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Quantitative Properties

Philosophy Compass, forthcoming

Abstract: *Two grams mass*, *three coulombs charge*, *five inches long* – these are examples of quantitative properties. Quantitative properties have certain structural features that other sorts of properties lack. What are the metaphysical underpinnings of quantitative structure? This paper considers several accounts of quantity, and assesses the merits of each.

Certain features of the world are perspicuously represented using a numerical scale. Some examples: amounts of mass, wavelengths of light, spatiotemporal distances. Call features of this sort *quantities*.

Quantities have two distinguishing characteristics. First, quantities within a family may be *ordered*; e.g., a 1g-mass object is less massive than a 2g-mass object, which is less massive than a 3g-mass object, and so on. Second, quantities within a family stand in *closeness* or *distance* relations to one another; e.g., a 2g-mass object is closer in mass to a 1g-mass object than it is to a 20g-mass object. Indeed, we can say exactly *how close* these objects are with respect to their masses: the distance in mass between a 2g-mass object and a 1g-mass object is one gram, and the distance in mass between a 2g-mass object and a 20g-mass object is eighteen grams.[1, 2]

The ordering and closeness relations behave in certain prescribed ways. For instance, if *a* is less massive than *b* and *b* is less massive than *c*, then *a* is less massive than *c*. And if the distance in mass between *a* and *b* is two grams, and the distance in mass between *b* and *c* is four grams, then the distance in mass between *a* and *c* is six grams. And so on.[3]

The ordering and closeness structure of quantities is mirrored by the ordering and closeness structure of numbers. We can exploit this symmetry, and

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[1] It's an interesting question how to treat properties such as *zero grams mass* (see Balashov 1999). In this paper I assume that *zero grams mass* belongs to the mass family.

[2] In earlier work, Brian Ellis argued that ordering suffices for quantity (Ellis 1960 and 1966, 24-38 and 74-89; c.f. Forge 1987), but he later recanted (Ellis 1987).

[3] See Krantz *et al.* (1971) for the canonical presentation of measurement theory. Their characterization of quantities employs two basic notions: *less than or equal to* ($\preceq$) and *concatenation* ($\circ$). Consider a family of quantities – for instance, the mass quantities. The *less than or equal to* predicate associated with the mass family ($\preceq_m$) generates an ordering. The *concatenation* predicate associated with the mass family ($\circ_m$) corresponds to a notion of "combining"; e.g., the concatenation of a 1g-mass object with a 2g-mass object yields a 3g-mass object. Intuitively, the *concatenation* predicate delivers the closeness structure of quantity: a 1g-mass object and a 3g-mass object are exactly two grams mass "apart."

use numbers to transparently represent quantitative structure.[4] For instance, a perspicuous numerical representation of the mass quantities might require that the number assigned to *x* is less than or equal to the number assigned to *y iff x* is less than or equal in mass to *y*.[5]

What are the metaphysical underpinnings of quantity, and how do they give rise to the ordering and distance relations distinctive of quantitative structure? This paper considers several accounts of quantity, and assesses the merits of each. Some criticisms I discuss have been raised in the literature, some are new. Throughout, I rely on *mass* as my case study; in most cases the extension to quantities like *charge*, which takes positive and negative values, is straightforward.[6]

1. Quantities as Proportions

In order to get a feel for the sorts of issues that arise when developing an account of quantity, let's begin with a simple proposal. Paradigmatic quantities – like quantities of *mass* or *charge* – stand in ratios or proportions to one another.[7] For instance, an 2g-mass object is twice as massive as a 1g-mass object, a 6g-mass object is three times as massive as a 2g-mass object, and so on. This observation may lead to the following thought: quantities are nothing "over and above" relations of proportion.

On one way of developing this view, facts about quantities are ultimately grounded in proportion relations among objects. So, suppose object *a* is twice as

[4] Of course, we could use a numerical scale to represent other sorts of properties as well. But the results would be somewhat contrived. Consider the determinable *being a shoe* and its determinates: *being a sneaker*, *being a dress shoe*, *being a sandal*, etc. One might represent these properties using a numerical scale by, say, mapping *being a sneaker* to the number 1, *being a dress shoe* to the number 2, and so on. But in these cases the representation is clearly artificial, and does not capture anything interesting about the nature of the determinates.
[5] The numerical assignment is generally required to be a homomorphism from $\langle A, \preceq, \circ \rangle$ into $\langle \mathbb{R}, \leq, + \rangle$, where *A* is the set of elements over which $\preceq$ and $\circ$ are defined (Krantz *et al.* 1971, 8-12). Note, however, that the adoption of this constraint as opposed to some other one is a conventional choice. "[E]very pair of representation and uniqueness theorems involves a choice of a numerical relational structure. This choice is essentially a matter of convention, although the conventions are strongly affected by considerations of computational convenience." (Krantz *et al.* 1971, 12)
[6] I put aside issues arising from more complex quantities, such as vectors. For some recent discussion, see *Dialectica*'s (2009) special issue on the philosophy of vectors; in particular, Beisbart (2009), Busse (2009), Forrest (2009), Leuenberger and Keller (2009), and Simons (2009).
[7] Arguably, not all quantities stand in proportions of this sort – for instance, *temperature*. Such quantities are commonly measured on an interval scale rather than a ratio scale. But in these cases the intervals between elements stand in proportions to one another; e.g., the temperature-distance between 5°C and 10°C is half the temperature-distance between 10°C and 20°C.

massive as object b. Then there is a relation – *twice as massive as* – that a bears to b. But there are no further facts about exactly how much mass a and b have, independently of how they are related to one another and other objects.

