Towards a Theory of Singular Thought About Abstract Mathematical Objects

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1: Introduction

Philosophical theorizing about singular thought and reference has tended to require causal contact with the object thought about and referred to,[1] thereby rendering such thought about and reference to abstract mathematical objects metaphysically impossible. After arguing that we should want a theory allowing for singular thought about and reference to mathematical abstracta, I describe three contemporary *mental files* theories of singular thought (theories saying that singular thought about and reference to a particular object requires possession of a mental store of information taken to be about that object): acquaintance theory, semantic instrumentalism, and semantic cognitivism. After showing that none of these can plausibly explain how we could think singularly about or refer to abstract mathematical objects, I argue for two claims intended to advance our understanding of singular thought about mathematical abstracta. First, that the conditions for possession of a file for an abstract mathematical object are the same as the conditions for possessing a file for an object perceived in the past – namely, that the agent retains information about the object. Thus insofar as we are able to have memory-based files for objects perceived in the past, we ought to be able to have files for abstract mathematical objects too. The second claim I argue for is that at least one recently articulated condition on a file's being a device for singular

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[1] As in Evans (1973, 1982), Bach (1987), Recanati (1993, 2012), Sawyer (2012).

thought – that it be capable of surviving a certain kind of change in the information it contains[2] – can be satisfied by files for abstract mathematical objects.

I begin with the distinction between singular and descriptive thought.

2: Singular Thought and Descriptive Thought

Consider the thoughts that an agent *S* would standardly express using the following two sentences:

(1) The tallest structure in Paris is made of wrought iron.

(2) *That* [points at the Eiffel Tower][3] is made of wrought iron.

If *S* thinks either (1) or (2),[4] they are thinking 'about' the Eiffel Tower in the sense that their thought is true (or not) in virtue of what the Eiffel Tower is like. Now, it seems that if *S* thinks (1) they are thinking about the Eiffel Tower in virtue of that structure satisfying a descriptive condition expressed by "the tallest structure in Paris" that is part of what (1) says. On the other hand, when *S* thinks (2) they are thinking about the Eiffel Tower in virtue of something different – perhaps something like their perceptually attending to the Eiffel Tower when thinking (2).

Thus while the thoughts (1) and (2) are both actually true or not in virtue of whether the Eiffel Tower is made of wrought iron, this comes about in two different ways. Many philosophers take this to result in an important difference in the *logical forms*[5] of the contents of the two thoughts. In particular, the logical form of (1) is:

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[2] Cf. Goodman 2016*a*, 2016*b*.

[3] I am using David Kaplan's (1989) convention of using square brackets to describe physical gestures made when uttering statements or thinking thoughts.

[4] The expression "thinks (1)" here abbreviates "thinks the thought that would be standardly expressed by making statement (1)". Thus "thinks either (1) or (2)" abbreviates "thinks either the thought that would be standardly expressed by making statement (1) or the thought that would be standardly expressed by making statement (2)".

[5] I am using "logical form" in Davidson's sense, where "to give the logical form of a sentence is to give its location

(i) ⌜$\exists x[\Phi x \,\&\, \forall y(\Phi y \rightarrow y = x) \,\&\, \Psi x]$⌝ [6]

where Φ and Ψ are metalinguistic variables ranging over object-language predicates. The logical form of (2), on the other hand, is:

(ii) ⌜α is Ψ⌝

where α is a metalinguistic variable ranging over object-language singular terms. Philosophers commonly mark this distinction by saying that if *S* thinks a thought with logical form (i) – such as (1) – they think a *descriptive* thought, whereas if *S* thinks a thought with logical form (ii) – such as (2) – they think a *singular* thought. This notion of singular thought in turn induces a notion of singular linguistic reference via the claim that the content of a thought and the content of a literal linguistic expression of that thought coincide.[7]

Once this distinction between singular and descriptive thought is in place, we can ask which entities we are capable of having singular thoughts about and which we are not. Before surveying three answers to that question I want to explain why it is important whether we can have singular thoughts about *abstract mathematical objects* such as natural and real numbers. (While some have argued that we cannot even think or speak *descriptively* about abstract mathematical objects (cf. e.g. Lear 1977, Jubien 1977), I here leave this question aside and assume a positive answer.)

Consider an intuitively true negative claim about the real number *e*, such as:

(3) It is not the case that *e* can be expressed as a fraction.

Our question is whether someone thinking (3) is having a singular thought or a descriptive thought

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in the totality of sentences, to describe it in a way that explicitly determines what sentences it entails and what sentences are entailed by it." (Davidson 2006, p.64.) This is not necessarily the same as the level of mental representation of the syntactic form postulated by certain Chomskyan linguistic theories known as LF.

[6] ⌜α is Φ⌝ abbreviates "the thought expressed by the sentence resulting from inserting the value of α into the argument place of the value of Φ".

[7] This is François Recanati's *congruence principle*: "in literal communication … the same state of affairs is represented by the speaker's thought and by the utterance which (literally) expresses that thought." (1993, p.54/5.)

about *e*. This depends on whether, as I'll say, "*e*" functions as a singular term or a (disguised) definite description in the mind and mouth of the agent. Now, if "*e*" functions *not* as a singular term but a disguised definite description, then the logical form of (3) is not

(iii) ⌜~(α is Φ)⌝.

Rather, the logical form of (3) is *ambiguous* between:

(iv) ⌜~∃*x*[Ψ*x* & ∀*y*(Ψ*y* → *y* = *x*) & *x* is Φ⌝

and

(iv*) ⌜∃*x*[Ψ*x* & ∀*y*(Ψ*y* → *y* = *x*) & ~(*x* is Φ)⌝.

That is, (3) is ambiguous between the negation scoping over the existential quantifier (iv) or vice-versa (iv*). But if the logical form of (3) is (iv), then (3) does not entail "∃*x*~(*x* can be expressed as a fraction)", but only the strictly weaker:

(4) ~∃*x*[*Px* & ∀*y*(*Py* → *y* = *x*) & *x* can be expressed as a fraction]

where "*P*" is the description abbreviated by "*e*" – e.g. "the limit of $(1 + 1/n)^n$ as *n* approaches infinity".[8] The problem here is that (4) is consistent with claims that ordinary speaker intuition says that (3) *unambiguously* contradicts, for instance:

(5) Every number is expressible as a fraction,

and:

(6) There are no numbers at all.[9]

[8] Obviously there are other candidates – and plausibly, different agents could use "*e*" to abbreviate different descriptions. Here I set aside any complications arising from this.

[9] One could appeal here to the fact that (4) conjoined with the axioms of real analysis entails "∃*x*~(*x* can be expressed

It looks like ordinary speaker intuition dictates that (3) unambiguously entails that there is at least one number that cannot be expressed as a fraction. To be sure, if the logical form of (3) is (iv*), then (3) is inconsistent with (5) and (6). But the point is that if (3) is *ambiguous* between (iv) and (iv*), then it is *ambiguous* whether (3) is consistent with (5) and (6), and ordinary speaker intuition appears to deny any such ambiguity. If that's right, then (3)'s logical form is (iii), rather than being ambiguous between (iv) and (iv*).

Thus ordinary speaker intuition favours reading "*e*" as a singular term rather than as a (disguised) definite description, and hence (3) as a singular rather than descriptive thought. And so, insofar as our account of singular thought and reference ought to respect data from ordinary speaker intuition as much as possible, we should ask after an account permitting "*e*" to function as a singular term – at least, until it has been shown that there is no such plausible account.

Why would there be no plausible such account? The philosophically naïve view of mathematical objects like the real number *e* has it that they are *abstract* objects. While it is not completely clear what it takes to be an abstract object, two things seem relatively clear, at least when it comes to abstract *mathematical* objects: such objects are metaphysically incapable of either spatiotemporal location or of being causal agents or patients. My ultimate goal is to preserve both this naïve view of abstract mathematical objects and the ordinary speaker intuitions about the logical forms and entailment relations stood in by thoughts like (3).

