

Maximality in finite-valued Łukasiewicz logics defined by order filters

Marcelo E. Coniglio¹, Francesc Esteva², Joan Gispert³ and Lluís Godó²

¹ Dept. of Philosophy - IFCH and
Centre for Logic, Epistemology and the History of Science,
University of Campinas, Brazil
`coniglio@cle.unicamp.br`

² Artificial Intelligence Research Institute (IIIA) - CSIC, Barcelona, Spain
`{esteva,godo}@iiia.csic.es`

³ Dept. of Mathematics and Computer Science, University of Barcelona, Spain
`jgispertb@ub.edu`

Abstract

In this paper we consider the logics L_n^i obtained from the $(n + 1)$ -valued Łukasiewicz logics L_{n+1} by taking the order filter generated by i/n as the set of designated elements. In particular, the conditions of maximality and strong maximality among them are analyzed. We present a very general theorem which provides sufficient conditions for maximality between logics. As a consequence of this theorem it is shown that L_n^i is maximal w.r.t. CPL whenever n is prime. Concerning strong maximality between the logics L_n^i (that is, maximality w.r.t. rules instead of axioms), we provide algebraic arguments in order to show that the logics L_n^i are not strongly maximal w.r.t. CPL, even for n prime. Indeed, in such case, we show there is just one extension between L_n^i and CPL obtained by adding to L_n^i a kind of graded explosion rule. Finally, using these results, we show that the logics L_n^i with n prime and $i/n < 1/2$ are ideal paraconsistent logics.

1 Introduction

In this paper we study the notion of maximality and strong maximality among finite-valued propositional logics. Recall the usual notion of maximality found in the literature: a propositional logic L_1 , that is a sublogic of another logic L_2 (in the sense of inclusion of their consequence relations over the same signature), is called maximal with respect to L_2 if, roughly speaking, L_1 extended with any theorem of L_2 which is not a theorem of L_1 , coincides with L_2 . Similarly, recall the stronger notion of strong maximality following [2, 4, 35]: L_1 is called strongly

maximal with respect to L_2 if, roughly speaking again, L_1 extended with a rule of inference valid in L_2 but not a valid in L_1 , coincides with L_2 .

The problem of finding and characterizing maximal sublogics (in both senses) of a given logic has already been addressed in the literature, specially in the context of paraconsistent logics, where being maximal with respect to classical logic is felt as a desirable or ideal feature, c.f. [3, 10]. Indeed, being maximal means that, while still allowing non-trivial inconsistent theories, it retains as much as possible of classical logic.

In the present paper we approach the general problem of characterizing maximality (not necessarily for paraconsistent logics) in two different scenarios. The first one considers a very general class of finite-valued logics, those defined by almost arbitrary finite logical matrices. In such a context, we provide a sufficient condition for a logic to be maximal w.r.t. another one with less truth-values under very general conditions. This result, inspired on the notion of recovery operators from paraconsistent logics, turns out to be very powerful and encompasses many maximality results scattered in the literature.

The second scenario considers a particular class of finite-valued logics, the class of n -valued Łukasiewicz logics L_n and their related logics defined by order filters. We show that these logics, for n being prime, are maximal but not strongly maximal with respect to classical logic. Actually, we show that each of these logics can always be uniquely extended with a sort of explosion inference rule such that the obtained logic is the unique one below classical logic, and hence strongly maximal.

The paper is structured as follows. After this introduction, we provide in Section 2 a very general condition for a finite matrix logic to be maximal w.r.t. another one with less truth-values, and we analyze in particular the case of 3-valued logics. In the rest of the paper we focus our attention on the class of finite-valued Łukasiewicz logics L_n^i defined by order filters. In Section 3 we identify which of these logics are maximal with respect to classical logic, while in Section 4 we study their status regarding the property of strong maximality. It is in Section 5 where we fully characterize, by algebraic techniques, conditions of strong maximality. Finally, in Section 6 the question of ideal paraconsistent logics (as introduced in [3]) will be analyzed in the present framework. Specifically, it will be shown that the logics L_n^i with n prime and $i/n < 1/2$ are ideal paraconsistent logics. In addition, the case L_3^1 will be discussed with more detail, and it will be argued that this logic constitutes the 4-valued version of the well-known 3-valued paraconsistent logic J_3 (see [20]). We finish in Section 7 with some conclusions and prospects of future research.

2 Maximality and recovery operators

Let us recall the usual notion of maximality of a (standard) logic with respect to another:

Definition 1. Let L_1 and L_2 two standard propositional logics defined over the same signature Θ such that L_1 is a proper sublogic of L_2 , i.e. such that

$\vdash_{L_1} \subsetneq \vdash_{L_2}$, where \vdash_{L_i} denotes the consequence relation of L_i (for $i = 1, 2$). Then, L_1 is said to be *maximal* w.r.t. L_2 if, for every formula φ over Θ , if $\vdash_{L_2} \varphi$ but $\not\vdash_{L_1} \varphi$, then the logic L_1^+ obtained from L_1 by adding φ as a theorem, coincides with L_2 .

By L_1^+ above we mean the logic whose consequence relation is obtained from the one of L_1 as follows: for every set of formulas $\Gamma \cup \{\psi\}$ over Θ ,

$$\Gamma \vdash_{L_1^+} \psi \quad \text{if } \Gamma, \{\sigma(\varphi) : \sigma \text{ is a substitution over } \Theta\} \vdash_{L_1} \psi.$$

Remark 1. It should be noticed that, according to the above definition, if L_1 is a proper sublogic of L_2 such that they validate the same formulas (that is: $\vdash_{L_1} \varphi$ iff $\vdash_{L_2} \varphi$, for every formula φ) then L_1 is maximal w.r.t. L_2 .

In this section, for the class of propositional logics induced by finite logical matrices, we will provide a very general sufficient condition for a logic to be maximal w.r.t. another one (see Theorem 1 below), its proof being inspired in the role played by the so-called recovery operators in paraconsistent and adaptive logics. Recall from [12] (see also [11, 10]) the definition of the class of paraconsistent logics called *Logics of Formal Inconsistency (LFIs)*: a given logic, say L , is an **LFI** if it is paraconsistent w.r.t. some negation, say \neg (that is, there exist formulas α and β such that β does not follow from $\{\alpha, \neg\alpha\}$ in L). In addition, there is a (primitive or definable) unary connective \circ in L (called a *consistency operator*) such that every formula β follows in L from a set of the form $\{\alpha, \neg\alpha, \circ\alpha\}$.¹ If L is an **LFI** which is sublogic of classical propositional logic (CPL), presented in the same signature of L ,² then the consistency operator \circ allows to recover CPL inside L by adding additional hypothesis concerning the consistency (or ‘classicality’, or ‘well-behavior’) of some formulas. Namely, for every (finite) set $\Gamma \cup \{\psi\}$ of formulas,

$$\Gamma \vdash_{\text{CPL}} \psi \quad \text{iff } (\exists \Lambda)[\Gamma, \{\circ\alpha : \alpha \in \Lambda\} \vdash_L \psi],$$

where Λ is a set of formulas. This is what is called a *Derivability Adjustment Theorem (DAT)*. The idea of DATs was proposed by Battens in the context of *Adaptive logics*, but this technique (as well as the notion of consistency operator) was already used by da Costa for his well-known hierarchy of paraconsistent systems C_n (see [18]).

A more interesting DAT (as, for instance, the ones obtained by da Costa) requires that the consistency (or well-behavior) operator \circ can just be applied to the propositional variables occurring in $\Gamma \cup \{\psi\}$. This suggests that, given two standard propositional logics L_1 and L_2 defined over the same signature Θ such that $\vdash_{L_1} \subseteq \vdash_{L_2}$, a DAT between both logics can be defined in terms of a *recovery operator* \circ (generalizing the idea of **LFIs**): for every (finite) $\Gamma \cup \{\psi\}$,

$$\Gamma \vdash_{L_2} \psi \quad \text{iff } \Gamma, \{\circ p_1, \dots, \circ p_m\} \vdash_{L_1} \psi,$$

¹This is a slightly simplified presentation of the original definition of **LFIs**.

²In this case, the formulas $\circ\alpha$ take the value 1 for every evaluation in CPL.

where $\{p_1, \dots, p_m\}$ is the set of propositional variables occurring in $\Gamma \cup \{\psi\}$.

The idea then is that if one of such recovery operators \circ_φ can be defined as a family of instances of a theorem φ of L_2 which is not derivable in L_1 , and if this process can be reproduced for any of such formulas φ , then it will follow that L_2 is maximal w.r.t. L_1 . To be more general, a finite recovery set $\bigcirc(p)$ of formulas depending only on one variable p will be considered instead of a single formula $\circ(p)$, following the original definition of **LFIs**. Actually, in Theorem 1 below some sufficient conditions are given in order to define such recovery sets, which will allow us to determine if one logic is maximal w.r.t. another.

In what follows, $\mathcal{L}(\Theta)$ will denote the term algebra generated by a propositional signature Θ from a fixed set $P = \{p_n : n \geq 1\}$ of propositional variables. If \mathbf{A} is an algebra over Θ then the set of homomorphisms from $\mathcal{L}(\Theta)$ to \mathbf{A} will be denoted by $Hom(\mathcal{L}(\Theta), \mathbf{A})$.

Given an algebra \mathbf{A} over Θ and a non-empty subset $F \subseteq A$, the pair $\langle \mathbf{A}, F \rangle$ is called a *logical matrix* [39]. The logic L defined by the matrix $\langle \mathbf{A}, F \rangle$ over $\mathcal{L}(\Theta)$ is given by the following consequence relation: for every set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}(\Theta)$,

$$\Gamma \vdash_L \varphi \text{ if, for all } e \in Hom(\mathcal{L}(\Theta), \mathbf{A}), e(\psi) \in F \text{ for all } \psi \in \Gamma \text{ implies } e(\varphi) \in F.$$

From now on, with no danger of confusion, given a logical matrix $\langle \mathbf{A}, F \rangle$ we will write $L = \langle \mathbf{A}, F \rangle$ to refer to the corresponding induced logic defined as above. We will also use the term *matrix logic* to refer a logic defined by a logical matrix.

Lemma 1. *Let $L_1 = \langle \mathbf{A}_1, F_1 \rangle$ and $L_2 = \langle \mathbf{A}_2, F_2 \rangle$ be two matrix logics defined over a signature Θ such that \mathbf{A}_2 is a subalgebra of \mathbf{A}_1 and $F_2 = F_1 \cap A_2$. Then $\vdash_{L_1} \subseteq \vdash_{L_2}$, that is: for every $\Gamma \cup \{\psi\}$, if $\Gamma \vdash_{L_1} \psi$ then $\Gamma \vdash_{L_2} \psi$.*

Proof. Assume that $\Gamma \vdash_{L_1} \psi$. Let $e \in Hom(\mathcal{L}(\Theta), \mathbf{A}_2)$ be an evaluation for L_2 such that $e[\Gamma] \subseteq F_2$. Let $\bar{e} : \mathcal{L}(\Theta) \rightarrow A_1$ such that $\bar{e}(\varphi) = e(\varphi)$ for every $\varphi \in \mathcal{L}(\Theta)$. Then $\bar{e} \in Hom(\mathcal{L}(\Theta), \mathbf{A}_1)$, so \bar{e} is an evaluation for L_1 such that $\bar{e}[\Gamma] \subseteq F_1$. By hypothesis, $\bar{e}(\psi) \in F_1$ and so $e(\psi) \in F_1 \cap A_2 = F_2$. This shows that $\Gamma \vdash_{L_2} \psi$. \square

After this previous lemma, we can state the main result on this section.

Theorem 1. *Let $L_1 = \langle \mathbf{A}_1, F_1 \rangle$ and $L_2 = \langle \mathbf{A}_2, F_2 \rangle$ be two distinct finite matrix logics over a same signature Θ such that \mathbf{A}_2 is a subalgebra of \mathbf{A}_1 and $F_2 = F_1 \cap A_2$. Assume the following:*

1. $A_1 = \{0, 1, a_1, \dots, a_k, a_{k+1}, \dots, a_n\}$ and $A_2 = \{0, 1, a_1, \dots, a_k\}$ are finite such that $0 \notin F_1$, $1 \in F_2$ and $\{0, 1\}$ is a subalgebra of \mathbf{A}_2 .
2. There are formulas $\top(p)$ and $\perp(p)$ in $\mathcal{L}(\Theta)$ depending at most on one variable p such that $e(\top(p)) = 1$ and $e(\perp(p)) = 0$, for every evaluation e for L_1 .
3. For every $k+1 \leq i \leq n$ and $1 \leq j \leq n$ (with $i \neq j$) there exists a formula $\alpha_j^i(p)$ in $\mathcal{L}(\Theta)$ depending at most on one variable p such that, for every evaluation e , $e(\alpha_j^i(p)) = a_j$ if $e(p) = a_i$.

Then, L_1 is maximal w.r.t. L_2 .

Proof. Let us begin by observing that the family of evaluations for L_1 which take values in A_2 for every propositional variable can be identified with the family of evaluations for L_2 .³

Notice that, by Lemma 1, $\vdash_{L_1} \subseteq \vdash_{L_2}$. Suppose that there is some formula $\varphi(p_1, \dots, p_m)$ such that $\vdash_{L_2} \varphi$ but $\not\vdash_{L_1} \varphi$ (otherwise the proof is done, by Remark 1). Then, $e(\varphi) \in F_2$ for every evaluation $e \in \text{Hom}(\mathcal{L}(\Theta), \mathbf{A}_2)$, but there is an homomorphism $e_0 \in \text{Hom}(\mathcal{L}(\Theta), \mathbf{A}_1)$ such that $e_0(\varphi) \notin F_1$. By the observation at the beginning of the proof (and by considering that $F_2 \subseteq F_1$), there exists a propositional variable p_i (for $1 \leq i \leq m$) such that $e_0(p_i) \notin A_2$. Consider now a substitution σ_0 such that

$$\sigma_0(p) = \begin{cases} \top(p_1) & \text{if } e_0(p) = 1, \\ \perp(p_1) & \text{if } e_0(p) = 0, \\ p_j & \text{if } e_0(p) = a_j \text{ (for } 1 \leq j \leq n) \end{cases}$$

and let $\gamma(p_1, \dots, p_n) = \sigma_0(\varphi)$. Observe that some of the variables p_j may not appear in γ , but at least one variable p_j (with $k+1 \leq j \leq n$) must occur in γ , by the hypothesis over e_0 . Now we can state two immediate facts:

Fact 1: Given an evaluation e for L_1 , if $e(p_j) \in A_2$ for every $1 \leq j \leq n$ then $e(\gamma) \in F_2$.

Proof: follows from the observation at the beginning of the proof, and by noting that γ is an instance of a tautology of L_2 .

