brouwer is somewhat torn between his commitment to a constructivist mathematics, in which the math is empirical, and an adherence to the traditional epistemology of mathematics, in which math is universal and atemporal. He doesn't desire that the conclusions of intuitionism actually change over time, but neither does he want to allow a universalisability.

11 Brouwer tends to use the opaque fleeing property in discussions aimed at more general audiences and not so much in his formal papers. (Compare with the conference address 1967a which, while still mathematically rigorous, is not so laden with formalities as some of his other writings.) On the other hand, as an intuitionist, Brouwer did not distinguish sharply between formal and 'everyday' modes of mathematics, and has been criticised for using a plain language style even in his formal presentations.

12 Technically, there are not just an infinite number of irrationals, but a non-denumerable or uncountable infinity of them, more irrationals than integers. A higher level infinity of points is required to create a sufficient density to constitute the continuum.

13 Though Charles Parsons, editor of Brouwer (1967c) and himself a mathematician, refers to the free choice sequence as 'a process in time' (notes to p. 446), this is merely heuristic, as they function atemporally. Free choice sequences imply the formal structure of time, for there is always a part of the sequence that has already been chosen, a part that has not yet been chosen, and the immediate choice to be made. However, proofs involving free choice sequences demonstrate that each of these three dimensions of time (past, future, present) exists all at once at different stages. That is, the proof treats a free choice sequence as having at the same time many different (but related) pasts, many different (but related) futures, etc. It is as though the free choice sequence establishes a notion of past per se, future per se, without having a wholly specific past or future. Phenomenologist van Atten (1999) argues that Brouwer is committed philosophically to the identification of a choice sequence with its specific moment of origin, but though possibly true in principle, this temporal localisation seems to be nearly irrelevant in practice.

14 As this essay offers primarily two examples, generalisations about the surd are relegated here to a footnote. In general, the surd can be recognised by the following phenomena: (1) a lack of official sanction or recognition, (2) a pressure to formalise one's results, (3) the conflation of theory and practice (or of epistemology and practical knowledge), (4) an emphasis on the contingent and contextual as opposed to the general case (nothing can be taken for granted, each result must be thought each time), (5) an insistence on thought as an activity and not just an arrival or end, (6) labour and genius side-by-side, (7) interdisciplinarity, (8) rediscovery of the simplest ideas as now problematic and complex again, (9) a suspicion of the abstract and a faith in the immediacy of experience, (10) activity or motion in the objects under consideration, (11) a willingness to let objects have their way, to work with them rather than to dominate them, (12) a sense of precariousness, a real risk that things might not work out, that they are still up in the air or in development, and (13) maximum libidinal investment.

15 Fourier analysis includes a large family of mathematical techniques. One primary distinction is between the Fourier transform, in which a signal is

represented as a continuous function of its component frequencies, and a Fourier series, in which a signal is represented as the sum of (an infinite number of) discrete sinusoidal components. While the Gibbs phenomenon does apply to the Fourier transform, the discussion here refers chiefly to the Fourier series.

16 This is not just a hypothetical example. Engineers routinely deal with discontinuities in signals, as the square wave, whose edges are discontinuities, is a frequent basis for signal construction. Other discontinuities occur when, for example, a signal is 'brick wall' filtered, or when a noise gate is applied that suddenly shuts off the signal when it falls below a certain level of amplitude. The Gibbs phenomenon is a genuine hurdle for engineers, whose audio and stereo component designs include attempts to combat it.

12
Deleuze in phase space

Manuel DeLanda

> The semantic view of theories makes language largely irrelevant to the subject. Of course, to present a theory, we must present it in and by language. That is a trivial point. In addition, both because of our own history – the history of philosophy of science which became intensely language-oriented during the first half of [the last] century – and because of its intrinsic importance, we cannot ignore the language of science. But in discussion of the structure of theories it can largely be ignored. (Van Fraassen 1989, 222)

Van Fraassen is perhaps the most important representative of the empiricist tradition in contemporary analytical philosophy. But why use a quotation from an analytical philosopher, however famous, to begin a discussion of the work of an author who many regard as a member of the rival continental school of philosophy? The answer is that Gilles Deleuze does not belong to that school, at least if the latter is defined not geographically but in terms of its dominant traditions (Kantian and Hegelian). As is well known, Deleuze himself argued for the superiority, in some respects, of Anglo-American, or empiricist, philosophy (Deleuze and Parnet 2002, ch. 2). In addition, Deleuze's work was in large part a sustained critique of language (or more generally, of representation) as the master key to philosophical thought and, as the opening quotation attests, van Fraassen is also a leader of the emerging faction of philosophers of science disillusioned with the linguistic approach. There are, then, several points of convergence between the two authors but also many divergences. This essay will explore both.