Now, it might be tempting to think that the ordering and closeness structure of quantity "falls out" of the pattern of proportion relations. After all, the thought goes, if a is twice as massive as b, then it automatically follows that b is less massive than a; and if a is twice as massive as b and b is twice as massive as c, then it automatically follows that b is closer in mass to c than to a. But in order to be justified in making any of these inferences, two further things must be done. First, one must say something about how the primitive predicates of our theory behave. Second, one must demonstrate how the behavior of these predicates give rise to the ordering and closeness structure of quantity.

So we need to fill in our simple proposal. Let's begin by laying down some axioms that govern the behavior of the posited relations. Here's a start:

(1) There is a unique mass-proportion relation M^* such that for all x, if x stands in any mass-proportion relation, then $M^*(x, x)$ (i.e., there is a unique *just as massive as* relation).

(2) For any mass-proportion relation M, there exists a mass-proportion relation N such that for any objects x and y, $M(x, y)$ *iff* $N(y, x)$ (i.e., every mass proportion relation has an inverse).

(3) For any mass-proportion relations M and N, there is a mass-proportion relation O such that, for any objects x, y, and z, if $M(x, y)$ and $N(y, z)$, then $O(x, z)$.

The next task is to show how the predicates governed by these axioms yield the ordering and closeness structure of quantity. For instance, we ought to be able to define the predicates "less massive than" and "closer in mass to" in terms of our proportion relations, and we ought to be able to show that the defined predicates behave in ways we expect. Without that, there's no connection between the mass proportion relations and these other predicates. And so any inference from "a is twice as massive as b" to "b is less massive than a" is unjustified.

An adequate theory of quantity in terms of proportion relations has to fill in these details. So let us now look at two proportional accounts of quantity that have been proposed in the literature.

2. Quantities as Proportions: A Three-Level Account

John Bigelow and Robert Pargetter (1988) present an account that builds on the simple proposal given above. They begin by positing first-order proportion relations such as *twice as massive as*. Then they add second-order relations that hold among these first-order relations. So they end up with a three-level theory: at level (1) there are objects, at level (2) there are first-order relations, and at level (3) there are second-order relations.

The purpose of adding these higher-order relations is to capture the ordering and closeness structure of quantity. Without these, they say, we cannot explain "the way that some mass-relations are in some sense 'closer' to one another than to others." (1988, 298) They write:

> To obtain an adequate theory of quantities, we must thus introduce a second stage of construction… We must also note that mass-relations can be more and less *similar* to one another. The relations "five times as long" and "six times as long" are in some sense *closer* to one another than either is to say "100 times as long"… Hence we need to recognize relations among relations…
> Thus level (3) relations of proportion classify level (2) determinates into equivalence classes. Within each such equivalence class of level (2), the level (3) proportions will also impose an *ordering*. And it is this ordering which explains how one thing can be *closer* to a second, in, say, mass than it is to a third. (Bigelow and Pargetter 1988, 299)

Does Bigelow and Pargetter's refinement of the simple proposal given in the previous section succeed in capturing the structure of quantity? The problem with the simple proposal was that it was incomplete: it did not specify what features the posited proportion relations have, and it did not demonstrate how these relations generate the ordering and closeness structure associated with quantity. So in order to improve upon the simple account, Bigelow and Pargetter's account needs to fill in these details.

In (1990, ch. 8), Bigelow and Pargetter make some suggestive remarks concerning the features of the first- and second-order relations they posit. They begin with a simple example – the relation between *grandparent of* and *parent of* – and use this as a model. Consider how *grandparent of* relates to *parent of*: if there exists an x, y, and z such that z is the parent of y and y is the parent of x, then z is a grandparent of x.[8] Intuitively, two successive applications of the *parent of* relation yields the *grandparent of* relation. According to Bigelow and Pargetter,

[8] Bigelow and Pargetter reverse the order of this conditional; they write, "if a is a grandparent of b, there is a c such that a is a parent of c and c is a parent of b." (1990, 355) While this doesn't matter in the grandparent-parent example, since it's plausible that the claim is a biconditional, it is problematic when applied to quantities. For instance, an object a may be four times as massive as b even if there is no c such that a is twice as massive as c and c is twice as massive as b.

this shows us that the *grandparent of* relation stands in a second-order relation of proportion to the *parent of* relation – the proportion 2:1 (1990, 355).

Let us see how this model applies to mass relations. Take *twice as massive as* and *four times as massive as*: if there exists an x, y, and z such that z is twice as massive as y and y is twice as massive as x, then z is four times as massive as x. Let's abbreviate this as follows:

If *twice as massive as*$^2(z, x)$, then *four times as massive as*(z, x).