The question of singular thought about mathematical abstracta has been pursued before, for instance in Kim (1977), Burge (2007), Azzouni (2010), and Hansen & Rey (2016). While it would take us too far afield to thoroughly compare my approach with these, I will briefly note

as a fraction)". But using this fact to explain ordinary speaker intuition requires crediting all ordinary speakers with belief in the axioms of real analysis, which doesn't seem very plausible.

some important differences between those approaches and my own. First, Kim argues that our lack of a firm philosophical grasp on the notion of causation impugns the legitimacy of appealing to causal connections as fixing reference. The claims that I will ultimately make are consistent with appeal to causal connections being perfectly legitimate in fixing the references of *some* singular thoughts. Second, both Azzouni (2010) and Hansen & Rey (2016) secure singular thought about mathematical abstracta as a specific instance of singular thought about *nonexistent* objects. The claims that I will make, on the other hand, are consistent with the object dependence of singular thought. Thus it is open to a believer in the kind of singular thought about mathematical abstracta that I will ultimately describe to hold that if a mathematician attempts to have a singular thought about the largest prime number they will inevitably fail because that number does not exist, and hence there is no such thought to be had. Third, Burge's (2007) view relies explicitly on the claim that subitizing – immediate perceptual recognition of the cardinality of a collection of visual or auditory objects – is a form of non-conceptual relation to the cardinal numbers in the subitizing range (up to 3 for human children and many animals, up to 4 for adults[10]). My claims require no such commitment (although they are consistent with it). Finally, Dickie's (2015) move of taking which entity a thought is about to be fundamentally connected to what *justifies* that thought, does on the face of it make room for an account of singular thought about mathematical abstracta (though she does not herself pursue this question). My claims, on the other hand, do not obviously take justification to play a fundamental role in securing singular thought, thus retaining the possibility of singular thought about mathematical abstracta without requiring the ability to have those thoughts be justified.

The next section describes three contemporary views of singular thought and what they say

[10] Cf. Dehaene (2011), pp.55-8.

concerning thought about mathematical entities like the number *e*.

3: Three Mental Files Theories of Singular Thought

All three theories of singular thought I'm about to describe assume a currently popular view of the relationship between singular thought, reference, and the way that systems of beliefs are structured: the *mental files* view of singular thought and reference.[11] A mental file is a mental store of information the thinking agent takes to be about a single individual.[12] For instance, my "Frege"-file contains information like "solved the problem of multiple generality" and "discovered the distinction between sense and reference" which I take to be about a single individual. Mental files theories of singular thought say that a thinker *S* thinks a singular thought about an object *o* if and only if, in the process of thinking about *o*, *S* deploys or activates a mental file referring to *o*.[13]

There are three things worth noting at this point. First, one motivation for a mental files view of singular thought is potential connections to notions of files proposed in both linguistics – in theorizing about (in)definiteness, anaphora and information structure – and in cognitive science, to study memory, perception, and attention.[14] Such apparent connections give mental files theories of singular thought the potential for integration into both cognitive linguistics and cognitive

[11] While the mental files view of singular thought is popular, it is not without its detractors. See in particular Dickie (ms) and Goodman (2016*a*, 2016*b*). Dickie argues that claims about mental files can be systematically replaced by claims about mental processes. Thus I conjecture that if my arguments go through on a mental files view of singular thought, they ought to go through on Dickie's 'process' view too (though it would take us too far afield to pursue this fully). I address the pertinent details of Goodman's view in due course.

[12] I stay officially neutral on whether the information comprising a mental file must only be conceptual or propositional, or whether it can include pictorial, auditory, or otherwise non-conceptual information. Recanati (2012) appears to allow only propositional information to comprise mental files. (p.37/8.)

[13] E.g. "[T]here are also non-descriptive senses or modes of presentation, and these, I claim, are mental files." (Ibid., p.40.) And: "mental files are 'about objects': like singular terms in the language, they refer, or are supposed to refer. They are, indeed, the mental counterparts of singular terms." (Ibid., p.35.)

[14] See ibid., p.vii and the references mentioned therein.

science as a whole – something which may be missing from other theories of singular thought.

The second thing to note is that the conditions under which a given file *refers* to specific object can vary between theories, as we'll soon see. The third concerns 'deploying' or 'activating' a mental file. Our question is whether we can have singular thoughts about abstract mathematical objects. Within mental files theories of singular thought this question becomes that of whether we can deploy files for mathematical abstracta when having thoughts. Nevertheless I leave aside the question of whether we can have files that we cannot deploy in thought, and will hereon assume that possession of a file for *o* entails that that file is deployable.

Hence the conditions on singular thought about *o* just are the conditions on possession of a mental file for *o*. And so, our question is now: which objects can we have mental files for? Here are three answers.

3.1: Acquaintance Theory

One view of singular thought, descended from a view once held by Russell, says that *S* is able to have singular thoughts about *o* if and only if *S* stands in at least one *acquaintance relation* to *o*. Russell himself thought we can be acquainted only with our own sense-data, universals, and (perhaps) ourselves.[15] Most contemporary acquaintance theorists permit acquaintance with ordinary concrete objects.[16] Here is the contemporary 'standard' acquaintance constraint on singular thought:

[15] Cf. Russell (1911): "We have acquaintance with sense-data, with many universals, and possibly with ourselves, but not with physical objects or other minds." (p.127.)

[16] E.g. Evans (1982), Bach (1987), Sawyer (2012), and Recanati (1993, 2012) (and others) allow acquaintance with ordinary concrete objects. Russell's claim that we can be acquainted with *universals* is not so often discussed, and I will not discuss it in what follows.

> One can be acquainted with an object O only by perception, memory, and communication chains. To have a singular thought about O, someone in one's linguistic community must have perceived O. (Jeshion 2010, p.109; cf. Bach 1987, chapter 2.)

Thus *S* can have singular thoughts about *o* only if *S* is currently perceiving *o*, remembers perceiving *o*, or has received a name for *o* from a communication chain originating in perception of *o*. For *S* to be able to have singular thoughts about *o* it is necessary that some member of *S*'s linguistic community has perceived *o* at some time.

François Recanati has recently articulated an acquaintance-based mental files theory of singular thought. For Recanati mental files contain information gained via acquaintance relations – which Recanati takes to be *causal* relations (Recanati 2012, p.34/5[17]) – and refer to the dominant causal source of the information they contain.[18] Hence *S* has a mental file for *o* only if *S* is acquainted with *o*. And thus it follows – importantly for us – that we cannot have mental files for individuals from which we are causally isolated.

Recall the naïve view that abstract mathematical objects like the real number *e* are incapable of spatiotemporal location or being causal agents or patients. If this is right then we cannot be acquainted with *e*. Hence Recanati's acquaintance-theoretic mental files view of singular thought, combined with the naïve view of mathematical abstracta, rules out singular thought about

[17] "Mental files are based on what Lewis calls 'acquaintance relations' [begin footnote 5] The paradigm is, of course, perceptual acquaintance, but the notion of acquaintance can be generalized 'in virtue of the analogy between relations of perceptual acquaintance and other, more tenuous, relations of epistemic rapport. […] In each case there are *causal chains from him* [the referent] *to me of a sort which permit the flow of information*." (My emphasis; Recanati quotes from Lewis 1999, p.380-1.) See also Hansen and Rey's remark that "it's a shame that Recanati relegated this expansion of (ER) relations to a footnote." (Hansen and Rey 2016, p.427.)

[18] "Mental files are 'about objects': like singular terms in the language, they refer, or are supposed to refer. What they refer to is […] the individual we are acquainted with (in the appropriate way), not the individual which best 'fits' the information." (Recanati 2012, p.35.) Note the intellectual debt to Evans (1973). There is a question of interpretation here: is reference a function of the information itself, or the relation through which the information flowed? The latter allows files containing no information to refer, whilst the former does not. I address this in footnote 37.

e.