Let us first of all clarify van Fraassen's position. What does it mean to say that in discussing the structure of scientific theories the language in which they are expressed is irrelevant? Or to put it differently, in what approach towards the nature of scientific theories is language itself crucial and why is that stance, according to van Fraassen, wrong? The stance in question is the so-called 'axiomatic approach' according to which a theory may be modelled as a set of axioms, a set of self-evident factual sentences, and all the theorems that may be derived from such axioms using deductive logic. Although there are many versions of this approach (some regarding the axioms purely syntactically, others treating them as part of natural language), what they all have in common is a disregard for the actual mathematical tools used by real scientists when creating a theory; tools such as the differential calculus. It is, of course, through the use of these non-linguistic tools that scientists create models of physical phenomena and it is these mathematical models which have become the object of intense interest for analytical philosophers. The question of whether the set of models which makes up a theory is axiomatisable or not, that is, whether they can be given a rigid hierarchical logical structure or not, has now become a less important concern. For all we know, a theory's models may constitute a heterogeneous population accumulated over time (Giere 1988, 82).

One may think that a theory's models may converge towards a neat logical structure over time but the truth is that the heterogeneity of this population has in fact increased, particularly in the twentieth century. While before most models used differential equations as their basis, suggesting that there may exist a general theory of models, in the last century many other kinds of equations (finite difference equations, matrix equations) were added to the reservoir of modelling resources available to scientists (Bunge 1979, 75). More recently digital computers have increased this reservoir even more with tools like cellular automata. Hence, rather than searching for a general modelling theory, what matters now is the detailed study of each type of model belonging to those increasingly *heterogeneous populations* we call scientific theories. In this essay I will explore only the oldest modelling technology, the one based on the differential calculus, partly because it is the one best understood and partly because it is the one actually discussed by both Deleuze and van Fraassen.

Differential equations are used to capture the rate of change of a given property in a physical system being modelled, as it relates to other properties which also change. Therefore, to use these equations one must first specify all the *relevant* ways in which a physical system can change. A simple system like a pendulum, for example, can change in only two ways, position and speed. We could, of course, explode the pendulum or melt it at high temperatures, and these would also constitute ways in which it can change; they would just not be significant ways of changing from the point of view of its intrinsic dynamics. The relevant ways of changing for a given system are referred to as its 'degrees of freedom' As the different properties of a system change, its overall state changes. This implies that a model of the system must capture all the different possible states in which it can exist. This set of states may be represented as a *space of possibilities* with as many dimensions as the system has degrees of freedom. This space is referred to, naturally, as 'state space' In this space each point represents one possible state for a physical system, the state it has at a given instant of time. As the states of a physical system change with time, that is, as the system goes through temporal sequences of states, its representation in state space becomes a continuous sequence of points, that is, a curve or a trajectory. A state space filled with such trajectories is called the 'phase portrait' of a system, or its 'phase space' for short (Abraham and Shaw 1985, 1: 20–1).

Each point in this space, each possible state, may have the same or different probabilities of existing. A space in which all the points are equally probable is a space without any structure, and it represents a physical system in which states change in a completely random way. Van Fraassen discusses two ways in which this space may be given more structure: through rules which restrict the areas that may be occupied (thus assigning different probabilities to different parts of the space, including forbidden areas with zero probability) and through rules which specify which states must follow other states, that is, rules governing trajectories. Van Fraassen refers to these two kinds of rules as 'laws of coexistence' (exemplified by Boyle's law of ideal gases) and 'laws of succession' (exemplified by Newton's laws of motion) (Van Fraassen 1989, 223). Both types of rules are given by equations and so, in a sense, for van Fraassen it is the equations that give us the structure of the space of possibilities. Deleuze, as I will argue shortly, gives a more original account of this structure, one that does not depend on the concept of 'law' But the main point remains the same in both accounts: if the possible states of phase space are all equiprobable, then no regularity may be discerned in the dynamics of the system. To the extent that this equiprobability is broken (by laws or by something else) then regularities may be detected. It is, of course, the latter case that is the most interesting to scientists and philosophers. The question to be answered shortly is whether the *immanent patterns of becoming* (or the recurring regularities) we observe in the real world are best thought of as 'laws' or as something else.

