Hermann Cohen's *Principle of the Infinitesimal Method*: A Defense[1]

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ABSTRACT: In Bertrand Russell's 1903 *Principles of Mathematics*, he offers an apparently devastating criticism of the neo-Kantian Hermann Cohen's *Principle of the Infinitesimal Method and its History* (*PIM*). Russell's criticism is motivated by his concern that Cohen's account of the foundations of calculus saddles mathematics with the paradoxes of the infinitesimal and continuum, and thus threatens the very idea of mathematical truth. This paper defends Cohen against that objection of Russell's, and argues that properly understood, Cohen's views of limits and infinitesimals do not entail the paradoxes of the infinitesimal and continuum. Essential to that defense is an interpretation, developed in the paper, of Cohen's positions in the *PIM* as deeply rationalist. The interest in developing this interpretation is not just that it reveals how Cohen's views in the *PIM* avoid the paradoxes of the infinitesimal and continuum. It also reveals some of what is at stake, both historically and philosophically, in Russell's criticism of Cohen.

1. Cohen's Principle of the Infinitesimal Method*, and Russell's reaction to it.*

By Bertrand Russell's lights, the account of calculus' foundations Hermann Cohen gave in his 1883 *Principle of the Infinitesimal Method and its History* (*PIM*) was a confused mess. Russell devotes an entire chapter of his 1903 *Principles of Mathematics* to criticizing the

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Marburg neo-Kantian's views on infinitesimals. Russell's most serious objections are driven by a simple point: as he sees it, Cohen's account of infinitesimals has been superseded by the modern concept of limits. That concept, developed earlier in the nineteenth century by Augustin-Louis Cauchy and Karl Weierstrass, demonstrates that calculus does not commit mathematics to the reality of infinitesimals. Yet Russell finds in Cohen the embarrassing insistence to the contrary that calculus *does* commit mathematics to infinitesimals. Russell thinks this commitment of Cohen's is disastrous. It entails the paradoxes of the infinitesimal and continuum, and so threatens the very idea of mathematical truth.[2]

I aim to show that this objection to Cohen misses the mark. Contrary to Russell's suggestion, Cohen's views in the *PIM* do not entail the paradoxes of the infinitesimal and continuum. Thus whatever other shortcomings Cohen's views might have from Russell's perspective, they are both coherent and, I will argue, deeply philosophically motivated.

Essential to my defense of Cohen will be a contrast between two possible interpretations of his views. Both possible interpretations are plausible: they both unify and explain a wide range of positions that Cohen takes in the *PIM*. But on the first of these interpretations, Cohen's views entail the paradoxes of the infinitesimal and continuum, whereas on the second

[2]Russell was not alone in taking a very dim view of Cohen's account of infinitesimals. Gottlob Frege found Cohen's *PIM* irredeemably confused and full of mathematical errors (1984 [1885]). Even Cohen's best-known student, Ernst Cassirer rejects almost all of the details of Cohen's views of calculus, likely at least partly in response to Russell's criticisms of them. See Giovanelli (2016) for an excellent survey of critical neo-Kantian responses to Cohen's *PIM*. Anglophone commentary has emphasized the ways that Cassirer's views differ from Cohen's in the *PIM*: see, in addition to Giovanelli (2016), Skidelsky (2009), Mormann & Katz (2014), and Oberdan (2014). For accounts that emphasize the continuity between Cassirer's and Cohen's views, see Moynahan (2003) and Seidengart (2012).

interpretation they do not. My defense of Cohen thus depends on a comparison of these two interpretations and an argument for why we should reject the first and accept the second. The comparison of these interpretations, and the measurement of them against Cohen's texts in the *PIM*, will take up most of the paper.

I begin in §2 by sketching Russell's principal objections to Cohen. Then I turn to Cohen, and in §3, I outline the project he calls the critique of knowledge and my strategy for interpreting him, in light of the historical method he uses throughout the *PIM*. §4 begins the survey of Cohen's positions in the *PIM* that the right interpretation of him must account for. §4 is concerned specifically with Cohen's positions on two concepts of the infinite and infinitesimal that recur throughout the *PIM*. §5 argues that Cohen's positions on these two concepts cannot be explained by his commitment to the principle of continuity, and that his commitment to that principle is itself one more position that needs to be explained by the right interpretation of him. §6 introduces the first of the two plausible interpretations of Cohen, and argues that it can explain his positions on the two concepts of the infinite and infinitesimal, his commitment to the principle of continuity, and his commitment to a peculiar definition of the concept of equality. The combined facts that this interpretation is so powerful at making sense of Cohen's text and that, on this interpretation, Cohen's views entail the paradoxes of the infinitesimal and continuum explain why Russell thought that Cohen's views did, in fact, entail those paradoxes.

However, §7 brings into view Cohen's position on a third concept of the infinite and infinitesimal, one that is distinct from the two surveyed in §4. I argue that Cohen's position on this third concept is inconsistent with the interpretation of his views considered in §6, and thus requires us to reject that interpretation as inconsistent with Cohen's text. §8 introduces a second interpretation of Cohen's views, and argues that it unifies and explains Cohen's positions on all three concepts of the infinite and infinitesimal, as well as his commitments to the principle of continuity and his peculiar concept of equality. For these reasons, I argue, this second interpretation is correct. §9 argues that, on this second interpretation, Cohen's deepest philosophical commitments do not entail the paradoxes of the infinitesimal and continuum. I conclude in §10 with some brief suggestions about what my interpretation of Cohen reveals about the contrast between Cohen's and Russell's positions, and about the larger philosophical and historical significance of Russell's objections to Cohen.

§2. Russell's principal objection to Cohen.

Russell has various objections to Cohen, but the most serious of them stem from one point: on Russell's interpretation of him, Cohen's commitment to infinitesimals is inconsistent with the modern mathematical concept of limits. More specifically, Russell thinks, Cohen's commitment to infinitesimals is inconsistent with one significant feature of the modern concept of limits: it does not appeal to any infinite or infinitesimal numbers, but instead appeals only to finite numbers. Thus, Russell insists, the modern concept of limits shows, contrary to Cohen's view, that calculus is not committed to infinitesimals.

On the modern view, the concept of the limit of a function may be understood this way. Consider a function f. As f's argument, x, approaches a, $f(x)$ approaches L just in case for any error term, ε, there is a distance, δ, such that if the distance between x and a is less than δ, then the distance between $f(x)$ and L will be less than ε. Or more carefully, $\lim_{x \to a} f(x) = L$ just in case for any $\varepsilon > 0$, there is a $\delta > 0$, such that if $0 < | x - a | < \delta$, then $| f(x) - L | < \varepsilon$. It follows from this definition that if L is the limit of a function near a, then as x approaches a, there is no smallest finite difference between $f(x)$ and L. For any candidate smallest difference one might imagine, the definition requires that we may specify a still smaller ε, and there will still be a δ such that if the distance between x and a is less than δ, then the distance between $f(x)$ and L will be less than our newly-specified, smaller ε.

It is important that on the modern concept of limits we do not say that as f's argument x approaches a and $f(x)$ approaches L, the difference between $f(x)$ and L becomes *infinitesimal*. Rather, we say that as x approaches a and $f(x)$ approaches L, the difference between $f(x)$ and L can be made to be smaller than any finite value, which is as small as we like. The concept thus involves no appeal to infinite or infinitesimal numbers.

This point -- that the modern concept of limits involves no appeal to infinite or infinitesimal numbers -- is essential to Russell's most serious criticisms of Cohen. At first glance, the objection is simply that the modern concept of limits demonstrates that calculus in no way depends on any commitment to infinitesimals, and so that strictly speaking there are no such things as infinitesimals. On the contrary, Russell insists, the modern concept of limits can be used to define the differential for calculus without appealing to infinitesimals (Russell 2010 [1903a]: 338-339). Yet Cohen insists that infinitesimals are necessary for the methods of calculus and the mathematically-precise physics that makes use of it.

But this gloss of Russell's objection does not capture the real significance the objection has for him. At issue is *why* it is so important for him that the modern concept of limits lets calculus do away with infinitesimals. Russell does not think infinitesimals are a perfectly unproblematic commitment for mathematics to have, but that it is just more parsimonious if mathematics can dispense with them.

On the contrary, Russell thinks a commitment to infinitesimals saddles mathematics with the paradoxes of the infinitesimal and continuum. Those paradoxes, Russell says, "are usually regarded by philosophers. . . as showing that the propositions of mathematics are not metaphysically true" (Russell 2010 [1903a]: 188). Thus for Russell, the great advantage of the modern concept of limits is that, in freeing calculus of any commitment to infinitesimals, it gives mathematics a path out of the paradoxes of the infinitesimal and continuum. In so doing, it decisively answers an otherwise persistent skepticism about mathematical truth.