Note as we're going past that Recanati's move of allowing *correctly expected* acquaintance to sustain singular thought won't work for *e*. If *e* is metaphysically incapable of entering into causal relations, then it is metaphysically impossible for one to be correct in expecting acquaintance with it. Thus even *expected* acquaintance theory renders singular thought about abstract mathematical objects metaphysically impossible. Unless, that is, we countenance some kind of *non-causal* perception, acquaintance, or 'intuition' of abstract mathematical objects. It would take us too far afield to fully address this move, so will confine myself to remarking that within Recanati's framework, this requires a non-physical notion of information flow, and it is not clear how such a notion would work (but see section 5 of Chudnoff (2014) for a sketch of some ideas).

I now turn to two more permissive views: semantic instrumentalism and semantic cognitivism.

3.2: Semantic Instrumentalism

According to semantic instrumentalism, an agent *S* can gain the ability to have singular thoughts about an object *o* by formulating a definite description "*D*" uniquely satisfied by *o*, and then thinking to themselves:

(7) Let "*dn*" refer to the (actual) satisfier of *D*.

If *S* then goes on to think ⌜*dn* is Φ⌝, *S* has a singular thought about *o*.[19] Here "*dn*" is device for

[19] Kaplan (1989), p.536: "There is nothing inaccessible to the mind about the semantics of direct reference, even when the reference is to that which we know only by description. What allows us to take various propositional attitudes toward singular propositions is not the form of our acquaintance with the object but is rather our ability to manipulate the conceptual apparatus of direct reference." See also Harman (1977), p.174: "If Mary believes there

singular thought introduced using a definite description – what philosophers call a *descriptive name.* Perhaps the most developed semantic instrumentalist theory is that described in Kaplan (1989). Kaplan's "dthat" is a device for introducing descriptive names: if *S* thinks to themselves "let "*dn*" refer to dthat(*D*)", they can then have singular thoughts by thinking ⌜*dn* is Φ⌝.[20] (Kaplan 1989, p.560, f/n 76.)

Transposing semantic instrumentalism into a mental files framework gets us that if *S* formulates a definite description "*D*" and thinks to themselves "let "*dn*" refer to the actual satisfier of "*D*" (or "let "*dn*" refer to "dthat(*D*)"), *S* thereby comes to possess a mental file for the actual satisfier of "*D*". If *S* deploys this "*dn*"-file when thinking about the actual satisfier of "*D*", those thoughts are singular. Thus the only constraint on the ability to have singular thoughts about a given object is that the agent be able to formulate a definite description satisfied by that object. So (a mental files version of) semantic instrumentalism says that because we can formulate a definite description uniquely satisfied by the real number *e* – such as "the limit of $(1 + 1/n)^n$ as *n* approaches infinity" – we can have singular thoughts about it.

So far, so good. However semantic instrumentalism has been widely judged implausible.[21] One reason why is that it appears to entail *voluntarism* about singular thought: to gain the ability to have singular thoughts about any given describable object *o*, it suffices that one *decide* to do so.[22] (Jeshion 2010, p.107, p.125.) Provided, that is, one is competent with the relevant semantic

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is a certain unique thing satisfying certain conditions C_1, C_2, C_3, she can introduce a new mental name *a* into her system by forming the beliefs that *a* is C_1, that *a* is C_2, and that *a* is C_3. This name functions as a name of the unique thing satisfying these conditions if there is one; otherwise it does not name anything. Moreover, the name continues to name this thing, as long as Mary uses it, even if nothing or something different should be[come] the unique thing satisfying those of her beliefs involving the name *a*."

[20] "Dthat" is an expression that when prefixed to a definite description yields a singular term referring to the satisfier of that description (if it has exactly one). (Kaplan 1989, p.521/2).

[21] "Almost all theorists think that Semantic Instrumentalism is false – indeed, wildly off." (Jeshion 2010, p.106-7.)

[22] Another way of putting this argument against semantic instrumentalism is Evans' (1982, p.50) invocation of Grice's (1969) dictum that we cannot gain new beliefs 'at the stroke of a pen'. Note also that there is a nearby argument

apparatus. Kaplan himself hedges on exact degree of mastery required here. (Kaplan 1989, p.536 f/n 61.) Note also that in a mental files framework for singular thought, voluntarism about singular thought is voluntarism about the initiation and persistence of mental files; we'll return to this in due course.

Nevertheless it is implausible that we can get ourselves into the position to have singular thoughts about an arbitrary given object solely by making a personal decision.[23] Take Kaplan's example of "Newman 1", a descriptive name introduced using the description "the (actual) first child born in the 22nd century". (Kaplan 1989, p.560 f/n 76.) Semantic instrumentalism says that I can acquire the ability to have singular thoughts about the first child born in the 22nd century by choosing to think to myself "let "Newman 1" refer to the (actual) first child born in the 22nd century" and subsequently thinking (e.g.) "Newman 1 will be Australasian".[24] But, the charge goes, it is implausible that singular thought can be had *so* cheaply. And insofar as voluntarism about singular thought is implausible, so is semantic instrumentalism. Therefore while semantic instrumentalism preserves both the naïve view of abstract mathematical objects and ordinary speaker intuitions about the logical forms of thoughts like (3), it does so at the cost of overall implausibility.

I now turn to semantic cognitivism.

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that semantic instrumentalism entails voluntarism about new knowledge – the view that one can gain new knowledge by deciding to do so (and being competent with the relevant semantic apparatus) – and this is implausible.

[23] "Semantic Instrumentalism supposes that we can *will* a singular intention. But how? By thinking harder, with more intensity, with feeling? This lacks plausibility." (Jeshion 2010, p.107.)

[24] Readers disliking the eternalism Kaplan's example apparently presupposes can exchange it for the perhaps more palatable "let "Oldman 1" refer to the first child born in the 16th century". (This example is modified from Jeshion 2002, p.72.)

3.3: Semantic Cognitivism

Robin Jeshion has recently argued that the difference between singular and descriptive thought is that in cases of singular thought the object thought about is *significant* to the thinking agent's sub-agential 'cognition'.[25] (Jeshion 2009, 2010. Jeshion's "sub-agential" is roughly synonymous with "sub-personal".) Jeshion's view is a mental files view: an agent *S* has a mental file for, and hence can have singular thoughts about, an individual *o* if and only if *o* is significant to *S*. (Jeshion 2009, p.394.) Hence Jeshion replaces the standard acquaintance constraint with the following significance condition:[26]

> *Significance Condition*: *S* can have singular thoughts about *o* if and only if *o* is significant to *S*.

The significance condition is supposed to allow for acquaintanceless singular thought without voluntarism about singular thought. First, significance does not require acquaintance. (Jeshion 2010, p.126.) Thus the real number *e* could be significant to some mathematicians despite the impossibility of acquaintance with it. Assuming an ordinary notion of significance here, it seems quite plausible that *e* is significant to many mathematicians. It plays foundational roles in many areas of mathematical study. Indeed, Euler's identity – the claim that $e^{i\pi} + 1 = 0$ – is frequently cited as a paradigm example of mathematical beauty. Hence it is highly plausible that some of the subjects of Euler's identity – including *e* – are significant to some mathematicians. (I leave aside whether the imaginary number *i* should qualify as an object.)

But, secondly, whether an object is significant is not a matter of voluntary decision. Rather,

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[25] I'm placing 'cognition' in scare-quotes to indicate that I am using that term to mean whatever Jeshion uses it to mean. From now on the phrase "significant to *S*" abbreviates "significant to *S*'s sub-agential 'cognition'".

[26] "Cognitivism dispenses with an acquaintance condition on singular thought, supplanting it with a significance condition." (2009), p.392. Cf. (2010), p.127/8.

it is a matter of agents' sub-agential 'cognition' – which is *not* under voluntary control. One cannot make an object significant by believing, deciding, or willing it so. And once an object *is* significant, one cannot voluntarily reduce it to insignificance.[27] Thus cognitivism rules out singular thought 'on the cheap'.