Intuitively, two successive applications of the *twice as massive as* relation yields the *four times as massive as* relation. So the *four times as massive as* relation is "twice" as large as the *twice as massive as* relation. Thus these two relations stand in the second-order relation of 2:1.

More generally, for any mass-relation M such that $\sim M(x, x)$ and for any natural numbers i, j, there exists a mass-relation N such that $M^i(x, y)$ *iff* $N^j(x, y)$. And when M and N are so related, they stand in the proportion relation of i:j. (For extension to the rational and real numbers, see Bigelow and Pargetter (1990, 352-363).)

While this is an interesting strategy, Bigelow and Pargetter's account runs into a number of difficulties – many of which stem from the fact that relations such as *twice as massive as* are multiplicative rather than additive. First, it seems that the first-order relations do not, in fact, have the features that Bigelow and Pargetter desire. Consider the congruence relation *just as massive as* (see Forge (1995, 598-600)). According to Bigelow and Pargetter, the proportion relation holding between *grandparent of* and *parent of* is 2:1, since two applications of the *parent of* relation yields the *grandparent of* relation (if *parent of*$^2(x, y)$, then *grandparent of*(x, y)). Now consider the relation between *twice as massive as* and *just as massive as*. Intuitively, *twice as massive as* is "twice" as much as *just as massive as*; they too stand in the proportion relationship of 2:1. However, successive applications of the *just as massive as* relation will not take us to a new relation: for any n whatsoever, if *just as massive as*$^n(x, y)$, then *just as massive as*(x, y). So it is not clear how Bigelow and Pargetter's account can capture the fact that *twice as massive as* and *just as massive as* stand in the proportion 2:1 – indeed, it is not clear whether the account captures the fact that these relations stand in any higher-order proportion at all.[9]

[9] Likewise, it seems that *just as massive as* and *half as massive as* also stand in the proportion relationship of 2:1. But successive applications of the *half as massive as* relation yield *one-fourth as massive as*, *one-eighth as massive as*, and so on. Again, it is not clear how the account can capture the fact that *just as massive as* and *half as massive as* stand in the proportion 2:1, or that they stand in any proportion at all.

Second, Bigelow and Pargetter do not demonstrate how their posited relations yield the ordering and closeness structure of quantity – and it isn't clear that these relations *could* yield this structure. Let's first consider the ordering relation, *less than or equal to in mass* ($\preceq_m$). Suppose that *c* is four times as massive as *a*, and *b* is twice as massive as *a*. It seems clear that *a* has less mass than *b*, and *b* has less mass than *c* (i.e., $a \preceq_m b \preceq_m c$). How can Bigelow and Pargetter capture this? One suggestion, in the spirit of their account, is to say that $a \preceq_m b \preceq_m c$ in virtue of the fact that *twice as massive as* and *four times as massive as* stand in the second-order relation of 1:2. Because 1 < 2, one might say, *twice as massive as* is "smaller" than *four times as massive as*. And if the relation *b* bears to *a* is smaller than the relation *c* bears to *a*, then $a \preceq_m b \preceq_m c$.

But this strategy won't work. To see why, consider the relation between *half as massive as* and *one-fourth as massive as*:

If *half as massive as*^{2}(*a*, *c*), then *one-fourth as massive as*(*a*, *c*).

(I.e., if there exists an *a*, *b*, and *c* such that *a* is half as massive as *b* and *b* is half as massive as *c*, then *a* is one-fourth as massive as *c*.)

Two applications of the *half as massive as* relation yields the *one-fourth as massive as* relation; so, *half as massive as* and *one-fourth as massive as* stand in the proportion relation of 1:2. Since 1 < 2, the present suggestion entails that *half as massive as* is "smaller" than *one-fourth as massive as*. And since the relation *b* bears to *c* is smaller than the relation *a* bears to *c*, $c \preceq_m b \preceq_m a$. So the present suggestion for recovering ordering is incoherent: it entails both $c \preceq_m b \preceq_m a$ and $a \preceq_m b \preceq_m c$. More generally, given the structural symmetry between the relations *twice as massive as*, *four times as massive as*, etc., and their inverses, it's not clear how any proposal using Bigelow and Pargetter's framework could capture the ordering relations among mass quantities.

Let's next consider the closeness relation, *closer in mass to*. Suppose that *a* has two grams mass, *b* has four grams mass, and *c* has eight grams mass. We want to recover the fact that *b* is closer in mass to *a* than to *c*. But *c* is twice as massive as *b*, and *b* is twice as massive as *a*. So the same relation, *twice as massive as*, holds between both pairs of objects. On Bigelow and Pargetter's account, then, the mass-distance between *b* and *a* is the same as the mass-distance between *c* and *b*. But that's the wrong result.[10]

[10] Or consider the following: suppose *c* has two grams mass, *b* has one gram mass, and *a* has one-half gram mass. Again, we want to capture the fact that *b* is closer in mass to *a* than to *c*. But here we run into even more trouble. For reasons given above, Bigelow and Pargetter's account entails that no

In sum, a number of challenges face Bigelow and Pargetter's three-level theory of quantity.[11] One might attempt to resuscitate their account by replacing the proportion relations (such as *twice as massive as*) with interval relations (such as *two grams more than*), and using these interval relations to characterize the ordering and closeness structure of quantity. But one then needs to demonstrate that the resulting account avoids analogues of the worries raised above. So let us look at a different strategy for developing a proportional account of quantity.