In contrast, Russell sees Cohen's claim that infinitesimals are necessary for calculus and the physics that uses it as entailing the view that calculus necessarily runs afoul the paradoxes of the infinitesimal and continuum. In this case, mathematics would have no response to the consequent skeptical threat to mathematical truth. Thus Russell insists that we must reject Cohen's view of infinitesimals, in favour of a view of calculus on which its foundation is provided by the modern concept of limits.

So much for Russell's objections. Let's turn to Cohen.

§3. Cohen's aims and method in the PIM.

Let's begin by considering Cohen's aims and method in the *PIM*, in order to head off two possible sources of confusion. The first possible source of confusion concerns the fact that Cohen identifies himself as Kantian, which might seem inconsistent with the interpretation of Cohen I will give in §§7 and 8, since on that interpretation Cohen's deepest philosophical commitments are rationalist. So before I can offer that rationalist interpretation of Cohen, I need to clarify the sense in which he is Kantian. The second possible source of confusion concerns the interpretive strategy I employ in §§4-8, and especially in §9, to articulate and contrast the two plausible interpretations of Cohen (and ultimately to argue that the right interpretation is rationalist). That interpretive strategy relies heavily on Cohen's accounts of the development of calculus in the history of mathematics, so we need to see why for Cohen the history of mathematics is essential to his philosophical aims.

In the *PIM*, Cohen aims to provide the foundation for calculus' central concepts, in particular, the concepts of limits and infinitesimals.[3] For Cohen, the validity of those concepts is secured by the fact that they are required by certain methodological principles that mathematical natural science is committed to. They are the principles that provide mathematical natural science with its rational foundations. Thus if mathematical natural science has rational foundations, then the concepts of limits and infinitesimals are valid; we could deny their validity only on pain of losing our grip on the rationality of mathematical natural science.

The method that Cohen employs to achieve these aims is one he elsewhere calls the *transcendental method*.[4] Philosophy done according to that method takes the fact of mathematical natural science as its starting point: that is, it begins with the theories of mathematical natural science as the objects of philosophical theorizing. Then, the method directs

[3]Cohen also aims both to determine what in scientific theories explains those theories' capacity to represent reality (that is, what Cohen calls the "principle of reality") and to develop a post-Kantian theory of intuition. A complete account of the *PIM*, including a complete account of how Cohen identifies the foundation of the concepts of limits and infinitesimals, would have to explain how those different aims relate to one another. Essential to that account would have to be an explanation of what "infinitesimal reality" is for Cohen, and why the infinitesimal concept is essential to the principle of reality. Space does not permit me to give that explanation here. In the absence of that fuller explanation, the interpretation of Cohen offered here must be considered partial.

[4]While Cohen uses the term 'transcendental' throughout the first (1871) edition of his *Kant's Theory of Experience* to indicate doctrines he takes to be characteristically Kantian, he does not use the term 'transcendental method' until his 1877 *Kant's Foundations of Ethics*. There, he sketches the method I describe above. Then in the second (1885) edition of *Kant's Theory of Experience*, he devotes an entire section of the long historical introduction to explaining how that method was the one Kant used to investigate Newtonian science.

philosophers to reflect on those theories, in order to identify in them the principles that provide mathematical natural science with its foundation (Cohen 1877: 24-6). Cohen calls those principles *formal conditions of the possibility of experience* (see Cohen 1871 and Cohen 1885 *passim*).

In the *PIM* (and in the second, 1885 edition of *Kant's Theory of Experience*), Cohen calls the investigation of knowledge that follows the transcendental method the *critique of knowledge* (*Erkenntnisskritik*). As Cohen puts it, "[c]ritique discovers purity in reason, to the extent that it discovers the conditions of certainty on which knowledge as science is based" (Cohen 1883: §8). To be clear, Cohen does not think the critique of knowledge, or any other philosophical investigation, *provides* or *secures* mathematical natural science with its rational foundations. On the contrary, the principles that do that are already contained in mathematical natural science, or as he puts it, "assumed" in it as a "latent basis" (Cohen 1883: §9). But it is the task of the critique of knowledge to *identify* those principles, and to exhibit them in their systematic connections to mathematical natural scientific concepts such as the concepts of limits and infinitesimals (Cohen 1883: §9).

It is no accident that Cohen uses the terms 'transcendental' and 'critique' to describe his philosophical method and the investigation that follows is. Cohen understands the transcendental method and the critique of knowledge to be distinctively Kantian. As he puts it in his 1898 introduction to F.A. Lange's *History of Materialism*, the "critical" perspective in philosophy was Kant's central insight (Cohen 1898: xvii). What is more, Cohen understands the transcendental method to be the essence or true spirit of Kant's philosophy, so that a commitment to the transcendental method is, in itself, sufficient to make one a Kantian. As he says in the 1885 edition of *Kant's Theory of Experience*, "Kant's originality and mission consists principally in his method. This method is the transcendental method" (Cohen 1885: 63).

Thus in Cohen's writings from the 1880s, including the *PIM*, he understands his Kantianism to consist principally in his commitment to the transcendental method and a critical perspective in philosophy. It follows that he does not take his Kantianism to consist in a commitment to any particular substantive doctrine of Kant's. This is important for my purposes, since (as I will argue in §§7 and 8) the substantive commitment that justifies Cohen's views on the infinitesimal concept in the *PIM* is as rationalist as it is Kantian, since (I will argue) that commitment just in the principle of sufficient reason. But since Cohen understands his Kantianism in terms of his commitment to the transcendental method, his rationalist commitment to the principle of sufficient reason is not inconsistent with his Kantianism as he understands it. (We will consider Cohen's rationalism in more detail in §8.)

There is another aspect of the transcendental method and the critique of knowledge that can cause confusion for Cohen's readers. For Cohen, the fact of science that the transcendental method takes as its starting point is not static: it changes with the evolving history of mathematical natural science. Cohen thus maintains that the formal conditions that provide mathematical natural science with its foundations are revealed most clearly in that evolving history. Consequently, the critique of knowledge's task of identifying those principles cannot be fulfilled without a sustained investigation of that history, including the history of calculus' foundational concepts.

The *PIM*'s brief forward is given over largely to explaining the historical orientation of Cohen's investigation. He says there that "[n]owhere was it so necessary for me, and nowhere did it seem so immediately useful, to pursue at the same time a systematically decisive thought along with its historical development" (Cohen 1883: iii). Indeed, Cohen suggests in an early section of the book that the critique of knowledge might *require* the historical analysis he takes to be a part of it:

> By seeing clearly the scientific background that led to the discovery of the infinitesimal concept, we reveal to ourselves most securely an understanding of that concept's meaning for the critique of knowledge. (Cohen 1883: §15)

Thus most of the *PIM* is comprised of a philosophically critical history of the contributions to mathematics, physics, and philosophy that culminated in Leibniz's and Newton's versions of calculus, as well as the early modern reception of it. It is this historical analysis that ultimately reveals the principles that secure mathematical natural science's rational foundations, and so too provide the rational foundation for the concepts of limits and infinitesimals. Consequently, if we are to understand Cohen's account of the foundations of calculus, we cannot avoid this historical analysis.

However, Cohen's historical analysis poses serious interpretive problems that any plausible account of his views in the *PIM* must face. Cohen does not ever aim to offer explicit arguments for his views, or even systematic definitions of the terms he uses to express those views. Instead, he explains his philosophical views only by working through the many historical episodes in which various constituent elements of his views were first articulated, often in reaction to historical views that stand in for various views that Cohen wants to reject. Typically, Cohen will describe a historical argument or exchange in the protagonists' vocabulary without clearly explaining which parts are irrelevant to his philosophical purposes.

Cohen's blend of history and systematic philosophical argument can be confusing. The mistake to avoid in interpreting Cohen is seeing his shifting arguments and vocabulary as consequences of underlying philosophical commitments that similarly shift over the course of the book's narrative, and thus as a symptom of the fact that Cohen has no stable, coherent views. Rather, interpreting the *PIM* is an exercise in reconstructing the general commitments that underlie Cohen's shifts in vocabulary from one episode to the next in his history, and that give that history its coherence as a philosophical narrative.

We will put this interpretive strategy to work in §9, where careful attention to Cohen's historical approach to philosophy is necessary for seeing how he avoids a potentially serious objection. But more generally, my strategy for interpreting Cohen in light of his historical approach to philosophy is this. In the next sections, I will examine his accounts of various historical episodes in the history of calculus' development. In the course of Cohen's exposition of the history of calculus, he considers over twenty different figures' concepts of the infinite, the infinitesimal, limits, and continua. Further, he takes positions on all of those concepts. The interpretive problem we face is to make unified sense of all of his positions on these various concepts. Fortunately, Cohen's positions follow a pattern. It is thus possible to identify the underlying philosophical commitments that explain and justify the pattern in Cohen's views -- even when Cohen himself never quite explicitly articulates the deepest of those philosophical commitments.