Returning to singular thought about abstract mathematical objects, cognitivism says that if a mathematician *M* formulates the description "the limit of $(1 + 1/n)^n$ as n approaches infinity", and *M*'s sub-agential cognition takes the satisfier of that description as significant, *M* will initiate a mental file containing information they take to be about that object. It seems plausible that this file could contain both deduced consequences of that description, such as "cannot be expressed as a fraction" and deductively unproven but inductively plausible conjectures about its satisfier, perhaps like "is such that its decimal expansion does not contain the sequence "777"". *M* can then use this file to have singular thoughts about *e*. But when I formulate the description "the first child born in the 22nd century", if the satisfier of that description is not significant to me, then even if I think to myself "let "Newman 1" refer to the actual first child born in the 22nd century", no file is initiated and singular thought is not possible.

One might wonder whether singular thoughts had in the course of a conversation require significance. Suppose a professional mathematician is regaling their lay friend with their latest discoveries concerning *e*. Are the layperson's resulting thoughts about *e* singular? It may be plausible that their thoughts are descriptive, insofar as they are thinking about *e only* as *whatever their interlocutor is talking about*. On the other hand, there are two ways to secure singularity of the listener's thoughts about *e*. First, in conversations between an expert and a layperson, the

[27] "An agent making a judgment "this is significant" is not sufficient for engendering the significance needed for singular thought." (2010, p.136.)

layperson *defers* to the expert, and so the layperson's singular thoughts are parasitic on the expert's standing ability to have such thoughts (which itself requires significance to the expert). Second, Jeshion has argued that receiving a name for an object through communication is (in general) by itself sufficient to render the referent significant to the listener (cf. Jeshion 2009, pp.383-5).

Thus of the theories of singular thought considered thus far, cognitivism is the leading contender for a plausible theory of singular thought about abstract mathematical objects. It permits singular thought about *e* but not about any definitely describable object whatsoever. However, cognitivism suffers from two serious defects.

3.4: Problems with Semantic Cognitivism

According to cognitivism, significance is necessary and sufficient for singular thought. But there are reasons for thinking that neither direction of this biconditional holds in full generality. First, some of our *perceptual demonstrative* thoughts are singular thoughts about insignificant objects. Second, there are compelling reasons for thinking that not all mental files initiated on the basis of significance are singular – there are also *descriptive* files for significant objects. The remainder of this section describes these objections in more detail. Sections 4 and 5 use these objections as springboards for two positive claims concerning singular thought about mathematical abstracta.

The reason for thinking we can have singular thoughts about insignificant objects is that many of our commonplace perceptual demonstrative thoughts are about objects that are *not* significant to us in any ordinary sense. Consider the following scenario. While working in my office I am briefly distracted by a speck of dirt on the window. I think to myself:

(S) That speck looks like it's on the inside surface of the window.

Cursory inspection reveals that the speck is indeed on the inside surface of the window. I resume writing, promptly forgetting about the speck. It seems incredible to say that this speck was *significant* to me. It seems a lot more accurate to say that I just *never cared* about it. But if the speck is not significant to me then Jeshion's significance condition entails that when I think (S) whilst visually attending to the speck, my thought is not singular. This is a problem because perceptual demonstrative thoughts are the home case of singular thought. (Murez & Recanati 2016, p.267.) If perceptual demonstrative thoughts are not singular, there may as well be no singular thoughts at all. And if demonstrative thought about insignificant objects *is* singular, then significance is not necessary for singular thought, *contra* Jeshion's significance condition.

Here is Jeshion's own response. She has claimed that if *S* perceives *o*, then *o* is significant to *S*:

> Objects that are directly perceived automatically count as significant insofar as they are all possible objects upon which an agent may act. Objects of perception are in this way automatically significant to the agent's cognitive system as a whole. (2014, p.83.)[28]

Now, if perception suffices for significance, then perceptual demonstrative thoughts are always about significant objects. Hence the necessity of significance for singular thought is consistent with the singularity of perceptual demonstrative thought.

However taking all perceived objects to be automatically significant has the following awkward consequence: it entails that the statement "this [points] is not significant to me" could not be used to express a truth in ordinary contexts. This seems wrong. In many ordinary contexts,

[28] Rachel Goodman has claimed that we should read Jeshion as taking the significance condition as not applying to perceptual demonstrative thought. (2016*b*, p.246-7.) While there may be reasons for this – in particular, that Jeshion discusses perceptual demonstrative thought and, so it seems, can't plausibly have been unaware of the problem – given that this contradicts what Jeshion herself says, and that she responds to the objection in the passage quoted by asserting the automatic significance of perceived objects, I will continue to read her significance condition as applying to all singular thought.

speakers and hearers are surrounded by objects of which "this [points] is insignificant" can be truly said. Now, it may be possible to avoid this by adopting some theoretically loaded notion of significance. However it would take us too far afield to pursue that line of thought here.

Here is a reason for thinking that the significance of an object is not *sufficient* for singular thought either. Rachel Goodman has recently argued that some mental files are *descriptive* files, devices for thinking of objects *as the bearers of certain properties* – such as being the tallest structure in Paris. Hence the contribution of such files to the contents of the thoughts they are used to have is a description, rather than an object. Thought from such files is not singular.[29] (Goodman 2016*a*, 2016*b*.) Goodman has argued there are cases where a file possessed on the basis of an object's significance can be descriptive in this way. Consider her 'de Mestral' example. (2016*a*, p.449.[30]) We are asked to imagine George de Mestral, who invented Velcro in 1948, fixating on the inventor of the zipper's usurpation of his deserved commercial success. de Mestral goes on to think that the inventor of the zipper probably lives in the largest mansion in town, holidays in the south of France, and a wealth of other jealousy-induced conjectures. Goodman contends that in this (fictional) case, de Mestral is thinking of the inventor of the zipper descriptively, despite both possessing a mental file for, and the significance to him, of that individual. If she is right, then it is open to an opponent of singular thought about abstract mathematical objects to concede the claim that thought about such objects may be thought from files possession of which is due to significance, whilst maintaining that this is still merely descriptive thought.

So, despite the fact that semantic cognitivism is the most promising candidate theory of singular thought countenancing such thought about abstract mathematical objects we've seen so

[29] Jeshion herself notes that earlier proponents of analysing thought in terms of mental files held that files can be used to have descriptive thoughts – in particular, Grice (1969) and Lewis (1979). (Jeshion 2010, p.132.)

[30] She also gives the 'aesthetically motivated collector' thought experiment in Goodman 2016*b*, p.249, which is intended to motivate the same conclusion.

far, it is both implausible as a theory of singular thought in general – because we can have singular thoughts about perceived but insignificant objects – and may fail to secure *singular* thought about abstract mathematical objects *even if* those objects can be significant. In the next two sections I use these two objections to semantic cognitivism as springboards for two claims intended to advance our understanding of singular thought about mathematical abstracta.

4: Stable Files and Information Retention

In this section I argue that the conditions on both thought about past perceived objects, and acquaintanceless thought about abstract mathematical objects, are the same – both require retention of information about the object of thought. I will also briefly conjecture that significance fares better than acquaintance as a condition on information retention, and hence on thought about both past perceived objects and abstracta.

We begin with a distinction, drawn from Recanati (2012), between *demonstrative files* and *stable files*.[31] *S* possesses a demonstrative file for *o* only so long as the perceptual relation facilitating flow of information from *o* into that file obtains.[32] Stable files, on the other hand, do not require for their existence any *ongoing* perceptual relation to the object: *S* can have a stable file for *o* without standing in any current perceptual link to *o*. Thus the requirements for singular thought bifurcate: *S* can have singular thoughts about *o* only if *S* possess *either* a demonstrative file for *o*, *or* a stable file for *o*. Singular thought about *o* without possession of a stable file for *o* requires a demonstrative file for *o*, which in turn requires a current perceptual link with *o* – but *not*

[31] Cf. chapters 6 and 7 in particular. Note that while I will speak of perceptual links rather than informational links more generally, this shouldn't affect my larger conclusions.