3. Quantities as Proportions: Relations to Numbers

A different way to pursue the idea that quantitative properties are grounded in proportion relations is offered by Brent Mundy (1988).[12] Mundy proposes that the underlying structure of quantity involves relations between ordered pairs of objects and numbers – where the number associated with each pair corresponds to, say, the mass-proportion that the objects stand in. So suppose a is a 4g-mass object and b is a 2g-mass object. Intuitively, the proportion of a's mass to b's mass is 2. On this account, there is a mass-proportion relation, R, such that $R(a, b, 2)$.[13] This relation satisfies the following axioms:[14]

For any objects x and y, if there is some object z and some numbers n and n' such that $R(x, z, n)$ and $R(y, y, n')$, then there is some number n'' such that $R(x, y, n'')$. (If x bears a mass proportion to something, and y bears a mass proportion to itself, then x bears a mass proportion to y.)

For any objects x and y, if there is some number n such that $R(x, y, n)$, then there is a number n' such that $R(y, y, n')$. (If x bears a mass proportion to y, then y bears a mass proportion to itself.)

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proportion relation at all holds between b and a, or between b and c. So we get the result that a, b, and c do not stand in *any* mass proportions to one another.

[11] For different criticisms of Bigelow and Pargetter's account, see Armstrong (1988), Ellis (2001, 66-67), and Forge (1995).

[12] My presentation of this account differs slightly from Mundy's, mostly in emphasizing the metaphysical commitments of the view.

[13] A variant of this account takes the relation R to hold among properties (as opposed to objects) and numbers. One disadvantage of this variant is that it quantifies over properties, and so requires a less parsimonious ontology. One advantage of this variant is that it is able to make more fine-grained discriminations among possibilities. For instance, it seems that everything at our world could have doubled in mass. *Prima facie*, a world where every object has twice the mass as corresponding objects in the actual world represents a genuine possibility. But this intuition is hard to capture on the original account, since the pattern of mass-relations at such a world is the same as the pattern at the actual world. See also section (4).

[14] See Mundy (1988, 8). Here I have simplified; Mundy's axioms are designed to allow for ratios to be infinite (although 0/0 is not defined). Since this complication isn't relevant to the metaphysical issues discussed here, I've ignored it.

For any objects x, y, z, and any numbers n, n', n'', if $R(x, y, n)$ and $R(y, z, n')$ and $n \cdot n' = n''$, then $R(x, z, n'')$.

The distinctive feature of this account is its direct appeal to numbers in grounding the structure of quantity – the ordering and distance structure of quantities is inherited from the ordering and distance structure of the real numbers:[15]

x is less than or equal in mass to y *iff* (*i*) there is some number n such that $R(x, y, n)$ and $0 \leq n \leq 1$, or (*ii*) there is no n such that $R(x, x, n)$ and there is no n such that $R(y, y, n)$ (i.e., both x and y have no mass).[16]

the distance in mass between x and z is y *iff* there is some number n such that (*i*) $R(x, y, n)$ and $R(z, y, n+1)$, or (*ii*) $R(y, x, n)$ and $R(z, x, n+1)$.[17, 18]

But there are several worries that arise for views that characterize quantitative structure in terms of relations to numbers. First, any such account hangs on the existence of numbers. And even granting that numbers exist, many accounts of numbers cannot plausibly be conjoined with views like this. For instance, on the view that numbers are "mental constructs" of some sort, this account has the implausible result that the quantitative structure of properties like *mass* is ultimately grounded in relations to our mental states (see Horsten (2012)). Another worry, raised by Hartry Field, concerns the use of numbers in ultimate explanations for physical phenomena. Numbers are causally irrelevant entities, he says, and thus are "extrinsic to the process to be explained." Illuminating explanations of physical phenomena "[do] not invoke functions to extrinsic and causally irrelevant entities" like numbers (1980, 43).

There are other ways to develop the core idea that quantities are fundamentally relational. The next two sections consider two relational accounts of quantity that do not directly invoke proportions. Although both are presented within a nominalist framework, they may be thought of as "relational accounts" because their primitive predicates are polyadic.

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[15] "[T]he real number system does not occur merely as an arbitrarily chosen numerical representational system, as it does in the representational theory of measurement. Both intuitively and within the context of the formal definition, the values of the ratio function must be understood as the real numbers themselves (up to isomorphism). When we say that x is double y, or is *pi* times y, we refer specifically to the actual real numbers 2 or *pi*." (Mundy 1988, 8-9)

[16] The second clause is needed for cases in which x and y have no mass. It is true that x is less than or equal in mass to y, even though there is no number n such that $R(x, y, n)$.

[17] The second clause is needed for cases in which y has no mass.

[18] "The distance between x and z is y" is captured by the *concatenation* predicate ($\circ(x, y, z)$) employed in standard measurement theory. See footnote 3.