§4. *Two concepts of the infinite and infinitesimal*.

Progress begins with noticing something that Cohen draws no attention to: all of the historical episodes he considers involve one of three different concepts of the infinite or infinitesimal. I will call those three concepts the *infinitesimal as arbitrarily small*, the *Galilean infinite*, and the *inexhaustible infinite*. In this section, I will consider the first two of these. I will return to the third in §7.

The infinitesimal as arbitrarily small. The first concept appears in passages where Cohen considers definitions of limits or infinitesimals that appeal to *arbitrarily small* finite magnitudes. For example, one might define an infinitesimal as "smaller than an arbitrarily small finite magnitude," as "smaller than an assignable finite value," or as "less than a finite value one's opponent specifies." Any of these phrases, or others like them, would also allow us to define the concept of a limit, if we took the phrases to describe the difference between a function's value $f(x)$ and the function's limit L when $f(x)$ has reached L. Cohen stresses how common a strategy this was in the seventeenth and eighteenth centuries for defining both limits and infinitesimals. Cohen also rejects this concept in every instance of its use that he considers.

Start with ancient geometry, which Cohen takes to be the origin of the concepts of limits and limit processes. On Cohen's account, Archimedes first introduced the idea of a limit to mathematics with his method of exhaustion for determining the value of the area under a curve (Cohen 1883: §35). The sum of the areas of a series of rectangles inscribed under the curve approximate the area under the curve, and the approximation is improved if the widths of the rectangles are halved and their number doubled. But then, if that doubling process continues indefinitely, the sum of the areas of the rectangles will approach the area under the curve indefinitely closely (without ever exceeding it). Thus Archimedes struck on the idea of a limit process. But, and crucially for Cohen, Archimedes understood his method of exhaustion to mean that the difference between the total area of the rectangles and the area under the curve could be reduced to "less than any assignable" value (Cohen 1883: §35).

Cohen rejects the method of exhaustion and the concept of limits it produces. As he puts it, the method "lacks nothing less than generality," and the limit concept it produces "lacks any creative positivity" (Cohen 1883: §35). He likewise rejects any concept of an infinitesimal magnitude that might be based on the method of exhaustion, since, as he sees it, that infinitesimal concept would be "devoid of fruitfulness" (Cohen 1883: §35).

Cohen's rejection of concepts that appeal to arbitrarily small values is even more clear in his discussion of analytic geometry in the early modern period. For example, in Diderot's *Encyclopedia*, D'Alembert defended a concept of limits on which a magnitude *A* is the limit of another magnitude *B* just in case *B* approaches *A* within any given magnitude, and *B* never exceeds *A*.[5] But Cohen rejects this concept of limits, insisting that it cannot provide the foundations that calculus needs (Cohen 1883: §2).

Likewise, Cohen rejects Bonaventura Cavalieri's method of indivisibles (now known as Cavalieri's Principle) for determining the areas of surfaces (and the volumes of solids). Cavalieri's method begins with lines of divisible, and thus finite and discrete, width, and then supposes they can be divided until one is left with lines of indivisible width. Then those lines can be used to measure continuous surface areas. Thus, on Cavalieri's method, if two figures are both between two parallel lines, then if every line that is parallel to those two lines intersects both figures in segments of equal length, the areas of the two figures are equal. However, Cohen insists, regardless of how many times the lines are divided, they are always of finite, discrete width. The best we can say, on Cavalieri's method, is that the lines can be divided so as to make their widths arbitrarily small. Cavalieri claims that they can nevertheless measure continuous surfaces because the lines are "continuous discretes" (Cohen 1883: §36). But Cohen's objection

[5] D'Alembert:

> One says that one quantity is the limit of another quantity when the second can approach the first more closely than by a given quantity, as small as one can imagine, but such that the approaching quantity can never exceed the quantity that it approaches. . . (D'Alembert 1996 [1765]: 130)

is that the magnitude of a continuous surface cannot be determined by appealing to lines of arbitrarily small width (and that appealing to an oxymoron like "continuous discretes" does not change that fact). As Cohen puts it, "the continuum cannot be exhausted by so indefinite a conception as ultimately discrete magnitudes" (Cohen 1883: §36).

One more example comes from Leibniz. In his correspondence with Pierre Varignon, Leibniz defines infinitesimals as the difference between 0 and a positive value that is less than an error term one's opponent specifies. That is, on this account, a magnitude is infinitesimal just in case it is less than a finite value one's opponent chooses, however small. Cohen rejects this suggestion decisively:

> With this connivance in the mathematical proof procedure, the unsteady path of the so-called logical foundation of the differential is set upon, a path which consists principally in the resolution of mistakes with still further mistakes. . . (Cohen 1883: §54)

For our purposes, the important point about these passages is that, in all of them, Cohen rejects concepts of limits and infinitesimals whose definitions appeal to arbitrarily small finite values. Nor are these passages the only ones that fit the pattern. On Cohen's account, Descartes' approach to determining the slope of a tangent on a curve (Cohen 1883: §39) and Fermat's method for the determining the maxima and minima of curves (Cohen 1883: §45) appeal to finite values that are arbitrarily small in the sense that they have a finite positive value, and yet may be treated as negligible. Cohen rejects them.

The Galilean infinite. Now consider the second concept of the infinite. For lack of a better term, I will call it the Galilean infinite, since of the figures Cohen discusses, Galileo offers the clearest expression of it. It is the concept of an infinite that is conceptually prior to the finite. That is, it expresses the idea that finite magnitudes must be defined ultimately by appeal to the infinite. It is worth observing that this concept of the infinite is thus flatly inconsistent with the modern concept of limits. If limits (or infinitesimals) are interpreted along the limes of the Galilean infinite, then they cannot be defined by appeal only to finite values. But that is exactly what the modern concept of limits does (and exactly why Russell thinks that modern concept is so significant). Cohen praises the Galilean infinite in every use of it that he considers.

Cohen praises Galileo for having the right concept of the relation between the infinite and the finite: "his genius was thus proved in this: constituting the finite out of the infinite" (Cohen 1883: §47). Or as Cohen also puts it, "[t]he infinite must be conceived independently of the finite, in order to be able to generate the finite from it" (Cohen 1883: §37). But this concept of the infinite is not, on Cohen's history, limited to the modern era. He thinks that despite some ancients' distrust of the irrational numbers, the Pythagoreans were right to conceive of number such that it is, "according to its scientific character, already thought in connection with continuity" (Cohen 1883: §43). That is, Cohen praises the idea that the concept of number must in the first instance be the concept of a continuous number series, not a discrete one. Further, Cohen praises the idea that the concept of a discrete number must be defined by appeal to a continuous magnitude. As he puts it, "the first definition of the discrete is further only possible through consideration of the continuum, whose discreteness it should become" (Cohen 1883: §37).

Two further examples come from Cohen's consideration of early modern applications of mechanical concepts to geometry -- a cross-pollination between physics and geometry that Cohen thinks was especially fruitful in the development of calculus. On Cohen's account, Johannes Kepler developed a mechanical method for determining the volumes of curved solids. First, Kepler conceived of curved lines as composed of an infinite number of points. Then he took the continuous motion of the defined figures to define the volumes of classes of curved solids (Cohen 1883: §38). For Cohen, Kepler's explicit appeals to the infinite and to continuous motion "provide a positive meaning for the concept of limit" (Cohen 1883: §38). That is, Cohen thinks Kepler got something right.

Further, as Cohen tells the history, Gilles de Roberval adapted Kepler's method to find a solution to the problem of determining the slope of a tangent on a curve. For Roberval, a curve is defined by the continuous motion of an infinitely small point that is driven in (at least) two directions. But then, if a curve is so defined, the slope of the tangent at any point on the curve will be given by the product of lines representing the directions the point is driven in (Cohen 1883: §39). As Cohen sees it, this solution to the tangent problem constituted a "positive turn" in calculus' development and the history of mathematics more generally.

The pattern is clear in these passages. Cohen praises mathematical concepts, definitions, or methods if they express the concept of the Galilean infinite.

§5. The principle of continuity.

If we want to know what underlying philosophical commitments unify and explain the positions Cohen takes on these two concepts of the infinite, we are now in a position to see some plausible candidates.