[32] "A demonstrative file exists only within a limited context: it exists only as long as the subject bears the demonstrative relation (whatever that relation is exactly) to some object *x* – a relation which makes it possible for the subject to focus his or her attention on *x*. *If x disappears from view for sufficiently long,* a change of context takes place and *the file comes out of existence*" (2012, p.68; my emphasis.)

o's significance to *S* (though of course perceived objects may also be significant). Hence we can have perceptual demonstrative singular thoughts about insignificant objects.

Nevertheless sometimes we think singularly about objects which we are not currently perceiving, but have done so in the past. According to Recanati for perception of *o* to result in the ability to have singular thoughts about *o* down the line, the demonstrative file for *o* must undergo *conversion* into a stable file for *o*.[33] Now, most people are not able to have singular thoughts about *every* object they have perceptually encountered in the past. So some demonstrative files are not converted into stable files. And while Recanati discusses some aspects of conversion between demonstrative files and (the various kinds of) stable files, he says almost nothing about what conversion requires.[34] Here are two ways of arguing that *retaining information* (taken to be) about *o* is necessary for possessing a mental file referring to *o*.

The first begins by noting that if the reference of a mental file is a function of some feature of the information it contains – as in Recanati's view where the reference of a mental file is a function of the genetic origin of the information it contains, namely the dominant causal source of that information[35] – then there is nothing fixing the reference of a mental file that is empty in the

[33] "When the contextual relation to the object is severed, the temporary file based on it disappears, but the information stored in the file does not disappear: it is transferred into the new file." (2012, p.62/3) And: "Conversion is the process through which information stored in a file is transferred into a successor file when the ER relation which sustains the initial file comes to an end." (Ibid., p.81)

[34] I say 'almost nothing' because Recanati does say that "when an object is encountered and some information about it is gained, that information is *typically* preserved in memory and made available when the object is encountered again and recognized…" (2012, p.81; my emphasis.) However saying that demonstrative files are *typically* converted into memory files does nothing to explain *why* this is so. Likewise, chapter 5 of *Mental Files in Flux* (Recanati 2016), is an extended discussion of conversion that does not mention the condition sunder which it takes place.

[35] Note that this is not the only feature of the information contained in a file that could fix that file's reference. Another is that the file refers to the satisfier of that information (if there is one), as in Goodman's descriptive files. Yet another is that a file refers to the satisfier, if there is one, of the description *from which that information was inferred*, rather than the satisfier of the information itself. These inferences may be invalid (we are fallible reasoners), in which case the satisfier of the original description – i.e. the referent of the file – need not be the satisfier of the inferred information contained in the file.

sense of containing no information.[36, 37] Thus if a mental file 'for' *o* contains no information, then it cannot actually refer to *o*. Likewise if a file formerly containing information deriving from *o* is emptied of every item of information it formerly contained, it cannot retain its former reference – at least, not according to Recanati's view. Therefore if the reference of a mental file is a function of some feature of the information it contains, then if a file is empty in the sense of containing no information, then it does not refer.

If empty files don't refer, then failing to retain any information taken to be about *o* results in lack of any mental file referring to *o*. Thus retention of information about *o* is necessary for having a stable file referring to *o*; and hence, necessary for the conversion of a demonstrative file into a stable file.

I now turn to the claim that retaining information (taken to be) about *o* *suffices* for possessing a stable file referring to *o*. Now, to retain information taken to be about a single object *just is* to retain a mental file. So if *S* retains information about *o* in the absence of a current perceptual link, then they are maintaining a stable file for *o*. And so, putting this together with the

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[36] *If* the referent of a file is a function of the information it contains. However there are more purely causal views according to which reference is fixed not by any feature of information contained in the file, but by causal relations between the agent and the referent. One main problem with purely causal views is how to account for reference change over time (as in Evans' (1973) 'Madagascar' case). The mental files theory I am describing deals nicely with reference change over time, and so I won't consider more purely causal views here; but see section 5.4 of Devitt (1981) for an attempt to account for reference change over time within a purely causal framework. (Note thought that Devitt's account requires allowing that both reference and truth can come in *degrees*, whereas as mine does not. I leave it to the reader to decide whether or not this counts against Devitt's approach.)

[37] This is where the question of interpretation mentioned in footnote 2 becomes relevant. An alternative interpretation of the Evans/Recanati 'dominant causal source' model is that reference is a function not of the information contained in the file, but the *information channel* through which that information was obtained. Thus a file refers to whatever object was in fact at the other end of that information channel (if there was one). This allows that a file could refer in spite of being empty of information. I do not have space to go into a full argument against this version of the 'dominant causal source' view here, but I will mention another reason why empty files cannot be used to refer, which this alternative is also susceptible to. The main idea is that *S*'s *a*-file refers to *o* only if *S* can use that *a*-file to have *beliefs* about *o*. If *S* uses their *a*-file to have the belief ⌜*a* is *F*⌝, then their *a*-file contains the information ⌜is *F*⌝, whence the file is *not* empty after all. Thus a file that is empty of information cannot be used to have beliefs. And we might think (but I won't argue here) that if a file cannot be used to have beleifs, then there is little reason to regard it as a device for reference.

conclusion of the previous several paragraphs, we have: *S*'s demonstrative file for *o* is converted into a stable file for *o* if, and only if, *S* retains some of the information from that former demonstrative file.

Before moving on, I want to make two remarks: one concerning the claim that information-retention suffices for stable file possession and hence singular thought, the other concerning the conditions under which information is retained.

For the first point, consider the following case discussed by Sutton (2004).[38] Suppose an agent *S* gains information through communication from one scientist[39] about a famous physicist named 'Feynman'. Somehow *S* comes to believe erroneously that there are *two* famous physicists so named – and that some of the things they've heard are true of one, some true of the other. Thus the agent uses two homonymous names "Feynman" and "Feynman", which *we* symbolize "Feynman$_1$" and "Feynman$_2$", to have singular thoughts. Unbeknownst to *S*, "Feynman$_1$" and "Feynman$_2$" actually co-refer. As time goes by, the agent forgets almost all of that information, until the only thing they believe about 'either' Feynman is that he is a famous physicist (and perhaps also that he is distinct from the 'other' Feynman). Sutton claims to feel the intuition that in this case, *S*'s belief "Feynman$_1$ ≠ Feynman$_2$" is a rationally believed empirical falsehood, rather than an irrationally believed logical contradiction. (Sutton 2004, p.97)

The worry is that Sutton's case is in tension with the claim that information retention suffices for memory-based singular thought. For *S* to have two distinct "Feynman"-files, files need to be distinguished by something *other* than informational content, informational channel,

[38] Thanks to an anonymous reviewer for bringing this case to my attention.
[39] The uniqueness of the channel transmitting information to *S* is not a feature of Sutton's case, but I have added it because it strengthens the point against the mental files-as-clusters view.

reference, or type-identical labels in *S*'s language of thought. But if a file is a *cluster* of information such that there is nothing to the file over and above the information comprising it, then there is no such distinguishing feature. Thus if *S* believes "Feynman$_1$ ≠ Feynman$_2$", they are deploying the same file twice over. But then *S*'s belief has the logical form ⌜α ≠ α⌝. And so *S* believes a logical falsehood, and is irrational in doing so (and possibly, it seems, unaware of this fact.)

One option is to just bite the bullet and reject Sutton's intuition that *S* really *is* rationally believing an empirical falsehood, thereby avoiding any need to invoke anything other than clusters of information. But for the reader agreeing with Sutton on the intuitive data I now briefly sketch an answer as to what could distinguish files sharing the same informational content, informational source, reference, and label.