Fact 2: Given an evaluation e for L_1 , if $e(p_j) = a_j$ for $1 \leq j \leq n$ then $e(\gamma) = e_0(\varphi) \notin F_1$.

Proof: Observe that, from the hypothesis, it follows that $e(\sigma_0(p_i)) = e_0(p_i)$ for every $1 \leq i \leq m$.

Now, for any propositional variable p , let $\alpha_j^i(p) = p$ for every $1 \leq j \leq n$, and let $\bigcirc(p)$ be the finite set of formulas

$$\bigcirc(p) = \{\gamma(\alpha_1^i(p), \dots, \alpha_n^i(p)) : k+1 \leq i \leq n\}.$$

Let e be an evaluation in L_1 . Observe the following:

(i) If $e(p) \in A_2$ then $e(\alpha_j^i(p)) \in A_2$ (since \mathbf{A}_2 is a subalgebra). For each $k+1 \leq i \leq n$ let e_i be an evaluation for L_1 such that $e_i(p_j) = e(\alpha_j^i(p))$, for every $1 \leq j \leq n$. Then $e_i(\gamma) \in F_2$, by Fact 1. But $e_i(\gamma) = e(\gamma(\alpha_1^i(p), \dots, \alpha_n^i(p)))$ and so $e(\gamma(\alpha_1^i(p), \dots, \alpha_n^i(p))) \in F_2$ for every $k+1 \leq i \leq n$. This means that $e[\bigcirc(p)] \subseteq F_1$ if $e(p) \in A_2$.

(ii) If $e(p) \notin A_2$ then $e(p) = a_i$ for some $k+1 \leq i \leq n$. From this, $e(\alpha_j^i(p)) = a_j$ for all $1 \leq j \leq n$. Let e' be an evaluation for L_1 such that $e'(p_j) = a_j$,

³This fact was used in the proof of Lemma 1.

for every $1 \leq j \leq n$. Then $e'(\gamma) = e(\gamma(\alpha_1^i(p), \dots, \alpha_n^i(p)))$. But, by Fact 2, $e'(\gamma) = e_0(\varphi) \notin F_1$ and so $e(\gamma(\alpha_1^i(p), \dots, \alpha_n^i(p))) \notin F_1$. Thus, $e[\bigcirc(p)] \not\subseteq F_1$ if $e(p) \notin A_2$. Equivalently, $e(p) \in A_2$ if $e[\bigcirc(p)] \subseteq F_1$.

From the observations (i) and (ii) it follows that

$$(*) \quad e[\bigcirc(p)] \subseteq F_1 \text{ iff } e(p) \in A_2.$$

Finally, let L_1^+ be the logic obtained from L_1 by adding φ (and all of its instances) as a theorem. As observed above,

$$\Gamma \vdash_{L_1^+} \psi \text{ iff } \Gamma, \{\sigma(\varphi) : \sigma \text{ is a substitution in } \mathcal{L}(\Theta)\} \vdash_{L_1} \psi.$$

Fact 3: Let $\Gamma \cup \{\psi\}$ be a finite a set of formulas in $\mathcal{L}(\Theta)$ depending on the variables p_1, \dots, p_t . Then

$$(**) \quad \Gamma \vdash_{L_2} \psi \text{ iff } \Gamma, \bigcirc(p_1), \dots, \bigcirc(p_t) \vdash_{L_1} \psi.$$

Proof: Assume that $\Gamma \vdash_{L_2} \psi$ and let $e \in \text{Hom}(\mathcal{L}(\Theta), \mathbf{A}_1)$ such that $e[\Gamma \cup \bigcup_{i=1}^t \bigcirc(p_i)] \subseteq F_1$. By (*), $e(p_i) \in A_2$ for every $1 \leq i \leq t$. Consider now an evaluation $\bar{e} \in \text{Hom}(\mathcal{L}(\Theta), \mathbf{A}_2)$ such that $\bar{e}(p) = e(p)$ if $p \in \{p_1, \dots, p_t\}$, and $\bar{e}(p) = 0$ otherwise. Then $\bar{e}(\beta) = e(\beta)$ for every β in $\mathcal{L}(\Theta)$ depending on the variables p_1, \dots, p_t . Thus, $\bar{e}[\Gamma] \subseteq F_1 \cap A_2 = F_2$ whence $\bar{e}(\psi) \in F_2$, by hypothesis. That is, $e(\psi) \in F_1$ and so $\Gamma, \bigcirc(p_1), \dots, \bigcirc(p_t) \vdash_{L_1} \psi$.

Conversely, assume that $\Gamma, \bigcirc(p_1), \dots, \bigcirc(p_t) \vdash_{L_1} \psi$ and consider an evaluation $\bar{e} \in \text{Hom}(\mathcal{L}(\Theta), \mathbf{A}_2)$ such that $\bar{e}[\Gamma] \subseteq F_2$. Define an evaluation $e \in \text{Hom}(\mathcal{L}(\Theta), \mathbf{A}_1)$ such that $e(p) = \bar{e}(p)$ for every variable p . Then $e(\beta) = \bar{e}(\beta)$ for every β in $\mathcal{L}(\Theta)$ and so $e[\Gamma] \subseteq F_1$ and also $e[\bigcirc(p_i)] \subseteq F_1$ for every $1 \leq i \leq t$, by (*). By hypothesis, $e(\psi) \in F_1$ and then $\bar{e}(\psi) \in F_1 \cap A_2$, that is, $\bar{e}(\psi) \in F_2$. This shows that $\Gamma \vdash_{L_2} \psi$, proving Fact 3.

Consider now a finite a set of formulas $\Gamma \cup \{\psi\}$ in $\mathcal{L}(\Theta)$ depending on the variables p_1, \dots, p_t . Suppose that $\Gamma \vdash_{L_2} \psi$. Then $\Gamma, \bigcirc(p_1), \dots, \bigcirc(p_t) \vdash_{L_1} \psi$, by Fact 3. But the latter implies that $\Gamma, \{\sigma(\varphi) : \sigma \text{ is a substitution in } \mathcal{L}(\Theta)\} \vdash_{L_1} \psi$, because each $\bigcirc(p_i)$ is a set of instances of φ . From this, it follows that $\Gamma \vdash_{L_1^+} \psi$, by definition of L_1^+ .

On the other hand, suppose that $\Gamma \vdash_{L_1^+} \psi$. Given that $\vdash_{L_1} \subseteq \vdash_{L_2}$ (by Lemma 1) and that $\vdash_{L_2} \varphi$ (by hypothesis) then $\Gamma \vdash_{L_2} \psi$, by definition of L_1^+ . This shows that L_1^+ coincides with L_2 and so L_1 is maximal w.r.t. L_2 . \square

In the next example we show an application of Theorem 1 in order to prove some maximality conditions for two logics related to the well-known 4-valued logic *FOUR* introduced by Belnap and Dunn [21, 5, 6].

Example 1. Consider Belnap-Dunn's matrix logic $\mathcal{BD} = \langle \mathfrak{M}_4, \{1, B\} \rangle$, where $\mathfrak{M}_4 = \langle M_4, \wedge, \vee, \neg \rangle$ is the algebra associated to the *logical lattice* M_4 (see Fig. 1) expanded with the De Morgan negation \neg defined as:

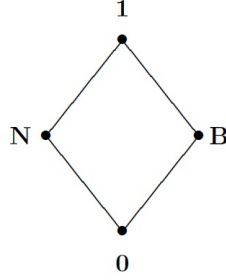


Figure 1: Lattice M_4 .

	\neg
1	0
B	B
N	N
0	1

Much later, De and Omori considered in [19] the expansion \mathcal{BD}^\sim of \mathcal{BD} by adding the strong negation \sim , given by the following table:

x	$\sim x$
0	1
N	B
B	N
1	0

On the other hand, before Belnap and Dunn's investigations, L. Monteiro already considered in 1963 (see [33]) the 4-valued algebra \mathfrak{M}_{4m} obtained from \mathfrak{M}_4 by adding a modal operator \square defined as follows:

	\square
1	1
B	0
N	0
0	0

This led to A. Monteiro to consider the variety **TMA** of *tetravalent modal algebras*, which is the one generated by \mathfrak{M}_{4m} (cf. [28]). As proven by Font and Rius in [23], the (degree-preserving) logic of **TMA** is characterized by the matrix logic $\mathcal{M}_B = \langle \mathfrak{M}_{4m}, \{B, 1\} \rangle$. Previous to [19] and with a different motivation, Coniglio and Figallo define in [16] the logic $\mathcal{M}_B^\sim = \langle \mathfrak{M}_{4m}^\sim, \{B, 1\} \rangle$, the expansion of \mathcal{M}_B with the strong negation \sim described above, characterizing the (degree-preserving) logic of the variety generated by \mathfrak{M}_{4m}^\sim (which was independently introduced by A. Monteiro in [32] and by G. Moisil in [31].)

By using Theorem 1, it is easy to show that both \mathcal{M}_B^\sim and \mathcal{BD}^\sim are maximal relative to CPL presented in the signature $\Theta = \{\wedge, \vee, \neg, \sim, \Box\}$ and $\Theta' = \{\wedge, \vee, \neg, \sim\}$ over the two-element Boolean algebra \mathfrak{B}_2 , respectively (where $\Box p$ is equivalent to p and $\neg p$ is equivalent to $\sim p$). Indeed, observe that \mathbf{B}_2 (expanded by \sim and \Box) is a subalgebra of \mathfrak{M}_{4m}^\sim , and $\top(p) = p \vee \sim p$ and $\perp(p) = p \wedge \sim p$ are as required. Notice that, since there are in M_4 just two values besides the ‘classical’ ones, namely $a_1 = N$ and $a_2 = B$, the formulas $\alpha_2^1(p) = \alpha_1^2(p) = \sim p$ are such that $e(\alpha_2^1(p)) = B$ if $e(p) = N$, $e(\alpha_1^2(p)) = N$ if $e(p) = B$. Therefore, it follows from Theorem 1 that \mathcal{M}_B^\sim is maximal relative to CPL presented over the signature Θ . Similarly, it also follows that \mathcal{BD}^\sim is maximal relative to CPL presented over the signature Θ' (the latter corresponding to [19, Theorem 3]). ■

As an immediate consequence of Theorem 1, it follows that any 3-valued logic which extends CPL and it can express the top and the bottom formulas, is maximal w.r.t. CPL.

Corollary 1. *Let \mathbf{A}_1 be an algebra defined over a signature Θ with domain $A_1 = \{0, 1/2, 1\}$, and consider the matrix logic $L_1 = \langle \mathbf{A}_1, F_1 \rangle$ where $0 \notin F_1$ and $1 \in F_1$. Further, let \mathbf{A}_2 be a subalgebra of \mathbf{A}_1 , with $A_2 = \{0, 1\}$, and assume that the matrix logic $L_2 = \langle \mathbf{A}_2, \{1\} \rangle$ is a presentation of classical propositional logic CPL over signature Θ such that L_2 is distinct from L_1 . Suppose additionally there are formulas $\top(p)$ and $\perp(p)$ in $\mathcal{L}(\Theta)$ on one variable p such that $e(\top(p)) = 1$ and $e(\perp(p)) = 0$, for every evaluation e for L_1 . Then, L_1 is maximal w.r.t. CPL (presented as L_2).*

Proof. Observe that L_1 and L_2 are matrix logics as in Lemma 1, since $\{1\} = F_1 \cap A_2$. Given that A_1 contains just one element out of $\{0, 1\}$, namely $a_1 = 1/2$, then Theorem 1 can be applied (since requirement (3) is satisfied by vacuity). As a consequence of Theorem 1, L_1 is maximal w.r.t. CPL (presented as L_2). □

In the next example some instances of Corollary 1 are analyzed, showing the strength of this result: indeed, several well-known 3-valued logics which are known to be maximal w.r.t. CPL fall inside the scope of Corollary 1.

Example 2. (1) Let us begin with Lukasiewicz 3-valued logic $\mathbf{L}_3 = \langle \mathbf{LV}_3, \{1\} \rangle$, where \mathbf{LV}_3 is the usual 3-valued algebra for \mathbf{L}_3 over $\Theta = \{\neg, \rightarrow\}$ with domain $\{0, 1/2, 1\}$. Let $\mathbf{L}_1^1 = \langle \mathbf{B}_2, F \rangle$ be a presentation of CPL, where \mathbf{B}_2 is the two-element Boolean algebra over Θ with domain $\{0, 1\}$ and $F = \{1\}$. It is easy to see that \mathbf{L}_3 satisfies the requirements of Corollary 1 by taking $\top(p) = (p \rightarrow p)$ and $\perp(p) = \neg(p \rightarrow p)$. This produces a new proof of the maximality of \mathbf{L}_3 w.r.t. CPL. In order to illustrate this fact consider by instance $\varphi(p_1) = p_1 \vee \neg p_1 := (p_1 \rightarrow \neg p_1) \rightarrow \neg p_1$, a formula which is valid in CPL but it is not valid in \mathbf{L}_3 . Indeed, any evaluation e_0 in \mathbf{L}_3 where $e_0(p_1) = 1/2$ is such that $e_0(\varphi) = 1/2$, a non-designated truth-value. By following the construction described in the proof of Theorem 1 (where $\alpha_1^1(p) = p$), it follows that $\gamma(p_1) = \varphi(p_1)$, and so $\circ(p) = p \vee \neg p$ is a recovery operator for \mathbf{L}_3 w.r.t. CPL defined in terms of φ .