The obvious place to start is a Leibnizian principle Cohen endorses explicitly and repeatedly: the principle of continuity. By Cohen's lights, the principle of continuity is centrally important for his theory of knowledge, "a fundamental form of the unity of consciousness" and "a special expression of the universal law of the unity of consciousness" (Cohen 1883: §40), and its significance for mathematics and philosophy is one of his explicit themes in the history in the

PIM. Indeed, Leibniz turns out to be the principal hero of that history, precisely because he not only put the principle of continuity to argumentative use, but because he articulated it explicitly and self-consciously. But despite the principle's centrality for Cohen's theory of knowledge, he says characteristically little about its actual content. The closest he comes to a definition of the principle of continuity is to say that it "signifies the presupposition: there are no jumps in consciousness" (Cohen 1883: §42). As with all transcendental-psychological sounding vocabulary in Cohen, his use of the word "consciousness" [*conscientia*] here does not refer to any state or activity in any actual mind. Rather, it refers to natural scientific "consciousness" -- that is, the theories of mathematically precise natural scientific knowledge, considered apart from the minds of any actual knowers. The principle of continuity is thus a requirement about how natural scientific theories represent their objects. In particular, it is the requirement that natural science represent the objects of its theorizing as *continuous*, or without "jumps." That is, it must represent mathematical and physical objects and processes as continuous.

However, despite the principle of continuity's centrality to Cohen's theory of knowledge, it cannot on its own explain his positions on either of the two concepts of the infinite we have seen so far. One potential problem concerns the principle of continuity's scope. It only concerns continuity, and says nothing one way or the other about the infinitely large. Thus, one might think, the principle of continuity cannot on its own explain Cohen's positions on concepts of the infinitely large.

If this concern about the principle of continuity's scope were the only reason to think it could not explain Cohen's positions on the three concepts of the infinite, then maybe it would be worthwhile to articulate a new version of the principle, one that was broadened so as to apply to the infinitely large. But actually, a more serious problem with the principle means this new version of it still would not help.

The more serious problem is this. The principle of continuity requires that mathematics and natural science represent their objects and processes as continuous. On its own, without further interpretation, the principle says nothing in any more detail about what it means for one of those objects (say, a magnitude) to be continuous. But that is what both of the concepts of the infinite we have seen so far do: they are both proposals for how we should understand the meaning of the infinite, infinitesimal, and *continuity*. Thus, for example, on the first of the two concepts, a magnitude is continuous just in case it is smaller than an arbitrarily small finite value. On its own, without further interpretation, the principle of continuity provides no reason to reject this interpretation of continuity. And so the principle cannot explain Cohen's rejection of it.

§6. *The priority of the infinite and continuous.*

However, if the principle of continuity cannot explain Cohen's assessments of the infinitesimal as arbitrarily small and the Galilean infinite, there is another principle that seems to do much better. In fact, as I will argue in §7, Cohen is *not* committed to this other principle and it does not ultimately explain his views in the *PIM.* Still, it is undeniable that this other principle provides an interpretation of Cohen that is, on its face, very plausible: if Cohen were committed to this principle, it would explain not just his positions on the infinitesimal as arbitrarily small and the Galilean infinite, but also his commitment to the principle of continuity, as well as his commitment to a peculiar concept of equality. What is more, this other principle provides an interpretation of Cohen that underscores exactly what Russell's objections to him were.

I will call this other principle the *priority of the infinite and continuous*. It is the principle that the concepts of the infinite, infinitesimal, and continuous are prior to the concepts of the finite and discrete. That is, the concepts of the infinite, infinitesimal, and continuous must define the concepts of the finite and discrete, and not the other way around.

The priority of the infinite and continuous rules out the concept of the infinitesimal as arbitrarily small. That concept defines the infinitesimal exclusively by appeal to finite numbers, thus violating the priority of the infinite and continuous. The priority of the infinite and continuous likewise entails the concept of the Galilean infinite, since that concept requires that finite numbers be defined by appeal to infinities or infinitesimals.

The priority of the infinite and continuous also entails the principle of continuity. To see why, consider a limit process once more. As a magnitude *B* approaches its limit *A*, the difference between them makes a continuous transition from a finite magnitude to zero. But if a limit process violates the principle of continuity, it defines a limit process that leaves a discontinuity in the transition of the difference between *A* and *B* from a finite magnitude to zero. The limit process thus determines its limit by appeal only to finite magnitudes. Further (as we will see momentarily), Cohen takes the difference between *A* and *B* in the limit to determine the magnitude of an infinitesimal. Consequently, the limit process also defines an infinitesimal magnitude by appeal only to finite magnitudes. But then it violates the priority of the infinite and continuous. Thus if Cohen were committed to the priority of the infinite and continuous, it would explain his commitment to the principle of continuity.

There is a further point to notice about the fact that the priority of the infinite and continuous justifies the principle of continuity. It entails a particular interpretation of continuity in a way that is important for Cohen. We observed above that the principle of continuity, on its own, does not require any particular interpretation of the meaning of continuity. Thus it might be consistent with the first concept of the infinite, the concept of the infinitesimal as arbitrarily small. On this interpretation, in a limit process, we could say that $f(x)$ has reached L just in case the difference between them was smaller than an arbitrarily small finite value. But now, if for Cohen the principle of continuity were ultimately justified by the priority of the infinite and continuous, no interpretation of continuity that violated the latter principle could satisfy the former. An interpretation of continuity that is consistent with the first concept of the infinite is thus ruled out.

Finally, there is one more otherwise puzzling position of Cohen's that the priority of the infinite and continuous seems to explain. Cohen has a very peculiar definition of equality. He repeatedly defines equality as "infinitesimal inequality" (Cohen 1883: §68; see also §58). (Russell calls this Cohen's "crowning mistake" [Russell 2010 (1903a): 346].) This is an odd enough concept that it is fair to ask why Cohen defends it.

That odd definition of equality functions to satisfy a demand that would be imposed by the priority of the infinite and continuous, if Cohen were committed to it. More specifically, it satisfies a demand that principle imposes on the definition of the mathematical quantities Kant had called "extensive magnitudes." Following Kant, Cohen thinks this kind of magnitude is essential for mathematical natural science, since it is the kind of magnitude that measures, for example, the units of space and time described by a mathematically precise physics. But the priority of the infinite and continuous requires that extensive magnitudes be defined by appeal to infinities or continua.

Cohen might satisfy this demand by exploiting the essential role of the concept of equality in Kant's definition of extensive magnitude. For Kant, extensive magnitudes are "homogeneous units" [*Gleichartigen*], that is, units that are interchangeable with one another (Kant 1997 [1781/1787]: A162/B203). But the units are interchangeable because they are all of equal size. Consequently, a magnitude is extensive just in case it belongs to a set of magnitudes that are all of equal size. Cohen accepts this Kantian definition, but then supplements it with a Leibnizian definition of equality. As Cohen puts it,

> with Leibniz, we recognize equality as an infinitesimal inequality. Thereby the limit concept corrects the concept of equality, but through the supposition of the infinitesimal. . . (Cohen 1883: §68; see also §58)

Cohen thus defines equality as *infinitesimal inequality*, a concept he then links to the concepts of limits and infinitesimals.

In Cohen's hands, Leibniz's definition of equality as infinitesimal inequality breaks down into two parts (Cohen 1883: §53). First, Cohen sees in Leibniz the use of the concept of a limit to define the size of infinitesimal magnitudes. Then second, Cohen follows Leibniz in stipulating that such an infinitesimal difference -- an infinitesimal inequality -- just is the relation of *equality*. For Cohen, the concepts of limits and infinitesimals thus provide the resources for an explication of the concept of equality. What is more, that appeal to infinitesimals ensures that the definition of extensive magnitudes satisfies the priority of the infinite and continuous.

At this point, it is clear that the priority of the infinite and continuous provides a powerful and attractive interpretation of Cohen. If Cohen were committed to that principle, it would unify and explain a number of his positions in the *PIM*: his rejection of the concept of the infinitesimal as arbitrarily small; his praise of the Galilean infinite; his commitment to the principle of continuity; and finally, his odd Leibnizian definition of equality as infinitesimal inequality.

At the same time, as powerfully as the priority of the infinite and continuous unifies various of Cohen's views, it also highlights why Russell objected so forcefully to those views. Russell wanted to be able to define calculus' foundational concepts by appeal exclusively to finite numbers, in order to avoid positing infinitesimals and so to avoid the paradoxes of the infinitesimal and continuum. But the priority of the infinite and continuous rules out that possibility. On the contrary, it requires that finite numbers be defined by appeal to infinities, infinitesimals, or continua. In so doing, it threatens to saddle mathematics with precisely the paradoxes that Russell wanted to avoid.

Thus if Cohen really were committed to the priority of the infinite and continuous, there would be no way for him to avoid those paradoxes. Given the number of passages in the *PIM* where Cohen praises views that are consistent with the priority of the infinite and continuous, it is hardly surprising that Russell took those views to be Cohen's last words on infinitesimals.

§7. A third concept of the infinite and infinitesimal.