The answer is to postulate that mental files have *addresses* in the cognitive system, whence *S*'s 'Feynman'-files are distinct in virtue of occupying distinct addresses. Note that the *object files* postulated in cognitive science are taken to have addresses: they occupy pre-existing 'slots' in working memory, of which there are 3 or 4. (Dehaene 2011, p.259.) There seems no reason to suppose that two of these slots couldn't contain that same information concerning the same object got from the same source and labelled with the same name.[40] Note also the similarity to Fodor's response to Kripke's "Paderewski" case.[41] For Fodor, *S*'s "Feynman"-files actually do have distinct labels, "Feynman$_1$" and "Feynman$_2$", in *S*'s *language of thought*. (Fodor 2008, pp.72-74.) And lastly, some thinkers (e.g. Recanati 2012, p.37) use 'address' and 'label' interchangeably.

Moreover, postulating different addresses for the two files is consistent with the claim that

[40] Recanati speaks in terms of files having 'addresses' (2012, p.37) but equates this with 'labels'.
[41] Kripke's 'Paderewski' case has *S* believe *distinct* things about two homonymously named individuals (that are really one). (Kripke 1979, p.265/6.)

retaining information is necessary and sufficient for retaining a file. We need only allow that whenever some information is retained as concerning a single individual, it is retained at an address, and if information and label are not retained, the address – or 'slot' – does not refer 'by itself'. (Whether the feature of having an address counts as lexical rather than semantical – as Sutton would have it – is independent of our concerns about memory-based singular thought and conversion, and so I take no stand on this.)

My second remark concerns the question of the conditions under which agents *do* retain information. This is an empirical question whose answers have empirical consequences to which those answers are responsible.[42] Hence any full defense of such an answer requires a review of the empirical literature too extensive to pursue here. Moreover, there is room to doubt whether cognitive scientific laws – such as that governing information retention – are amenable to any kind of non-trivial armchair analysis in terms of folk-psychological notions like significance or acquaintance.[43] In light of such concerns, I will ultimately only briefly discuss the relative prospects of acquaintance and significance in explaining information retention at the end of this section, and I officially refrain from defending any particular philosophical analysis of the conditions under which information is retained.

Nevertheless, even if no non-trivial folk-psychological analysis of information retention is forthcoming, we have still established an important result: retention of information about *o* is necessary and sufficient for possessing a stable file for *o*. This is important because this is a

[42] Thanks to an anonymous reviewer for bringing this point to my attention.

[43] Cf. Hansen and Rey (2016), p.433: "Specific causal relations are no doubt relevant in many cases; but why think they are present in *all*? Why think there's any *general* solution to puzzles of this sort beyond occasion-relative pragmatics and forensics? Indeed, *pace* the recent resurgence of interest in traditional "metaphysics," why think that there's a general satisfactory account of *all* the multitude of "things" that we are able to think about? It's hard not to suspect that the majority of such issues are really just matters of pragmatics and forensics." (Emphasis original.)

condition *not* on having a stable file for a *past perceived* object, but on having a stable file *simpliciter*. But note that *both* thought about past perceived objects *and* acquaintanceless thought about mathematical abstracta are thought from stable files. Therefore information retention sustains *both* the capacity for memory-based thought about past perceived objects, *and* thought about objects with which we are not, or even *can*not, be acquainted. And so, even lacking a philosophical or cognitive scientific analysis of the conditions under which information is retained (in terms of significance or acquaintance or whatever), we have: given that we *do* possess stable files for past perceived objects, we *can* possess stable files for abstract mathematical objects.

Therefore one potential explanation of the ordinary speaker intuitions about the logical forms and inferential behaviour of thoughts like "it is not the case that *e* can be expressed as a fraction" cited in section 2 is that in addition to demonstrative files, *descriptions* can be converted into stable files. The result is a stable file about the satisfier of the description at the time of conversion – just as the referent of a perception-derived stable file is the reference of the converted demonstrative file at the time of conversion.[44]

Here's how a description could be converted into a stable file, resulting in a device for acquaintanceless singular thought. An agent *S* coins a definite description *D* they believe is uniquely satisfied, such as "the limit of $(1 + 1/n)^n$ as n approaches infinity". *S* then infers (deductively, inductively or abductively) from that description and perhaps some relevant background beliefs some information about *e* – e.g. that its decimal expansion probably does not contain "777" – that, for whatever reason, is retained by *S*. Such an occurrence certainly seems *possible*, if not fairly pedestrian. Given that *S* is retaining a store of information about *e*, *S*'s

[44] *Modulo* concerns about reference change over time, which I leave aside.

cognition converts *D* into a stable file that refers to the unique satisfier of *D*. And so, given that possession of a stable file referring to an object *o* is necessary for being able to have singular thoughts about *o*, *S* is now part of the way towards have singular thoughts about the satisfier of *D* (what else is required is the topic of the next section) – even if it is metaphysically impossible for any member of *S*'s linguistic community to perceive of otherwise be causally related to the satisfier of *D*.

Note that reducing the question of possession of a stable file to that of information retention ensures that if information retention is not under voluntary control, then neither is possession of stable files. It certainly seems as though information retention is not under voluntary control. Many of us are familiar with the frustration of having forgotten some important item of information, or of involuntarily retaining some information one would rather forget. But, if information retention is not under voluntary control, then neither is stable file possession. Stable files cannot be had 'on the cheap'. Given that acquaintanceless singular thought and memory-based thought are both thought from stable files, the ability to have singular thoughts about an object in virtue of a *past* perceptual link with it is no more under voluntary control than the ability to have acquaintanceless singular thoughts (as in Kaplan's "Newman 1" case). The claim that there is no singular thought 'on the cheap' applies just as much to memory-based singular thought as it does to acquaintanceless singular thought. This is consistent with both capacities being rooted in the same capacity: information retention.

At this point a cognitivist might claim that information retention is itself triggered and constrained by the *significance* of the object thought about. There are two things to say about this. One is that this runs afoul of the scepticism about the non-trivial use of folk-psychological concepts in cognitive scientific laws mentioned above. Setting this aside, the other thing is to note that this

proposal seems, on the face of it, more plausible than saying it is *acquaintance* that so triggers and constrains retention of information. Here are two reasons for this. First, it is exactly why we retain information about only *some* objects with which we have been perceptually acquainted that is at question. Second, it seems that explaining information retention in terms of *continued* acquaintance is circular. It is not clear how being acquainted with an object through memory is anything other than retaining information about it gained previously. But this is what we are trying to explain. Thus insofar as the notion of acquaintance cannot give any useful analysis of information retention, it may be the case that significance fares better as a folk-psychological analysis.

Nevertheless, whether or not significance can explain possession of a stable file we saw in section 3.4 that some stable files can only be used to have descriptive thoughts. Hence it is still open to an acquaintance theorist to maintain that even if acquaintance does not constrain retention of information and hence stable file possession, acquaintance is nevertheless required for possession of a stable file that can be used to have *singular* thoughts. In the next, final section I describe a further condition that a stable file must satisfy for it to be a device for singular thought (from Goodman 2016*a*) to do with the kinds of information changes a file is capable of surviving. I then argue for the possibility of a file for an abstract mathematical object satisfying that condition, thus rendering possible singular thought about abstract mathematical objects.

5: Descriptive Files

If everything I've said so far is correct, then information retention is necessary for acquaintanceless singular thought. But is it sufficient? According to Goodman some stable files – *descriptive files*

– are devices for thinking about individuals *merely as possessors of certain properties*. Hence their contribution to the thoughts they are used to have are definite descriptions like "the tallest structure in Paris" rather than objects like the Eiffel Tower. For all I've said so far, stable file-based thought about mathematical abstracta may be of the former variety.

In this final section I argue that if *all* files for abstract mathematical objects are descriptive files, then an intuitively possible situation ought not be possible. In particular, it ought not be possible for it to seem to a mathematician that their thoughts about the mathematical entities they study to involve deploying the same files before and after a change in belief about what those entities really are. My example will be reconceiving numbers as sets. However I will argue that it is highly plausible that this *is* a way that things could seem to a mathematician. And if this is right, then it is plausible that we can have *singular* stable files for mathematical abstracta. To begin I examine some relevant details concerning descriptive files.