Thus, \mathbf{L}_3 plus φ coincides with CPL. Notice that the truth-table of the recovery operator \circ is as follows:

	\circ
1	1
1/2	1/2
0	1

(2) Consider now the logic $\mathbf{L}_2^1 = \langle \mathbf{LV}_3, \{1, 1/2\} \rangle$. As it is well known, the matrices of \mathbf{L}_3 are functionally equivalent to that of the 3-valued paraconsistent logic \mathbf{J}_3 , introduced by da Costa and D'Ottaviano, see [20]. This means that \mathbf{L}_2^1 coincides with \mathbf{J}_3 up to language. By item (1) and Corollary 1 it follows that \mathbf{L}_2^1 is maximal w.r.t. CPL. This constitutes a new proof of the maximality of \mathbf{J}_3 (and all of its alternative presentations, such as **LF11** or **MPT**, see [17]) w.r.t. CPL. A generalization of \mathbf{J}_3 to \mathbf{L}_4 , called \mathbf{J}_4 , will be proposed in Subsection 6.2. As an illustration of how the technique of the proof works, let $\varphi(p_1) = \neg((\neg p_1 \rightarrow p_1) \wedge (p_1 \rightarrow \neg p_1))$. It is easy to see that $\varphi(p_1)$ is valid in CPL but it is not valid in \mathbf{L}_2^1 : any evaluation e_0 in \mathbf{L}_2^1 with $e_0(p_1) = 1/2$ is such that $e_0(\varphi) = 0$. Then, by the proof of Theorem 1 (where $\alpha_1^1(p) = p$), $\gamma(p_1) = \varphi(p_1)$ and so $\circ(p) = \neg((\neg p \rightarrow p) \wedge (p \rightarrow \neg p))$ is a recovery operator for \mathbf{L}_2^1 w.r.t. CPL defined in terms of instances of φ . This means that \mathbf{L}_2^1 plus φ coincides with CPL. The truth-table of \circ is as follows:

	\circ
1	1
1/2	0
0	1

(3) In an unpublished draft, J. Marcos [29] (see also [11, Section 5.3]) proposes a family of 8,192 logics which are 3-valued and paraconsistent, belonging to the hierarchy of **LFIs**. Among these logics, there is \mathbf{J}_3 (whose truth-tables can define the matrices of all the other logics in the family) and Sette's logic \mathbf{P}^1 (see [37]), whose truth-tables are definable by the matrices of any of the logics in the family. All these logics are maximal w.r.t. CPL presented in the signature $\{\wedge, \vee, \rightarrow, \neg, \circ\}$ such that $\circ\varphi$ is valid for every φ (that is, algebraically, $\circ(x) = 1$ for all $x \in \{0, 1\}$). The proof of maximality of all these logics w.r.t. CPL follows easily from Corollary 1 by taking $\top(p) = p \rightarrow p$ and $\perp(p) = p \wedge \neg p \wedge \circ p$.

(4) Let \mathbf{I}^1 be the 3-valued paracomplete logic introduced by A.M. Sette and W.A. Carnielli in [38]. It is defined over $\Theta = \{\rightarrow, \neg\}$ with domain $\{0, 1/2, 1\}$ and designated value 1, and whose operations are given by the tables below.

\rightarrow	1	1/2	0
1	1	0	0
1/2	1	1	1
0	1	1	1

	\neg
1	0
1/2	0
0	1

Once again, the maximality of \mathbf{I}^1 w.r.t. CPL follows from Corollary 1 by taking $\top(p) = p \rightarrow p$ and $\perp(p) = \neg(p \rightarrow p)$.

(5) Let $G_{n+1} = \langle \mathbf{G}_{n+1}, \{1\} \rangle$ be the $(n+1)$ -valued Gödel logic defined over the algebra \mathbf{G}_{n+1} for $\Theta = \{\wedge, \vee, \rightarrow, \neg\}$ with domain $\{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$ such that $x \wedge y = \min\{x, y\}$; $x \vee y = \max\{x, y\}$; $x \rightarrow y = 1$ if $x \leq y$ and $x \rightarrow y = y$ otherwise; and $\neg x = 1$ if $x = 0$ and $\neg x = 0$ otherwise. In particular, G_3 is defined over $\{0, 1/2, 1\}$ with the following tables for \rightarrow and \neg :

\rightarrow	1	1/2	0
1	1	1/2	0
1/2	1	1	0
0	1	1	1

	\neg
1	0
1/2	0
0	1

Clearly G_3 falls within the scope of Corollary 1 (where $\top(p) = p \rightarrow p$ and $\perp(p) = p \wedge \neg p$) and so it is maximal w.r.t. CPL presented over Θ . Observe that for $n \geq 3$ the algebra \mathbf{G}_{n+1} does not have enough expressive power to define all the formulas α_j^i in order to apply Theorem 1. For instance, in \mathbf{G}_4 there are no formulas $\alpha_2^1(p)$ and $\alpha_1^2(p)$ such that $e(\alpha_2^1(p)) = 2/3$ if $e(p) = 1/3$ and $e(\alpha_1^2(p)) = 1/3$ if $e(p) = 2/3$. ■

Example 3. In [3] the authors introduced the notion of *ideal paraconsistent logics*. Together with this, they presented a family \mathcal{M}_{n+2} of $(n+2)$ -valued matrix logics (with $n \geq 2$) which are ideal paraconsistent (and so, from the very definition, they are also maximal w.r.t. CPL, see Definition 4 in Section 6). The fact that all these logics are maximal w.r.t. CPL (as proved in [3]) can also be proved by applying Theorem 1, as it will be shown in what follows.

Given $n \geq 2$ consider the algebras \mathbf{A}_{n+2} over the signature $\Theta = \{\neg, \diamond, \supset\}$ with domain $A_{n+2} = \{0, 1, a_1, \dots, a_n\}$ such that the operations are defined as follows: $\neg 0 = 1$, $\neg 1 = 0$ and $\neg x = x$ otherwise; $\diamond 0 = 1$, $\diamond 1 = 0$, $\diamond a_i = a_{i+1}$ if $i < n$ and $\diamond a_n = a_1$; $x \supset y = 1$ if $x \notin D = \{1, a_1\}$ and $x \supset y = y$ otherwise. The logic \mathcal{M}_{n+2} is defined by the logical matrix $\langle \mathbf{A}_{n+2}, D \rangle$ for every $n \geq 2$. Let us see that the conditions of Theorem 1 are satisfied for every logic \mathcal{M}_{n+2} w.r.t. CPL. It is easy to see that $\{0, 1\}$ is a subalgebra of \mathbf{A}_{n+2} and so, by Lemma 1, \mathcal{M}_{n+2} is a sublogic of CPL presented in the signature Θ in which \diamond coincides with negation and 1 is the designated value. In addition, it is easy to see that, given a propositional variable p , the formulas $\top(p) = (p \supset \diamond p) \supset (p \supset \diamond p)$ and $\perp(p) = \neg \top(p)$ are such that $e(\top(p)) = 1$ and $e(\perp(p)) = 0$, for every evaluation e . Consider now the formulas $\alpha_j^i(p) = \diamond^{j-i} p$ if $i < j$ and $\alpha_j^i(p) = \diamond^{n-i+j} p$ if $i > j$, where $\diamond^0 p = p$ and $\diamond^{i+1} p = \diamond \diamond^i p$, for every i . An easy computation shows that $e(\alpha_j^i(p)) = a_j$ if $e(p) = a_i$, for every $i \neq j$. Therefore, the conditions of Theorem 1 are fulfilled and so each logic \mathcal{M}_{n+2} is maximal w.r.t. CPL.

The question of ideal paraconsistent logics in the present framework will be treated again in Section 6. ■

The examples given above show the value of Theorem 1 in order to state maximality of logics under certain hypothesis concerning the expressive power of the given logics. Indeed, several proofs of maximality found in the literature can be easily obtained as a consequence of Theorem 1: for instance, the ones given for the 3-valued paraconsistent logic \mathbf{P}^1 in [37, Proposition 11], for the 3-valued

logic I^1 in [38, Proposition 17] and for J_3 (formulated as the equivalent logic **LF1**) in [13, Theorem 4.6], respectively. It is worth noting that all the examples of maximality of a logic L_1 w.r.t. another logic L_2 given in this section, as well as the examples to be given in the rest of the paper, are non-vacuous in the sense of Remark 1. Indeed, in all the examples of maximality presented here the set of theorems of L_1 is strictly contained in the set of theorems of L_2 , thus the notion of maximality holds in a non-trivial way. For instance, the formula $p \rightarrow \circ p$ is a theorem of CPL which does not hold in any of the logics presented in Example 2(3), while the formula $p \rightarrow \neg \diamond p$ is a theorem of CPL which does not hold in any of the systems \mathcal{M}_{n+2} presented in Example 3. On the other hand, it should be observed that the set of designated values may not play a relevant role with respect to maximality, for instance, when analyzing maximality with respect to CPL (recall e.g. Corollary 1 or see Proposition 2 in next section).

As observed above, Theorem 1 cannot be applied to logics which do not have enough expressive power, as seen in Examples 2(5) for Gödel's logics G_n (with $n \geq 4$). This is not the case for finite-valued Łukasiewicz logics, as it will be shown in the next section.

3 Maximality between finite-valued Łukasiewicz logics induced by order filters

In the rest of the paper we will deal with matrix logics based on the family of finite-valued Łukasiewicz logics L_n with $n \geq 2$. The $(n+1)$ -valued Łukasiewicz logic can be semantically defined as the matrix logic

$$L_{n+1} = \langle \mathbf{LV}_{n+1}, \{1\} \rangle,$$

where $\mathbf{LV}_{n+1} = (LV_{n+1}, \neg, \rightarrow)$ with $LV_{n+1} = \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$, and the operations are defined as follows: for every $x, y \in LV_{n+1}$,

$$\neg x = 1 - x \text{ (Łukasiewicz negation)}$$

$$x \rightarrow y = \min\{1, 1 - x + y\} \text{ (Łukasiewicz implication)}$$

The following operations can be defined in every algebra \mathbf{LV}_{n+1} :

$$x \otimes y = \neg(x \rightarrow \neg y) = \max\{0, x + y - 1\} \text{ (strong conjunction)}$$

$$x \oplus y = \neg x \rightarrow y = \min\{1, x + y\} \text{ (strong disjunction)}$$

$$x \vee y = (x \rightarrow y) \rightarrow y = \max\{x, y\} \text{ (lattice disjunction)}$$

$$x \wedge y = \neg((\neg x \rightarrow \neg y) \rightarrow \neg y) = \min\{x, y\} \text{ (lattice conjunction)}$$

Observe that L_2 is the usual presentation of classical propositional logic CPL as a matrix logic over the two-element Boolean algebra \mathbf{B}_2 with domain $\{0, 1\}$ with signature $\{\neg, \rightarrow\}$.

The logics \mathbf{L}_n can also be presented as Hilbert calculi that are axiomatic extensions of the infinite-valued Łukasiewicz logic \mathbf{L}_∞ . Recall that \mathbf{L}_∞ is algebraizable and the class MV of all MV-algebras is its equivalent quasivariety semantics [36, 14]. Since algebraizability is preserved by finitary extensions then every finite valued Łukasiewicz logic \mathbf{L}_n is also algebraizable, and we will denote by MV_n its corresponding subvariety of algebras.

In this section, finite-valued Łukasiewicz logics with a set of designated values possibly different to $\{1\}$ will be studied from the point of view of maximality. First, some notation will be introduced.

For every $i \geq 1$ and for every $x \in \mathbf{LV}_{n+1}$, ix will stand for $x \oplus \dots \oplus x$ (i -times), while x^i will stand for $x \otimes \dots \otimes x$ (i -times).

For $1 \leq i \leq n$ let

$$F_{i/n} = \{x \in \mathbf{LV}_{n+1} : x \geq i/n\} = \left\{ \frac{i}{n}, \dots, \frac{n-1}{n}, 1 \right\}$$

be the order filter generated by i/n , and let

$$\mathbf{L}_n^i = \langle \mathbf{LV}_{n+1}, F_{i/n} \rangle$$

be the corresponding matrix logic. From now on, the consequence relation of \mathbf{L}_n^i is denoted by $\models_{\mathbf{L}_n^i}$. Observe that $\mathbf{L}_{n+1} = \mathbf{L}_n^n$ for every n . In particular, CPL is \mathbf{L}_1^1 (that is, \mathbf{L}_2). If $1 \leq i, m \leq n$, we can also consider the following matrix logic:

$$\mathbf{L}_m^{i/n} = \langle \mathbf{LV}_{m+1}, F_{i/n} \cap \mathbf{LV}_{m+1} \rangle.$$

Since $F_{i/n} \cap \mathbf{LV}_{m+1} = F_{j/m}$ for some $1 \leq j \leq m$, $\mathbf{L}_m^{i/n} = \mathbf{L}_m^j$ for that j . It is interesting to notice that some of these logics are paraconsistent, and some are not. Indeed, it is easy to prove the following characterization.

Proposition 1. *The logic \mathbf{L}_n^i is paraconsistent w.r.t. \neg iff $i/n \leq 1/2$.*

Proof. \mathbf{L}_n^i is paraconsistent w.r.t. \neg iff there exists $x \in \mathbf{LV}_{n+1}$ such that $x \geq i/n$ and $\neg x \geq i/n$, iff $i/n \leq x \leq (n-i)/n$ for some $x \in \mathbf{LV}_{n+1}$, iff $i/n \leq (n-i)/n$, iff $2i \leq n$. \square

Thus, for instance, for $n = 5$ it follows that \mathbf{L}_5^1 and \mathbf{L}_5^2 are paraconsistent, while \mathbf{L}_5^3 , \mathbf{L}_5^4 and $\mathbf{L}_5^5 = \mathbf{L}_6$ are explosive. By its turn, if $n = 3$ then \mathbf{L}_3^1 is paraconsistent, while \mathbf{L}_3^2 and $\mathbf{L}_3^3 = \mathbf{L}_4$ are explosive. The paraconsistent logics of this form which are maximal w.r.t. CPL will be analyzed with more detail in Section 6.

Theorem 1 can be used in order to analyze the maximality of the logic \mathbf{L}_n^i w.r.t. $\mathbf{L}_m^{i/n}$ whenever $m|n$ (taking into account that \mathbf{LV}_{m+1} is a subalgebra of \mathbf{LV}_{n+1} iff $m|n$). In particular, the maximality of certain instances of \mathbf{L}_n^i w.r.t. CPL can be obtained by using Theorem 1.

The following examples deal with the algebras \mathbf{LV}_n which, as observed above, can define a meet operator \wedge such that, for any order filter F , $(a \wedge b) \in F$ iff $a, b \in F$. Because of this, a recovery operator $\circ(p)$ will be considered instead of a recovery set $\bigcirc(p)$, consisting of the conjunction of all of its members.

Example 4. Let us first consider the case of \mathbf{LV}_4 . For $1 \leq i \leq 3$ let $\mathbf{L}_3^i = \langle \mathbf{LV}_4, F_{i/3} \rangle$. Then $F_{1/3} = \{1/3, 2/3, 1\}$, $F_{2/3} = \{2/3, 1\}$ and $F_{3/3} = F_1 = \{1\}$. As in the previous example, it can be proved that each \mathbf{L}_3^i satisfies the requirements of Theorem 1 w.r.t. CPL and so each \mathbf{L}_3^i is maximal w.r.t. CPL, presented as $\text{CPL} = \langle \mathbf{B}_2, \{1\} \rangle$. Indeed, \mathbf{B}_2 is a subalgebra of \mathbf{LV}_4 and $\top(p) = (p \rightarrow p)$ and $\perp(p) = \neg(p \rightarrow p)$ are as required. Finally, the formulas $\alpha_2^1(p) = \alpha_1^2(p) = \neg p$ are such that $e(\alpha_2^1(p)) = 2/3$ if $e(p) = 1/3$, $e(\alpha_1^2(p)) = 1/3$ if $e(p) = 2/3$. (Observe that there are in \mathbf{LV}_4 just two ‘non-classical’ values: $a_1 = 1/3$ and $a_2 = 2/3$.)