I will postpone answering this question until later since it involves switching from talk of models to talk of reality. That is, we can discuss phase space as a means for studying the characteristics of certain mathematical models and leave it at that, or we can go on to argue that, to the extent that these models manage to capture the dynamical nature of a real system (the recurring regularities in its behaviour), phase space may also provide insights into physical reality. This latter question, the question of the ontological status of different features of phase space, is highly controversial. For example, discussing the *ontological status* of possible, yet unrealised, states of a system is a task fraught with danger. Philosophers studying modal logic, the branch of logic which deals with such concepts as possibility and necessity, have struggled to clarify the issue for decades with little success. So for the time being let us simply view phase space as a useful tool in the exploration of models. Even if phase space had no implications for our understanding of reality, now that models have replaced linguistic statements as the 'stuff' out of which scientific theories are made, these spaces have become important philosophical objects in their own right. So first of all let us explore the question of what kind of mathematical objects these spaces are supposed to be.

A mathematical space is characterised by a set of points and by a definition of proximity between points, in other words, by the relations which define a given subset of points as a neighbourhood. If proximity is defined via a minimum length (e.g. all points less than a given distance away from a centre form a neighbourhood) the space is said to be metric (or 'striated' in Deleuze's terms). If some other criterion is used to specify which points are nearby other points the space is said to be non-metric (or 'smooth' in Deleuzian terminology). Euclidean geometry is the prime example of a striated space, while projective, differential and topological geometries exemplify smooth spaces (Deleuze and Guattari 1987, 361). But how can proximity be defined without rigid lengths? In differential geometry, for example, one takes advantage of the fact that the calculus operates on equations expressing rates of change and that one of its operators (differentiation) gives as its output an instantaneous value for that rate of change. The points that form a space can then be defined not by distances from a fixed coordinate system (as in the striated case) but by the instantaneous rate at which *curvature* changes at a given point. Some parts of the space will not be changing at all, other parts may be changing slowly, while others may be changing very fast. A differential space, in effect, becomes *a field of rapidities and slownesses*, and via these infinitesimal relations one can specify neighbourhoods without having to use rigid lengths. A space so defined is called a 'manifold' or 'multiplicity'. Given that phase spaces are used to study models made out of differential equations it should come as no surprise that these spaces are always manifolds.

To obtain a better idea of how spatial representations can give insight into the nature of a model we need to discuss how a model works. Any equation, whether differential or not, has numerical solutions, that is, sets of values for its unknown variables which make the equation come out true. Each such solution represents one state of the system being modelled (one point in phase space). But in order to learn about a physical system, scientists need to know more than just a few numerical solutions: they must have a sense of the pattern formed by all the solutions of a given equation. This pattern, which in some cases can be given by yet another equation, is called an 'analytic' or an 'exact' solution. Without being exactly solvable an equation is of limited value as a model. The main incentive driving the use of phase spaces was the resistance which some recalcitrant equations offered to being solved exactly, such as differential equations in which there are interactions between the variables (that is, equations which are *nonlinear*). This is one reason why most models used in classical physics are either linear, or if they are nonlinear, used only for ranges of values within which their behaviour can be linearised. This, of course, limits the kinds of physical phenomena which may be modelled this way, and worse, it may lead to the false idea that the world is in fact linear. As the mathematician Ian Stewart puts it:

> Classical mathematics concentrated on linear equations for a sound pragmatic reason: it couldn't solve anything else. . . . So docile are linear equations that classical mathematicians were willing to compromise their physics to get them. So the classical theory deals with *shallow* waves, *low*-amplitude vibrations, *small* temperature gradients [that is, linearizes nonlinearities]. So ingrained became the linear habit that by the 1940s and 1950s many scientists and engineers knew little else. . . . Linearity is a trap. The behaviour of linear equations . . . is far from typical. But if you decide that only linear equations are worth thinking about, self-censorship sets in. Your textbooks fill with triumphs of linear analysis, its failures buried so deep that the graves go unmarked and the existence of the graves goes unremarked. As the 18th century believed in a clockwork world, so did the mid-20th in a linear one. (Stewart 1989, 83)