However, despite the undeniable plausibility of an interpretation of Cohen that sees the priority of the infinite and continuous as the unifying commitment underlying his views, that interpretation is not ultimately right. To see why, we need to consider Cohen's discussions of a third concept of the infinite.

The inexhaustible infinite. In two significant passages Cohen considers concepts of the infinite and infinitesimal according to which they are, to borrow a phrase that Varignon uses in his correspondence with Leibniz, “inexhaustible.” On this concept of the infinite, it expresses the idea that there is no largest finite number. And on the corresponding concept of the infinitesimal, it expresses the idea that there is no smallest finite number. Since there are no largest or smallest finite numbers, the finite numbers never “exhaust” the infinite or infinitesimal, or even number itself. Cohen accepts these concepts.

The first example follows a discussion of ancient theories of rational and irrational numbers, in which Cohen claims,

> [t]he ancients excluded the fraction from the concept of number. And yet already for them, the irrational numbers objectively disrupt finite discretion. . . . Thus the invitation to abandon the discrete lies in the concept of irrational numbers. The irrationals set the task of carrying out an unbroken series, which no rational fractions can represent. This series thus excludes the discrete. . . . (Cohen 1883: §43)

Cohen is concerned here with the ancients' discoveries of rational and irrational numbers, and ultimately of infinitesimal magnitudes and the continuum. His discussion suggests that he sees these discoveries as inevitable once mathematicians had first conceived of number itself (Cohen 1883: §§43-44). For, Cohen seems to think, even just the concept of the whole numbers implicitly contains the concept of irrational numbers, and the concept of a continuous, "unbroken" series of numbers.

His discussion continues in the next section of the book, where Cohen again takes up the idea that there was an inevitable progression from the concept of whole numbers to the discovery of the reals and the continuum. That progression was inevitable, he now argues,

> Since the numbers presuppose a condition: stop at no point! The unit may never be completed. Each unit is thus arbitrary, not only as a symbol of number, but also as a number magnitude; except only that unit that excludes all discreteness from its concept. . . (Cohen 1883: §44; *sic* in regards to Cohen's bizarre punctuation.)

For Cohen, even just the concept of the whole numbers implicitly presupposes the condition that the wholes "stop at no point!" The injunction to "stop at no point!" thus yields the concept of an infinite series, a series that cannot be exhausted by finite numbers alone. Further, if the injunction to "stop at no point!" yields an infinite series when applied to the addition of a successor to each number on the whole line, that is not its only application. When the injunction is applied to the introduction of cuts *within* units on the whole line, it yields the concept of a continuous series, or as Cohen puts it here, an "unbroken series" (Cohen 1883: §43). Thus on his account, once the ancients divided the wholes and discovered the rationals, the discovery of the reals was inevitable, despite the ancients' distrust of them: for the reals "set the task of carrying out an unbroken series, which no rational fractions can represent." (Cohen 1883: §43).

Here, in Cohen's hands, the concepts of the infinite and of continuous or infinitesimal magnitudes express the idea that the numbers "stop at no point!" -- that is, that there are no largest or smallest finite numbers. Cohen not only accepts this idea, in this passage he makes argumentative use of it.

Cohen endorses another version of this idea in his discussion of Leibniz's and Varignon's correspondence (Cohen 1883: §§54-55). As we have seen, Cohen rejects Leibniz's official characterization of infinitesimal magnitudes in that correspondence, on which they are less than a finite error term one's opponent specifies. In contrast, Cohen highlights a different kind of claim that Leibniz makes about the infinite. Cohen says this other claim of Leibniz's has a "thoroughgoing validity." It is Leibniz's claim that "the infinite, taken in a rigorous sense, must have its source in the unterminated, otherwise I see no way of finding an adequate ground for distinguishing it from the finite" (Cohen 1883: §55). That is, the infinite is the "unterminated," an idea Cohen thinks Varignon captures with the "evocative term" that he uses to characterize infinitesimals -- namely, "inexhaustible" (Cohen 1883: §55). But if the infinite is "unterminated" and "inexhaustible," it is the concept that there is no largest finite number. Likewise, if the infinitesimal is "inexhaustible," it is the concept that there is no smallest finite number.

Notice that the concept of the inexhaustible infinite characterizes the infinite and infinitesimal exclusively by appeal to finite numbers. On this concept of the infinite, we may define an infinite magnitude this way: a magnitude is infinite just in case no finite number is large enough to equal its size. Likewise, we may define an infinitesimal this way: a magnitude is infinitesimal just in case no finite number is small enough to equal its size. Crucially, the right-hand sides of these two definitions do not appeal to any infinities, infinitesimals, or continua, and instead appeal only to finite numbers.

Here is the important point for our purposes. The concept of the inexhaustible infinite is inconsistent with the priority of the infinite and continuous. The inexhaustible infinite characterizes the infinite and infinitesimal exclusively by appeal to finite numbers. But remember, the priority of the infinite and continuous requires that concepts of finite numbers be defined by appeal to infinities, infinitesimals, or continua, and not the other way around. Since the concept of the inexhaustible infinite makes no such appeal, it violates the priority of the infinite and continuous.

Consequently, if Cohen were committed to the priority of the infinite and continuous, he would reject the concept of the inexhaustible infinite. But when Cohen considers the concept of the inexhaustible infinite -- in his discussion of the ancients' discovery of the reals and the continuum and his discussion of Leibniz and Varignon -- he endorses it. It follows that Cohen cannot be committed to the priority of the infinite and continuous. Consequently, that principle cannot explain any of his views in the *PIM*.

The fact that the inexhaustible infinite rules out the priority of the infinite and continuous raises the threat that Cohen's views on the infinite and infinitesimal are incoherent. He accepts the inexhaustible infinite, and thus cannot accept the priority of the infinite and continuous. But he praises the Galilean infinite, which is entailed by the priority of the infinite and continuous.

For now, we can note that this is not yet a logical inconsistency, since Cohen need not hold the priority of the infinite and continuous to have a positive view of the Galilean infinite. But this logical point is unsatisfying, and we will have to return to this potential incoherence in §9.

In the meantime, we need to find another principle that unifies and explains Cohen's positions on not just the concept of the infinitesimal as arbitrarily small and the Galilean infinite, but on the inexhaustible infinite as well. What is more, if that other principle is going to explain Cohen's views better than the priority of the infinite and continuous, it must also explain his commitment to the principle of continuity and the requirement he satisfies with his definition of equality as infinitesimal inequality.

§8. The principle of sufficient reason.

The principle that does this work is the principle of sufficient reason -- that is, the principle that, for any fact, there must be a reason or explanation for that fact. This principle expresses the commitment that in science, we aim to explain what we assert, rather than accepting any parts of our theories as in-principle unexplainable. The principle is, in effect, an injunction against arbitrariness. Despite the fact that Cohen never explicitly invokes the principle of sufficient reason by name, there are both textual and systematic reasons for seeing it as the deepest commitment underlying all of his principal arguments in the *PIM*.

At various points throughout the *PIM*, Cohen suggests a connection between continuity and reason. He claims that ultimately discrete magnitudes are, or risk being, in some sense *arbitrary*. For example, he claims repeatedly that extensive magnitudes, which may be discrete, are arbitrary (Cohen 1883: §§68, 78, 92). He claims conversely that continuous magnitudes, that is, infinitesimals, are not arbitrary, drawing an explicit contrast between the arbitrary and the continuous (Cohen 1883: §45). He claims, in a passage we have seen already, that the discrete

> must be defined free of all arbitrariness and intellectual unsteadiness. The first definition of the discrete is further only possible through consideration of the continuum, whose discreteness it should become. (Cohen 1883: §37)

And further:

> Each unit is thus arbitrary, not only as a symbol of number, but also as a number magnitude; except only that unit that excludes all discreteness from its concept. . . (Cohen 1883: §44).

So by Cohen's lights, discrete magnitudes are arbitrary, but continuous magnitudes are not. Indeed, continuous magnitudes are not arbitrary, because they are, he thinks, *rational*. In Cohen's discussion of Leibniz's correspondence with Varignon, he says approvingly of Leibniz that Leibniz "gives clear validity to the whole power of reason, which lies in continuity" (Cohen 1883: §55). As Cohen sees it, what "lies in continuity" is "the whole power of reason."

In addition to the textual evidence that Cohen sees a connection between continuity and reason, there is powerful systematic evidence that the principle of sufficient reason is the deepest commitment driving Cohen's arguments. The principle of sufficient reason ultimately explains Cohen's positions on all three concepts of the infinite. Likewise, it justifies Cohen's commitment to the principle of continuity, and explains how that latter principle must be understood, so as to cohere with his positions on the three concepts of the infinite. Finally, it explains Cohen's commitment to the requirement he satisfies with his definition of equality as infinitesimal inequality. Let's take these different points in turn.