Fix $1 \leq i \leq 3$. Thus, given a theorem $\varphi(p_1, \dots, p_m)$ of CPL which is not valid in \mathbf{L}_3^i , consider the formula $\gamma(p_1, p_2)$ as in the proof of Theorem 1. Then, the formula $\circ(p) = \gamma(p, \neg p) \wedge \gamma(\neg p, p)$ defines an operator (in terms of a conjunction of instances of φ) which allows to recover classical logic inside \mathbf{L}_3^i . ■

From Komori’s characterization of axiomatic extensions of (infinite-valued) Lukasiewicz logic \mathbf{L}_∞ [27], it directly follows that the logic \mathbf{L}_{n+1} is maximal w.r.t. CPL iff n is a prime number. By adapting our previous arguments, we can obtain the following extension of this classical result for logics matrix logics over \mathbf{LV}_{n+1} with (almost) arbitrary filters.

Proposition 2. *Let $n \geq 2$ and $\emptyset \neq F \subseteq \mathbf{LV}_{n+1}$. Then, the logic $L = \langle \mathbf{LV}_{n+1}, F \rangle$ is maximal w.r.t. CPL provided that $0 \notin F$ and n is a prime number.*

Observe that, as a direct consequence, all the logics \mathbf{L}_q^i with q prime are maximal w.r.t. classical logic.

Corollary 2. *Let q be a prime number, and $1 \leq i \leq q$. Then, the logic \mathbf{L}_q^i is maximal w.r.t. CPL.*

Remark 2. Note that, for a given prime q , if $i < j$ the set of theorems of \mathbf{L}_q^j is strictly included in the set of theorems of \mathbf{L}_q^i . However this does not contradict the fact that both logics are maximal w.r.t. CPL, since their consequence relations are in fact incomparable. For example, the set of theorems of \mathbf{L}_3^2 is included in the set of theorems of \mathbf{L}_3^1 , but the inclusion is strict: $\models_{\mathbf{L}_3^1} (p \vee \neg p) \otimes (p \vee \neg p)$ while $\not\models_{\mathbf{L}_3^2} (p \vee \neg p) \otimes (p \vee \neg p)$. It suffices to consider an evaluation e such that $e(p) = 1/3$; then $e((p \vee \neg p) \otimes (p \vee \neg p)) = 1/3 \not\geq 2/3$. On the other hand, \mathbf{L}_3^2 is not a sublogic of \mathbf{L}_3^1 : $p \models_{\mathbf{L}_3^2} (p \otimes p) \oplus (p \otimes p)$ but $p \not\models_{\mathbf{L}_3^1} (p \otimes p) \oplus (p \otimes p)$. In order to see this, consider an evaluation e such that $e(p) = 1/3$; then $e((p \otimes p) \oplus (p \otimes p)) = 0$.

Next examples exploit the fact that \mathbf{L}_{n+1} is a sublogic of \mathbf{L}_{m+1} iff m divides n , considering additional filters as designated values, and obtaining maximality in some cases.

Example 5. Now, the logics associated to the algebra \mathbf{LV}_5 will be analyzed. For $1 \leq i \leq 4$ let $\mathbf{L}_4^i = \langle \mathbf{LV}_5, F_{i/4} \rangle$ such that $F_{1/4} = \{1/4, 1/2, 3/4, 1\}$, $F_{2/4} = F_{1/2} = \{1/2, 3/4, 1\}$, $F_{3/4} = \{3/4, 1\}$, and $F_{4/4} = F_1 = \{1\}$. Since 2 divides 4 then \mathbf{LV}_3 is a subalgebra of \mathbf{LV}_5 and \mathbf{L}_5 is a sublogic of \mathbf{L}_3 . We will prove that, indeed, any \mathbf{L}_4^i (for $1 \leq i \leq 4$) is maximal w.r.t. $\mathbf{L}_2^{i/4} = \langle \mathbf{LV}_3, F_{i/4} \cap \mathbf{LV}_3 \rangle$, by using Theorem 1.

By Lemma 1, each \mathbf{L}_4^i is a sublogic of $\mathbf{L}_2^{i/4}$. \mathbf{LV}_3 is a subalgebra of \mathbf{LV}_5 and $\top(p) = (p \rightarrow p)$ and $\perp(p) = \neg(p \rightarrow p)$ are as required. Let $a_1 = 1/2$, $a_2 = 1/4$ and $a_3 = 3/4$, and consider the formulas $\alpha_1^2(p) = p \oplus p$, $\alpha_3^2(p) = \alpha_2^3(p) = \neg p$, and $\alpha_1^3(p) = p \otimes p$. Finally, let $\alpha_i^i(p) = p$ for $i = 2, 3$. Then, the formulas α_j^i defined above are such that $e(\alpha_j^i(p)) = a_j$ if $e(p) = a_i$, for $i = 2, 3$ and $j = 1, 2, 3$.

Fix $1 \leq i \leq 4$. Thus, given a theorem $\varphi_i(p_1, \dots, p_{m_i})$ of $\mathbf{L}_2^{i/4}$ which is not valid in \mathbf{L}_4^i , consider the formula $\gamma_i(p_1, p_2, p_3)$ as in the proof of Theorem 1. Then, the formula

$$\circ_i(p) = \gamma_i(p \oplus p, p, \neg p) \wedge \gamma_i(p \otimes p, \neg p, p)$$

defines a recovery operator (in terms of a conjunction of instances of φ_i) which allows to recover $\mathbf{L}_2^{i/4}$ inside \mathbf{L}_4^i . This shows that the latter is maximal w.r.t. the former. ■

Example 6. Consider now the case of \mathbf{LV}_7 . Since 2 and 3 divide 6, it follows that \mathbf{LV}_3 and \mathbf{LV}_4 are subalgebras of \mathbf{LV}_7 and so $L = \langle \mathbf{LV}_7, F \rangle$ is a sublogic of both $\langle \mathbf{LV}_3, F \cap \mathbf{LV}_3 \rangle$ and $\langle \mathbf{LV}_4, F \cap \mathbf{LV}_4 \rangle$ for any non-trivial filter F of \mathbf{LV}_7 . However, it is not possible to prove the maximality of L by applying Theorem 1 since, for every formula $\alpha(p)$ and every evaluation e in \mathbf{LV}_7 , $e(\alpha(p)) \neq 1/2$ if $e(p) \in \{1/3, 2/3\}$ (since \mathbf{LV}_4 is a subalgebra), while $e(\alpha(p)) \notin \{1/3, 2/3\}$ if $e(p) = 1/2$ (since \mathbf{LV}_3 is a subalgebra). ■

As another example of application of Theorem 1, we can obtain the following maximality condition of a logic \mathbf{L}_n^i with respect to a logic $\mathbf{L}_m^{i/n}$.

Proposition 3. *Let $1 \leq i, m \leq n$. Then $\mathbf{L}_n^i = \langle \mathbf{LV}_{n+1}, F_{i/n} \rangle$ is maximal w.r.t. $\mathbf{L}_m^{i/n} = \langle \mathbf{LV}_{m+1}, F_{i/n} \cap \mathbf{LV}_{m+1} \rangle$ if the following condition holds: there is some prime number q and $k \geq 1$ such that $n = q^k$, and $m = q^{k-1}$.*

Proof. We recall that \mathbf{LV}_{n+1} is singly generated by any element $0 < \frac{l}{n} < 1$ such that l and n are mutually prime [24, Lemma 1.2]. Then, since q is prime, $\mathbf{LV}_{q^{k+1}} \setminus \mathbf{LV}_{q^{k-1}+1} = \{0 < \frac{r}{q^k} < 1 : r \text{ and } q \text{ are mutually prime}\}$ and therefore all conditions of Theorem 1 are satisfied. □

4 On strong maximality and explosion rules in the logics \mathbf{L}_q^i

Along this section q will denote a prime number.

In the previous section we have seen that all the logics of the form $\mathbf{L}_q^i = \langle \mathbf{LV}_{q+1}, F_{i/q} \rangle$ are maximal w.r.t. CPL. However, there are maximal logics that are not maximal w.r.t. CPL in an stronger sense, as firstly considered in [4, 3] in the context of paraconsistent logics, or in [35] in a more general context of belief revision techniques for change of logics.

Definition 2. Let L_1 and L_2 two standard propositional logics defined over the same signature Θ such that L_1 is a proper sublogic of L_2 , i.e. such that $\vdash_{L_1} \subsetneq \vdash_{L_2}$. Then, L_1 is said to be *strongly maximal* w.r.t. L_2 if, for every finitary rule $\varphi_1, \dots, \varphi_n / \psi$ over Θ , if $\varphi_1, \dots, \varphi_n \vdash_{L_2} \psi$ but $\varphi_1, \dots, \varphi_n \not\vdash_{L_1} \psi$, then the logic L_1^* obtained from L_1 by adding $\varphi_1, \dots, \varphi_n / \psi$ as structural rule, coincides with L_2 .

By L_1^* above we mean the logic whose consequence relation $\vdash_{L_1^*}$ is the minimal extension of \vdash_{L_1} such that $\sigma(\varphi_1), \dots, \sigma(\varphi_n) \vdash_{L_1^*} \sigma(\psi)$ for any substitution σ over Θ (see e.g. [39, 3]).

For instance, as observed in [19, Remark 14], the logic \mathcal{BD}^\sim introduced in Section 2, that is maximal w.r.t. CPL, it is not strongly maximal relative to CPL. Thus, a natural question is whether a given logic is strongly maximal w.r.t. another logic. In particular, in this section, we are interested in studying the status of the logics $L_q^i = \langle \mathbf{LV}_{q+1}, F_{i/q} \rangle$ with q prime in relation to the notion of strong maximality w.r.t. CPL. We will show that the answer is negative, as each of them admits a proper extension by a finitary rule related to the explosion law w.r.t. Lukasiewicz negation. In fact, in Section 5.2 it will be shown that such proper extensions are strongly maximal w.r.t. CPL.

Remark 3. By using the techniques presented in [15], a sound and complete Hilbert calculus for each L_q^i (where $i < q$) can be defined from the one for L_{q+1}^{\leq} (the degree-preserving counterpart of L_{q+1}) by adding additional inference rules. The negative feature of such approach is that these Hilbert calculi have “global” inference rules, that is, inference rules such that one of its permises need to be a theorem of L_{q+1} . By a general result by Blok and Pigozzi (see Theorem 4.3 in [8]) and from Theorem 2 in Section 5 below, a standard Hilbert calculus for L_q^i (for $i < q$) can be obtained from the usual one for $L_q^q = L_{q+1}$ by means of translations. That is, such calculi have no “global” inference rules. The negative side of this approach is that the resulting axiomatization is obtained by translating connectives from the other logic, and so the resulting calculus can appear as very artificial. As an alternative, it seems that a direct method for defining a sound and complete “more natural” Hilbert calculus for each L_q^i over a suitable signature can also be obtained by means of a ‘separation’ technique for the truth-values, similar to the one used in Subection 6.2 to define an alternative axiomatization for L_3^1 . To verify this conjecture is left as an open problem. Anyway, from now on it will be assumed the existence of a standard Hilbert calculus H_q^i which is sound and complete for the logic L_q^i , where $i < q$. Of course H_q^q will stand for the usual axiomatization of L_{q+1} .

According to the notation introduced at the beginning of Section 3, $i\alpha$ is an abbreviation for the formula $\alpha \oplus \dots \oplus \alpha$ (i -times), and the consequence relation of L_n^i is denoted by $\vdash_{L_n^i}$. Recall the following basic property of \mathbf{LV}_{n+1} .

Lemma 2. *For every $1 \leq i \leq n$ and $x \in \mathbf{LV}_{n+1}$: $ix < i/n$ iff $x = 0$. Thus, $e(i\alpha) < i/n$ iff $e(\alpha) = 0$ for every evaluation e in \mathbf{LV}_{n+1} and every formula α .*

From now on, \perp will denote any formula of the form $\neg(p \rightarrow p)$, for a propositional variable p . Observe that $e(\perp) = 0$ for every evaluation in \mathbf{LV}_{n+1} ,

every $n \geq 1$ and every propositional variable p . This is why the choice of p is inessential for a concrete construction of \perp .

Consider, for $1 \leq i \leq q$, the i -explosion law

$$(exp_i) \frac{i(\varphi \wedge \neg\varphi)}{\perp}.$$

It is not hard to prove that this rule is not derivable in any H_q^i , the sound and complete Hilbert calculus given for the logic L_q^i (see Remark 3).

Corollary 3. *For every $1 \leq i \leq q$, the rule (exp_i) is not derivable in H_q^i .*

Proof. We first observe that if p, p' are two different propositional variables, then $i(p \wedge \neg p) \not\vdash_{L_q^i} p'$ for $1 \leq i \leq n$. Indeed, let e be an evaluation in \mathbf{LV}_{n+1} such that $e(p) \notin \{0, 1\}$ and $e(p') = 0$. Since $e(p \wedge \neg p) \neq 0$ then $e(i(p \wedge \neg p)) \geq i/n$, by Lemma 2. Hence, $i(p \wedge \neg p) \not\vdash_{L_q^i} p'$. Finally, the corollary then follows from soundness and completeness of H_q^i w.r.t. L_q^i . \square

However, the i -explosion rule is clearly admissible in H_q^i since it is a *passive* rule, that is: for no instance of the (exp_i) rule, the premisses can be a theorem of H_q^i . Indeed, for any classical evaluation over $\{0, 1\}$ it is the case that $e(\varphi) \in \{0, 1\}$ for every formula φ and so $e(i(\varphi \wedge \neg\varphi)) < i/n$, by Lemma 2. This leads us to consider the following definition.

Definition 3. \bar{H}_q^i is the Hilbert calculus obtained from H_q^i by adding the i -explosion rule (exp_i) . We will denote by $\vdash_{H_q^i}$ and $\vdash_{\bar{H}_q^i}$ the notions of proof associated to the Hilbert calculi H_q^i and \bar{H}_q^i , respectively.

The following is a characterization of $\vdash_{\bar{H}_q^i}$ in terms of $\vdash_{H_q^i}$.

Proposition 4. *Let $\Gamma \cup \{\varphi\}$ be a set of formulas. Then $\Gamma \vdash_{\bar{H}_q^i} \varphi$ iff either $\Gamma \vdash_{H_q^i} \varphi$, or $\Gamma \vdash_{H_q^i} i(\psi \wedge \neg\psi)$ for some formula ψ .*

Proof. ‘Only if’ part: Suppose that $\Gamma \vdash_{\bar{H}_q^i} \varphi$ such that $\Gamma \not\vdash_{H_q^i} \varphi$. Then, any derivation in \bar{H}_q^i of φ from Γ must use the rule (exp_i) . Let $\varphi_1 \dots \varphi_n$ be a derivation in \bar{H}_q^i of φ from Γ . Thus, there exists $1 \leq m < n$ such that $\varphi_m = i(\psi \wedge \neg\psi)$ for some formula ψ , allowing so the first application of (exp_i) in the given derivation. This means that $\Gamma \vdash_{H_q^i} i(\psi \wedge \neg\psi)$, since it was assumed that (exp_i) was not applied before φ_m in the given derivation.