Henri Poincaré, the mathematician who pioneered the use of phase space, was motivated by the desire to overcome these limitations. In essence he asked himself 'If there is no analytical way of capturing the pattern of all numerical solutions, can there be a round-about way?' Dealing with differential manifolds, instead of equations, Poincaré was able to bring further mathematical resources, geometrical resources, to bear on the answer to the question. Specifically, he was able to study the manifolds and figure out whether they contained special, or singular, points. One thing that made these points special was that they remained unaltered if the manifold was transformed in a variety of ways. That is, these *singularities* constituted the most stable and characteristic aspect of the manifold (its topological invariants). In addition, in some cases (when the physical system is not isolated from its surroundings) these singularities have a direct effect on the many trajectories representing sequences of states in the physical system: the trajectories become attracted to the singularities. Thus, given that the set of trajectories is the geometrical counterpart to the numerical solutions of the equation, and that their behaviour in phase space is governed by these special points, *the distribution of singularities* gives us information about the pattern of all the solutions. This is not exactly an analytical solution, but it is the next best thing. By the time he was finished, Poincaré had discovered several types of point singularities (dips, nodes, focal points, centres), loop singularities (called 'periodic attractors') and even caught a disturbing glimpse of the fractal singularities that would later on be referred to as 'chaotic attractors' (Barrow-Green 1997, 32–8). Using Poincaré's discoveries we can now phrase more precisely one of the points of divergence between van Fraassen and Deleuze's views on phase space: while for van Fraassen what give structure to the space of possibilities are laws (of coexistence and of succession) for Deleuze it is the distribution of singularities itself. (Although there are singularities which are not attractors, in what follows I will restrict my discussion to those which are).

This is, in a nut shell, the reason why phase space commands so much attention today: as philosophers of science have turned away from the linguistic to the mathematical expression of scientific concepts, the study of differential equations (and of the behaviour of their solutions) has become top priority, and the best way to study these equations (particularly if they are nonlinear) is using the geometrical methods pioneered by Poincaré. But this still does not explain in what sense phase space is important outside the philosophy of science, that is, what insights such spaces may yield about the physical reality studied by scientists. If these models were nothing but mathematical constructs there would be no reason to think that they may throw some light on the nature of reality. But many of these models actually work, that is, they manage to capture the regularities in the behaviour of real systems. Let me give an admittedly oversimplified example. Let us assume we have a laboratory where we can manipulate real physical systems, that is, we can restrict their degrees of freedom (by screening out other factors) and we can place a system in a given state and then let it run spontaneously through a sequence of states. Let us assume that we can also measure with some precision the values of the degrees of freedom (say, temperature, pressure and volume) at each of those states.

After several trials we generate data about the system starting it in different initial states. The data will consist, basically, of sequences of numbers giving the values of temperature, pressure and volume which the system takes as it evolves from different initial conditions. (These number series may be plotted, turning them into a trajectory.) We then run our mathematical model, giving it the same values for initial conditions as our laboratory runs, and generate a set of phase space trajectories. Finally, we compare the two sets of curves. If the trajectories display geometrical similarity (or approximate similarity, since the precision of measurements is always finite) we will have evidence that the model works. As one analytical philosopher puts it, 'we can say that a dynamical theory is approximately true just if the modelling geometric structure approximates (in suitable respects) to the structure to be modelled: a basic case is where trajectories in the model closely track trajectories encoding physically real behaviours (or, at least, track them for long enough)' (Smith 1998, 72).