Consider the first concept of the infinite: the concept of the infinitesimal as arbitrarily small. That was Leibniz's considered account of infinitesimals in his correspondence with

Varignon. Likewise, Archimedes' and D'Alembert's concepts of limits depended on the idea that the values of a function $f(x)$ reach the function's limit L when the difference between $f(x)$ and L is less than an arbitrarily small finite number. Perhaps ironically, there is a very Leibnizian-sounding argument against this concept of limits and infinitesimals. For any finite value we choose as the smallest value, any smaller than which will count as infinitesimal, it is possible to ask: why that value and not some other, smaller finite value? There is no mathematical reason to choose any finite value rather than any other. So whatever finite value we choose, our choice will violate the principle of sufficient reason. That principle thus rules out the concept of the infinitesimal as an arbitrarily small value. Thus Cohen's commitment to that principle entails his rejection of the concept that the infinitesimal is an arbitrarily small value.[6]

Now recall the second concept of the infinite that Cohen considers: the Galilean infinite, on which the infinite is conceptually prior to the finite. Cohen did not find this concept only in Galileo, but in the Pythagoreans as well, and in Kepler's and Roberval's applications of mechanical concepts to analytic geometry. Since on this concept the infinite is conceptually prior to the finite, it rules out concepts of the infinite, infinitesimal, and continuity whose definitions appeal ultimately to finite numbers. It thus rules out concepts of the infinite, infinitesimal, and

[6] Notice the tenor of Cohen's criticism of Leibniz here. Since Cohen rejects Leibniz's account of infinitesimals on the grounds that it violates the principle of sufficient reason, he is, from his point of view, rejecting Leibniz's concept for being insufficiently *Leibnizian*. Cohen's assessment of Leibniz on this point is at odds with the consensus in the contemporary Leibniz literature. Following Hidé Ishiguro (1972/1991), Samuel Levey (1998), most Leibniz scholars now take Leibniz's idea that an infinitesimal is less than an arbitrarily small finite number to be an analysis of Leibniz's concept of continuity, and they see no inconsistency between that view and Leibniz's principle of sufficient reason. I will not here try to adjudicate this disagreement between Cohen and contemporary Leibniz scholars.

continuity whose definitions appeal ultimately to *arbitrary* finite numbers. Consequently, concepts of the Galilean infinite will never violate the principle of sufficient reason`s injunction against arbitrariness, as the concept of the infinitesimal as arbitrarily small does. Concepts of the Galilean infinite thus satisfy the principle of sufficient reason, and Cohen praises them for that reason.

Finally, recall the third concept of the infinite: the inexhaustible infinite, according to which there are no largest or smallest finite numbers. This was the concept of the infinite that Cohen found in the ancients' discovery of the reals, in Leibniz's characterization of the infinite as "unterminated," and in Varignon's characterization of infinitesimals as "inexhaustible." This concept of the infinite does not just satisfy the principle of sufficient reason; it is entailed by it. For, consider any finite number we might take as a candidate for either the largest or smallest finite number. For any number we consider, there will be no mathematical reason to choose that number, as opposed to any other. Any number we consider will be arbitrary. Thus, on pain of violating the principle of sufficient reason, there can be no largest or smallest finite numbers. Here is the explanation for why, as Cohen puts it, "the numbers presuppose a condition: stop at no point!" Stopping at any finite number would violate the principle of sufficient reason. So, since the concept of the inexhaustible infinite is entailed by the principle of sufficient reason, Cohen accepts it.

The principle of sufficient reason also justifies his commitment to the principle of continuity. Consider, for example, a limit process. It aims to represent the continuous transition from a positive value to zero of the difference between the values of a function $f(x)$ and L. Now suppose that limit process violates the principle of continuity. If it does that, then the transition of the difference between $f(x)$ and L from a positive value to zero will not be continuous. That is, the limit process will leave a finite difference between $f(x)$ and L, before that difference jumps to zero. But for any positive finite difference left between $f(x)$ and L, there will be no reason for it to be that difference, as opposed to a smaller one. And that arbitrariness violates the principle of sufficient reason. Thus the limit process must satisfy the principle of continuity, on pain of violating the principle of sufficient reason.

There are two further points to notice about the fact that the principle of sufficient reason justifies the principle of continuity for Cohen. First, the principle of sufficient reason rules out a particular interpretation of continuity, namely, any interpretation of continuity according to which it is explicated by any appeal to arbitrary finite numbers. On this interpretation, in a limit process, we could say that $f(x)$ has reached L just in case the difference between them was smaller than an arbitrarily small finite value. But now, since for Cohen the principle of continuity is ultimately justified by the principle of sufficient reason, that interpretation of continuity is ruled out.

Second, since the principle of sufficient reason entails the principle of continuity for Cohen, the principle of continuity is, in just that sense, a demand of reason. It is entailed by the view that in mathematics, as in science more generally, we must have reasons for what we assert, and that whatever is arbitrary does not constitute a reason. That is ultimately *why* Cohen calls the principle of continuity "the special expression of the universal law of the unity of consciousness" and a "fundamental form of the unity of consciousness" (Cohen 1883: §40): it is ultimately a demand of reason.

Finally, the principle of sufficient reasons entails the requirement that Cohen satisfies with his peculiar definition of equality as infinitesimal inequality. Remember, Cohen accepts Kant's definition of extensive magnitudes as homogenous magnitudes, that is, as magnitudes that are all of equal size. These are the magnitudes that, for Cohen, constitute the units of space and time. The principle of sufficient reason imposes a constraint on the use of this concept of magnitude. In particular, it entails that mathematical natural science can represent extensive magnitudes only if it also represents infinitesimal magnitudes. To see why, suppose natural science could represent extensive magnitudes without representing infinitesimals. In that case, it would have to represent a smallest finite magnitude. But for any candidate finite magnitude, there can be no mathematical reason for why that magnitude is smallest, and not some other, smaller magnitude – and that would violate the principle of sufficient reason. Thus on pain of violating that principle, mathematical natural science can represent extensive magnitudes only if it also represents infinitesimals.

Cohen's Leibnizian definition of equality satisfies this demand. The definition of extensive magnitudes makes use of the concept of equality, since a magnitude is extensive just in case it belongs to a set of magnitudes that are all of equal size. But now, Cohen insists, the concept of equality must be defined by appeal to infinitesimal magnitudes. Thus, for Cohen, the concept of extensive magnitudes can be defined only by appeal to the concept of infinitesimal magnitudes. Consequently, his definition of equality as infinitesimal inequality satisfies the demand, imposed by the principle of sufficient reason, that mathematical natural science can represent extensive magnitudes only if it also represents infinitesimal magnitudes.

I have argued that the principle of sufficient reason is the deepest commitment underlying some of Cohen’s most characteristic doctrines in the *PIM*: it explains his positions on all three concepts of the infinite that he discusses in his history; it justifies his commitment to the principle of continuity; and it explains his commitment to the Leibnizian definition of equality as infinitesimal inequality.

The interpretation of Cohen I have defended is thus deeply rationalist. To be sure, Cohen's rationalism does not consist in a commitment to innate ideas. Nor does it consist in a foundationalist epistemological view on which knowledge must be derived from metaphysical first principles that are themselves somehow known with perfect certainty. Likewise, Cohen's rationalism does not consist in the view that philosophical doctrines must be demonstrated using anything like a geometrical method. Rather, Cohen's rationalism is defined by his commitment to the principle of sufficient reason considered as a fundamental *methodological* commitment.[7] His rationalism consists in the view that in mathematical natural science, as in all knowledge in general, we must have reasons for what we assert, and that the demand for reasons never reaches an end. On this view, number, and especially the infinitesimal, conceived on the model of the inexhaustible infinite, become exemplars of reason itself. Since explanation in mathematical natural science never comes to an end, explanation, no less than number itself, may "stop at no point!" On Cohen's rationalism, the demand for explanations in mathematical natural science is thus inexhaustible.

Of course, one of the historical sources of Cohen's rationalism is Leibniz, who takes the principle of sufficient reason to be an injunction requiring that we seek reasons for any claim we make in philosophy, natural science, or mathematics. At the same time, Cohen's commitment to the principle of sufficient reason as a methodological constraint on all inquiry is arguably even more similar to a view of Kant's. Since Cohen maintains that the demand for explanation never comes to an end, his view recalls Kant's conception of reason in its regulative use. For Kant,

[7] On this point, I am in complete agreement with Frederick Beiser's recent claim that Cohen is a *methodological rationalist* (Beiser 2019: 3-4, 112-113). See Beiser's discussion especially for his account of how Cohen's rationalism is anti-foundationalist.

reason in its regulative use always seeks unconditioned grounds for whatever is the case, but he also thinks that demand for unconditioned grounds can never be satisfied within the limits of possible experience. Consequently, for Kant that demand always gives direction to future inquiry. Likewise for Cohen, since the demand for reasons never comes to an end, it is a demand that always gives direction to future inquiry.