‘If’ part: Suppose that $\Gamma \vdash_{H_q^i} \varphi$. Then, clearly $\Gamma \vdash_{\bar{H}_q^i} \varphi$. Now, suppose that $\Gamma \vdash_{H_q^i} i(\psi \wedge \neg\psi)$ for some formula ψ . Then $\Gamma \vdash_{\bar{H}_q^i} \perp$, by using (exp_i) . But $\perp \vdash_{L_q^i} \varphi$ and so $\perp \vdash_{H_q^i} \varphi$, by completeness of H_q^i w.r.t. L_q^i . This means that $\Gamma \vdash_{\bar{H}_q^i} \varphi$. \square

The following question is how to characterize semantically the logic \bar{H}_q^i with respect to L_q^i , the original semantics for H_q^i . The answer will be obtained in the

next section by algebraic arguments (Theorem 7 and Remark 4). Indeed, it will be shown there that \bar{H}_q^i is sound and complete w.r.t. \bar{L}_q^i where, for every i and n with $1 \leq i \leq n$,

$$\bar{L}_n^i = \langle \mathbf{L}\mathbf{V}_{n+1} \times \mathbf{L}\mathbf{V}_2, F_{i/n} \times \{1\} \rangle$$

such that $\mathbf{L}\mathbf{V}_2$ is the two-element Boolean algebra \mathbf{B}_2 with domain $\{0, 1\}$.

5 Translations, equivalent logics and strong maximality

5.1 Preliminaries

Blok and Pigozzi introduce the notion of equivalent deductive systems in [8] (see also [9]). Two propositional deductive systems S_1 and S_2 in the same language \mathcal{L} are equivalent iff there are two translations τ_1, τ_2 (finite subsets of \mathcal{L} -propositional formulas in one variable) such that:

- $\Gamma \vdash_{S_1} \varphi$ iff $\tau_1(\Gamma) \vdash_{S_2} \tau_1(\varphi)$,
- $\Delta \vdash_{S_2} \psi$ iff $\tau_2(\Delta) \vdash_{S_1} \tau_2(\psi)$,
- $\varphi \dashv\vdash_{S_1} \tau_2(\tau_1(\varphi))$,
- $\psi \dashv\vdash_{S_2} \tau_1(\tau_2(\psi))$.

From very general results stated in [8] it follows that two equivalent logic systems are indistinguishable from the point of view of algebra, provided that one of them is algebraizable. Indeed, in such case if one of the systems is algebraizable then the other will be also algebraizable w.r.t. the same quasivariety. By applying this fact to the systems of the form L_n^i studied in the previous sections, several results on relative maximality between these systems and classical logic will be obtained in the next Subsection 5.2. Actually, these results will be generalized in Subsection 5.3 to obtain relative maximality results among the systems L_n^i . However, for the sake of self containment, we prefer to leave the results of Subsection 5.2 with their simpler proofs as well.

In the rest of this subsection, we provide the necessary preliminaries that will be needed in the subsequent subsections.

We recall that L_∞ is algebraizable and the class MV of all MV-algebras is its equivalent quasivariety semantics [36, 14]. Since algebraizability is preserved by finitary extensions then every finite valued Łukasiewicz logic is also algebraizable.

Now we can prove that the deductive systems L_n^i and L_n^j are in fact equivalent in the above sense. First, observe that, by the McNaughton functional representation theorem [30], for every $n \geq 2$ and every $1 \leq m \leq n$ there is an MV-term $\lambda_{m,n}(p)$ such that for every $a \in [0, 1]$,

$$\lambda_{m,n}(a) = \begin{cases} 0, & \text{if } a \leq \frac{m-1}{n}; \\ na - (m-1), & \text{if } \frac{m-1}{n} < a < \frac{m}{n}; \\ 1, & \text{if } \frac{m}{n} \leq a. \end{cases}$$

Lemma 3. *The restrictions of the $\lambda_{i,m}$ and $\lambda_{n,n}$ functions on \mathbf{LV}_{n+1} are the characteristic functions of the order filters $F_{i/n}$ and F_1 respectively, i.e. for each $a \in \mathbf{LV}_{n+1}$,*

$$\lambda_{i,n}(a) = \begin{cases} 0, & \text{if } a < \frac{i}{n} \\ 1, & \text{if } a \geq \frac{i}{n} \end{cases} \quad \lambda_{n,n}(a) = a^n = \begin{cases} 0, & \text{if } a < 1 \\ 1, & \text{if } a = 1 \end{cases}$$

Theorem 2. *For every $n \geq 2$ and every $1 \leq i, j \leq n$, \mathbf{L}_n^i and \mathbf{L}_n^j are equivalent deductive systems.*

Proof. It is enough to prove that for every $1 \leq i \leq n-1$, \mathbf{L}_n^i is equivalent to $\mathbf{L}_n^n = \mathbf{L}_{n+1}$. Let the translations τ and σ be given by $\tau = \{\lambda_{i,n}(p)\}$ and $\sigma = \{\lambda_{n,n}(p)\}$. It is easy to check that for every set of formulas $\Gamma \cup \{\varphi\}$,

$$\begin{aligned} \Gamma \vDash_{\mathbf{L}_n^i} \varphi & \text{ iff } \{\tau(\psi) : \psi \in \Gamma\} \vDash_{\mathbf{L}_n^n} \tau(\varphi) \\ \Gamma \vDash_{\mathbf{L}_n^n} \varphi & \text{ iff } \{\sigma(\psi) : \psi \in \Gamma\} \vDash_{\mathbf{L}_n^i} \sigma(\varphi) \\ \varphi \not\vDash_{\mathbf{L}_n^i} \sigma(\tau(\varphi)) & \text{ and } \varphi \not\vDash_{\mathbf{L}_n^n} \tau(\sigma(\varphi)). \end{aligned}$$

Thus, \mathbf{L}_n^i and $\mathbf{L}_n^n = \mathbf{L}_{n+1}$ are equivalent deductive systems. \square

From the equivalence among \mathbf{L}_n^i and \mathbf{L}_{n+1} , we can obtain, by translating the axiomatization of the finite valued Łukasiewicz logic \mathbf{L}_{n+1} , a calculus sound and complete with respect \mathbf{L}_n^i that we denote by \mathbf{H}_n^i (see [8, Theorem 4.3]).

Since \mathbf{L}_∞ is algebraizable and the class MV of all MV-algebras is its equivalent quasivariety semantics, finitary extensions of \mathbf{L}_∞ are in 1 to 1 correspondence with quasivarieties of MV-algebras. Actually, there is a dual isomorphism from the lattice of all finitary extensions of \mathbf{L}_∞ and the lattice of all quasivarieties of MV . Moreover, if we restrict this correspondence to varieties of MV we get the dual isomorphism from the lattice of all varieties of MV and the lattice of all axiomatic extensions of \mathbf{L}_∞ . Since $\mathbf{L}_{n+1} = \mathbf{L}_n^n$ is an axiomatic extension of \mathbf{L}_∞ , \mathbf{L}_{n+1} is an algebraizable logic with the class $MV_n = \mathcal{Q}(\mathbf{LV}_{n+1})$, the quasivariety generated by \mathbf{LV}_{n+1} , as its equivalent variety semantics. It follows from the previous theorem and from [8] that \mathbf{L}_n^i , for every $1 \leq i \leq n$, is also algebraizable with the same class of MV_n -algebras as its equivalent variety semantics. Thus, the lattices of all finitary extensions of \mathbf{L}_n^i are isomorphic, and in fact, dually isomorphic to the lattice of all subquasivarieties of MV_n , for all $0 < i < n$.

Therefore maximality conditions in the lattice of finitary (axiomatic) extensions correspond to minimality conditions in the lattice of subquasivarieties (subvarieties). Thus, given two finitary extensions L_1 and L_2 of a given logic \mathbf{L}_n^i , where K_{L_1} and K_{L_2} are its associated MV_n -quasivarieties, L_1 is strongly maximal with respect L_2 iff K_{L_1} is a minimal subquasivariety of MV_n among those MV_n -quasivarieties properly containing K_{L_2} . Moreover, if L_1 and L_2 are axiomatic extensions of \mathbf{L}_n^i , then K_{L_1} and K_{L_2} are indeed MV_n -varieties. In that case, L_1 is maximal with respect L_2 iff K_{L_1} is a minimal subvariety of MV_n among those MV_n -varieties properly containing K_{L_2} .

All the axiomatic extensions of \mathbf{L}_∞ are characterized by Komori in [27], where it is shown that every axiomatic extension is finitely axiomatizable and depends only on two finite sets of natural numbers I, J not both empty. Moreover, Panti proved in [34] that every axiomatic extension can be axiomatized relative to \mathbf{L}_∞ by a single axiom $\gamma_{I,J}$ with a single propositional variable. For the case of finite valued Lukasiewicz logics, Komori's characterization depends on just a finite set of natural numbers in the following sense: given $n > 1$, every axiomatic extension of \mathbf{L}_{n+1} is of the form

$$\bigcap_{1 \leq j \leq k} \mathbf{L}_{m_j+1}$$

for some natural number k where $m_j|n$ for every $1 \leq j \leq k$. Moreover, from the equivalence of Theorem 2, it follows that every axiomatic extension of \mathbf{L}_n^i is of the form

$$\bigcap_{1 \leq j \leq k} \mathbf{L}_{m_j}^{i/n}$$

for some natural number k where $m_j|n$ for every $1 \leq j \leq k$, and it is axiomatized by a single axiom $\gamma_{m_1, \dots, m_k}^{i/n}$ which depends on one variable. We denote by $\mathbf{H}_{m_1, \dots, m_k}^{i/n}$ the calculus obtained from \mathbf{H}_n^i by adding the axiom $\gamma_{m_1, \dots, m_k}^{i/n}$. Note that for every $m \geq 1$ such that $m|n$, the calculus $\mathbf{H}_m^{i/n}$ is the same logic as \mathbf{H}_m^j , where j is the natural number such that $F_{j/m} = F_{i/n} \cap \mathbf{L}V_m$.

The lattice of all axiomatic extensions \mathbf{L}_∞ is fully described also by Komori in [27], thus from the equivalence of Theorem 2, we can obtain the following maximality conditions for all axiomatic extensions of \mathbf{L}_n^i .

Theorem 3. *Let $0 < i, m \leq n$ be natural numbers such that $m|n$. If L is an axiomatic extension of \mathbf{L}_n^i , then*

- *L is maximal with respect to $\mathbf{L}_m^{i/n}$ iff $L = \mathbf{L}_m^{i/n} \cap \mathbf{L}_{q^{k+1}}^{i/n}$ for some prime number q with $q|n$ and a natural $k \geq 0$ such that $q^k|m$ and $q^{k+1} \nmid m$.*

Proof. Using the equivalence of Theorem 2 we obtain that the lattice of axiomatic extensions of \mathbf{L}_n^i is isomorphic to the lattice of axiomatic extensions of \mathbf{L}_{n+1} . As mentioned above, every axiomatic extension of \mathbf{L}_{n+1} is characterized by a finite set $\{m_1, \dots, m_k\}$ where all of its elements are divisors of n . Given two such sets $\{m_1, \dots, m_k\}$ and $\{n_1, \dots, n_s\}$, we define the following relation among finite subsets of divisors of n : $\{m_1, \dots, m_k\} \preceq \{n_1, \dots, n_s\}$ iff for every $1 \leq i \leq k$ there is $1 \leq j \leq s$ such that $m_i|n_j$. This relation \preceq is the dual order of the lattice of axiomatic extensions of \mathbf{L}_{n+1} in the following sense: $\{m_1, \dots, m_k\} \preceq \{n_1, \dots, n_s\}$ iff $\bigcap_{1 \leq j \leq s} \mathbf{L}_{n_j+1} \leq \bigcap_{1 \leq i \leq k} \mathbf{L}_{m_i+1}$. Clearly,

$\{m\} \preceq \{m, q\}$ and $\{m, q\} \not\preceq \{m\}$ if q is a prime number such that $q|n$ and $q \nmid m$; Similarly, $\{m\} \preceq \{m, q^{k+1}\}$ and $\{m, q^{k+1}\} \not\preceq \{m\}$ if q is a prime number such that $q|n$, $q^k|m$ and $q^{k+1} \nmid m$. Moreover if $\{m\} \preceq \{m_1, \dots, m_k\}$ and $\{m_1, \dots, m_k\} \not\preceq \{m\}$, then there is m_i such that $m|m_i$. If $m \neq m_i$ then there

is a prime number q such that $mq|m_i|n$. Thus $\{m, q\} \preceq \{m_1, \dots, m_k\}$ if $q \nmid m$ and $\{m, q^{k+1}\} \preceq \{m_1, \dots, m_k\}$ if $q^k|m$ and $q^{k+1} \nmid m$. If $m = m_i$, then there is an m_j with $1 < j < k$ such that $m_j \nmid m$. If there is a prime number q such that $q|m_j|n$ such that $q \nmid m$, then $\{m, q\} \preceq \{m_1, \dots, m_k\}$. Otherwise, there are a prime number q and a natural $k > 0$ such that $q^{k+1}|m_j|n$, $q^k|m$ and $q^{k+1} \nmid m$, then $\{m, q^{k+1}\} \preceq \{m_1, \dots, m_k\}$. Duality and Theorem 2 close the proof. \square

As a corollary we obtain that the sufficient condition of Proposition 3 is also necessary.

Corollary 4. *Let $1 \leq i, m \leq n$. Then $L_n^i = \langle \mathbf{LV}_{n+1}, F_{i/n} \rangle$ is maximal w.r.t. $L_m^{i/n} = \langle \mathbf{LV}_{m+1}, F_{i/n} \cap LV_{m+1} \rangle$ if and only if there is some prime number q and $k \geq 1$ such that $n = q^k$, and $m = q^{k-1}$.*

The task of fully describing the lattice of all finitary extensions of L_∞ , isomorphic to the lattice of all subquasivarieties of MV , turns to be an heroic task since the class of all MV -algebras is Q -universal (see [1]). For the finite valued case it is much simpler, since MV_n is a locally finite discriminator variety (cf. [7, 25]). Any locally finite quasivariety is generated by its critical algebras (see [22]), where an algebra A is said to be *critical* iff it is a finite algebra not belonging to the quasivariety generated by all its proper subalgebras. A description of all critical MV -algebras can be found in [25].