One important thing distinguishing descriptive from singular files is the kinds of change in contained information which they can survive.[45] Singular files can, in principle, survive the loss of any piece of information – including the description that fixed its reference (if that's how its reference was originally fixed).[46] For descriptive files, on the other hand, one piece of information is *privileged*: the description it contributes to the content of the thoughts it is used to have. A descriptive file cannot survive a change in which description is privileged in this way.

Now, Goodman distinguishes *purely* descriptive files from *holistically* descriptive files. A

[45] This is not the *only* difference between descriptive and singular reference for Goodman. (Cf. Goodman 2016*a*, pp.445-6.) However it would take us too far afield to address the other differences and their connections with the difference in terms of persistence conditions here.

[46] Modulo concerns regarding *sortalism* about reference – the claim that to think about an entity requires not being mistaken about what *kind* of entity it is, in some sense of 'kind'. I here set aside such concerns – they shouldn't have any impact on the arguments I'm going to give.

purely descriptive file cannot survive a change of the (single) contributed description. A holistically descriptive file contributes a cluster – i.e. disjunction – of descriptions, and can survive piecemeal, 'ship of Theseus'-style changes of individual members of that disjunction. Suppose that *S* has a holistically descriptive α-file. At time t_1 *S*'s α-file contributes the disjunction $P \vee Q \vee R$; at t_2 the disjunction $A \vee Q \vee R$; at t_3, $A \vee B \vee R$; and at t_4, $A \vee B \vee C$. If the α-file is holistically descriptive then it can survive these changes. (I take is that this kind of change is supposed to be a descriptive counterpart to a singular file's undergoing reference-change over time, as in Evans' (1973) "Madagascar" example.) But consider now the *simultaneous* replacement of *every* member of the contributed cluster with a new description not previously in the cluster. For instance at t_1 the contribution of *S*'s α-file to the contents of *S*'s thoughts is $P \vee Q \vee R$ but at the very next moment t_2 it is $A \vee B \vee C$. This is not piecemeal, ship of Theseus-style change; this is *wholesale* change. While Goodman does not explicitly say so, it seems that in cases of such simultaneous wholesale changes in contributed clusters of description, the file does not survive. Rather, the file ceases to exist, and a new one may be initiated; the α-files *S* deploys at t_1 and t_2 are *distinct* files (perhaps bearing the same label).

Here is a reason for thinking descriptive files cannot survive wholesale information change. Goodman explicitly uses the phrase 'ship of Theseus-style change'.[47] But the ship of Theseus definitely *cannot* survive wholesale simultaneous replacement of all its component parts. Now, she does also say that for descriptive files, the 'lack of any informational link' with the object the file is about – we might wonder whether she means *causal* informational link – entails that the only candidate mechanism constraining how the cluster of descriptions the file contributes to the

[47] "holistically descriptive files are such that, in principle, they allow for complete information overhaul – that is, they allow for *'ship of Theseus-style' changes*." (Goodman 2016*a*, p.458; my emphasis.)

content of the thoughts it is used to have can change over time is inference from the information contained in the file. (Goodman 2016*a*, p.459.) Thus it *might* be in principle possible for a holistically descriptive file to undergo simultaneous wholesale change in its contributed cluster of descriptions, provided every member of the new cluster is *simultaneously* inferred from old information in the file. At any rate, whether or not such a change is psychologically possible, it remains that the case I'm about to describe does not involve using the information contained in the file at t_i as a base for inference.

Now to the argument that if all our files for abstract mathematical files are purely or holistically descriptive then it should not be possible for it to seem to a mathematician that their thoughts about deploying files for natural numbers have the same content before and after a change in belief about whether (e.g.) natural numbers are sets or not. Consider a mathematical community that starts out taking the natural numbers to be accurately described by the Peano-Dedekind axioms. The members of this community take the number 0 to be the only number that is not a successor; they take 1 to be the successor 0; 2 to be the successor of 1; and so on. Call this the *Peano-Dedekind conception* of the natural numbers.

At a certain point this community comes to agree that the natural numbers are better described by the Frege-Russell definition of the natural numbers as sets of equinumerous sets, implemented in Quine's New Foundations set theory with urelements (NFU).[48] Note that we can stipulate that this agreement is *not* the result of inference from the Peano-Dedekind conception itself. It could be due to advances in set theory plus an obsession with ontological parsimony. Thus the members of this community come to take 0 to be the set of sets equinumerous to the set of non-

[48] The use of NFU is solely to avoid paradox in the example. NFU is both consistent (relative to Peano arithmetic) and also countenances a universal set. Nothing (else) of substance hangs on this choice.

self identicals, i.e. $\{x: x \text{ 1-1 } \{y \neq y\}\}$. They take the number 1 to be the set $\{x: x \text{ 1-1 } \{0\}\}$, the number 2 to be $\{x: x \text{ 1-1 } \{0, 1\}\}$, and so on. Call this the *Frege-Russell* conception of the natural numbers.

Consider now the claim that, after the move, it *subjectively appears* to members of this community that when thinking and speaking about the natural numbers, they *really were* thinking and speaking about sets of equinumerous sets all along, appearances to the contrary notwithstanding. So it appears to them that the number 2 they have been thinking and speaking about really always was the set $\{x: x \text{ 1-1 } \{0, 1\}\}$; that insofar as 2 was taken as the successor of 1, this appears to be at best a confused or incomplete conception; and that even when subscribing to the Peano-Dedekind conception, when they added 3 and 4 to get 7, 3 really was a member of a member of 4, despite their ignorance of this.

Now, my point here is not about whether this community's 'hidden essentialism' about abstract mathematical objects is correct or ultimately philosophically justifiable. Rather, it is about whether it can *subjectively appear* to an individual member of this mathematical community that they were thinking and speaking of sets all along. For as I will now argue, taking all stable files for abstract mathematical objects to be either purely or holistically descriptive entails that it should be *impossible* for it to *seem* to a mathematician that abstract mathematical objects have such 'hidden essences'. And so, if it *can* seem to a mathematician that mathematical objects have hidden essences, then – whether or not it's plausible that this seeming could be correct – we *can* have stable files for such objects that are not purely or holistically descriptive.

Here is what the subjective appearance of hidden essentialism in the above example amounts to in a singular/descriptive mental files framework. Consider a member of the above community M who at time t_i operates with the Peano-Dedekind conception. Hence M's "0"-file

contains information like "is the only number that is not a successor" and "is the identity element for addition". *M* then at a later time t_j adopts the Frege-Russell conception. After t_j, *M*'s "0"-file contains set-theoretic information like "is the set of sets equinumerous to $\{y: y \neq y\}$".

Now, if it seems to *M* that numbers were sets all along, then it seems to *M* that their token thoughts and utterances of (e.g.) "1 is the number of the Earth's moons" have the same *content* both before and after the adoption of the Frege-Russell conception – just as it seems to Henry Cavendish that his token thoughts of "water is wet" had the same content before and after he discovered that water = H_2O. Now, given that are talking about subjective appearances of identity of content, the notion of content operating here is a *narrow* one according to which "Hesperus = Hesperus" and "Hesperus = Phosphorus" have different content. This contrasts with content as singular proposition expressed – *wide* content – in which case "Hesperus = Hesperus" and "Hesperus = Phosphorus" express the same proposition (assuming that "Hesperus" and "Phosphorus" are both singular terms). In the narrow sense of content, for tokenings ⌜α is Φ⌝ and ⌜β is Φ⌝ to express the same content, the α-file must be the same file as the β-file.[49] And hence, if it seems to *M* that their tokenings of "1 is the number of the Earth's moons" before and after the move have the same (narrow) content, then it seems to *M* that they deploy the same file twice over when tokening that thought at those different times.