The rationalism of Cohen's views makes it possible to defend his views against one of Russell's most important objections to them. I turn to that defense now

§9. The PIM *and the paradoxes of the infinitesimal and continuum.*

Remember, Russell objects to Cohen's commitment to infinitesimals because he thinks that commitment will saddle mathematics with the paradoxes of the infinitesimal and continuum. Those paradoxes, in turn, seem to show "that the propositions of mathematics are not metaphysically true" (Russell 2010 [1903a]: 188), and thus threaten a kind of skepticism about mathematical knowledge.

However, contrary to Russell's interpretation of him, Cohen's views do not entail the paradoxes of the infinitesimal and continuum. I have argued in §8 that Cohen's deepest philosophical commitment, the commitment that underlies and unifies his most important doctrines in the *PIM*, is the principle of sufficient reason. But that principle does not entail the paradoxes of the infinitesimal and continuum. As we have seen, the principle of sufficient reason does entail the concept of the inexhaustible infinite. But that concept of the infinite defines the infinite and infinitesimal exclusively by appeal to finite numbers. Consequently, it avoids the paradoxes of the infinitesimal and continuum. Further, although Russell is concerned about Cohen's commitment to infinitesimals, we can now see that Cohen conceives of infinitesimals on the model of the inexhaustible infinite. He can thus define infinitesimal magnitude by appeal exclusively to finite numbers and thereby avoid the paradoxes of the infinitesimal and continuum. If this much is right, neither Cohen's view of infinitesimals nor the deepest philosophical commitments underlying that view runs afoul of those paradoxes.

However, this might seem too quick. After all, Cohen *does* repeatedly praise concepts of the infinite, infinitesimal, and continuity that entail those paradoxes. The Galilean infinite requires that finite number concepts be defined by appeal to concepts of the infinite, infinitesimal, and continuous, and not the other way around. It thus entails the paradoxes of the infinitesimal and continuum. These are surely the passages in the *PIM* that lead Russell to interpret Cohen as saddling mathematics with those paradoxes.

The problem here is not simply that Cohen praises concepts that entail the paradoxes of the infinitesimal and continuum -- although to be sure, that *is* a problem. The problem is also that Cohen endorses concepts that do *not* entail those paradoxes: namely, concepts that express the inexhaustible infinite. In §7, we noted the possibility that Cohen's views on the infinite and infinitesimal might be inconsistent. Here, finally, we seem to see exactly what the inconsistency might be: he both praises concepts that entail the paradoxes of the infinitesimal and continuum and endorses concepts that do not. If this is right, then far from defending Cohen against Russell's objections to him, we have uncovered an incoherence at the center of Cohen's views in the *PIM*.

The problem here is Cohen's praise of the Galilean infinite. It entails the paradoxes of the infinitesimal and continuum and is inconsistent with the inexhaustible infinite, which does not entail those paradoxes. If we want to save Cohen from this apparent incoherence, and do so in a way that avoids the paradoxes of the infinitesimal and continuum, it would be convenient if we could turn a blind eye to Cohen's praise of the Galilean infinite.

Considerations of charity might suggest we do exactly that. As we have just seen in the previous section, Cohen's deepest philosophical commitment, the commitment underlying many of the most characteristic doctrines of the *PIM*, is the principle of sufficient reason. That is the commitment that matters most for Cohen, and that many of his central doctrines depend on. But while the Galilean infinite is consistent with the principle of sufficient reason, it is not entailed by that principle. (In this respect, the Galilean infinite differs from the inexhaustible infinite.) Thus Cohen could, if pressed, abandon the Galilean infinite without adverse consequences for any of his underlying commitments or other important doctrines in the *PIM*. That would resolve the apparent incoherence in his views and avoid the paradoxes of the infinitesimal and continuum.

But this is too easy. The fact is that Cohen did *not* abandon the Galilean infinite, and on the contrary, he praises it in every passage he discusses it. To wave these passages away in the name of charity treats Cohen's text far too cavalierly. We cannot solve the problems Cohen's text presents by simply ignoring the inconvenient parts of it.

We need a way of resolving this apparent incoherence that is better grounded in the text of the *PIM*. But to get this resolution, we must confront one of the principal reasons that Cohen's writing is so hard to understand: namely, the way he uses history in the course of defending his own philosophical views.

As I indicated in §3, Cohen very often does not give systematic philosophical arguments for the views he means to defend. Instead, he very often explains his own views by working through various historical episodes in which different elements of his views were first articulated, often in reaction to various historical views that stand in for views that Cohen wants to reject. But often Cohen will describe these historical arguments without clearly explaining which parts of the views or arguments at issue are relevant to his own philosophical purposes. Further, it is very common in the *PIM* (and Cohen's other historical writings) for Cohen to praise (or criticize) very different, and in some cases, inconsistent, theories.

To be sure, one possible interpretive response to these passages is to think that wherever Cohen praises inconsistent historical views, he is lapsing into incoherence. But given how often Cohen does this, such an interpretive response requires us to view Cohen as having massively incoherent views, on a variety of topics central to the *PIM*, throughout the entirely of the book.

If we want to avoid this dire interpretation, we should regard the relevant passages differently. When Cohen praises (or criticizes) a view, he is not endorsing (or rejecting) *every aspect* of the view in its entirety. Rather, he intends to endorse (or reject) only certain aspects or features of the view, and does not intend to endorse (or reject) other aspects of it. Cohen's discussions of historical arguments or exchanges thus often raise a question for his readers: when he praises (or criticizes) a theory, what aspects of that theory does he intend to endorse (or reject)? Cohen often does not answer this question explicitly, leaving it up to his readers to tease out of his discussions the best answers we can.

One example of this comes from Cohen's discussions of Leibniz's and Newton's accounts of the continuity of motion. Cohen describes each figures' views in that figure's own terminology, placing each view in the context of its author's own mechanics. Cohen thus praises the way Leibniz makes the concept of the differential fundamental to his account of motion

(§§57a-58), while also giving a positive assessment of Newton's appeal to fluents in *his* mechanics (§64). Is Cohen contradicting himself here? No. But to see why he is not, we need to understand exactly what *aspects of* Leibniz's and Newton's views he intends to endorse. In these passages, Cohen does not explicitly tell us what those aspects are. However, his larger argumentative aims in the *PIM* suggest that all he means to endorse is a point of agreement between Leibniz and Newton: namely, that motion is continuous (and that the relevant sense of continuity is a product of thought, and is not something abstracted from sensible intuition).[8] There is thus no need to see any incoherence in Cohen on these points.

A second example comes from Cohen's discussion of Leibniz in relation to the idea of intensive reality, that is, the idea that intensive or infinitesimal magnitudes are necessary for mathematical natural science's representation of reality. Although this idea is essential to Cohen's overall aims in the *PIM*, he does not introduce or defend it with much careful systematic detail. Rather, he traces its historical origins and development, and he intends that history to reveal the idea's truth. However, read in a certain light, Cohen's account of Leibniz's views on intensive reality appear incoherent. Leibniz's conception of monads unifies the concepts of reality and intensive or infinitesimal magnitude. But in one passage Cohen praises Leibniz's conception (§54), while in other passages he criticizes the same view sharply (§§51, 54). He does not explicitly explain his difference in attitudes in either of these two passages. Is Cohen's assessment of Leibniz thus incoherent?

No. Leibniz's conception of monads combines (at least) two different ideas. One is the view that infinitesimals are necessary for the representation of reality. This is the view Cohen intends to endorse. The other is the view that the concept *substance* (as expressed in Leibniz's idea

[8] See especially §64 on this point.

of a monad) is an essential part of a philosophical account of reality. This is a view Cohen intends to reject. However, this more careful account of Cohen's assessment of Leibniz is not one that Cohen articulates explicitly. Rather, as his readers, we must tease that account out of Cohen's larger discussion of intensive reality. In particular, it is revealed by the different contexts in which Cohen praises and criticizes Leibniz's view. When he *praises* Leibniz's view, he is drawing a contrast between Leibniz's view and Descartes' view that physical reality is essentially extension. Against Descartes, Cohen agrees with Leibniz that infinitesimal magnitudes are necessary for the representation of reality (§54, cf. §29). When Cohen *criticizes* Leibniz's view, he is drawing a contrast between Leibniz's view and Kant's view that substance and reality are distinct categories. Against Leibniz, Cohen agrees with Kant that substance and reality are not the same concepts (§54). Once we attend to this context, we see that Cohen's views are coherent

Like these two examples, Cohen's discussions of the Galilean infinite are best understood as passages where his historical mode of arguing produces an appearance -- but *only* an appearance -- of incoherence. On analogy to the two examples above, we can dispel the appearance of incoherence by distinguishing between different ideas contained within the concept of the Galilean infinite.