Theorem 4. [25, Theorem 2.5] *An MV -algebra A is critical if and only if A is isomorphic to a finite MV -algebra $\mathbf{LV}_{n_0+1} \times \dots \times \mathbf{LV}_{n_{l-1}+1}$ satisfying the following conditions:*

1. For every $i, j < l$, $i \neq j$ implies $n_i \neq n_j$.
2. If there exists n_j , $j < l$ such that $n_i|n_j$ for some $i \neq j$, then n_j is unique.

Moreover the following result characterizes the inclusion among locally finite quasivarieties.

Lemma 4. [25, Lemma 2.9] *Let $\mathfrak{F} = \{\mathbf{LV}_{n_{i_1}+1} \times \dots \times \mathbf{LV}_{n_{i_l(i)}+1} : i \in I\}$ and $\mathfrak{G} = \{\mathbf{LV}_{m_{j_1}+1} \times \dots \times \mathbf{LV}_{m_{j_l(j)}+1} : j \in J\}$ be two finite families of critical MV -algebras. Then it holds that*

$$\mathcal{Q}(\mathfrak{F}) \subseteq \mathcal{Q}(\mathfrak{G})$$

if, and only if, for every $i \in I$ there exists a non-empty $H \subseteq J$ such that:

1. For any $1 \leq k \leq l(i)$ there are $j \in H$ and $1 \leq r \leq l(j)$ such that $n_{i_k}|m_{j_r}$.
2. For any $j \in H$ and $1 \leq r \leq l(j)$ there exists $1 \leq k \leq l(i)$ such that $n_{i_k}|m_{j_r}$.

5.2 Strong maximality among logics \mathbb{L}_q^i , $\bar{\mathbb{L}}_q^i$, and classical logic

As a direct application of Lemma 4, we have the following particular case that will be used later.

Corollary 5. *Consider the following two sets of one critical MV-algebra each: $\{\mathbf{LV}_{q+1} \times \mathbf{LV}_2\}$ and $\{\mathbf{LV}_{k+1}\}$, where q is a prime number such that $q > 1$. Then*

$$\mathcal{Q}(\{\mathbf{LV}_{q+1} \times \mathbf{LV}_2\}) \subseteq \mathcal{Q}(\mathbf{LV}_{k+1})$$

if and only if $q|k$.

Proof. The two families of critical algebras above correspond in Lemma 4 to take $I = \{1\}$ and $J = \{1\}$, with $n_{11} = q$, $n_{12} = 1$, $m_{11} = k$. Then one can check that these values satisfy the two conditions of the lemma only in the case that $q|k$. \square

Now, for any $k > 1$, we are able to provide a full description of the minimal subquasivarieties of $MV_k = \mathcal{Q}(\mathbf{LV}_{k+1})$ strictly containing the variety of Boolean algebras.

Theorem 5. *Let $k > 1$. The set of all minimal subquasivarieties of $MV_k = \mathcal{Q}(\mathbf{LV}_{k+1})$ among those strictly containing the class of all the Boolean algebras $\mathbf{B} = \mathcal{Q}(\mathbf{LV}_2)$ is*

$$M^k = \{\mathcal{Q}(\mathbf{LV}_{q+1} \times \mathbf{LV}_2) : q > 1 \text{ prime, } q|k\}.$$

Proof. By Lemma 4 and the previous Corollary 5, every $K \in M^k$ is a subquasivariety of $\mathcal{Q}(\mathbf{LV}_{k+1})$ strictly containing \mathbf{B} . Moreover, for every $K_1, K_2 \in M^k$, if $K_1 \neq K_2$ then $K_1 \not\subseteq K_2$ and $K_2 \not\subseteq K_1$.

On the other hand, let K be a minimal subquasivariety of $\mathcal{Q}(\mathbf{LV}_{k+1})$ strictly containing \mathbf{B} . Since $K \neq \mathbf{B}$, it must contain a critical algebra C that, by Theorem 4, it must be such that $C \cong \mathbf{LV}_{m_1+1} \times \cdots \times \mathbf{LV}_{m_s+1}$, where $m_i|k$ for every $1 \leq i \leq s$, and $m_j > 1$ for some $1 \leq j \leq s$. Hence, for every prime number q such that $q|m_j$, and hence $q|k$, we have $\mathbf{LV}_{q+1} \times \mathbf{LV}_2 \in \mathcal{Q}(C) \subseteq K$, and thus $\mathcal{Q}(\mathbf{LV}_{q+1} \times \mathbf{LV}_2) \subseteq K$. Since we are assuming the minimality of K , it must be $\mathcal{Q}(\mathbf{LV}_{q+1} \times \mathbf{LV}_2) = K$. \square

Theorem 6. *If $q > 0$ is a prime number, then $\mathcal{Q}(\mathbf{LV}_{q+1} \times \mathbf{LV}_2)$ is axiomatized by the MV quasi-identities plus:*

- $\gamma_q(x) \approx 1$ (the identity axiomatizing $\mathcal{V}(\mathbf{LV}_{q+1})$)
- $q(x \wedge \neg x) \approx 1 \Rightarrow y \vee \neg y \approx 1$

Proof. It is easy to check that $\mathbf{LV}_{q+1} \times \mathbf{LV}_2$ satisfies these two quasi-identities. Since the MV-identities and $\gamma_q(x) \approx 1$ axiomatize $\mathcal{V}(\mathbf{LV}_{q+1})$, and $\mathcal{V}(\mathbf{LV}_{q+1})$ is a locally finite quasivariety, it is enough to prove that every critical MV-algebra

$C \in \mathcal{V}(\mathbf{LV}_{q+1})$ where the quasi-equation $q(x \wedge \neg x) \approx 1 \Rightarrow y \vee \neg y \approx 1$ holds, belongs to $\mathcal{Q}(\mathbf{LV}_{q+1} \times \mathbf{LV}_2)$.

Let C be a critical MV-algebra satisfying the axiomatization, then C is such that $C \cong \mathbf{LV}_{m_1+1} \times \cdots \times \mathbf{LV}_{m_k+1}$ satisfying conditions of Theorem 4. Moreover, every $1 \leq i \leq k$, is such that either $m_i = 1$ or $m_i = q$ because \mathbf{LV}_{m_i+1} belongs to $\mathcal{V}(\mathbf{LV}_{q+1})$. If there is $c \in C$ such that $q(c \wedge \neg c) = 1$ then, by the second quasi-equation of the above axiomatization, $b \vee \neg b \approx 1$ for any $b \in C$. Thus we have $C \in \mathbf{B} \subseteq \mathcal{Q}(\mathbf{LV}_{q+1} \times \mathbf{LV}_2)$. Otherwise, recalling that either $m_i = 1$ or $m_i = q$ for every i , if for every $c \in C$ one has $q(c \wedge \neg c) \neq 1$ then $m_i = 1$ for some $1 \leq i \leq k$. In that case, by the characterization of critical algebras (Theorem 4), we have $C \cong \mathbf{LV}_2$ or $C \cong \mathbf{LV}_{q+1} \times \mathbf{LV}_2$. If $C \cong \mathbf{LV}_2$, then trivially $C \in \mathcal{Q}(\mathbf{LV}_{q+1} \times \mathbf{LV}_2)$. If $C \cong \mathbf{LV}_{q+1} \times \mathbf{LV}_2$, then clearly $C \in \mathcal{Q}(\mathbf{LV}_{q+1} \times \mathbf{LV}_2)$. \square

Above, note that the identity $y \vee \neg y \approx 1$ corresponds to the previously mentioned Panti's axiom $\gamma_{I,J}(y)$, with $I = \{1\}$ and $J = \emptyset$, axiomatizing CPL as an axiomatic extension of \mathbf{L}_{n+1} for any $n > 1$.

Finally, we obtain the following characterization result about strong maximality of logics $\bar{\mathbf{L}}_q^j$ with respect to classical logic.

Theorem 7. *Let $q > 1$ be a prime number. Then, for every j such that $0 < j \leq q$:*

- $\bar{\mathbf{L}}_q^j$ is strongly maximal with respect to CPL and it is axiomatized by \mathbf{H}_q^j plus the rule $j(\varphi \wedge \neg \varphi) / (\psi \vee \neg \psi)^q$.
- \mathbf{L}_q^j is strongly maximal w.r.t. $\bar{\mathbf{L}}_q^j$.

Proof. By using the equivalence of Theorem 2 and the algebraizability of \mathbf{L}_{q+1} , the lattice of subquasivarieties of $\mathcal{V}(\mathbf{LV}_{q+1})$ is dually order isomorphic to the lattice of all finitary extensions of \mathbf{L}_{q+1} . Clearly $\text{CPL} = \mathbf{L}_2$ is the finitary extension of $\bar{\mathbf{L}}_q^j$ corresponding to the subvariety $\mathcal{Q}(\mathbf{LV}_2)$ of $\mathcal{Q}(\mathbf{LV}_{q+1} \times \mathbf{LV}_2)$, and $\bar{\mathbf{L}}_q^j$ is the finitary extension of \mathbf{L}_q^j corresponding to the subquasivariety $\mathcal{Q}(\mathbf{LV}_{q+1} \times \mathbf{LV}_2)$ of $\mathcal{V}(\mathbf{LV}_{q+1})$.

By Theorem 4, the only critical algebras of $\mathcal{V}(\mathbf{LV}_{q+1})$ are \mathbf{LV}_{q+1} , \mathbf{LV}_2 and $\mathbf{LV}_2 \times \mathbf{LV}_{q+1}$ and, by Lemma 4, all its subquasivarieties are $\mathcal{Q}(\mathbf{LV}_2) \subsetneq \mathcal{Q}(\mathbf{LV}_2 \times \mathbf{LV}_{q+1}) \subsetneq \mathcal{Q}(\mathbf{LV}_{q+1})$. Therefore, by Theorem 5, $\bar{\mathbf{L}}_q^j$ is strongly maximal with respect to CPL, while \mathbf{L}_q^j is strongly maximal with respect to $\bar{\mathbf{L}}_q^j$.

Finally, the axiomatization of $\bar{\mathbf{L}}_q^j$ follows from Theorem 6 and the facts that $j\varphi \models_{\bar{\mathbf{L}}_q^j} q\varphi$ holds for every formula φ and that the equation $(qx)^q = qx$ is valid in the class MV_q . \square

From the above proof, it readily follows the next corollary.

Corollary 6. $\bar{\mathbf{L}}_q^j$ is the unique strongly maximal logic w.r.t. CPL above \mathbf{L}_q^j . In fact, $\bar{\mathbf{L}}_q^j$ is the only logic between \mathbf{L}_q^j and CPL.

Remark 4. It is worth noting that the rule $j(\varphi \wedge \neg\varphi)/(\psi \vee \neg\psi)^q$ exactly corresponds to the explosion rule (exp_j) introduced in Section 4. Indeed, the rule $j(\varphi \wedge \neg\varphi)/(\psi \vee \neg\psi)^q$ is clearly derivable from (exp_j). On the other hand, assuming $j(\varphi \wedge \neg\varphi)$, by this rule it follows that $(\psi \vee \neg\psi)^q$ for every ψ . Hence the logic becomes CPL because the translation of the classical axiom $\psi \vee \neg\psi$ is precisely $(\psi \vee \neg\psi)^q$, and thus \perp follows from $j(\varphi \wedge \neg\varphi)$. This does not come as a surprise, since as we have proved above, \mathbb{L}_q^j is strongly maximal w.r.t. $\bar{\mathbb{L}}_q^j$ and so the latter is the only proper extension of \mathbb{L}_q^j (with a finitary rule) properly contained in CPL.

As a corollary of the previous remark, it follows the completeness of $\bar{\mathbb{H}}_q^j$.

Corollary 7. $\bar{\mathbb{H}}_q^j$ is sound and complete w.r.t. $\bar{\mathbb{L}}_q^j$.

5.3 Strong maximality with respect to systems \mathbb{L}_n^i

Next theorems are generalizations of Theorems 5, 6 and 7 respectively.

Theorem 8. Let $n > 0$ and $k > 1$. The set of all minimal subquasivarieties of $MV_{nk} = \mathcal{Q}(\mathbf{LV}_{nk+1})$ among those strictly containing $\mathcal{Q}(\mathbf{LV}_{n+1})$ is

$$M_n^{nk} = \left\{ \mathcal{Q}(\{\mathbf{LV}_{n+1}, \mathbf{LV}_{q+1} \times \mathbf{LV}_2\}) : q \text{ prime, } q|k \text{ and } q \nmid n \right\} \cup \left\{ \mathcal{Q}(\{\mathbf{LV}_{n+1}, \mathbf{LV}_{q^{r+1}+1} \times \mathbf{LV}_2\}) : q \text{ prime, } q|k, q^r|n \text{ and } q^{r+1} \nmid n \right\}.$$

Proof. By Lemma 4, every $K \in M_n^{nk}$ is a subquasivariety of $\mathcal{Q}(\mathbf{LV}_{nk+1})$ strictly containing $\mathcal{Q}(\mathbf{LV}_{n+1})$. Moreover, for every $K_1, K_2 \in M_n^{nk}$, if $K_1 \neq K_2$ then $K_1 \not\subseteq K_2$ and $K_2 \not\subseteq K_1$.

Let K be a minimal subquasivariety of $\mathcal{Q}(\mathbf{LV}_{nk+1})$ strictly containing $\mathcal{Q}(\mathbf{LV}_{n+1})$. Trivially, $\mathbf{LV}_{n+1} \in K$. Since $K \neq \mathcal{Q}(\mathbf{LV}_{n+1})$, then it must contain a critical algebra $C \cong \mathbf{LV}_{m_1+1} \times \cdots \times \mathbf{LV}_{m_s+1}$ such that $m_i|nk$ for every $1 \leq i \leq s$ and $m_j \nmid n$ for some $1 \leq j \leq s$. If there is a prime number $q|m_j$ such that $q \nmid n$, then $\mathbf{LV}_{q+1} \times \mathbf{LV}_2 \in \mathcal{Q}(C) \subseteq K$. Otherwise, there is a prime q such that $q|m_j$ and $q^r|n$, and for some $r \geq 1$, $q^{r+1} \nmid n$ and $q^{r+1}|m_j$, whence $\mathbf{LV}_{q^{r+1}+1} \times \mathbf{LV}_2 \in \mathcal{Q}(C) \subseteq K$. Thus, in both cases K contains some $K_i \in M_n^{nk}$, from which it follows that $K \in M_n^{nk}$ since we are assuming minimality of K . \square

Theorem 9. For every $n > 0$.