4. Nominalism: LESS and CONG

In *Science Without Numbers* (1980), Hartry Field develops a nominalistic account of quantity inspired by Hilbert's theory of geometry. Field's account introduces a two-place *less than or equal to* predicate (LESS) and a four-place *congruence* predicate (CONG) for every family of quantities.[19] So, *mass-LESS* and *mass-CONG* predicates apply to objects with mass. To say that an object *b* is mass-LESS than *a* means, intuitively, that the mass of *b* is less than that of *a*. And to say that *ab* is mass-CONG to *cd* means, intuitively, that the difference between the masses of *a* and *b* is the same as the difference between the masses of *c* and *d*.

The axioms governing these LESS and CONG predicates are slightly modified versions of those given by Krantz *et al.* for an "absolute difference structure" (Krantz *et al.* 1971, 143-150, and Field 1980, 58). These axioms ensure that the LESS and CONG predicates behave as we would expect. Intuitively, LESS captures the ordering structure quantities, and CONG captures the closeness or distance structure of quantities. For instance, suppose that *ab* mass-CONG *bc* (and it is not the case that *ab* mass-CONG *ac* – ensuring that *a*, *b*, and *c* all differ in mass). Then the distance in mass between *a* and *c* is, intuitively, twice the distance in mass between *a* and *b*.

One worry one might raise about this account concerns the axioms governing these LESS and CONG predicates. These axioms make certain "existence assumptions" – in order for a system to satisfy these axioms, there have to exist certain objects with certain features. These existence assumptions rule out certain *prima facie* possibilities. One existence assumption is that there must be exist at least two massive objects. So, worlds with a single massive object are ruled out by this account. Another existence assumption is a "solvability condition" – intuitively, any interval *xy* can be "copied" inside a larger interval.[20] Consider the following example. It seems there is a possible world with just three massive objects – a 1g-mass object, a 2g-mass object, and a 5g-mass object. But such a world does not satisfy the solvability condition, and so is ruled out by Field's account.[21]

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[19] For quantities that do not require a unique ordering, LESS is replaced by the three-place predicate of *betweenness*. One can define *betweenness* in terms of LESS in the following way: *b* is between *a* and *c iff a* LESS *b* LESS *c* or *c* LESS *b* LESS *a* (Field 1980, 56).

[20] Axiom 5 of Definition 8: if $cd \preceq ab$, then there exists d' between *a* and *b* such that ad' CONG *cd* (Krantz *et al*. 1971, 172-173).)

[21] Melia (1998, 68) argues that Field's account requires an infinite number of massive objects in order to recover the standard representation of mass facts. But it seems that Field's account only requires the following. Take the objects with the largest and smallest masses. If there are any objects with masses between those, then their masses must form a chain with equal-sized links between the

Another worry one might raise about this account is that it lacks the resources to discriminate between distinct metaphysical possibilities. The strength of this worry depends on the details of the case – after all, it may be more important to discriminate between some metaphysical possibilities than others.

Here is one sort of "indiscrimination worry." It seems that two worlds might differ only in that everything at one world has double the mass of the corresponding objects in the other. But on Field's account this isn't a genuine possibility. Any such worlds are exactly alike in their patterns of LESS and CONG relations, and so are exactly alike with respect to the mass facts that obtain.

Note that this particular indiscrimination worry is not specific to Field's account. All first-order relational accounts have trouble making sense of uniform shifts of this sort. Consider an account according to which quantitative structure is grounded in proportion relations, such as *twice as massive as*. On this account, too, we cannot make sense of the possibility that everything has doubled in mass. Or consider an account according to which quantitative structure is grounded in interval relations, such as *two grams more than*. On this account, we cannot make sense of the possibility that everything might have been, say, five grams more massive.

Here is another sort of indiscrimination worry – one that *is* specific to Field's account. It seems possible for there to be a world, w_1, in which *a* and *b* are the only massive objects, and *a* is twice as massive as *b*. It also seems possible for there to be a world, w_2, in which *a* and *b* are the only massive objects, and *a* is three times as massive as *b*. Worlds w_1 and w_2 are exactly alike with respect to their patterns of LESS and CONG relations. And thus they are exactly alike with respect to the constraints these relations place on numerical assignments of mass. But if they are exactly alike with respect to the constraints these relations place on numerical assignments of mass, then it cannot be the case that these worlds differ with respect to the masses of *a* and *b*. So it seems we cannot discriminate between the two possibilities we started out with.

Here is another way to bring out the worry. Consider again world w_1, where *a* and *b* are the only massive objects. Given the pattern of LESS and CONG relations, there are very few constraints on the numerical representations of *a* and *b* at w_1 – indeed, nearly *any* numerical assignment that yields the result that the number assigned to *a* is greater than the number assigned to *b* will do.[22] But that seems wrong. For it seems that two worlds may be alike with respect to

smallest and greatest masses. (Again, the key axiom is Axiom 5 (c.f. footnote 20).) Either way, however, Field's account rules out many plausible possibilities.