There are two ideas in particular we must distinguish. As we saw in §6, the Galilean infinite includes the concept of the priority of the infinite and continuous, that is, the requirement that the infinite, infinitesimal, and continuous are conceptually prior to concepts of the finite. However, as we saw in §8, another feature of the Galilean infinite is that it is consistent with the principle of sufficient reason. These two aspects of the Galilean infinite come apart. When Cohen praises the Galilean infinite, the aspect of it we should interpret him as endorsing is its consistency with the principle of sufficient reason *and nothing more*. We should *not* interpret him as endorsing the priority of the infinite and continuous. Indeed, as we saw in §8, Cohen *cannot* be endorsing the priority of the infinite and continuous, precisely because it is ruled out by another concept that Cohen endorses -- namely, the inexhaustible infinite.

To be sure, this interpretation of Cohen depends on taking a charitable attitude towards his text, insofar as it seeks to avoid attributing incoherent views to him. From this perspective, it is a virtue of this interpretation that it makes his views on the infinite and infinitesimal internally consistent. Likewise, it is a virtue of this interpretation that it allows Cohen to avoid the paradoxes of the infinitesimal and continuum: the priority of the infinite and continuous is what entails those paradoxes, but this interpretation does not attribute that principle to him.

However, while this interpretation does depend on a degree of charity in reading Cohen's text, it does not invoke charity as justification for treating Cohen's text cavalierly, or for simply ignoring the inconvenient passages in the *PIM*. On the contrary, this is an interpretation of Cohen that comes into view only when we attend closely to his characteristic mode of arguing in the *PIM*, and in particular, to the way he uses the history of mathematics, physics, and philosophy in the exposition and defense of his own philosophical views.

So contrary to Russell's suggestion, Cohen's deepest views in the *PIM* do not saddle mathematics with the paradoxes of the infinitesimal and continuum. But then, what are the real sources of disagreement between them? To be sure, Cohen's views of the foundations of calculus are still very different than the ones provided by the modern concept of limits that Russell endorses. Russell would certainly argue that the modern concept of limits has decisive advantages over a view like Cohen's, even if Cohen's view avoids the paradoxes of the infinitesimal and continuum. Not least of those advantages is that the modern concept of limits, because it consists of a universal quantifier ranging over a conditional statement, is not committed to the existence of infinitesimals at all. Consequently, it avoids the paradoxes of the infinitesimal and continuum not by appeal to any potentially controversial philosophical interpretation of the infinitesimal concept (the way Cohen's does), but simply by dispensing with the need to posit infinitesimals at all. From Russell's perspective, that is a decisive advantage.

§10. Cohen, Russell, and Two Paradigms of Reason

However, I will conclude by considering a different point of contrast between Cohen and Russell, one that is brought into view by the interpretation of Cohen I have defended. There is a deep point of disagreement between Cohen and Russell over their attitudes towards the principle of continuity and the principle of sufficient reason. As a first approximation, the disagreement is that Cohen accepts those principles while Russell rejects them. Russell rejects the principle of continuity explicitly (Russell 1903b: 196). While his views on the principle of sufficient reason in the period immediately prior to his writing the *Principles of Mathematics* are more muddled, he nevertheless pointedly refuses to endorse it. Russell claims (without argument) that the principle of sufficient reason itself is too vague and uninformative to be evaluated, and thus refuses to endorse it on its own (Russell 1958 [1900]: 30-32).

But the disagreement between Cohen and Russell is not just about the truth or falsity of two principles. Rather, the disagreement is about the status of those principles in philosophy and science, and their status as paradigms of reason itself.

For Cohen, the principle of continuity and, ultimately, the principle of sufficient reason are methodological commitments contained in mathematical natural science and revealed in that science's evolving history. They are the methodological commitments that provide the rational foundations for mathematical natural science, including the rational foundation for the concepts of limits and infinitesimals. Consequently, Cohen takes those principles to be foundational principles of reason. The principle of continuity is thus “a fundamental form of the unity of consciousness” (Cohen 1883: §40), and is a demand of “the whole power of reason” itself (Cohen 1883: §55). The principle of sufficient reason expresses the demand that everything ought to be explained rationally. This is a conception of reason rooted in the seventeenth and eighteenth centuries, in Leibniz’s, and before him Spinoza’s, philosophical systems, as well as in Kant’s conception of reason in its regulative use. Cohen’s *PIM*, a whig history of mathematics that culminates in the seventeenth century with Leibniz, reflects a commitment to just that early modern conception of reason.

This conception of reason ultimately explains why Cohen rejects the view that Russell would later defend, namely, that formal logic alone can secure the rational foundation of mathematics, including the concepts of limits and infinitesimals (Cohen 1883: 2). On Cohen’s view, it is methodological principles such as the principle of continuity and the principle of sufficient reason, and not formal logic, that constitute the paradigms of reason.[9] Thus those principles, and not formal logic, provide mathematics with its rational foundations.[10]

[9]At least to the extent that Cohen denies that formal logic can, on its own, secure the foundations of mathematics, his view anticipates the one developed by Cassirer. See (Heis 2010).

[10] There is an important question here about how Cohen conceives of logic, and how his conception of it informs his rejection of the view that *formal* logic provides the foundations of mathematics. It is impossible to state Cohen’s views on this simply, because his views changed multiple times over the course of his life, and during the 1880s, when he wrote the *PIM*, his views were in flux. In his first, 1871 edition of *Kant’s Theory of Experience*, Cohen follows Kant in maintaining that transcendental (rather than formal) logic is necessary to provide mathematics with its foundations, and that transcendental logic does not abstract away from space and time as the forms of sensible intuition (Cohen 1871: 81ff). In Cohen’s 1902 *Logic of Pure Knowledge*, he still maintains that transcendental logic

But what is more, on Cohen's view it is not just that formal logic cannot on its own *secure* the rational foundations of mathematical natural science. Indeed, on its own logic cannot even *identify* what does secure those rational foundations. For Cohen it falls to the critique of knowledge to identify the principles that, as necessary methodological commitments of mathematical natural science, provide that science with its rational foundations. It likewise falls to the critique of knowledge to exhibit those principles in their systematic connections to mathematical and scientific concepts such as the concepts of limits and infinitesimals.

Russell affirms just the view that Cohen denies. On Russell's view, the rational foundations of mathematics consist precisely in the fact that mathematics can be expressed in formal logical terms. Thus the very fact that the modern concept of limits can be expressed in formal logical terms is sufficient to provide its rational foundations.

Russell's view of how mathematical concepts are provided their rational foundation is ultimately explained by the fact that he has a very different conception of reason than Cohen has. For Russell, what determines the limits of rational thought is not any number of rationalist philosophical principles, nor any other philosophical principles expressed in natural language. For him, the limits of rational thought are determined by the formal logic first invented by Gottlob Frege and that Russell himself would, assisting A.N. Whitehead, continue to develop in

is required to identify the foundations of knowledge, but that now means something entirely different for him, since he now rejects Kant's doctrine that space and time are forms of sensible intuition (Cohen 1902: 5ff). In the *PIM*, in 1883, his views on these points are in transition (and possibly not coherent). What is clear is that he insists that formal logic cannot be the ground of mathematics, and that only a transcendental inquiry (namely, the critique of knowledge) can identify those grounds. See Beiser (2019: chs 4, 6, 8, and 11) and especially Edel (1988: chs. 2.2, 5.1, and 6) for much more detailed accounts of the evolution of Cohen's views on these issues.

their 1910-13 *Principia Mathematica*. For Russell, these logicians' achievement constitutes a decisive advance in the resources logicians, scientists, and philosophers have at their disposal for *reasoning*, not just about mathematics but about anything whatsoever. In that sense, for Russell, the truths of formal logic define the truths of reason itself. It thus follows for him that expressing mathematical concepts in terms of formal logic is all that is required to provide their rational foundation. This conception of reason would become much more important in the twentieth century, not just in Russell's writing but in the writing of philosophers who were, in different ways, inspired by him.

If this much is right, then the advantage of the rationalist interpretation of Cohen I have defended is clear. It shows how Russell misses the mark when he suggests that Cohen's views run afoul of the paradoxes of the infinitesimal and continuum. But more importantly, it suggests at least some of what is at stake in the disagreement between them. Russell's criticism of Cohen, and the responses that were open to Cohen to make, point to the differences in how each conceives of reason itself. This episode thus begins to show how, at the turn of the twentieth century, a centuries-old conception of reason was giving way to a newer conception, one that would, in Russell's wake, come to dominate the early analytic vision of philosophy.

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