- If q is a prime number such that $q \nmid n$, then $\mathcal{Q}(\{\mathbf{LV}_{n+1}, \mathbf{LV}_{q+1} \times \mathbf{LV}_2\})$ is axiomatized by the MV identities plus
 - $\gamma_{\{n,q\},\emptyset}(x) \approx 1$ (the identity axiomatizing $\mathcal{V}(\{\mathbf{LV}_{n+1}, \mathbf{LV}_{q+1}\})$)
 - $nq(x \wedge \neg x) \approx 1 \Rightarrow \gamma_{\{n\},\emptyset}(y) \approx 1$.
- If q is a prime number such that $q^r|n$ and $q^{r+1} \nmid n$, $\mathcal{Q}(\{\mathbf{LV}_{n+1}, \mathbf{LV}_{q^{r+1}+1} \times \mathbf{LV}_2\})$ is axiomatized by the MV identities plus
 - $\gamma_{\{n,q^{r+1}\},\emptyset}(x) \approx 1$ (the identity axiomatizing $\mathcal{V}(\{\mathbf{LV}_{n+1}, \mathbf{LV}_{q^{r+1}+1}\})$)

$$- nq(x \wedge \neg x) \approx 1 \Rightarrow \gamma_{\{n\}, \emptyset}(y) \approx 1.$$

Proof. We prove the first item, the other is proved in a analogous way. It is easy to check that \mathbf{LV}_{n+1} and $\mathbf{LV}_{q+1} \times \mathbf{LV}_2$ satisfy all the quasi-identities. Since the MV-identities with $\gamma_{\{n,q\}, \emptyset}(x) \approx 1$ axiomatize $\mathcal{V}(\{\mathbf{LV}_{n+1}, \mathbf{LV}_{q+1}\})$ and $\mathcal{V}(\{\mathbf{LV}_{n+1}, \mathbf{LV}_{q+1}\})$ is a locally finite quasivariety, it is enough to prove that every critical MV-algebra $C \in \mathcal{V}(\{\mathbf{LV}_{n+1}, \mathbf{LV}_{q+1}\})$ where the quasiequation $nq(x \wedge \neg x) \approx 1 \Rightarrow \gamma_{\{n\}, \emptyset}(y) \approx 1$ holds, belongs to $\mathcal{Q}(\{\mathbf{LV}_{n+1}, \mathbf{LV}_{q+1} \times \mathbf{LV}_2\})$. Therefore, let C be a critical MV-algebra satisfying the axiomatization. Then, C is such that $C \cong \mathbf{LV}_{m_1+1} \times \cdots \times \mathbf{LV}_{m_r+1}$ satisfying conditions of Theorem 4, and moreover for every $1 \leq i \leq k$, either $m_i | n$ or $m_i = q$ because $C \in \mathcal{V}(\{\mathbf{LV}_{n+1}, \mathbf{LV}_{q+1}\})$. If there is $c \in C$ such that $nq(c \wedge \neg c) = 1$ then, by the second quasi-equation of the axiomatization above, $\gamma_{\{n\}, \emptyset}(b) \approx 1$ for any $b \in C$, thus $C \in \mathcal{V}(\{\mathbf{LV}_{n+1}\}) = \mathcal{Q}(\{\mathbf{LV}_{n+1}\}) \subseteq \mathcal{Q}(\{\mathbf{LV}_{n+1}, \mathbf{LV}_{q+1} \times \mathbf{LV}_2\})$. If for every $c \in C$, $nq(c \wedge \neg c) \neq 1$ then $m_i = 1$ for some $1 \leq i \leq k$. In that case, by the characterization of critical algebras (Theorem 4), either $C \cong \mathbf{LV}_2$ or $C \cong \mathbf{LV}_{m+1} \times \mathbf{LV}_2$. If $C \cong \mathbf{LV}_2$, then trivially $C \in \mathcal{Q}(\{\mathbf{LV}_{n+1}, \mathbf{LV}_{q+1} \times \mathbf{LV}_2\})$. Otherwise, if $C \cong \mathbf{LV}_{m+1} \times \mathbf{LV}_2$, since $C \in \mathcal{V}(\{\mathbf{LV}_{n+1}, \mathbf{LV}_{q+1}\})$, either $m | n$ or $m = q$. If $m | n$ then $C \in \mathcal{V}(\{\mathbf{LV}_{n+1}\}) = \mathcal{Q}(\{\mathbf{LV}_{n+1}\}) \subseteq \mathcal{Q}(\{\mathbf{LV}_{n+1}, \mathbf{LV}_{q+1} \times \mathbf{LV}_2\})$. If $m = q$ then $C \cong \mathbf{LV}_{q+1} \times \mathbf{LV}_2 \in \mathcal{Q}(\{\mathbf{LV}_{n+1}, \mathbf{LV}_{q+1} \times \mathbf{LV}_2\})$. \square

If $1 \leq i, m \leq n$, by analogy with $\mathbf{L}_m^{i/n}$, we define the matrix logic

$$\bar{\mathbf{L}}_m^{i/n} = \langle \mathbf{LV}_{m+1} \times \mathbf{LV}_2, (F_{i/n} \cap \mathbf{LV}_{m+1}) \times \{1\} \rangle.$$

Then we have the following generalization of Theorem 7.

Theorem 10. *Let $0 < i \leq n$ be natural numbers and let q be a prime number. Then we have:*

- *If $q \nmid n$ then, for every j such that $(i-1)q < j \leq iq$, $\mathbf{L}_n^i \cap \bar{\mathbf{L}}_q^{j/nq}$ is strongly maximal with respect to \mathbf{L}_n^i , and it is axiomatized by $\mathbf{H}_{n,q}^{j/nq}$ plus the rule $j(\varphi \wedge \neg \varphi) / \gamma_n^{j/nq}(\psi)$.*
- *If $q^r | n$ and $q^{r+1} \nmid n$ then, for every j such that $(i-1)q < j \leq iq$, $\mathbf{L}_n^i \cap \bar{\mathbf{L}}_{q^{r+1}}^{j/nq}$ is strongly maximal with respect to \mathbf{L}_n^i , and it is axiomatized by $\mathbf{H}_{n,q^{r+1}}^{j/nq}$ plus the rule $j(\varphi \wedge \neg \varphi) / \gamma_n^{j/nq}(\psi)$.*

Recall that in the above rules $\gamma_n^{j/nq}(\psi)$ refers to the axiom in one variable that axiomatizes $\mathbf{L}_n^{j/nq}$ relative to \mathbf{L}_{nq}^j . Moreover, every finitary extension of some \mathbf{L}_k^j is strongly maximal with respect to \mathbf{L}_n^i iff it is of one of the two preceding types.

Proof. Notice that $\mathbf{L}_n^i = \mathbf{L}_n^{j/nq}$ for every j such that $(i-1)q < j \leq iq$. Thus, \mathbf{L}_n^i is an extension of \mathbf{L}_{nq}^j . Now, by using the equivalence of Theorem 2 and the algebraizability of $\mathbf{L}_{nq} + 1$, the lattice of subquasivarieties of $\mathcal{V}(\mathbf{LV}_{nq+1})$ is dually order isomorphic to the lattice of all the finitary extensions of \mathbf{L}_{nq}^j . Moreover, $\mathbf{L}_n^i \cap \bar{\mathbf{L}}_q^{j/nq}$ and $\mathbf{L}_n^i \cap \bar{\mathbf{L}}_{q^{r+1}}^{j/nq}$ are the finitary extensions of \mathbf{L}_{nq}^j associated

to $\mathcal{Q}(\{\mathbf{LV}_{n+1}, \mathbf{LV}_{q+1} \times \mathbf{LV}_2\})$ and $\mathcal{Q}(\{\mathbf{LV}_{n+1}, \mathbf{LV}_{q^{r+1}+1} \times \mathbf{LV}_2\})$, respectively. Hence, they are strongly maximal with respect to \mathbf{L}_n^i , by Theorem 8. The axiomatization follows from Theorem 9 and the facts that $j \varphi \dashv\vdash_{\mathbf{L}_{nq}^j} nq \varphi$ holds for every formula φ and that the equation $(nqx)^{nq} = nqx$ is valid in the class MV_{nq} .

Finally, the last statement of this theorem follows from Theorem 8 and Theorem 2. \square

6 An application to ideal paraconsistent logics

As mentioned in Example 3, Arieli et al. have introduced in [3] the concept of *ideal paraconsistent logics*. We recall here this notion.

Definition 4 (c.f. [3]). Let L be a propositional logic defined over a signature Θ (with consequence relation \vdash_L) containing at least a unary connective \neg and a binary connective \rightarrow such that:

- (i) L is paraconsistent w.r.t. \neg (or simply \neg -paraconsistent), that is, there are formulas $\varphi, \psi \in \mathcal{L}(\Theta)$ such that $\varphi, \neg\varphi \not\vdash_L \psi$;
- (ii) \rightarrow is an implication for which the deduction-detachment theorem holds in L , that is, $\Gamma \cup \{\varphi\} \vdash_L \psi$ iff $\Gamma \vdash_L \varphi \rightarrow \psi$, for every set for formulas $\Gamma \cup \{\varphi, \psi\} \subseteq \mathcal{L}(\Theta)$.
- (ii) There is a presentation of CPL as a matrix logic $L' = \langle \mathbf{A}, \{1\} \rangle$ over the signature Θ such that the domain of \mathbf{A} is $\{0, 1\}$, and \neg and \rightarrow are interpreted as the usual 2-valued negation and implication of CPL, respectively.
- (iv) L is a sublogic of CPL in the sense that $\vdash_L \subseteq \vdash_{L'}$, that is, $\Gamma \vdash_L \varphi$ implies $\Gamma \vdash_{L'} \varphi$, for every set for formulas $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}(\Theta)$.

Then, L is said to be an *ideal paraconsistent logic* if it is maximal w.r.t. L' , and every proper extension of L over Θ is not \neg -paraconsistent.

An implication connective satisfying the above condition (ii) will be called *deductive implication* in the rest of the paper.⁴

Thus, a \neg -paraconsistent logic L with a deductive implication is ideal if it is maximal w.r.t. CPL (presented over the signature Θ of L) and, if L'' is another logic over Θ properly containing L , with $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}(\Theta)$ such that $\Gamma \vdash_{L''} \varphi$ but $\Gamma \not\vdash_L \varphi$, then the logic obtained from L by adding Γ/φ as an inference rule is not \neg -paraconsistent.

As already noticed, the logics \mathbf{L}_n^i with $i/n \leq 1/2$ are paraconsistent. In this section, using the results of the previous sections, we study the status of the logics \mathbf{L}_n^i in relation to ideal paraconsistency. Namely, in the following subsection, we will show that the logics of the form \mathbf{L}_q^i , where q is prime and $i/q \leq 1/2$ are ideal paraconsistent, while in subsection 6.2 the special case of \mathbf{L}_3^1 , renamed as \mathbf{J}_4 , is analyzed in more detail.

⁴Such an implication is called *deductive* in [11, 16] and *proper* in [3].

6.1 The ideal paraconsistent logics \mathbf{L}_q^i

By combining Proposition 1 with Corollary 2 we know a logic \mathbf{L}_q^i is \neg -paraconsistent and maximal w.r.t. CPL, provided that q is prime and $i/q \leq 1/2$. From now on we will assume this is the case when referring to a logic \mathbf{L}_q^i .

Recall that $\overline{\mathbf{H}}_q^i$ is the Hilbert calculus obtained from the calculus \mathbf{H}_q^i for \mathbf{L}_q^i by adding the i -explosion rule (exp_i). Since $\varphi \wedge \neg\varphi \vdash_{\mathbf{H}_q^i} i(\varphi \wedge \neg\varphi)$, the logic $\overline{\mathbf{H}}_q^i$ is explosive. Then, taking into account Corollary 6, it follows that every proper extension of \mathbf{L}_q^i defined over its signature is either $\overline{\mathbf{L}}_q^i$ or CPL, and hence not \neg -paraconsistent.

In addition, by Lemma 3, we know there is a definable unary connective \sim_q^i such that, for every evaluation e , $e(\sim_q^i p) = 0$ if $e(p) \geq i/q$, and $e(\sim_q^i p) = 1$ otherwise, for every propositional variable p .⁵ This is a kind of “classical” negation defined on \mathbf{L}_q^i . Using this negation, one can define in turn a new implication \Rightarrow_q^i by stipulating $\varphi \Rightarrow_q^i \psi = \sim_q^i \varphi \vee \psi$. In fact, one can easily check that \Rightarrow_q^i is a deductive implication on \mathbf{L}_q^i in the sense of Definition 4 and that over $\{0, 1\}$ it coincides with the classical implication. All the above considerations lead to the following result.

Proposition 5. *Let q is a prime number, and let $1 \leq i < q$ such that $i/q \leq 1/2$. Then, \mathbf{L}_q^i is a $(q + 1)$ -valued ideal paraconsistent logic.⁶*

Therefore we have a large family of examples of ideal paraconsistent logics. In particular, for each prime q , all the logics in the set $PC_{q+1} = \{\mathbf{L}_q^i : i < q/2\}$ are $(q + 1)$ -valued ideal paraconsistent logics. Moreover, if we consider “the more theorems a paraconsistent logic has, the more well-behaved is the logic” as a valid further criterion, then we can still refine the set PC_{q+1} . Indeed, if we denote by $Th(L)$ the set of theorems of a logic L then, as noticed in Remark 2, we have the strict inclusions $Th(\mathbf{L}_q^i) \subsetneq Th(\mathbf{L}_q^j) \subsetneq Th(\mathbf{CPL})$ whenever $i > j$. Therefore the logic $\mathbf{J}_{q+1} = \mathbf{L}_q^1$ appears to be the “best” ideal logic in the set PC_{q+1} ,⁷ since it is the logic in that set having the biggest set of theorems from classical logic.

Finally, it is worth mentioning that all the paraconsistent logics of the form \mathbf{L}_n^i are, indeed, **LFI**s (recall Section 2):

Proposition 6. *Suppose that $i/n \leq 1/2$. Then, the logic \mathbf{L}_n^i is an **LFI** w.r.t. \neg and where the consistency operator is defined as $\circ\alpha = \sim_n^i(\alpha \wedge \neg\alpha)$.*

Proof. Straightforward. □

⁵Namely, $\sim_q^i p = \neg\lambda_{i,q}(p)$.

⁶Strictly speaking, in this claim we implicitly assume that the signature of \mathbf{L}_q^i has been changed by adding the definable implication \Rightarrow_q^i as a primitive connective.

⁷We have chosen the name \mathbf{J}_{q+1} to denote the logic \mathbf{L}_q^1 inspired in the 3-valued case, where the ideal paraconsistent logic \mathbf{J}_3 coincides with \mathbf{L}_2^1 .