[22] With one exception: some massive object must be assigned 0 (Krantz *et al.* 1971, 148-150).

their pattern of LESS and CONG relations, yet differ with respect to their mass *closeness* relations (e.g., it seems *a* is *closer in mass* to *b* at world w_1 than it is at world w_2). So, there seem to be some more informative and substantive constraints on the numerical assignments of mass than the LESS and CONG predicates deliver. And this suggests that the LESS and CONG predicates are not fine-grained enough to capture the structure of quantities. [23]

Now, some of these counterintuitive results might be avoided if one adopts an ontology of concrete possibilia and allows the relevant relations to hold among objects located at different worlds. But this move causes problems of its own. Following Lewis (1986*b*), let us say that a concrete world is spatiotemporally isolated – no objects located at different worlds stand in any spatiotemporal relations to one another. The axioms governing the LESS relation entail that the relation is *connected*; i.e., for any possible objects *x* and *y*, either *x* LESS *y* or *y* LESS *x*.[24] And this applies to spatiotemporal distances as well as to mass.[25] So, if we allow concrete possible worlds, then it follows these worlds cannot be spatiotemporally isolated from one another. [26]

To the extent one finds these results counterintuitive, that is a mark against Field's approach to quantity. But perhaps the account's virtues – simplicity, elegance, and parsimony – outweigh these costs.

5. Nominalism: Variable Degree Predicates

A different nominalist account of quantity has been proposed by Mundy (1989). This theory is presented as an application of a non-standard categorial logic that allows predicates of variable degree. Here is a rough sketch of the proposal. For every family of quantities there is a corresponding primitive predicate $\preccurlyeq$, intuitively understood as "less than or equal to." (In the case of mass, the

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[23] See Dasgupta (2013), Hawthorne (2006), Liggins (2003), Melia (1998), Meyer (2009), Mundy (1987), and Swoyer (1987, 245-246) for some discussion of these and related objections.
[24] See Field (1980, 120).
[25] In the case of spatiotemporal distances, it is natural to replace LESS with the three-place *betweeness* predicate (see footnote 19). One can prove from the axioms given by Krantz *et al.* for an absolute difference measurement structure that worlds cannot be spatiotemporally isolated, as follows: Suppose A is the set of possible objects and $\preceq$ is a relation on $A \times A$. Axiom 1 of Definition 8 states that $< A \times A, \preceq >$ is a weak order (for all $a, b, c, d \in A$, either $ab \preceq cd$ or $cd \preceq ab$). Axiom 5 states that if $cd \preceq ab$, then there exists d' between a and b such that ad' CONG cd (Krantz *et al.* 1971, 172). Suppose a and b are located at different worlds. By Axiom 5, there exists a point between them (*mutatis mutandis* $ab \preceq cd$).
[26] For other problems situating Field's account within a broadly Lewisian framework, see Eddon (2013).

predicate $\preccurlyeq_m$ is understood as "less than or equal to in mass.") This predicate applies to sequences of objects. For example, consider the following proposition:

John, Ted, Harry $\preccurlyeq_m$ Earth, Moon

Intuitively, this proposition says that the sum of the masses of John, Ted, and Harry is less than the sum of the masses of the Earth and the Moon.

The predicate $\preccurlyeq$ is a variable degree predicate because the sequences of objects it relates may be of any (finite) length. It may, for instance, hold between an object a and the null sequence: $\preccurlyeq_m a$ (intuitively, the mass of a is non-negative).

The axioms governing $\preccurlyeq$ are given in Mundy (1989, 133). And with these axioms in place, we can recover the ordering and closeness structure of quantities. Here is one way to do this:

the sum of the masses of $x_1, x_2, \ldots, x_n$ is less than or equal to the sum the masses of $y_1, y_2, \ldots, y_n$ *iff* $x_1, x_2, \ldots, x_n \preccurlyeq_m y_1, y_2, \ldots, y_n$.

the distance in mass between the sum of the masses of $x_1, x_2, \ldots, x_n$ and the sum of the masses of $z_1, z_2, \ldots, z_n$ is the sum of the masses of $y_1, y_2, \ldots, y_n$ *iff* $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \preccurlyeq_m z_1, z_2, \ldots, z_n$ and $z_1, z_2, \ldots, z_n \preccurlyeq_m x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$.[27]

As with Field's account, all facts about quantity are ultimately grounded in relations among objects. But there are some interesting differences. Suppose my desk has 40kgs mass and my chair has 20kgs mass. Consider a world w_1 where the only objects are my desk and my chair. On Field's account, there are hardly any substantive constraints on the assignment of numerical values to the objects at w_1. On Mundy's (1989) account, though, this isn't the case. On this account, the following propositions are true at w_1:

(1) Chair, Chair $\preccurlyeq_m$ Desk

(2) Desk $\preccurlyeq_m$ Chair, Chair

Intuitively, (1) says that the sum of the mass of the chair and itself (i.e. double the mass of the chair) is less than or equal to the mass of the desk. And (2) says that the mass of the desk is less than or equal to twice the mass of the chair. Together,

[27] Note that this characterization makes use of a notion of *summation*, and one might wonder whether this is legitimate (as I go on to note). Here is another way to recover the ordering and closeness structure of quantity. First, define a function ϕ that assigns numbers to objects in a way that reflects $\preccurlyeq_m$, such as the following:

$x_1, x_2, \ldots, x_n \preccurlyeq_m y_1, y_2, \ldots, y_n$ *iff* $(\phi(x_1) + \phi(x_2) + \ldots + \phi(x_n)) \leq (\phi(y_1) + \phi(y_2) + \ldots + \phi(y_n))$

Then use ϕ to capture the ordering and distances structure of quantity:

(1) x is less than or equal in mass to y *iff* $\phi(x) \leq \phi(y)$

(2) the distance in mass between x and z is y *iff* $\phi(x) + \phi(y) = \phi(z)$

(See also Mundy 1988 and 1989 for more discussion.)

they entail that the desk is exactly twice as massive as the chair. So the mass facts at w_1 guarantee that whatever number is assigned to the mass of the desk must be twice the number assigned to the mass of the chair. This could not be captured on Field's account.