Thus, it is the success in practice of models based on differential equations that motivates an ontological analysis of phase space. But this analysis will closely depend on our prior ontological commitments which may have nothing to do with the success of mathematical models or even with anything related to scientific practice. The most basic commitment is expressed by the belief that there is, or that there is not, a world which exists independently of our minds. If the world is believed to be not mind-independent, if ideas in our heads (whether transcendental categories or arbitrary signifiers) are believed to shape the very contents of the world, then one has idealist ontological commitments. If, on the other hand, the world is assumed to be autonomous then a variety of commitments are possible. One may believe in the mind-independence of objects of direct experience (pets, automobiles, buildings) but assume that entities like oxygen, electrons, causal relations and so on, are merely theoretical constructs. Ontological commitments of this sort are typically associated with positivism and empiricism, though different philosophers will draw the line of what is 'directly observable' in different places. (Van Fraassen, for instance, seems to believe that objects perceived through telescopes, but not microscopes, count as directly experienced (Van Fraassen 1980, 16).) Finally, if one rejects direct experience as the criterion of autonomous reality one is said to have realist ontological commitments, although what exactly one assumes are the contents of this mind-independent world varies enormously. Religious realists will count spiritual entities, such as angels or demons, as part of these contents, while materialists will not.

To return to phase space, the first candidates for ontological evaluation, given their critical role in the testing of a model, are the trajectories. These, as I said, represent possible sequences of states. The problem here is, of course, the status of possibilities in general. Empiricists are particularly sceptical about possible entities. Quine, one of the most famous representatives of this school, is well known for the fun he pokes at these entities. As he writes: 'Take, for instance, the possible fat man in the doorway; and again, the possible bald man in the doorway. Are they the same possible man, or two possible men? How do we decide? How many possible men are there in that doorway? Are there more possible thin ones than fat ones? How many of them are alike? Or would their being alike make them one?' (Quine 1979, 177). In other words, it seems impossible to *individuate* possible entities, to assert their identity in the midst of all the possible variations. There just do not seem to be enough constraints within a possible world to know whether we are dealing with one or several entities as we modify the details. But, it may be argued, this is a problem only for *linguistically* specified possible worlds. The target of Quine's ridicule is the modal logician who believes that the fact that people can understand counterfactual sentences like; 'If J.F.K. had not been assassinated the Vietnam War would have ended sooner.' implies the objective existence of possible worlds. (Clearly, the possible world where J.F.K survived does have reality, just not mind-independent reality.) On the other hand, realist philosophers like Ronald Giere have argued that while Quine's sceptical remarks are valid for counterfactuals the extra constraints which structure phase space can overcome these limitations:

> As Quine delights in pointing out, it is often difficult to individuate possibilities. [But] many models in which the system laws are expressed as differential equations provide an unambiguous criterion for individuating the possible histories of the model. They are the trajectories in state-space corresponding to all possible *initial conditions*. Threatened ambiguities in

> the set of possible initial conditions can be eliminated by explicitly restricting the set in the definition of the theoretical model. (Giere 1985, 83–4)

Let us assume for a moment that Giere is right and that within the restricted world of phase space the possible histories of a system are indeed well individuated, avoiding the inherent ambiguities in counterfactual sentences. Why should we commit ourselves to assert the existence of these well-defined possibilities (and other forms of physical modality)? Empiricists like van Fraassen would still deny the need for such commitment to modalities given that for him the point of building theoretical models is simply to achieve *empirical adequacy*, that is, to increase our ability to make predictions and to control outcomes in the laboratory. For this purpose all that matters is that we generate one single trajectory for a given initial condition, then try to reproduce that particular combination of values for the degrees of freedom in the laboratory, and finally observe whether the sequence of *actual states* matches that predicted by the trajectory. Given the one trajectory we associate with the actual sequence in an experiment, the rest of the population of trajectories is merely a useful fiction, that is, ontologically unimportant. Giere refers to this ontological stance towards modalities as 'actualism' (Giere 1985, 44).

But as he goes on to argue, this ontological stance misses the fact that the population of trajectories as a whole displays certain regularities in the possible histories of a system, global regularities which play a role in shaping any one particular actual history. In the terms I used above, the space of possibilities has structure, and this structure is not displayed by any one single trajectory. For Giere, understanding a system is not knowing how it actually behaves in this or that specific situation, but knowing *how it would behave* in conditions which may in fact not occur. And to know that, we need to use the global information embodied in the population of possible histories. Van Fraassen may reply, of course, that this information is given by the laws of succession that control the evolution of trajectories. This would seem to commit him, however, to assert the existence of another modal property, necessity, and that would bring him back to square one since necessity and possibility are interdefinable (if something necessarily exists, for example, then it is not possible that it would not exist). But when van Fraassen speaks of laws he does not refer to the objective patterns of becoming shaping sequences of states in the laboratory (or in nature) but to mathematical rules constraining trajectories in the model. Thus, the debate between analytical realists and empiricists seems to offer only two alternatives: either be ontologically committed to traditional modalities or reject the latter but lose your ability to explain why there are recurrent regularities in the world. This alternative, however, is a trap, and the significance of Deleuze's realist approach is precisely that it supplies us with an escape route.