6.2 The four-valued ideal paraconsistent logic J_4

As mentioned in Remark 3, we know from Theorem 4.3 in [8] that it is possible to obtain a standard (that is, without “global” inference rules) Hilbert calculus for a logic L_n^i for $i < n$ from the usual one for L_{n+1} by using translations. However, the calculi obtained in this manner can lack an intuitive meaning since they are defined in terms of the implication connective \rightarrow of L_{n+1} , that is naturally associated to the filter $F_1 = \{1\}$ but not to the filter $F_{i/n} = \{i/n, \dots, 1\}$, which is the one at work in L_n^i . Actually, the implication naturally associated to the filter $F_{i/n}$ is \Rightarrow_n^i , which was considered above, for which modus ponens (MP) and the deduction-detachment theorem hold.

In this section we focus on the particular case of the (ideal paraconsistent) logic $J_4 = L_3^1$. J_4 can be considered as a generalization to four values of the paraconsistent 3-valued logic J_3 introduced by da Costa and D’Ottaviano in [20] and briefly mentioned in Example 2. For this logic a more natural signature Σ will be considered for describing it axiomatically in terms of a deductive implication connective (in the sense of Definition 4 item (ii)) and a unary connective $*$ representing the square operation $x \otimes x$, which can be seen as a kind of ‘truth stresser’ (see e.g. [26]). A soundness and completeness result for this calculus proved by using a ‘separation’ technique for truth-values will be presented. Note that dealing with logics $J_q = L_q^1$ for a prime $q > 3$ appears to be much more complicated, and certainly it lies outside the scope of this paper.

The signature Σ that will be used in the rest of the section is given by two unary connectives $*$ (square) and \neg (negation), plus a binary connective \vee for disjunction. Abusing the notation, we formally define next J_4 over this signature, and we will show later that it is an equivalent presentation of L_3^1 .

Definition 5. J_4 is the matrix logic $\langle \mathbf{A}_4, F_{1/3} \rangle$ over Σ , where the algebra is $\mathbf{A}_4 = (\mathbf{LV}_4, \vee, \neg, *)$, with operations defined by the tables below:

\vee	1	2/3	1/3	0
1	1	1	1	1
2/3	1	2/3	2/3	2/3
1/3	1	2/3	1/3	1/3
0	1	2/3	1/3	0

	\neg	$*$
1	0	1
2/3	1/3	1/3
1/3	2/3	0
0	1	0

Observe that \neg is Łukasiewicz negation in \mathbf{LV}_4 , while $*x = x \otimes x$ (with \otimes being Łukasiewicz strong conjunction) and \vee is the lattice join in \mathbf{LV}_4 .

In this signature Σ the following derived connectives can be defined (as usual, the corresponding operators will be denoted using the same symbol):

- $\Delta(p) = **p$;
- $\sim p = \Delta(\neg p)$;
- $p \Rightarrow r = \sim p \vee r$;
- $p \Leftrightarrow r = (p \Rightarrow r) \wedge (r \Rightarrow p)$;
- $p \wedge r = \neg(\neg p \vee \neg r)$;
- $\nabla(p) = \neg\sim p$;
- $\alpha_{1/3}(p) = \nabla(p) \wedge \sim *p$;
- $\beta_{1/3}(p) = \alpha_{1/3}(p) \wedge * \neg p$.

It is easy to see that Δ is Monteiro-Baaz *Delta-operator* and \sim is Gödel negation ($\sim x = 1$ if $x = 0$, and 0 otherwise). Note that \sim actually coincides with $\sim_{\frac{1}{3}}$, and thus \Rightarrow is nothing but $\Rightarrow_{\frac{1}{3}}$. Furthermore, $\nabla(x) = 0$ if $x = 0$, and 1 otherwise; $\alpha_{1/3}(x) = 1$ and $\beta_{1/3}(x) = 1/3$ if $x = 1/3$, and 0 otherwise.

It is worth to remark that Łukasiewicz implication is definable from these operators in the following way:

$$p \rightarrow r = ((\nabla(\neg p) \vee r) \wedge (\neg p \vee \nabla(r))) \wedge \neg \beta_{1/3}(r) \vee ((\sim p \wedge \alpha_{1/3}(r)) \vee (\alpha_{1/3}(p) \wedge \alpha_{1/3}(r))).$$

Then, the following result follows easily:

Proposition 7. *The algebras \mathbf{LV}_4 and \mathbf{A}_4 are functionally equivalent.*

This means that the proposed operators over Σ constitute an alternative presentation of the algebra \mathbf{LV}_4 underlying \mathbf{L}_4 . Next we define an axiomatic system for \mathbf{J}_4 .

Definition 6. The Hilbert calculus \mathbf{H}_4 for the logic \mathbf{J}_4 , defined over the signature Σ , is given as follows:

Axiom schemas: those of CPL over the signature $\{\vee, \Rightarrow, \sim\}$ plus

- (Ax1) $\neg\sim\alpha \Rightarrow \alpha$
- (Ax2) $\alpha \vee \neg\alpha$
- (Ax3) $\neg\neg\alpha \Leftrightarrow \alpha$
- (Ax4) $\neg(\alpha \vee \beta) \Rightarrow \neg\alpha$
- (Ax5) $\neg(\alpha \vee \beta) \Rightarrow \neg\beta$
- (Ax6) $\neg\alpha \Rightarrow (\neg\beta \Rightarrow \neg(\alpha \vee \beta))$
- (Ax7) $*\alpha \Rightarrow \alpha$
- (Ax8) $*(\alpha \vee \neg\alpha)$
- (Ax9) $*\alpha \Rightarrow \sim * \neg\alpha$
- (Ax10) $**\alpha \Leftrightarrow \sim \neg\alpha$
- (Ax11) $\neg * \alpha \Leftrightarrow \neg\alpha$
- (Ax12) $*(\alpha \vee \beta) \Leftrightarrow (*\alpha \vee *\beta)$

Inference rule:

$$(MP) \frac{\alpha \quad \alpha \Rightarrow \beta}{\beta}$$

Observe that, since (MP) is the only inference rule, H_4 satisfies the deduction-detachment theorem w.r.t. the implication \Rightarrow : $\Gamma \cup \{\alpha\} \vdash_{H_4} \beta$ iff $\Gamma \vdash_{H_4} \alpha \Rightarrow \beta$, for every set of formulas $\Gamma \cup \{\alpha, \beta\}$. On the other hand, it can be proved that $*(\alpha \Rightarrow \beta) \Rightarrow (*\alpha \Rightarrow *\beta)$ is derivable in H_4 , which gives additional support to consider $*$ as a truth stresser. Soundness of H_4 can be proved straightforwardly.

Proposition 8 (Soundness of H_4). *The calculus H_4 is sound w.r.t. J_4 , that is: $\Gamma \vdash_{H_4} \varphi$ implies that $\Gamma \vDash_{J_4} \varphi$, for every finite set of formulas $\Gamma \cup \{\varphi\}$.*

In order to prove completeness, since H_4 is a finitary Tarskian logic, one can use the technique of maximal consistent sets of formulas. Indeed, for any set of formulas $\Gamma \cup \{\varphi\}$, if $\Gamma \not\vdash_{H_4} \varphi$ then, by Lindenbaum-Los theorem, Γ can be extended to a maximal set Λ such that $\Lambda \not\vdash_{H_4} \varphi$. We will call the set Λ *maximal non-trivial with respect to φ* in H_4 . Maximal sets w.r.t. a formula enjoy remarkable properties which directly follow from the axioms and rules of H_4 .

Proposition 9. *Let Λ be a maximal set non-trivial with respect to φ in H_4 . Then, Λ is closed, i.e. for every formula ψ , $\Lambda \vdash \psi$ iff $\psi \in \Lambda$. Moreover, for any formulas α and β the following conditions hold:*

- (1) $\alpha \vee \beta \in \Lambda$ iff $\alpha \in \Lambda$ or $\beta \in \Lambda$;
- (2) $\alpha \notin \Lambda$ iff $\sim\alpha \in \Lambda$;
- (3) $\alpha \Rightarrow \beta \in \Lambda$ iff $\alpha \notin \Lambda$ or $\beta \in \Lambda$;
- (4) $\alpha \notin \Lambda$ implies $\neg\alpha \in \Lambda$;
- (5) $\alpha \in \Lambda$ iff $\neg\neg\alpha \in \Lambda$;
- (6) $\neg\sim\alpha \in \Lambda$ implies $\alpha \in \Lambda$;
- (7) $\neg(\alpha \vee \beta) \in \Lambda$ iff $\neg\alpha \in \Lambda$ and $\neg\beta \in \Lambda$;
- (8) $*\alpha \in \Lambda$ implies $\alpha \in \Lambda$;
- (9) $*(\alpha \vee \beta) \in \Lambda$ iff $*\alpha \in \Lambda$ or $*\beta \in \Lambda$;
- (10) $**\alpha \in \Lambda$ iff $\neg\alpha \notin \Lambda$;
- (11) $\neg*\alpha \in \Lambda$ iff $\neg\alpha \in \Lambda$;
- (12) $*\alpha \notin \Lambda$ iff $*\neg\alpha \in \Lambda$.

Next we prove a Truth Lemma for H_4 .

Lemma 5 (Truth Lemma for H_4). *Let Λ be a maximal set of formulas non-trivial with respect to φ in H_4 . Consider the following evaluation e_Λ of propositional variables for J_4 :*

$$(T) \quad e_\Lambda(\gamma) = \begin{cases} 1 & \text{iff } \gamma \in \Lambda, \text{ and } \neg\gamma \notin \Lambda \\ 2/3 & \text{iff } \gamma \in \Lambda, \neg\gamma \in \Lambda, \text{ and } *\gamma \in \Lambda \\ 1/3 & \text{iff } \gamma \in \Lambda, \neg\gamma \in \Lambda, \text{ and } *\gamma \notin \Lambda \\ 0 & \text{iff } \gamma \notin \Lambda. \end{cases}$$

Then, (T) holds for every complex formula γ .

Proof. The proof is done by induction on the complexity of the formula γ . If γ is atomic then (T) holds by hypothesis. Now, suppose (T) holds for every

formula with complexity $\leq n$ (induction hypothesis – IH) and let γ be a formula with complexity n . In order to prove (T) from (IH) by analyzing all the possible cases (namely, $\gamma = \neg\alpha$ or $\gamma = *\alpha$ or $\gamma = \alpha \vee \beta$), each item of Proposition 9 should be used.⁸ The details are left to the reader. \square

Theorem 11 (Completeness of H_4). *The calculus H_4 is complete w.r.t. J_4 , that is: $\Gamma \vDash_{J_4} \varphi$ implies that $\Gamma \vdash_{H_4} \varphi$, for every finite set of formulas $\Gamma \cup \{\varphi\}$.*

Proof. Let $\Gamma \cup \{\varphi\}$ be a set of formulas in the language of J_4 such that $\Gamma \not\vdash_{H_4} \varphi$. By Lindenbaum-Los, there exists a set Λ maximal non-trivial with respect to φ in H_4 such that $\Gamma \subseteq \Lambda$. Let e_Λ be the evaluation defined as in the Truth Lemma 5. Then, it follows that $e_\Lambda(\gamma) \in F_{1/3}$ iff $\gamma \in \Lambda$, for every formula γ . Therefore e_Λ is an evaluation such that $e_\Lambda[\Gamma] \subseteq F_{1/3}$ but $e_\Lambda(\varphi) = 0$ since $\varphi \notin \Lambda$, hence $\Gamma \not\vdash_{J_4} \varphi$. \square

Recall that, from Theorem 7 and Remark 4, the Hilbert calculus \bar{H}_4 obtained from H_4 by adding the explosion rule

$$(exp_1) \frac{\varphi \wedge \neg\varphi}{\perp}$$

(see Definition 3) is the axiomatization of the (only) proper extension of H_4 which is strongly maximal w.r.t. CPL, and that is semantically characterized by the matrix logic

$$\bar{J}_4 = \langle \mathbf{A}_4 \times \mathbf{A}_2, F_{1/3} \times \{1\} \rangle,$$

where \mathbf{A}_2 is the Boolean algebra over $\{0, 1\}$ in the signature Σ , where the operator $*$ is defined as $*x = x$.

7 Conclusions

In this paper we have been concerned with the study of maximality and strong maximality conditions among finite-valued Łukasiewicz logics L_n^i with order filters as designated values. In particular, we have characterized the conditions under which a logic L_n^i is maximal w.r.t. CPL and its unique extension \bar{L}_n^i by an inference rule is strongly maximal w.r.t. classical logic. This allows us to show that, although they are not strongly maximal w.r.t. CPL, the logics L_n^i with n prime and $i/n \leq 1/2$ are in fact ideal paraconsistent logics. Thus, they provide interesting and well-motivated examples of ideal paraconsistent logical systems which are $(n + 1)$ -valued, in contrast with the $(n + 2)$ -valued logics \mathcal{M}_{n+2} presented in [3] and reproduced here in Example 3, whose definition is somewhat *ad hoc*.

⁸Observe that it is enough to prove the ‘only if’ part of (T), since the four conditions on the right-hand side are pairwise incompatible, and $e(\gamma)$ can only take one of the values $0, 1, 1/3, 2/3$. Thus, if, for instance, the first condition on the right-hand side of (T) holds for a given formula γ then the other 3 conditions are false and so $e_\Lambda(\gamma) \notin \{1/3, 1, 0\}$, by the ‘only if’ part of (T). Hence, $e_\Lambda(\gamma)$ must be $2/3$. This shows that the ‘if’ part of (T) follows from the ‘only if’ part.

As for future work, there are several interesting problems that we leave open in this paper. Concerning maximality, a natural question is how to obtain a stronger version of Theorem 1 which give us sufficient conditions to guarantee that a given matrix logic L_1 is *strongly maximal* w.r.t. another matrix logic L_2 . On the other hand, notice that the study of strong maximality developed in Section 5 was heavily based on results on the algebraic semantics associated to these systems by means of the Blok and Pigozzi’s techniques. Thus, another interesting issue to be explored in future work is to obtain more examples of strong maximality for different families of algebraizable logics

Another question raised here is the axiomatization of the ideal paraconsistent logics J_{q+1} for $q > 3$ in a “natural” signature containing a deductive implication. As it was shown in Subsection 6.2, the signature $\Sigma = \{\vee, \neg, *\}$ is suitable for the case $q = 3$. Moreover, besides being apt for axiomatizing $J_4 = L_3^1$, it can be proved that the (non-paraconsistent) logic L_3^2 can also be axiomatized over Σ in a relatively simple way. Note that $\alpha \Rightarrow \beta = \neg\alpha \vee \beta$ defines a deductive implication in L_3^2 .

The fact that Lukasiewicz implication is definable in Σ justifies the convenience of using that signature for dealing with the case $q = 3$. However, this property does not hold for any prime $q > 3$. Indeed, there are primes q in which Lukasiewicz implication of L_{q+1} cannot be defined over Σ , e.g. $q = 17$. The study of the fragments of L_q^i in the signature Σ is thus a different but closely related problem, which deserves future research.

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