Here is another example. Suppose that my chair is 20kgs and my sofa is 50kgs. Now consider a world w_2 where the only objects are my chair and my sofa. At w_2, the following propositions are true:

(3) Chair, Chair, Chair, Chair, Chair $\preccurlyeq_m$ Sofa, Sofa

(4) Sofa, Sofa $\preccurlyeq_m$ Chair, Chair, Chair, Chair, Chair

We may understand (3) and (4) in the following way: the mass of the chair "five times over" is equal to the mass of the sofa "twice over." In other words, the chair's mass is exactly 2/5th the sofa's mass. Again, this cannot be captured on Field's account.

This nominalist account is more discriminating than Field's. However, it also faces some challenges. First, like Field's account, it cannot distinguish between our world and a world where everything has, say, twice as much mass. Second, one might argue that the theory is not as ideologically parsimonious as it first appears. For example, one might argue that the intended interpretation of propositions such as "Huey, Dewey, Louie $\preccurlyeq_m$ Mickey, Minnie" implicitly employs a notion of *summation* – the sum of the masses of Huey, Dewey, and Louie is less than the sum of the masses of Mickey and Minnie.[28] Finally, one might object to the use of variable degree predicates in characterizing the fundamental structure of the world. See Oliver and Smiley (2004) and MacBride (2005) for an overview of the objections against multigrade predicates, and some responses.

6. A Non-Relational Account: Structural Universals

Nearly all the proposals considered so far have been purely relational. In this section, we'll examine an account that takes the structure of quantity to be grounded solely in fundamental properties. In the next section, we'll look at a "combination" account that posits first-order properties in addition to second-order relations.

[28] It is tempting to respond that, as long as one accepts something along the lines of unrestricted composition, one can interpret such propositions as making claims about the relative masses of fusions: the fusion of Huey, Dewey, and Louie is less massive than the fusion of Mickey and Minnie. But this line of response is ineffective, since expressions in a sequence are repeatable. Claims such as "Chair, Chair $\preccurlyeq_m$ Desk," for instance, cannot be understood in this way, since it is not possible for one thing to comprise two distinct proper parts of a fusion.

A non-relational theory of quantity has some intuitive appeal. Take a property like *length*. It seems that if the relation *longer than* holds between two objects, it does so *because* of some intrinsic features of the objects – in particular, their lengths.[29] Similarly, one might say that if the relation *less than* holds between two properties, it does so because of some intrinsic features of the properties.

One non-relational account of quantity has been proposed by D. M. Armstrong. According to Armstrong, quantitative properties are *structural universals* – universals that have other universals as constituents, where *constituency* is something akin to *parthood*.[30] And, Armstrong says, this constituency structure underlies the structure associated with quantity. First, the constituency structure organizes quantitative properties into families: mass universals have only other mass universals as constituents, charge universals have only other charge universals as constituents, and so on. Second, the constituency structure orders the quantities within a family: for all universals *m* and *n*, *m* is less than *n iff n* has *m* as a constituent.

But Armstrong's account runs into trouble capturing the closeness structure of quantity. Suppose we have three mass universals – *A*, *B*, and *C* – where *A* is a constituent of *B*, and *B* is a constituent of *C*. Is *B* closer to *A* or to *C*? One natural thought is to compare the number of constituents that *B* has and *A* lacks to the number of constituents that *C* has and *B* lacks. The difference in the number of constituents of these universals will determine how far apart they are, and thus whether the distance between *A* and *B* is greater or less than the distance between *B* and *C*.

But for this proposal to work, we must assume that *A*, *B*, and *C* all have a finite number of constituents. And, in general, this won't be true. Consider *mass*. *Mass* is continuous – for any two quantities of mass, there is a third quantity between them – and thus every mass universal has an infinite number of constituents. So it will not do any good to "count up" the number of constituents that *B* has and *A* lacks, or that *C* has and *B* lacks. For there are infinitely many

[29] Armstrong: "Whatever the exact nature of the relations that hold together those families of properties which constitute a quantity, it is plausible that, like the part-whole relation, they are *internal*, flowing from the nature of the terms… The relations flow internally from the properties and are explained by them." (1987, 315) Swoyer: "[F]acts about length and mass are supposed to be somehow grounded in facts about relations like *being longer than* or *being a greater mass than*." (1987, 245)

[30] See Armstrong (1978, 116-131), (1988), and (1989, 101-107). It is not clear whether Armstrong construes *constituency* as fundamental. If so, then this proposal would not count as purely non-relational.

constituent universals they have in common, and infinitely many constituent universals they do not. So the constituency structure that Armstrong posits is not rich enough to recover the closeness structure of quantity.[31]

7. A Combination Account: Properties and Relations

Suppose we hold onto the idea that there are intrinsic, monadic quantitative properties like *three grams mass*. And suppose we also hold onto the idea that there is something fundamentally relational about quantities. Could we then avoid some of the worries for purely relational theories, but still retain the intuitive idea that quantities are intrinsic?