Deleuze is not an actualist but he is not a realist about traditional modalities either. Rather, he creates a new form of physical modality to account for both the regularities in the models and the immanent patterns of becoming in nature. This new modality he refers to as 'virtuality'. Let me first discuss how he derives this notion at the level of models. As I mentioned above there is an alternative to laws when it comes to specifying the structure of spaces of possibilities: the distribution of singularities or attractors. While each of the trajectories that fill phase space is a solution (obtained through the integration operator) to an equation, the equation itself, or more exactly, the singularities that govern the behaviour of its solutions, specify a problem. The key to this alternative interpretation lies in not subordinating the problem to its solutions (the integral curves or trajectories) but to become aware of its relative autonomy. What happens, for example, if we examine phase space without any trajectories? I mentioned before that this space may be viewed as a field of rapidities and slownesses. This field is technically referred to as a 'vector field' because each rapidity and slowness with which the curvature changes at any given point can be assigned a direction. Now, while it is true that we become aware of the existence of singularities by observing that the integral curves become attracted to certain special places, the singularities are topological features (invariants) of the vector field itself. As Deleuze writes:

> Already Leibniz had shown that the calculus expressed problems which could not hitherto be solved or, indeed, even posed. One thinks in particular of the role of the regular and the singular points which enter into the complete determination of a species of curve. No doubt the specification of the singular points (for example, dips, nodes, focal points, centres) is undertaken by means of the form of integral curves, which refers back to the solutions of the differential equations. There is nevertheless a complete determination with regard to the existence and distribution of these points which depends upon a completely different instance – namely, the field of vectors defined by the equation itself. . Moreover, if the specification of the points already shows the necessary immanence

> of the problem in the solution, its involvement in the solution which covers it, along with the existence and the distribution of points, testifies to the transcendence of the problem and its directive role in relation to the organisation of the solutions themselves. (Deleuze 1994, 177)

Thus, the first step in this alternative interpretation consists in sharply differentiating these two components of phase space, the population of trajectories and the vector field, a step that, to my knowledge, has not been taken by any analytical philosopher. Indeed, in the analytical literature there seems to be no awareness of the role which vector fields play in the modelling process. There are, for example, events in phase space (referred to as 'bifurcations') which change one distribution of singularities into another, topologically inequivalent, one. These are very important events, as far as the modelling process is concerned, because they capture abrupt changes in the dynamics of real systems, like the change from one regime of flow (periodic or convective) to another (turbulent) in a flowing liquid. But in order to model such critical transitions, scientists must operate not on the trajectories but on the vector field itself. In particular, they must perturb the system by adding a small vector field to the main one and check the resulting distribution of singularities for topological equivalence (Abraham and Shaw 1985, 2: 37–41). So let us add vector fields to the list of things that must be given an ontological interpretation. At first, this would seem to bring us back to the endless and fruitless discussions which, from the time of Leibniz to the early nineteenth century, tended to surround the notion of an 'infinitesimal quantity', since each vector in the field is one such infinitesimal. These entities were eliminated from mathematics via the notion of a limit, a notion which presupposes only the concept of number and nothing else. But what must be given an ontological interpretation is not the vectors themselves but *the topological invariants of the entire field*, and these have nothing whatsoever to do with infinitesimals.