However, *M*'s number-files being purely or holistically descriptive precludes *M* deploying the same "1"-file when tokening "1 is the number of Earth's moons" before and after the move. This is because if *M*'s number-files are purely or holistically descriptive, then they cannot survive

[49] Recanati claims that what accounts for the difference in content between "Hesperus = Hesperus" and "Hesperus = Phosphorus" is that one file is deployed twice when thinking the former, but two files are deployed once each in the latter. (Recanati 2012, p.42.)

(wholesale) changes in the (disjunctions of) descriptions they contribute to the thoughts they are used to have. The most plausible candidates for which (disjunctions of) descriptions *M*'s number-files contribute at a given time t_i are those expressing *M*'s conception at t_i of what the numbers are. Thus prior to t_j the best contenders for the (disjunctions of) descriptions contributed by *M*'s number-files are disjunctions of Peano-Dedekind descriptions like "the number that is not a successor", "the additive identity", and so forth. After t_j the best contenders for the contributed (disjunctions of) descriptions are disjunctions of Frege-Russell descriptions like "the set of sets equinumerous to $\{x: x \neq x\}$" and "is a member of 1", etc. But if all of *M*'s number-files are either purely or holistically descriptive, *and* prior to t_j the (disjunctions of) descriptions fixing the references of *M*'s number-files are (disjunctions of) Peano-Dedekind descriptions whereas after t_j they are (disjunctions of) Frege-Russell descriptions, then those files contribute *entirely different* (disjunctions of) descriptions before and after t_j. We know that descriptive files cannot survive (wholesale, non-inferential) change in the (disjunctions of) their contributed descriptions. Thus *M* has distinct number-files before and after t_j. And, if *M* has distinct number-files at t_i and t_j, then *M*'s tokenings of "1 is the number of the Earth's moons" have different contents at t_i and t_j.

Thus if it *seems* to *M* that their tokenings of "1 is the number of the Earth's moons" at t_i and t_j do have the same content, then it seems to *M* that they are deploying the same number-files before and after t_j. But if all *M*'s number-files are purely or holistically descriptive then this seeming is deceptive, then *M* is *mistaken* about the identity of their number-files at these different times. However there are reasons to hold that a thinking agent cannot be mistaken about whether they are deploying distinct files on two occasions versus the same file twice over. For if this kind of mistake were possible, then one could be mistaken about whether *it appears to one* that a tokening of "Aristotle is Aristotle" deploys the same "Aristotle"-file twice over and hence is

trivially true (and its denial irrational), or deploys distinct "Aristotle"-files and hence could be empirically false (the Stagyrite was not a shipping magnate). This is not, it seems, a kind of mistake about subjective appearance one should be able to make.[50] If that's right, then if it *seems* to *S* that they are redeploying the same mental file twice over, then that *is* what they are doing. Hence the identity of mental files is *transparent* to the thinker.

Note that the impossibility of being mistaken about whether one is deploying distinct files or the same file twice over is consistent with being mistaken about whether two deployments of the same make the same contribution to the (wide) content of the thoughts they are used to have. Just as *S*'s "Madagascar"-file may contribute to the (wide) content of *S*'s thoughts an area of continental Africa on one occasion of deployment, but the large coastal African island on another, is consistent with *S* both being unaware of this change and having introspective access to whether they are deploying the same "Madagascar"-file on both occasions. Likewise, that *S*'s deployments of their α-file contribute $A \vee B \vee C$ on one occasion, and $P \vee Q \vee R$ on another, is also consistent with *S* being unaware of this. Thus, that *S* has introspective access to whether they are deploying the same file twice over is consistent with *S* lacking introspective access to whether that file has the same semantic value on different occasions.

Now, if it seems to *M* that their thinkings of "1 is the number of the Earth's moons" before and after the move from the Peano-Dedekind to the Frege-Russell conception of the natural numbers have the same content, then it seems to *M* that they are deploying the same file on both occasions. Therefore if mental file identity is transparent to the thinker, then *M* is deploying the

[50] "[M]ost philosophers of mind accept the … thesis that you have *transparent* access to the content of your own thoughts: provided you're minimally rational, you simply cannot mistake one conceptual content for another" (Schroeter 2007, p.597; quoted in Recanati 2012, p.117.) Recall also Sutton's (2004) intuition discussed in section 4.

same files both before and after the move. This latter claim is inconsistent with *M*'s number-files being purely or holistically descriptive because in that case, their number-files would not have survived the move.

At this point one could object that this is not a wholesale change in contributed disjunctions of descriptions, but a rapid piecemeal change. However to maintain this is tantamount to denying that there can be wholesale change in contributed disjunctions of descriptions. We are yet to see any argument for this; hence I leave it aside.

It remains to establish that it *can* seem to *M* that they token the same content when thinking "1 is the number of the Earth's moons" before and after the move. One reason to think it can is that we do not have to look far to see philosophers and mathematicians making claims that, on the most obvious reading, entail that such seemings are possible. Here are three philosophical examples:

> Natural numbers are sets. They are finite von Neumann ordinals. (Steinhart 2002, p.343.)
>
> My view is that numbers are certain nondistributive formal properties of multitudes. (Simons 2007, p.233.)
>
> I maintain that numbers are not abstract ("Platonic") entities, but such familiar things as sticks and stones, apples and books. (Zemach 1985, p.225.)

In the absence of a claim that philosophers and mathematicians disagreeing with these views think falsely when having arithmetical thoughts, these claims are most naturally read as claims about what numbers really were all along. But if all our number-files are purely or holistically descriptive, then they cannot survive a reconception of what the numbers are. This entails that adoption of any of the above views commits one to the claim that prior to the reconception, one's arithmetical thoughts were mostly false. I submit that this result is implausible.

Moreover it is not just philosophers that make such claims about what mathematical entities have always really been. Working mathematicians make them too. For instance:

> All branches of mathematics are developed, consciously *or unconsciously*, in set theory. (Levy 1979, p.3; my emphasis.)

The most natural reading of this statement has it that if *M* was 'unconsciously' doing set theory before the move to the Frege-Russell conception, and consciously thereafter, then *M* was thinking about sets all along, but has only recently come to realise this. Hence the author of *Basic Set Theory* appears to subscribe to a view entailing that one can maintain the same mental files for mathematical objects across reconceptualizations.

Thus insofar as we should take seriously the claims and consequential commitments of working mathematicians and philosophers, we should allow that it can seem to at least some mathematicians and philosophers that they have thoughts with the same contents before and after a reconceptualization of what the relevant objects are (such as reconceiving numbers as sets). And as we've seen, this in conjunction with the transparency of mental file identity entails that we can have mental files for numbers that are neither purely nor holistically descriptive.

Finally, we can preserve the transparency of mental file identity for these mathematicians and philosophers if we regard our files for abstract mathematical objects as capable of being singular, even though their references were originally fixed descriptively. For then there is no barrier to claiming that the same number-files persist through the move from the Peano-Dedekind to the Frege-Russell conception. Hence when *M* takes themselves to be deploying the same file in thinking and uttering "1 is the number of the Earth's moons" before and after adopting the Frege-Russell conception, they are correct in this (regardless of whether or not they are correct in thinking that the file refers to the same (or any) object on both deployments, or in thinking they should

adopt the new conception in the first place). Therefore we should regard it possible that we can have singular files for, and thus singular thoughts about, abstract mathematical objects – though what exact cognitive-scientific laws govern the initiation and maintenance of such files is a question I leave for another time.

If everything I've said so far is correct, then we have established two important claims about acquaintanceless singular thought about abstract mathematical objects. First, to have such thoughts it is necessary that we can have stable files for such objects. I have argued that this requires no more than what is required by our ability to have memory-based singular thoughts. Second, for those stable files to be devices for *singular* thought requires that they be able to survive simultaneous wholesale change in the descriptions fixing their reference; something which, I hope to have shown, is intuitively possible.

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