One way to develop this idea has been proposed by Mundy (1987). We begin with the array of intrinsic, monadic quantitative properties. To these we add two second-order relations: *less than or equal to* (≲) and *sum of* (*). ≲ generates an ordering over properties; e.g., *one gram mass* is less than *two grams mass*, which is less than *three grams mass*, and so on. And * corresponds to a notion of "summation" over properties; e.g., the sum of *one gram mass* and *two grams mass* is *three grams mass*. Intuitively, this gives us the distance or closeness structure of mass quantities - distance between *one gram* and *three grams* is two grams. The axioms governing ≲ and * guarentee that these relations behave in appropriate ways (for details, see Mundy (1987, 37-40) and Krantz *et al.* (1971, 72-75)).[32]

This theory avoids many of the worries raised for the other accounts considered above. It recovers the $\preceq$ and $\circ$ relations distinctive of quantitative structure without conflating intuitively distinct possibilities or requiring implausible existence assumptions. But it has some costs as well. One worry is that it assumes a platonist conception of properties – every possible quantitative property exists, whether or not it is instantiated by some actual object.

Another worry is that there seems to be some tension between this account and combinatorial theories of possibility. If one adopts an account like Mundy's, it's natural to assume that the second-order relations hold necessarily. For example, if *two grams mass* is less than *three grams mass*, then it is necessarily so;

[31] For a more detailed discussion of Armstrong's account, see Eddon (2007); for worries concerning the coherence of structural universals, see Lewis (1986*a*).

[32] Among them is an existence assumption, analogous to the one required for standard measurement theory, which requires that for any properties X and Y such that $X \lesssim Y$ or $Y \lesssim X$, there exists some property Z which is the sum of X and Y (i.e., $*(X, Y, Z)$). One might also add the requirement that for any properties X and Y such that $X \lesssim Y$, there exists some property Z such that the sum of X and Z is Y (i.e., $*(X, Z, Y)$). The latter assumption ensures that there aren't any "gaps" – for instance, if *three grams mass* is less than *seven grams mass*, then there exists the property *four grams mass*. See also Krantz *et al.* (1971, 81-85).

there is no possible world where *three grams mass* is less than *two grams mass* instead.

Advocates of combinatorial theories of possibility tend to reject necessary relations of this sort. But whether this version of Mundy's view is compatible with a combinatorial theory of possibility depends on the particular principle of recombination that one adopts. For instance, this view *is* compatible with David Lewis's "cut and paste" principle of recombination, according to which "anything can coexist with anything else, at least provided they occupy distinct spatiotemporal positions." (Lewis 1986*b*, 88) But it is not compatible with a recombination principle along the following lines (where all the quantifiers over relations are restricted to fundamental relations):

(RP) For any first-order relation R, there are some possible objects $x_1, x_2, \ldots, x_n$ such that possibly $R(x_1, x_2, \ldots, x_n)$ and possibly $\sim R(x_1, x_2, \ldots, x_n)$; and for any second-order relation S, there are some first-order relations $R_1, R_2, \ldots, R_n$ such that possibly $S(R_1, R_2, \ldots, R_n)$ and possibly $\sim S(R_1, R_2, \ldots, R_n)$.

In order to reconcile a view like Mundy's (1987) second-order account with a recombination principle like RP, one must deny that the second-order relations hold necessarily. But this combination of views has some unpalatable consequences.

First, it seems implausible that, say, *one gram mass* could be greater than *two grams mass*. Second, and more importantly, this combination of views makes it hard to make sense of modal claims that require cross-world comparisons. Suppose my desk has a mass of 40kgs. It seems my desk could have been more massive than it actually is. Assessed in terms of possible worlds, this claim is true *iff* there is a possible world at which the mass of my desk is greater than the mass of my desk at the actual world. Now consider a possible world *w* where my desk has a mass of 42kgs. Further suppose that, at *w*, 42kgs is less than 40kgs. Is my desk more massive at *w* than at the actual world, or less massive? Given a recombination principle of the sort above, there are no stable ordering or distance relations that hold among quantitative properties instantiated at different worlds. And so there does not seem to be any fact of the matter concerning the actual mass of my desk relative to its mass at *w*. This indeterminacy will infect any account that makes use of cross-world comparisons – such as Lewisian accounts of resemblance and duplication, Lewis/Stalnaker-style accounts of counterfactuals, and Humean analyses of causation.[33, 34]

[33] See Lewis (1973*a*), (1973*b*), (1983), Stalnaker (1968), *inter alia.*

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[34] Many thanks to Ted Sider and an anonymous referee, both of whom provided lengthy and enormously helpful comments. Special thanks to Chris Meacham for extensive comments and discussion.

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