A clue to the modal status of these invariants, that is, of the distribution of singularities, is the fact that, as is well known, trajectories in phase space always approach an attractor *asymptotically*, that is, they approach it *indefinitely close but never reach it* (Abraham and Shaw 1985, 1: 35–6). Although the sphere of influence of an attractor, its basin of attraction, is a subset of points of phase space, and therefore a set of possible states, the attractor itself is not a possible state since it can never become actual. In other words, unlike trajectories which represent possible histories which may or may not be actualised, attractors can never be actualised since no point of a trajectory ever reaches the attractor itself. Despite their lack of actuality, attractors are nevertheless real since they have definite effects. In particular, they confer on trajectories a certain degree of stability, called 'asymptotic stability' (Nicolis and Prigogine 1989, 65–71). Small shocks may dislodge a trajectory from its attractor but as long as the shock is not too large to push it out of the basin of attraction, the trajectory will spontaneously return to the stable state defined by the attractor. It is in this sense that singularities represent only the long-term tendencies of a system, never a possible state. Thus, it seems, we need a new form of physical modality, distinct from possibility and necessity, to account for this double status of singularities: real in their effects but incapable of ever being actual. This is what the notion of virtuality is supposed to achieve. As Deleuze argues:

> The virtual is opposed not to the real but to the actual. *The virtual is fully real in so far as it is virtual.* Indeed, the virtual must be defined as strictly a part of the real object – as though the object had one part of itself in the virtual into which it plunged as though into an objective dimension.
>
> The reality of the virtual consists of the differential elements and relations along with the singular points which correspond to them. The reality of the virtual is structure. We must avoid giving the elements and relations that form a structure an actuality which they do not have, and withdrawing from them a reality which they have. (Deleuze 1994, 208–209)

Deleuze uses his analysis of phase space as evidence for the real existence of virtual entities, entities he calls 'virtual multiplicities' Unlike essences or eternal archetypes, virtual multiplicities do not exist in some transcendent space but in a space which is fully immanent to that of matter and energy. And unlike the instantiations of essences, which typically resemble the model of which they are imperfect copies, multiplicities do not bear any resemblance to the processes which actualise them. This is, in part, a consequence of defining multiplicities exclusively in terms of topological invariants (singularities, dimensionality), that is, in terms of features so abstract that they are compatible with many features of metrically defined entities. Thus, the simplest virtual multiplicity, one defined by a single point singularity (a minimum) may be actualised in a variety of processes yielding a variety of actual entities with very different metric properties: soap bubbles, crystals of a variety of shapes, light rays and, indeed, certain mathematical objects. I said above that the term 'multiplicity' is synonymous with 'manifold', that is, both refer to differential geometry spaces with a variable number of dimensions. But once the term 'virtual' is attached to it, the term 'multiplicity' designates a real entity of which mathematical spaces are only one actualisation among others. This implies that a given phase space and the physical system it models may both be actualisations of the same virtual multiplicity. This, in turn, implies that the crucial relation between model and reality is not one of resemblance (phase space trajectories resembling plotted series of laboratory measured values) but one of *co-actualisation*. Whatever resemblance one does obtain between plots and trajectories is a consequence of this deeper isomorphism. Thus, unlike other philosophers who cannot explain the success of mathematical models, falling back on the physicist Eugene Wigner's thesis about 'the unreasonable effectiveness of mathematics in the natural sciences' (Wigner 1979, 222), Deleuze is at least in a position to offer a hypothesis to explain that unreasonable success.

On the other hand, by postulating the existence of virtual entities, Deleuze does flirt with essentialism. The thesis of divergent actualisation, which removes the resemblance between archetypes and their realisations, can only take us so far: essences may be topological rather than metric, but they remain essences. Thus, many other aspects of this novel realist ontology must be developed to ensure that essentialism is kept at bay. To begin with, it is not enough to simply affirm that the space formed by virtual multiplicities is immanent rather than transcendent: specific mechanisms of immanence must be given to show how that space is continuously constructed. In addition, a theory of non-metric time must be created, to complement the non-metric nature of virtual space, so that multiplicities may acquire their own historicity and may therefore be distinguished from eternal archetypes. Deleuze made some progress towards both of these goals and, elsewhere, I have tried to show how his valuable insights may be extended, but clearly much work remains to be done (DeLanda 2002). Yet even at this early stage in the development of his ontology, its sheer originality, particularly when contrasted with the more traditional empiricist and realist approaches, must be acknowledged. It is to be hoped that as language and representation cease to be the centre of the philosophical universe a renewed materialism free from the trappings of dialectics will take their place. In this long term project the line of flight away from the linguistic turn sketched by Gilles Deleuze will surely remain one of the most viable and promising routes.

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